The Ground Axiom, the Ultrapower Axiom, and Ultimate L

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Outline

Pluralism and nonpluralism in mathematics

Nonpluralism: there is a unique mathematical reality.

Pluralism:

- *Ontological pluralism*: there are multiple equally valid mathematical realities.
- *Truth-value pluralism*: certain mathematical propositions have no determinate truth-value, "can be" either true or false.

Goals:

1. Argue against a form of ontological pluralism, namely Steel's multiverse view.
2. Present an optimistic scenario for truth-value nonpluralism in set theory involving the axiom $V = \text{Ultimate } L$.

Nonpluralism for $\mathbb{N}$

- Most mathematicians are nonpluralists about the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.
  - Ontological nonpluralism: the natural numbers are uniquely determined up to isomorphism (by the second-order Peano axioms).
  - Truth-value nonpluralism: mathematicians can generally expect to settle (that is, prove or refute) any statement of interest about $\mathbb{N}$ on the basis of justified axioms (for example, the first-order Peano axioms).
    - There are no plausible candidates for statements about $\mathbb{N}$ that cannot be settled on the basis of justified axioms.
- There are many views about pluralism for higher-order mathematical objects like $P(\mathbb{N})$, the set of all subsets, or *powerset*, of $\mathbb{N}$. These views amount to *drawing the line* between the domains for which nonpluralism holds and those for which pluralism holds.

Views on pluralism: drawing the line

Extreme views:

- *Gödel:* ontological nonpluralism for the universe of sets V (e.g., for $\mathbb{N}$, $P(\mathbb{N})$, $P(P(\mathbb{N}))$, $P^3(\mathbb{N})$, $\ldots \bigcup_{n\in\mathbb{N}} P^n(\mathbb{N}), \ldots$).
  - Gödel concludes that all statements about V have determinate truth-values, whether or not we can settle them on the basis of well-justified axioms.
- *Hamkins:* ontological pluralism for $\mathbb{N}$: there are multiple equally valid "natural numbers structures."
  - Hamkins conjectures that number theoretic propositions with no determinate truth-value will be discovered.

Middle ground:

- *Feferman:* all statements about $\mathbb{N}$ have determinate truth-values, not all statements about $P(\mathbb{N})$ do.
- *Martin:* all statements about $\mathbb{N}$ and $P(\mathbb{N})$ have determinate truth-values, open question for $P(P(\mathbb{N}))$ and beyond.

Arguments for pluralism about V

Why are pluralist views about $P(\mathbb{N})$, $P(P(\mathbb{N}))$, and V so much more common than pluralist views about $\mathbb{N}$? After all, these structures are also uniquely determined by second order axioms.

- *Argument for truth-value pluralism:* most set theoretic propositions *can't* be settled on the basis of accepted axioms, namely ZFC.
  - The most famous example is Cantor's Continuum Hypothesis (CH), the oldest problem in set theory, proved independent of the ZFC axioms by Gödel and Cohen.
  - Since then, with a few notable exceptions, *every* major open problem in set theory has turned out to be independent of ZFC.
- *Argument for ontological pluralism:* the technique for proving independence results, called *forcing*, can be construed as providing a glimpse of alternate mathematical realities.

The forcing technique

Forcing is a technique for extending models of ZFC, introduced by Cohen to prove the unprovability of the Continuum Hypothesis.

- Suppose M is a countable model of ZFC and $\mathbb{P} \in M$ is a *forcing notion* (i.e., a partially ordered set).
- Build a *forcing extension* of M via $\mathbb{P}$, a model N of ZFC with $M \subseteq N$ whose properties are tightly controlled by the choice of $\mathbb{P}$.
  - For example, one might have $M \vDash 2^{\aleph_0} = \aleph_1$ while $N \vDash 2^{\aleph_0} = \aleph_2$.
- Since $\mathbb{P} \in M$, M has partial access to N; e.g., M often contains the theory of N with parameters from M.

If instead M is uncountable, one usually cannot build the model N outright. For example, the universe V has no nontrivial extensions whatsoever. Still, any M has partial access to what looks like the theories of its forcing extensions.

Pluralism from forcing

Arguments that the forcing technique substantiates an ontologically pluralist perspective on set theory [paraphrasing Hamkins]:

- *Mathematical practice:* It is often useful to reason as if forcing extensions of V actually exist, even though from the nonpluralist perspective they do not.
- *Multiple concepts of set:* Forcing seems to show that there are many concepts of set, embodied by various models of ZFC, none of which can lay claim to being the one true concept.
- *Indistinguishability:* If there is no good theoretical reason for preferring a model of set theory to one of its forcing extensions, we shouldn't begin with a foundational framework (e.g., ZFC) that purports off the bat to identify a preferred set theoretic universe.

