The HOD conjecture and the structure of elementary embeddings.

The question: is every set definable from an ordinal? Does $V = HOD$?
Independent of ZFC + large cardinals

HOD conjecture: More tractable version of this question "is V close to HOD"?
- Related to inner model theory

Point of talk:
HOD conjecture's relationship to large cardinals / constraints on large cardinals.

Jensen's covering lemma.

Exactly one of the following holds:

① Every uncountable set of ordinals is contained in a constructible set of the same cardinality
$\Rightarrow$ if λ is a singular cardinal, λ is singular in L, and $\lambda^{+L} = \lambda^+$

② There is an elementary $j : L \longrightarrow L$.
$\Rightarrow$ Every uncountable cardinal is inaccessible in L.
$\Rightarrow$ Uncountable cardinals are

order indiscernible in L

Beyond the constructible universe

Intuitively

1. means V is close to L
2. means V is very far from L

Large cardinal axioms imply 2

All sufficiently strong theories imply 2.

Inner model program. Construct "analogs of L" that approximate V "better."

Analogs of L: canonical inner model of large cardinal axioms.

Current status: $\boxed{\text{Woodin limit of Woodins}}$ < κ is κ^+ supercompact

The inner model problem

Are there canonical models for all large cardinal axioms?

- Any canonical model should be definably wellordered
  $\Rightarrow$ contained in HOD
- Maybe HOD covers V.

Test question. Is V close to HOD?

The HOD dichotomy

Suppose κ is extendible: for all $\lambda > \kappa$ there is $j : V_\lambda \longrightarrow V_{\lambda'}$ s.t. $\kappa = \mathrm{crit}(j)$, $j(\kappa) > \lambda$.

Theorem (Woodin). Exactly one of the following holds:

1. Every set of ordinals of size $\geq \kappa$ is contained in an OD set of the same size
   $\Longrightarrow$ Every singular cardinal $\lambda \geq \kappa$ is singular in HOD and $\lambda^{+\mathrm{HOD}} = \lambda^+$
2. Every regular $\delta \geq \kappa$ is measurable in HOD.

The HOD conjecture.

HOD hypothesis: There is a proper class of regular cardinals that are not measurable in HOD.

No large cardinal can refute it. i.e. no large cardinal axiom implies 2. Because all large cardinals are consistent w/ $V = \mathrm{HOD}$.

HOD conjecture (Woodin) The HOD

hypothesis is provable from an extendible.

Choiceless large cardinals & HOD

<u>Theorem</u> (Kunen) There is no elementary embedding from V to V.

<u>Question</u> What if AC is dropped?

Under ZF + large cardinals beyond choice, V has to be far from HOD.

<u>Theorem</u> (Woodin). If there is a $j: V \to V$ and a proper class of extendibles, then the HOD conjecture is false.

<u>Sketch of Woodin's theorem</u>

1. If $j: V \longrightarrow V$ & λ is a limit of extendibles, then λ^+ is measurable in HOD.
2. If there is a proper class of extendible cardinals, one can class force the Axiom of Choice, while preserving all the extendible cardinals.

The forcing is weakly homogeneous & definable so it shrinks HOD.

Remark: In the model you have

$j : \mathrm{HOD} \to \mathrm{HOD}$.

Embeddings of HOD.

Open Can there be an elementary embedding from HOD to HOD?

Fact. No such embedding is definable.

Sketch. If $j : \mathrm{HOD} \to \mathrm{HOD}$ is definable, then $j \subseteq \mathrm{HOD}[G]$, and you can't force $j : V \to V$.

Theorem (HOD hypothesis) If there is a strongly compact cardinal, there is no $j : \mathrm{HOD} \to \mathrm{HOD}$.

Embeddings of HOD, cont.

Could there be a full analog of Jensen's covering lemma above extendibles?

I.e. either HOD covers V or there is a $j : \mathrm{HOD} \to \mathrm{HOD}$.

- not the right hypotesis

Q. Assume κ is extendible & HOD hyp fails

Q. Assume κ is extendible + HOD hyp fails.
For $\delta \geq \kappa$, is there a $j: \mathrm{HOD} \cap V_\delta \to \mathrm{HOD} \cap V_\delta$?

Embeddings of HOD, cont.

Def. $j: M \to N$ is κ-elementary if for all $\sigma \in P_\kappa(M)$, $M[\sigma] \models \varphi$ iff $N[j(\sigma)] \models \varphi(j(\sigma))$.

$$\begin{array}{ccc} M[\sigma] & \longrightarrow & N[j(\sigma)] \\ \cup\!\!\mid & & \cup\!\!\mid \\ M & \xrightarrow[j]{} & N \end{array}$$

Theorem. If κ is supercompact, either:
1. HOD hypothesis
2. For all regular $\delta \geq \kappa$, there is κ-elementary $j: N \to \mathrm{HOD} \cap V_\delta$

N is an inner model of V_δ

Constructibly Jonsson cardinals

Def. λ is Jonsson if for any $f: [\lambda]^{<\omega} \to \lambda$ there is some $A \subsetneq \lambda$ w/ $|A| = \lambda$ and $f``[A]^{<\omega} \subseteq A$.

