Generalizations of the Ultrapower Axiom

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Abstract

1 The Ultrapower Axiom and its limitations

If M and N are transitive models of ZFC, an elementary $j : M \to N$ is called an *ultrapower embedding* if there is a set $X \in M$ and a point $a \in j(X)$ such that every element of N is definable in N from parameters in $j[M] \cup \{a\}$; j is an *internal ultrapower embedding* if in addition j is definable (from parameters) over M.

The Ultrapower Axiom (UA) asserts that if $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are ultrapower embeddings, then there is an inner model N that admits internal ultrapower embeddings $i_0 : M_0 \to N$ and $i_1 : M_1 \to N$ such that $i_0 \circ j_0 = i_1 \circ j_1$.

The Ultrapower Axiom is a convenient setting for developing the theory of large cardinals, settling many questions that are independent of ZFC. For example, Magidor showed that it is consistent with ZFC that the least strongly compact cardinal is the least measurable cardinal and it is consistent with ZFC that the least strongly compact cardinal is the least supercompact cardinal. Assuming UA, one can prove it is the latter possibility that holds.

But UA is for the most part only useful for the analysis large cardinal hypotheses that can be formulated in terms of ultrafilters; e.g., measurability, strong compactness, and supercompactness. It seems powerless in the face of large cardinal axioms formulated in terms of extenders; e.g., tall cardinals, strong cardinals, superstrong cardinals, Woodin cardinals.

Proposition 1.1. *UA does not decide whether the first tall cardinal is strong.* □

Moreover, many questions that initially appear to pertain solely to ultrafilters are intractable under UA since they seem to require the analysis of derived extenders. For example, UA implies that the Mitchell order wellorders the class of normal ultrafilters. But the Mitchell order on arbitrary countably complete ultrafilters is mysterious because of the derived extender problem.

In fact, UA suffices to show that the Mitchell order is linear on Dodd sound ultrafilters. A countably complete uniform ultrafilter U on κ is *Dodd sound* if the function $E : P(\kappa) \to M_U$ defined by $E(A) = j_U(A) \cap [\mathrm{id}]_U$ belongs to M_U. The function E can be identified with a finite set of (relativized) extenders derived from U as in the definition of a Dodd sound mouse.

In the known inner models, Schlutzenberg has established that every ultrafilter can be decomposed as a finite linear iteration of Dodd sound ultrafilters, which combined with the linearity of the Mitchell order on Dodd sound ultrafilters yields a sort of classification of countably complete ultrafilters. It seems impossible to carry out such a classification under UA.

A related question comes from the analysis of elementary embeddings. Kunen proved that if $V = L[U]$, then every elementary embedding of the universe is given by iterating the unique normal ultrafilter. Under UA, one can show the same holds in V_κ where κ is the second measurable cardinal. Moreover, this theorem generalizes. A cardinal κ is *μ-measurable* if there is an elementary embedding $j : V \to M$ with critical point κ such that the ultrafilter U on κ derived from j using κ belongs to M. If κ is the least μ-measurable cardinal, then V_κ satisfies that every elementary embedding is an iteration of normal ultrafilters.

This can be further generalized via the following recursive definition. The *Radin sequence* derived from an elementary embedding $j : V \to M$ with critical point κ is the longest sequence $\langle U_\beta \rangle_{\beta<\alpha}$ such that $U_0 = \kappa$, and for all positive $\beta < \alpha$, U_β is the ultrafilter on V_κ derived from j using $\langle U_\xi \rangle_{\xi<\beta}$. An ultrafilter U is *α-normal* if $U = U_\alpha$ where $\langle U_\beta \rangle_{\beta\leq\alpha}$ is the Radin sequence of j_U. A cardinal κ is *α-Radin* if there is an inner model M and an elementary $j : V \to M$ with critical point κ such that the Radin-sequence of j has length α.

Under the Ultrapower Axiom, if κ is the least $(n+1)$-Radin cardinal, then in V_κ, every elementary embedding of the universe is given by an iteration of n-normal ultrafilters. Does this extend to the transfinite? What if κ is the least ω-Radin cardinal?