Steel's multiverse view

The form of ontological pluralism I want to argue against is *Steel's multiverse view*.

- Steel grants ZFC plus large cardinal axioms (LCA).
    - More on large cardinal axioms later, but note that they are preserved by (set) forcing.
- Steel holds that forcing undermines the contention that there is a unique background concept of set using the indistinguishability argument.
    - ZFC + LCA exhaust the extent of truth-value nonpluralism; forcing motivates truth-value pluralism for questions like CH.
- Steel proposes an alternate foundational framework (a form of ontological pluralism) to provide an ontology to this form of truth-value pluralism.

Steel's multiverse framework

Multiverse language:

- Two sorts of objects: sets and worlds.
- One binary relation $\in$, for membership.

Multiverse axioms:

- If W is a world, then $(W, \in)$ is a model of ZFC + LCA.
- Every set belongs to some world.
- For any world W and any forcing notion $\mathbb{P} \in W$, there is a world that is a forcing extension of W via $\mathbb{P}$.
- If $(M, \in)$ is a model of ZFC and some world is a forcing extension of M, then M is a world.
- Any two worlds have a common forcing extension.

Steel's multiverse continued

- Steel's framework stays as close as possible to the nonpluralist view while granting that forcing creates an ambiguity about what we mean by the "true universe of sets."
  - On this view, only questions about the multiverse, not about individual worlds, are meaningful.
  - This prevents asking indeterminate questions like CH.
- Want to argue that Steel's multiverse view is undermined by recent results in set theory.

The Weak Absolutist Thesis

The Weak Absolutist Thesis expresses a weak form of nonpluralism:

Weak Absolutist Thesis

The multiverse has a definable world.

- Equivalent to the statement that the multiverse has a *minimum* world.
  - This minimum world is the *unique* definable world.
  - Every other world in the multiverse is a forcing extension of it.
  - Arguably, this world, if it exists, is the preferred point of the multiverse. Indeed, it is the *only* world that can be specified.
- If the thesis is false it's impossible to select the "one true universe."
- If the thesis is true: undermines the original reason for moving to a multiverse since the existence of a preferred, uniquely specifiable world eliminates the supposed ambiguity in what we mean by "the universe of sets."

Large cardinal axioms

Large cardinal axioms (LCA) are axioms that assert the existence of ever larger levels of infinity.

Modern large cardinal axioms achieve this by asserting the existence of elementary embeddings from the universe of sets V into a transitive class $M \subseteq V$. (M is *transitive* if all the elements of M are subsets of M.)

- *Axiom of Measurable Cardinals:* there is a nontrivial elementary embedding from V into a transitive class.
- *Axiom of Extendible Cardinals:* for all sufficiently large cardinals λ, there is a $j : V \to M$ such that $j(\lambda) > \lambda$ and $P(j(\lambda)) \subseteq M$.

Usuba's Theorem

Theorem (Usuba)

Assume there is a world in the multiverse that satisfies the Axiom of Extendible Cardinals. Then the multiverse has a minimum world.

- The Weak Absolutist Thesis is a *theorem* (of ZFC + LCA).
- Evidence for the nonpluralist view of the universe of sets. [Maybe just evidence against Steel's multiverse view.]
- Evidence for Hamkins-Reitz's *Ground Axiom* (GA):

Ground Axiom (GA)

V is the minimum world of the multiverse.

Completing ZFC

- The Ground Axiom does very little to ameliorate incompleteness in ZFC + LCA; for example, it does not settle the Continuum Hypothesis.
- The rest of the talk centers around a speculative argument for truth-value nonpluralism about V.

Question

Are there justifiable axioms that can be added to ZFC + LCA to yield a complete picture of V?

- The incompleteness theorems show that we cannot hope for a foundational theory that is literally complete – even for number theory.
- But are there *semi-complete axioms* for V, in the sense that the first-order Peano axioms are semi-complete for $\mathbb{N}$?

The Inner Model Program

An *inner model* is a transitive model M of ZFC that contains the ordinals.

Gödel's *constructible universe*, denoted L, is the smallest inner model. The axiom $V = L$ asserts that every set belongs to L. Assuming $V = L$, one can settle essentially all independent questions in set theory; i.e., $V = L$ is semi-complete. But $V = L$ is widely believed to be false:

Theorem (Scott)

Assume the Axiom of Measurable Cardinals. Then $V \neq L$.