Fact. L has no Jonsson cardinals.

Def. λ is constructibly Jonsson if for any constructible $f: [\lambda]^{<\omega} \to \lambda$, there is some $A \subsetneq \lambda$ w/ $|A| = \lambda$ and $f``[A]^{<\omega} \subseteq A$.

Fact. There is $j: L \to L$ iff there is a constructibly Jonsson iff every cardinal is constructibly Jónsson.

Definably Jonsson cardinals

Def. λ is ω-Jonsson if $f : [\lambda]^\omega \to \lambda$, there is some $A \subsetneq \lambda$, $|A| = \lambda$, $f``[A]^\omega \subseteq A$.

Theorem (Erdös–Hajnal) There are no ω-Jonsson cardinals.

Def. λ is definably ω-Jonsson if for all OD $f : [\lambda]^\omega \to \lambda$, there is some $A \subsetneq \lambda$, $|A| = \lambda$, $f``[A]^\omega \subseteq A$.

Theorem Suppose κ is strongly compact. The HOD hypothesis fails $\iff$ every regular $\lambda \geq \kappa$ is definably ω-Jonsson.

Uniqueness of definable embeddings

Theorem (Woodin) Suppose $j_0, j_1 : V \to M$ are definable. $j_0(\alpha) = j_1(\alpha)$ for all ordinals α.

Proof. Assume j_0, j_1 are Σ_n counterexamples. Let α be the least ordinal s.t. there are Σ_n-definable $j_0, j_1 : V \to N$ s.t. $j_0(\alpha) \neq j_1(\alpha)$. α is definable without parameters so $j_0(\alpha) = j_1(\alpha)$ $\Rightarrow \Leftarrow$.

[illegible]

Uniqueness of undefinable embeddings

What about undefinable embeddings?

Proving this requires giving another proof of Kunen.

Woodin: if ~~the~~ strong HOD hypothesis holds, then if $j_0, j_1 : V \to M$, $j_0(\alpha) = j_1(\alpha)$ for all α.

Theorem. If $j_0, j_1 : V \to M$, then $j_0(\alpha) = j_1(\alpha)$ for all $\alpha \in \mathrm{Ord}$.

Local uniqueness.

There can exist $j_0, j_1 : V_\lambda \to M$ s.t. $j_0 \restriction \lambda \neq j_1 \restriction \lambda$.

Def If δ is regular and $\lambda > \delta$, $j_0, j_1 : V_\lambda \to M$ are δ-similar if $j_0(\delta) = j_1(\delta)$ and $\sup\{j_0(\alpha) : \alpha < \delta\} = \sup\{j_1(\alpha) : \alpha < \delta\}$.

Theorem. If κ is extendible & $\delta \geq \kappa$ is regular the HOD hypothesis holds iff for all suff large $\lambda \geq \delta$, if $j_0, j_1 : V_\lambda \to M$ are δ-similar, then $j_0(\alpha) = j_1(\alpha)$ for $\alpha < \delta$.

Below extendibles.

N is <u>ω-club amenable</u> if the ω-club filter F on any ordinal satisfies $F \cap N \in N$.

<u>Theorem</u> Suppose κ is strongly compact and N is ω-club amenable. Either

① All N-regular $\delta \geq \kappa$ satisfy $cf(\delta) = |\delta|$.

$\Rightarrow$ if λ is singular cardinal, λ is singular in N and $\lambda^{+N} = \lambda^+$

② All suff large regular cardinals are measurable in N.

<u>The covering question</u>

Suppose κ is strongly compact and the HOD hypothesis holds. Does HOD cover V?

<u>Def.</u> M has the <u>λ-cover property</u> if whenever A is a set of ordinals of size $< \lambda$, A is covered by a set of ordinals in M of size $< \lambda$.

<u>Theorem.</u> (HOD hypothesis) For all strong

limits $\lambda \geq$ least strongly compact, HOD has the λ-cover property

Large cardinals in HOD

If κ is extendible & HOD hypothesis holds, then large cardinals $\geq \kappa$ are downwards absolute to HOD.

In fact, HOD is a <u>weak extender model of κ is supercompact</u>: for all $\lambda \geq \kappa$, there is a normal fine κ-complete ultrafilter on $P_\kappa \lambda$, say $\mathcal{U}$, s.t. $\mathcal{U} \cap \mathrm{HOD} \in \mathrm{HOD}$ and $P_\kappa \lambda \cap \mathrm{HOD} \in \mathcal{U}$

Smaller large cardinals.

