The HOD conjecture and the structure of elementary embeddings.

The question: Is every set definable from an ordinal? Does \( V = \text{HOD} \)? Independent of ZFC + large cardinals.

HOD conjecture: More tractable version of this question "is \( V \) close to \( \text{HOD} \)?" - Related to inner model theory.

Point of talk:
HOD conjecture’s relationship to large cardinals / constraints on large cardinals.

Jensen’s covering lemma.

Exactly one of the following holds:

1. Every uncountable set of ordinals is contained in a constructible set of the same cardinality.

   \[ \Rightarrow \] If \( \lambda \) is a singular cardinal, \( \lambda \) is singular in \( L \), and \( \lambda^{+L} = \lambda^+ \).

2. There is an elementary \( j : L \rightarrow L \).

   \[ \Rightarrow \] Every uncountable cardinal is inaccessible in \( L \).

   \[ \Rightarrow \] Uncountable cardinals are
Beyond the constructible universe

Intuitively

1. means \( V \) is close to \( L \)
2. means \( V \) is very far from \( L \)

Large cardinal axioms imply 2.
All sufficiently strong theories imply 2.

Inner model program.
Construct analogs of \( L \) that approximate \( V \) better.

Analog of \( L \): canonical inner model of large cardinal axioms.

Current status: Woodin limit of Woodins, \( \psi \) is supercompact

The inner model problem

Are there canonical models for all large cardinal axioms?

- Any canonical model should be definably wellordered
  \( \Rightarrow \) contained in \( HOD \)
- Maybe \( HOD \) covers \( V \).

Test question. Is \( V \) close to \( HOD \)?
The HOD dichotomy

Suppose $\theta$ is extendible: for all $\lambda < \kappa$ there is $j: V_{\theta} \rightarrow V_{\lambda}$ s.t. $\kappa = \text{crit}(j)$, $j(\theta) > \lambda$.

Theorem (Woodin). Exactly one of the following holds:

1. Every set of ordinals of size $\geq \theta$ is contained in an OD set of the same size.
   \[ \xrightarrow{\Rightarrow} \text{Every singular cardinal } \lambda \geq \theta \text{ is singular in } \text{HOD} \text{ and } \lambda^{+}_{\text{HOD}} = \lambda^{+} \]

2. Every regular $\geq \theta$ is measurable in HOD.

The HOD conjecture.

HOD hypothesis: There is a proper class of regular cardinals that are not measurable in HOD.

No large cardinal can refute it, i.e. no large cardinal axiom implies 2 because all large cardinals are consistent by $V = \text{HOD}$.

HOD conjecture (Woodin) The HOD
The hypothesis is provable from an extendible.

**Choiceless Large Cardinals & HOD**

**Theorem (Kunen)** There is no elementary embedding from $V$ to $V^*$.

**Question** what if AC is dropped?

Under $ZF +$ large cardinals beyond choice, $V$ has to be far from $HOD$.

**Theorem (Woodin)** If there is a $j : V \rightarrow V$ and a proper class of extendibles, then the HOD conjecture is false.

**Sketch of Woodin’s theorem**

1. If $j : V \rightarrow V$ & $\lambda$ is a limit of extendibles, then $\lambda^+$ is measurable in HOD.

2. If there is a proper class of extendible cardinals, one can close force the Axiom of Choice, while preserving all the extendible cardinals.

The forcing is weakly homogeneous & definable so it shrinks HOD.

**Remark:** In the model, you have
j : HOD → HOD.

Embeddings of HOD.

Open Can there be an elementary embedding from HOD to HOD?

Fact. No such embedding is definable.
Sketch. If j : HOD → HOD is definable, then j ∈ HOD[α], and you can’t force j : V → V.

Theorem (HOD hypothesis) If there is a strongly compact cardinal, there is no j : HOD → HOD.

Embeddings of HOD, cont.

Could there be a full analog of Jensen’s covering lemma, above extendibles?

I.e. either HOD covers V or there is a j : HOD → HOD.
* not the right hypothesis
(4) Assume \( \kappa \) is regular and \( \text{HOD} \) is ill-founded. For \( \sigma \geq \kappa \), is there a \( j : \text{HOD}\cap V_{\kappa} \rightarrow \text{HOD}\cap V_{\kappa} \)?