2 The Extender Power Axiom and its failure

If M and N are transitive models of ZFC, an elementary $j : M \to N$ is an *extender embedding* if there is a set $X \in M$ such that every element of N is definable in N from parameters in $j[M] \cup j(X)$; j is an *internal extender embedding* if in addition j is definable over M.

Note that $j : M \to N$ is an extender embedding if and only if there is an M-extender E such that $N = \text{Ult}(M, E)$, and j is an internal extender embedding if and only if such an E belongs to M.

The Extender Power Axiom (EPA) asserts that if $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are extender embeddings, then there is an inner model N admitting internal extender embeddings $i_0 : M_0 \to N$ and $i_1 : M_1 \to N$ such that $i_0 \circ j_0 = i_1 \circ j_1$.

The main theorem of this section is due to the author and Woodin:

Theorem 2.1 (Goldberg-Woodin)**.** *Suppose κ is 2^κ-supercompact. Then the Extender Power Axiom is false.*

The proof of this theorem used Woodin's counterexample to UBH from [**?**].

Here we will prove the theorem from a slightly weaker hypothesis. The proof is due to the author and uses a completely different idea involving stationary tower forcing.

Lemma 2.2. *If $j_0, j_1 : M \to N$ are elementary embeddings that are definable over a generic extension of M, then $j_0 \restriction \text{Ord} = j_1 \restriction \text{Ord}$.*

Theorem 2.3. *If there is a cardinal κ such that every κ-complete filter on κ extends to a κ-complete ultrafilter, then the Extender Power Axiom is false.*

Proof. Let $\mathbb{P}_{<\kappa}$ be the full stationary tower forcing at κ. Then the Boolean completion $\mathbb{B}$ of $\mathbb{P}_{<\kappa}$ has cardinality at most 2^κ. Moreover the proof of [?, ??] shows that if $G \subseteq \mathbb{P}_{<\kappa}$ is V-generic, then $V[G]_\kappa \subseteq \text{Ult}(V, G)$. In particular, if $\text{crit}(j_G) > \beth_\alpha$, then $V_{\omega+\alpha} = V_{\omega+\alpha}^{\text{Ult}(V,G)} = V[G]_{\omega+\alpha}$. It follows that $\mathbb{P}_{<\kappa}$ is κ-nice in the sense of Hamkins [?]. Combining the proof of [?, ??] with the result [?, ??], one obtains that there is a κ-complete ultrafilter U on $\mathbb{B}$.

By the basic theory of Boolean ultrapowers, there is (in V) an M_U-generic filter H on $j_U(\mathbb{B})$. Letting $G = H \cap j_U(\mathbb{P}_{<\kappa})$, one can form in $M_U[G]$ the ultrapower of M_U by G, which we will denote by $k : M_U \to P$.

Assume towards a contradiction that the Extender Power Axiom holds. Let $h : M_U \to N$ and $i : P \to N$ be internal extender embeddings witnessing this, so $h \circ j_U = i \circ k \circ j_U$. We claim that $i \circ k = h$. By Lemma 2.2, $i \circ k \restriction \text{Ord} = h \restriction \text{Ord}$. Since $h \circ j_U = i \circ k \circ j_U$, $i \circ k \restriction j_U[V] = h \restriction j_U[V]$. Since every element of M_U is definable from parameters in $j_U[V] \cup \text{Ord}$, it follows that $i \circ k = h$.

In particular, this means that k is close to M_U: that is, every ultrafilter derived from k belongs to M_U. But it is easy to see that no condition in the stationary tower forcing $\mathbb{P}_{<\kappa}$ can force that $j_{\dot{G}}$ is close to V. Using the elementarity of j_U, this contradicts that k is the ultrapower of M_U by the M_U-generic filter on H on $j_U(\mathbb{P}_{<\kappa})$. ☐

3 Woodin's theorem and Strong Comparison

Woodin has recently proved a very surprising theorem on the relationship between the Ultrapower Axiom and the descriptive inner model program.