<u>Theorem</u> (Cheng-Friedman-Hamkins). It's consistent that κ is supercompact but not weakly compact in HOD.

<u>Def.</u> κ is <u>distributively supercompact</u> if for all $\lambda \geq \kappa$, there is a forcing extension with no new $<\kappa$-sequences in which κ is λ-supercompact.

Thm. (HOD hypothesis) If κ is supercompact then κ is distributively supercompact in HOD.

On weak extender models

Def. An inner model M is supercompact at κ if for all $\lambda \geq \kappa$, there is a normal fine κ-complete $\mathcal{U}$ on $P_\kappa \lambda$ s.t. $P_\kappa \lambda \cap M \in \mathcal{U}$.

(HOD hypotesis $\Longleftrightarrow$ HOD is supercompact)

Theorem. TFAE
- M is supercompact
- M extends to a w.e.m. with no new $<\kappa$-sequences.

Back to unique embeddings

Theorem. Suppose κ is supercompact. TFAE:

1. HOD hypotesis.
2. For all regular $\delta \geq \kappa$, for suff large λ, for X_0, X_1 second-order elementary in V_λ,

<u>Remark.</u> If R is a wellorder of V_λ
② holds for substructures of (V_λ, R).

<u>Definability of ultrafilters</u>

<u>Ultrapower Axiom</u>:

<u>Theorem</u> (UA)

<u>Theorem</u> (AD)

<u>Theorem</u> (HOD hypothesis)

<u>Theorem</u> (NBG)

<u>Definability from ultrafilters</u>

<u>Def.</u> $CD(\kappa) =$

$HCD(\kappa) =$

<u>Theorem.</u> Suppose κ is strongly compact.

On the first-order theory of HOD

Theorem (HOD hypothesis). Suppose κ is supercompact.

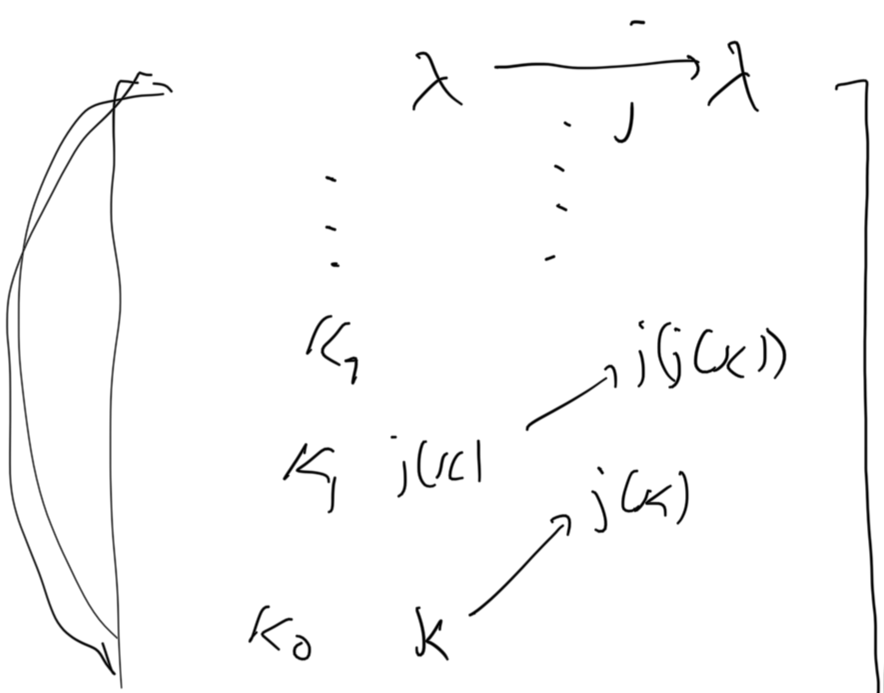
Conclusions.

- HOD conjecture is equivalent to local version of true features of embeddings provable in ZFC
- Negation of HOD conjecture is

equivalent to weak versions of $j: HOD \longrightarrow HOD$, traces of large cardinals beyond ZFC

<u>Next time.</u> Proof of

1. uniqueness of elementary embeddings
2. HOD conj $\iff$ local uniqueness of embeddings



$V_{\lambda+1}$ $\quad P(\lambda)$

λ-DC

if $j: L(V_{\lambda+1}) \to L(V_{\lambda+1})$ w $\mathrm{crit}(j) < \lambda$, then in an inner model, this is an elementary embedding $j: V_{\lambda+2} \longrightarrow V_{\lambda+2}$