**Embeddings of HOD, cont.**

**Def.** \( j : M \rightarrow N \) is \( \kappa \)-elementary if for all \( \sigma \in \mathcal{P}_\kappa(M) \), \( M[\sigma] = \emptyset \iff N[j[\sigma]] = \emptyset \).

\[
\begin{array}{ccc}
M[\sigma] & \longrightarrow & N[j[\sigma]] \\
\cup l & \downarrow & \cup l \\
M & \longrightarrow & N
\end{array}
\]

**Theorem.** If \( \kappa \) is supercompact, either:

1. HOD hypothesis
2. For all regular \( \sigma \geq \kappa \), there is a \( \kappa \)-elementary \( j : N \rightarrow \text{HOD} \cap V_{\kappa} \)

**Constructively Jonsson Cardinals**

**Def.** \( \lambda \) is Jonsson if for any \( f : [\lambda]^{<\omega} \rightarrow \lambda \) there is some \( A \subseteq \lambda \) with \( |A| = \lambda \) and \( f''[A]^{<\omega} \subseteq A \).

**Fact.** \( L \) has no Jonsson cardinals.

**Def.** \( \lambda \) is constructively Jonsson if for any constructible \( f : [\lambda]^{<\omega} \rightarrow \lambda \) there is some \( A \subseteq \lambda \) with \( |A| = \lambda \) and \( f''[A]^{<\omega} \subseteq A \).

**Fact.** There is \( j : L \rightarrow L \) if and only if there
is a constructibly Jonsson. If every cardinal is constructibly Jonsson.

**Definably Jonsson Cardinals**

**Def.** $\lambda$ is $\omega$-Jonsson if $f : [\kappa]^{\omega} \rightarrow \lambda$, there is some $A \subseteq \lambda$, $|A| = \lambda$, $f''[A]^{\omega} \subseteq A$.

**Theorem (Erdős-Hejhal)** There are no $\omega$-Jonsson cardinals.

**Def.** $\lambda$ is **definably $\omega$-Jonsson** if for all OD $f : [\kappa]^{\omega} \rightarrow \lambda$, there is some $A \subseteq \lambda$, $|A| = \lambda$, $f''[A]^{\omega} \subseteq A$.

**Theorem** Suppose $\kappa$ is strongly compact. The $\text{HD hypothesis}$ fails $\iff$ every regular $\aleph_\omega$ is definably $\omega$-Jonsson.

**Uniqueness of definable embeddings**

**Theorem (Woodin)** Suppose $j_0, j_1 : V \rightarrow M$ are definable. $j_0(\alpha) = j_1(\alpha)$ for all ordinals $\alpha$.

**Proof** Assume $j_0, j_1$ are $\Sigma_0$ counterexamples. Let $\alpha$ be the least ordinal s.t. there are $\Sigma_0$-definable $j_0, j_1 : V \rightarrow N$ s.t. $j_0(\alpha) \neq j_1(\alpha)$. $\alpha$ is definable without parameters. So $j_0(\alpha) = j_1(\alpha)$.
Uniqueness of undefinable embeddings

What about undefinable embeddings?

Proving this requires giving another proof of Kunen.

Woodin: if the HOD hypothesis holds, then if \( j, j_1 : V \rightarrow M \), \( j(\alpha) = j_1(\alpha) \) for all \( \alpha \).

Theorem. If \( j, j_1 : V \rightarrow M \), then \( j_0(\alpha) = j_1(\alpha) \) for all \( \alpha \in \text{Ord} \).

Local uniqueness.

There can exist \( j, j_1 : V_\lambda \rightarrow M \) s.t.

\( j_0 \upharpoonright \lambda \neq j_1 \upharpoonright \lambda \).

Def \( \xi \geq \delta \) is regular and \( \lambda > \delta \), \( j, j_1 : V_\lambda \rightarrow M \) are \( \delta \)-similar if \( j_0(\delta) = j_1(\delta) \) and

\( \sup \xi j_0(\alpha) : \alpha < \delta \bar{\xi} = \sup \xi j_1(\alpha) : \alpha < \delta \bar{\xi} \).

Theorem. If \( \kappa \) is extendible & \( \delta \geq \kappa \) is regular, the HOD hypothesis holds if & only for all
suff large \( \lambda \geq \delta \), if so, \( j_0, j_1 : V \rightarrow M \) are \( S \)-similar, then \( j_0(\alpha) = j_1(\alpha) \) for \( \alpha < \delta \).