Theorem 3.1 (Woodin)**.** *Assume* AD^+ *and* $V = L(P(\mathbb{R}))$*. Then the Ultrapower Axiom holds in* HOD*.*

Sketch. We sketch the proof under the further assumption of $\text{AD}_\mathbb{R}$ plus $\text{cf}(\Theta) = \omega$. If $\text{AD}_\mathbb{R}$ is not assumed, the proof is quite different.

We will establish that Woodin's principle Weak Comparison is true in HOD. So we must show that if $H_0, H_1 \in \text{HOD}$ are finitely generated models of ZFC that are Σ_2-elementarily embeddable in HOD, then either $H_0 \in H_1$, $H_1 \in H_0$, or there is a model N that admits close elementary embeddings $i_0 : H_0 \to N$ and $i_1 : H_1 \to N$.

The main idea is to show that given any countable model H of ZFC in HOD that is Σ_2-elementarily embeddable in HOD, it is possible to iterate H to a model H^* such that $R = \mathbb{R} \cap \text{HOD}$ is the set of reals of a symmetric collapse of H^* at the supremum λ of the Woodin cardinals of H^*, and, moreover, the sets of reals in the derived model of H^* at λ is a Wadge initial segment of the $<\Theta$-universally Baire sets of HOD.

This iteration can be constructed as a Neeman-style R-genericity iteration of H. Enumerate R as $\{x_n : n < \omega\}$. Let $H_0 = H$. Then there are models $H_n \in \text{HOD}$ and Neeman genericity iterations $\mathcal{T}_n \in \text{HOD}$ of H_n with x_n generic over its last model H_{n+1}; the critical points of the branch embeddings of the $\mathcal{T}_n$ increase to Θ^H and the branches of the $\mathcal{T}_n$ are chosen to be realizable.

Let H^* be the direct limit H^* of the linear system of H_n, and let $j : H \to H^*$ be the direct limit map. The proof of the derived model theorem shows that the sets of reals in the derived model D of H^* form a Wadge initial segment of the universally Baire sets of HOD.

Finally, there is an internal extender embedding $i : H^* \to \text{HOD}^D$ given by the converse to the derived model theorem (and the assumption that $\text{cf}(\Theta) = \omega$). The composition $i \circ j$ is a close embedding from H to HOD^D.

Now assume further that every element of H is definable in H from ordinal parameters less than γ for some $\gamma < \Theta^H$. Assume further that the genericity iteration of H is constructed so that its critical point is above γ. We claim that $D = H^D(S \cup R)$, where $S \subseteq D$ is a countable set in HOD. For this, we let $S = (i \circ j)[\gamma] \cup \{A_n\}_{n<\omega}$ where the sets $A_n \subseteq R$ are cofinal in the Wadge order of D and $\langle A_n \rangle_{n<\omega} \in \text{HOD}$. Such a sequence $\langle A_n \rangle_{n<\omega}$ exists in HOD since D satisfies $\text{cf}(\Theta^D) = \omega$ and HOD is a model of AC. Note that S is a countable set in HOD since $\gamma < \omega_1^{\text{HOD}}$, $D \in \text{HOD}$, and $(i \circ j)[\gamma] = i[\gamma] \in D$. (Here we use a key property of the embedding of V into the derived model of HOD: the embedding restricted to any ordinal less than Θ belongs to the target model.)

Note that every set in $P(R) \cap D$ belongs to $H^D(\{A_n\}_{n<\omega} \cup R)$, and hence $\Theta^D \subseteq H^D(S \cup R)$. Also $(i \circ j)[H] \subseteq H^D((i \circ j)[\gamma])$, and so

$$\text{HOD}^D \subseteq H^D(i \circ j[H] \cup \Theta^D) \subseteq H^D(S \cup R)$$

since all the generators of $i \circ j$ lie below Θ^D. Finally since D satisfies $V = L(P(\mathbb{R}))$, every element of D is definable in D from elements of $\text{Ord}^D \cup R \cup (P(R) \cap D)$, and so $D \subseteq H^D(S \cup R)$, as claimed.