The *inner model program* is a research program in set theory that seeks to construct generalizations of L that are compatible with large cardinals.

Ultimate L

- There is a mounting body of evidence that it is possible to construct a generalization of L that satisfies *all* true large cardinal axioms. This hypothesized inner model is called *Ultimate L*.
- If so, the corresponding generalization of the axiom $V = L$, namely Woodin's axiom $V =$ Ultimate L, cannot be refuted by large cardinal axioms.
- ZFC + LCA + $V =$ Ultimate L would then be a semi-complete theory.
- But is the axiom $V =$ Ultimate L true?

Template for justifying axioms

To justify the axiom A:

1. Prove from large cardinals that there is an inner model M satisfying the axiom A that is close to V in some sense.
2. Argue that $V = M$.

- Very broad [arguably too broad] framework encompasses the widely accepted argument for the Axiom of Foundation, the argument we have sketched for GA, and Woodin's speculative argument for $V =$ Ultimate L, to be discussed later.
- *Obvious Issue:* what if inner models of A and inner models of $\neg$A provably exist in close proximity to V?

Case: Ground Axiom

Let's start by seeing how this template can be used to formulate the argument for the Ground Axiom.

- Notion of closeness: say M is close to V if V is a forcing extension of M.
  - M is then very close to V; e.g., for all sufficiently large cardinals λ, $(\lambda^+)^M = \lambda^+$ and $(2^\lambda)^M = 2^\lambda$.

Theorem (Usuba)

Assume the Axiom of Extendible Cardinals. Then there is an inner model M satisfying GA *such that V is a forcing extension of M.*

- GA implies no inner model $M \neq V$ is close to V.
  - This rules out the Obvious Issue: no large cardinal axiom can prove that there is a model of $\neg$GA close to V, since GA implies no such model exists and GA is consistent with all large cardinals.

Case: $V = \text{Ultimate } L$

- Notion of closeness: say M is close to V if M is a *weak extender model.*
  - This means roughly that M inherits all large cardinals from V.
  - For all sufficiently large singular cardinals λ, $(\lambda^+)^M = \lambda^+$.

Conjecture (Ultimate L Conjecture)

Assume the Axiom of Extendible Cardinals. Then there is a weak extender model satisfying $V = \text{Ultimate } L$.

- Unlike GA case, LCA imply there is a weak extender model satisfying $\neg(V = \text{Ultimate } L)$.
  - Is there a stronger notion of closeness such that $V = \text{Ultimate } L$ is equivalent to "No inner model $M \subsetneq V$ is close to V" yet one can prove there is an inner model of $V = \text{Ultimate } L$ close to V?
- Failing this, $V = \text{Ultimate } L$ needs further justification.

Extrinsic case for $V =$ Ultimate L: recovery results

The case for $\mathsf{AD}^{L(\mathbb{R})}$ suggests a new template for justifying the axiom $V =$ Ultimate L. Recall that case:

1. *A posteriori evidence:* $\mathsf{AD}^{L(\mathbb{R})}$ has plausible structural consequences for sets of reals:
   - All sets of reals in $L(\mathbb{R})$ are Lebesgue measurable and have the Baire Property.
   - $\Game^{\mathbb{R}}\Pi^1_1$ has the Uniformization Property.
2. *Recovery result:* $\mathsf{AD}^{L(\mathbb{R})}$ follows from these consequences.

There is hope that there is a similar case for $V =$ Ultimate L:

1. Isolate plausible consequences of $V =$ Ultimate L.
   - For example, GA is a consequence of $V =$ Ultimate L, but there are also independent reasons to believe it.
2. Prove $V =$ Ultimate L from the conjunction of its plausible consequences.

Outline of the remainder of the talk

- Introduce a (conjectured) consequence of $V =$ Ultimate L called the Ultrapower Axiom (UA).
- Present an extrinsic case for UA.
- Present the case that there might be a recovery result for $V =$ Ultimate L from UA.

The Ultrapower Axiom

- UA is a combinatorial principle motivated by the methodology of inner model theory.
- UA isolates a simple feature of large cardinals that holds in all the known inner models.
- UA is therefore expected to hold in Ultimate L and to be a consequence of the axiom $V = \text{Ultimate } L$.

Large cardinals and ultrapowers

Large cardinal axioms are usually formulated in terms of elementary embeddings from the universe of sets into an inner model. The simplest such embeddings are called *ultrapower embeddings*.

Suppose M and N are transitive models of ZFC and $j : M \to N$ is an elementary embedding.