Below extendibles.

\( N \) is \underline{w-club amenable} if the w-club filter \( F \) on any ordinal satisfies \( \text{FINEN} \).

**Theorem.** Suppose \( x \) is strongly compact and \( N \) is \( w \)-club amenable. Either

1. **All** \( N \)-regular \( S \subseteq x \) satisfy \( \text{cf}(S) = |S| \).

    \( \implies \) if \( \lambda \) is singular cardinal, \( x \) is singular in \( N \) and \( \lambda^{+N} = \lambda^{+} \)

2. All suff large \( \Delta \) regular cardinals are \underline{measurable} in \( N \).

**The covering question**

Suppose \( x \) is strongly compact and the \( \text{HOD} \) hypothesis holds. Does \( \text{HOD} \) cover \( V \)?

**Def.** \( M \) has the \( \lambda \)-cover property if whenever \( A \) is a set of ordinals of size \( < \lambda \), \( A \) is covered by a set of ordinals in \( M \) of size \( < \lambda \).

**Theorem.** (\( \text{HOD} \) hypothesis) \underline{For all strong...}
Large cardinals in HOD

If $\kappa$ is extendible & HOD hypothesis holds, then large cardinals $\lambda \leq \kappa$ are downwards absolute to HOD.

In fact, HOD is a weak extender model of $\kappa$ is supercompact. For all $\lambda \leq \kappa$, there is a normal $\kappa$-complete ultrafilter on $\text{P}_{\kappa}\lambda$, say $\mathcal{U}$, i.e.

$\forall \lambda \in \text{HOD} \cap \text{HOD} \quad \text{and} \quad \text{P}_{\kappa}\lambda \in \text{HOD} \subseteq \mathcal{U}$

Smaller large cardinals.

**Theorem** (Cheong-Friedman-Hemachandra). It's consistent that $\kappa$ is supercompact but not weakly compact in HOD.

**Def.** $\kappa$ is **distributively supercompact** if for all $\lambda \leq \kappa$, there is a forcing extension with no new $\kappa$-sequences in which $\kappa$ is $\lambda$-supercompact.
Thm. [HOD hypothesis]. \( \text{If } \kappa \text{ is supercompact, then } \kappa \text{ is traditionally supercompact in } \text{HOD.} \)

**On weak extender models**

**Def.** An inner model \( M \) is supercompact at \( \kappa \) if for all \( \mathcal{U} \), there is a normal fine \( \kappa \)-complete \( \mathcal{U} \)-ultrafilter \( P \) on \( \mathcal{P}(\kappa) \) s.t. \( P \mathcal{U} \cap M \in \mathcal{U} \).

\( \text{HOD hypothesis } \implies \text{HOD is supercompact} \)

**Theorem.** TFAE

- \( M \) is supercompact
- \( M \) extends to a u.w.m. with no new \( \kappa \)-sequences.

**Back to unique embeddings**

**Theorem.** Suppose \( \kappa \) is supercompact.

TFAE:

1. HOD hypothesis.
2. For all regular \( S \subset \kappa \), for suff \( \varepsilon \gg \kappa \), for \( X_0, X_1 \) second-order elementary in \( V_\varepsilon \).
Remark. If $R$ is a wellorder of $V_\alpha$
(2) holds for substructures of $(V_\alpha, R)$.

**Definability of ultrafilters**

**Ultrapower Axiom:**

**Theorem (UA)**

**Theorem (AD)**

**Theorem (TOD hypothesis)**

**Theorem (NBG)**

**Definability from ultrafilters**

**Def.** $CD(\alpha) =

$HCD(\alpha) =

**Theorem.** Suppose $\alpha$ is strongly compact.
On the first-order theory of "HOD"

Theorem (HOD hypothesis). Suppose it is supercompact.

Conclusions.

* HOD conjecture is equivalent to local version of true features of embeddings provable in ZFC

* Negation of HOD conjecture is
equivalent to weak versions of $j : \text{HOD} \rightarrow \text{HOD}$, traces of large cardinals beyond ZFC

Next time. Proof of:
1. Uniqueness of elementary embeddings
2. $\text{HOD} \cong j$, $j$ implies local uniqueness of embeddings