Now returning to our finitely generated models H_0 and H_1, we have genericity iterations $j_0 : H_0 \to H_0^*$ and $j_1 : H_1 \to H_1^*$ and further internal extender embeddings $i_0 : H_0^* \to \text{HOD}^{D_0}$ and $i_1 : H_1^* \to \text{HOD}^{D_1}$. Since $D_n = H^{D_n}(S \cup R)$ for a countable set $S = \{a_k\}_{k<\omega}$, D_n is coded by the sequence of truth predicates

$$T^n_m = \text{Th}^{D_n}_{\Sigma_m}(R \cup \{a_k\}_{k<m})$$

Since $T^n_m \in D_n$ for all $m < \omega$, if $P(R) \cap D_0 \subsetneq P(R) \cap D_1$, then in fact $\langle T^0_m \rangle_{m<\omega} \in D_1$ and hence $D_0 \in D_1$.

Similarly, we obtain that either $D_0 \in D_1$, $D_1 \in D_0$, or $D_0 = D_1$. It follows that either $\text{HOD}^{D_0} \in \text{HOD}^{D_1}$, $\text{HOD}^{D_1} \in \text{HOD}^{D_0}$, or $\text{HOD}^{D_0} = \text{HOD}^{D_1}$, which proves Weak Comparison. □

As a corollary to this theorem, one has:

Theorem 3.2 (Woodin). *If V = Ultimate L, then the Ultrapower Axiom holds.* □

The proof motivates a new comparison principle analogous to Weak Comparison. Let us call this *Strong Comparison at* λ: if H_0 and H_1 are countable transitive models of ZFC admitting Σ_2-elementary embeddings $\pi_n : H_n \to V$ with $\lambda \in \text{ran}(\pi_n)$, then there are close embeddings $j_n : H_n \to H_n^*$ with arbitrarily large critical point below $\pi_n^{-1}(\lambda)$, and internal extender embeddings $i_0 : H_0^* \to N_0$ and $i_1 : H_1^* \to N_1$ such that either $N_0 \in N_1$, $N_1 \in N_0$, or $N_0 = N_1$.

The sketch of Woodin's theorem given above can be adapted to show that if $\text{AD}_{\mathbb{R}}$ holds, Θ is regular, and $V = L(P(\mathbb{R}))$, then Strong Comparison holds in HOD at Θ.

Say an extender E is *good* if M_E is closed under κ_E-sequences and in M_E, there are no uniform ultrafilters on cardinals in (κ_E, ν_E).

Theorem 3.3. *Assume V = Ultimate L. If E_0 and E_1 are good extenders with the same critical point, either E_0 and E_1 are comparable in the Mitchell order or one is an initial segment of the other.*

Sketch. We prove the theorem from a strong version of the axiom V = Ultimate L stating that there is a proper class of Woodin cardinals and if φ is true a true Σ_2-sentence, then there is a pointclass $\Gamma \subseteq \text{uB}$ such that $L(\Gamma, \mathbb{R})$ satisfies $\text{AD}_{\mathbb{R}}$ plus Θ is regular and $\text{HOD}^{L(\Gamma,\mathbb{R})} \vDash \varphi$.

Assume towards a contradiction that the theorem fails. Then by our assumption, there is a pointclass $\Gamma \subseteq \text{uB}$ such that $L(\Gamma, \mathbb{R})$ satisfies $\text{AD}_{\mathbb{R}}$ plus Θ is regular and the theorem fails in $\text{HOD}^{L(\Gamma,\mathbb{R})}$. Work in $L(\Gamma, \mathbb{R})$. Woodin's arguments establish that HOD satisfies Strong Comparison at Θ. Let γ be a Σ_2-reflecting cardinal of HOD such that $V_\gamma^{\text{HOD}} \vDash \text{ZFC}$, and let $S \preceq V_\gamma^{\text{HOD}}$ be the set of definable elements of V_γ^{HOD}.