- j is an *ultrapower embedding* if for some $a \in N$, every element of N is definable in N from parameters in $\text{ran}(j) \cup \{a\}$.
- j is *internal* to M if $N \subseteq M$ and j is definable over M.

The definition of UA

Ultrapower Axiom (UA)

For any ultrapower embeddings $j_0 : V \to M_0$ and $j_1 : V \to M_1$, there are ultrapower embeddings $i_0 : M_0 \to N$ and $i_1 : M_1 \to N$ with $i_0 \circ j_0 = i_1 \circ j_1$.

$$\begin{tikzcd}
& N & \\
M_0 \arrow[ur, "i_0"] & & M_1 \arrow[ul, "i_1"'] \\
& V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}$$

Countably complete ultrafilters

- Suppose X is a set. A set $U \subseteq P(X)$ is a *(countably complete) ultrafilter* on X if there is an inner model M, an elementary embedding $j : V \to M$, and a point $a \in j(X)$ with

$$U = \{Y \subseteq X : a \in j(Y)\}$$

- There is an emerging extrinsic case for UA arguing that UA is a *regularity principle* for ultrafilters, with consequences analogous to the consequences of AD for subsets of the reals.

Determinacy and Ultrafilters

Suppose δ is an ordinal. A function $f : P(\delta) \to P(\delta)$ is:

- a *reduction* if $f(A) \cap \alpha$ depends only on $A \cap \alpha$
- a *contraction* if $f(A) \cap (\alpha + 1)$ depends only on $A \cap \alpha$

for any $A \subseteq \delta$ and $\alpha < \delta$.

- If $X, Y \subseteq P(\delta)$, f is a *reduction from X to Y* if for all $A \subseteq \delta$, $A \in X$ if and only if $f(A) \in Y$. Similarly for contractions.

Theorem (AD; Wadge)

If $X, Y \subseteq P(\omega)$, there is either a reduction from X to Y or a contraction from Y to $P(\omega) \setminus X$.

Theorem (UA)

If U, W are ultrafilters on δ, there is either a reduction from U to W or a contraction from W to $P(\delta) \setminus U$.

Analogy with AD, continued

For $X, Y \subseteq P(\delta)$, write $X \leq_L Y$ if there is a reduction from X to Y. A set $X \subseteq P(\delta)$ is *self-dual* if $X \leq_L P(\delta) \setminus X$.

Theorem (AD)

$\leq_L$ is a prewellorder of the self-dual subsets of $P(\omega)$.

Theorem (UA)

$\leq_L$ is a wellorder of the ultrafilters on δ.

- Are these consequences of UA equivalent to UA?
- Does AD imply UA?

UA above a strongly compact

An uncountable cardinal κ is *strongly compact* if the infinitary logic $L_{\kappa,\kappa}$ satisfies the Compactness Theorem.

- Assume UA and let κ be the least strongly compact cardinal.
- One can alter the universe below κ and preserve UA.
  - For example, for any $\lambda < \kappa$, there is a forcing extension satisfying UA such that $V \neq \text{HOD}_{V_\lambda}$ and GCH fails above λ.

Above κ, however, UA seems to determine everything about V:

Theorem (UA)

$V = \text{HOD}_X$ for some $X \subseteq \kappa$.

As a consequence, V is a forcing extension of HOD.

Theorem (UA)

For all $\lambda \geq \kappa$, $2^\lambda = \lambda^+$.

Recovery Conjecture

These results motivate the conjecture that UA uniquely determines the universe up to forcing below the least strongly compact. More precisely:

- The Ground Axiom is destroyed by any forcing below the least strongly compact (or more generally by any set forcing).
- The Ultrapower Axiom is probably destroyed by any alteration of the universe above the least strongly compact cardinal.

Conjecture (Recovery Conjecture)

Assume the Ultrapower Axiom, the Ground Axiom, and the Axiom of Extendible Cardinals. Then $V = \text{Ultimate } L$.

Conclusion

1. Usuba's Theorem argues against Steel's multiverse perspective.
   - Rules out a form of ontological pluralism motivated by forcing.
   - Provides evidence for the Ground Axiom.
2. There is a scenario for a semi-complete axiom for set theory: $V =$ Ultimate L.
   - Ultimate L Conjecture implies this axiom is consistent with large cardinal axioms.
   - Recovery Conjecture would give an extrinsic case for $V =$ Ultimate L similar to the one for $\text{AD}^{L(\mathbb{R})}$.
   - These conjectures, if true, make an argument for truth-value nonpluralism about V.

Thank you