Let H be the transitive collapse of S and let E_0 and E_1 be counterexamples to the theorem in H. Let $H_n = \text{Ult}(H, E_n)$, and note that H_n is elementarily embeddable into V_γ^{HOD}. Therefore we can apply strong comparison to obtain close embeddings $j_n : H_n \to H_n^*$ with $\text{crit}(j_n) > \lambda_{E_n}$ and internal extender embeddings $i_n : H_n^* \to P_n$ such that $P_0 \in P_1$, $P_1 \in P_0$, or $P_0 = P_1$. In fact, we must have $P_0 = P_1$ (since e.g., if $P_0 \in P_1$, one would have $H_0 \in H_1$). Denote this model by P.

Let $\kappa = \kappa_{E_0} = \kappa_{E_1}$. Note that if $i_0(\kappa) \leq i_1(\kappa)$, then for all $A \subseteq \kappa$, $i_0(A) = i_1(A) \cap i_0(\kappa)$:

$$i_0(A) = i_0(j_0(A) \cap \kappa) = i_1(j_1(A)) \cap i_0(\kappa) = i_1(A) \cap i_0(\kappa)$$

Suppose first that $i_0(\kappa) = i_1(\kappa)$. Then a symmetric argument shows that the short extender of i_0 is equal to the short extender of i_1; let us call this extender F. Assume without loss of generality that $\nu_{E_0} \leq \nu_{E_1}$. By the goodness of i_0, i_1, $i_0 \restriction \nu_{E_0} = j_F \restriction \nu_{E_0} = i_1 \restriction \nu_{E_0}$. Therefore for $a \in [\nu_{E_0}]^{<\omega}$,

$$(E_0)_a = D(j_{E_0}, a) = D(i_0 \circ j_0 \circ j_{E_0}, i_0(a)) = D(i_1 \circ j_1 \circ j_{E_1}, i_1(a)) = (E_1)_a$$

and so E_0 is an initial segment of E_1, contrary to assumption.

Suppose instead that $i_0(\kappa) < i_1(\kappa)$. Let F be the short extender of i_0. Then $F \in H_1^*$ and hence $i_0 \restriction \nu_{E_0} \in H_1^*$. Note that for $a \in [\nu_{E_0}]^{<\omega}$, $(E_0)_a = D(j_{E_0}, a) = D(i_1, i_0(a))$, and so $E_0 \in H_1^*$. But since $\text{crit}(j_1) > \lambda_{E_1}$, $E_0 \in H_1$, and hence $E_0 \lhd E_1$, contrary to assumption. $\square$

The same argument generalizes to supercompact extenders:

Theorem 3.4. *Assume $V =$ Ultimate L and E_0 and E_1 are extenders with critical point κ such that M_{E_0} and M_{E_1} are closed under κ^+-sequences and no cardinal in (κ^+, ν_n) carries a countably complete uniform ultrafilter in M_{E_n}. Then either E_0 and E_1 are comparable in the Mitchell order or one is an initial segment of the other.*

More comparison arguments of this sort yield information about Radin sequences:

Theorem 3.5. *Assume $V =$ Ultimate L and let κ be the least ω-Radin cardinal. Then in V_κ, every elementary embedding of the universe is given by an iteration of ultrafilters each of which is n-normal for some $n < \omega$.*

4 Questions

Assume $V =$ Ultimate L.

- Does GCH hold?
- Is every precipitous ideal atomic?
- Is every irreducible ultrafilter Dodd sound?

- Is every λ-strongly compact cardinal either λ-supercompact or a limit of λ-supercompact cardinals?

More vaguely:

- Is there a more elegant way to formulate strong comparison principles; is there a revised version of EPA that is true assuming $V =$ Ultimate L?
- Can we extend the Mitchell order result past good extenders?
- Is there an abstract analysis of iteration trees under $V =$ Ultimate L?
- Develop the theory of rank-to-rank cardinals assuming $V =$ Ultimate L.