The consistency of large cardinal axioms
Large cardinals beyond choice
Periodicity phenomena
Ultrafilters
On the consistency of choiceless cardinals

Even ordinals and the Kunen inconsistency

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Outline

1. The consistency of large cardinal axioms

2. The Kunen inconsistency

3. Large cardinals beyond the axiom of choice

4. Choiceless rank-into-rank axioms
   - ▶ Periodicity phenomena
   - ▶ Analogies with determinacy
   - ▶ Ultrafilter structure
   - ▶ Speculation on consistency

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Large cardinals

▶ Large cardinal axioms are set theoretic principles asserting the existence of larger and larger levels of Cantor's sequence of transfinite cardinals, cardinals so large that neither their existence nor their consistency can be proven from the standard axioms of set theory (ZFC).

▶ Consistency of a set theoretic principle is typically established by using some large cardinal axiom to build a model of the principle. Conversely, the strength of a principle can be measured in terms of the large cardinal axioms required to prove its consistency.

▶ The consistency of an arbitrary theory can be reduced in this way to the consistency of large cardinal axioms.

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The consistency of large cardinal axioms

- ▶ Are large cardinal axioms consistent?
    - ▶ One cannot hope to establish this mathematically (Gödel's incompleteness theorem).
    - ▶ There is evidence, however, that many of the large cardinal axioms currently studied *are* consistent.
- ▶ Structure theory:
    - ▶ Determinacy and large cardinals
    - ▶ Forcing and large cardinals
    - ▶ Inner models for large cardinals
- ▶ Truth (?)
    - ▶ Large cardinals really exist in some platonic sense.

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The Kunen inconsistency

- ▶ Troubling fact: large cardinals have been proposed that later turned out to be inconsistent for subtle reasons.
- ▶ Large cardinal axioms are formulated in terms of elementary (truth-preserving) embeddings from the universe of sets V into an inner model M.
    - ▶ Axioms increase in strength as one requires M to satisfy more and more closure properties.
    - ▶ Reinhardt proposed the ultimate such embedding axiom: the existence of an elementary embedding from V to V.

Theorem (Kunen)

There is no elementary embedding from V to V.

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Huge cardinals

Definition

Suppose $j : V \to M$ is an elementary embedding.

- ▶ $\mathrm{crit}(j)$ is the least cardinal κ such that $j(\kappa) \neq \kappa$.
- ▶ The *critical sequence of j* is defined by $\kappa_0(j) = \mathrm{crit}(j)$; for all $n < \omega$, $\kappa_{n+1}(j) = j(\kappa_n(j))$; and $\kappa_\omega(j) = \sup_{n<\omega} \kappa_n(j)$.

A cardinal κ is *n-huge* if there is an elementary embedding $j : V \to M$ such that M is closed under $\kappa_n(j)$-sequences.

Note: $\kappa_\omega(j)$ is the least $\lambda \geq \mathrm{crit}(j)$ such that $j(\lambda) = \lambda$.

Theorem (Kunen)

There are no ω-huge cardinals.

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Beyond the Axiom of Choice

▶ Kunen's theorem seems to show that the large cardinal hierarchy comes to an abrupt halt at the level of ω-huge cardinals.

▶ The proof relies heavily on the Axiom of Choice (AC).

▶ It is unknown whether Kunen's theorem can be proved without AC.

▶ In fact, there is a seemingly endless hierarchy of extremely strong principles beyond the Kunen inconsistency that do not seem to be inconsistent in the choiceless context.

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Some choiceless large cardinals

- V is stratified by the *cumulative hierarchy* $\langle V_\alpha \rangle_{\alpha \in \mathrm{Ord}}$, where $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$, and $V_\gamma = \bigcup_{\beta < \gamma} V_\beta$ for limit γ.
- Weakest choiceless axioms: for some α, there is an elementary embedding from $V_{\alpha+2}$ to $V_{\alpha+2}$.
- A cardinal λ is *definably Berkeley* if for any $\eta < \lambda \leq \alpha$, there is an embedding $j : V_\alpha \to V_\alpha$ such that $\eta < \mathrm{crit}(j) < \lambda$.
    - If $j : V \to V$, then $\kappa_\omega(j)$ is definably Berkeley.
- A cardinal λ is *Berkeley* if for any $\eta < \lambda \leq \alpha$ and $A \subseteq V_\alpha$, there is an embedding $j : (V_\alpha, A) \to (V_\alpha, A)$ such that $\eta < \mathrm{crit}(j) < \lambda$.

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No Berkeley cardinals under AC

- ▶ Suppose λ is the least Berkeley cardinal.
  - ▶ So for each $\alpha < \lambda$, there exist $\beta_\alpha \geq \lambda$ and $A_\alpha \subseteq V_{\beta_\alpha}$ such that there is no $j : (V_{\beta_\alpha}, A_\alpha) \to (V_{\beta_\alpha}, A_\alpha)$ with $\mathrm{crit}(j) \leq \alpha$.
- ▶ Let $\beta = \sup_{\alpha < \lambda} \beta_\alpha$ and fix an elementary embedding $j : (V_\beta, \langle \beta_\alpha, A_\alpha \rangle_{\alpha < \lambda}) \to (V_\beta, \langle \beta_\alpha, A_\alpha \rangle_{\alpha < \lambda})$ with $\mathrm{crit}(j) < \lambda$.
  - ▶ For $\alpha \geq \mathrm{crit}(j)$, $j(\alpha) \neq \alpha$: otherwise $j \restriction V_{\beta_\alpha}$ is an embedding $i : (V_{\beta_\alpha}, A_\alpha) \to (V_{\beta_\alpha}, A_\alpha)$ with $\mathrm{crit}(i) \leq \alpha$.
  - ▶ Thus $\lambda \leq \kappa_\omega(j)$.
- ▶ Now let $\beta_n = \beta_{\kappa_n(j)}$ and $A_n = A_{\kappa_n(j)}$ and fix $k : (V_\beta, \langle \beta_n, A_n \rangle_{n < \omega}) \to (V_\beta, \langle \beta_n, A_n \rangle_{n < \omega})$ with $\mathrm{crit}(k) < \lambda$.
  - ▶ k restricts to an embedding from (V_{β_n}, A_n) to (V_{β_n}, A_n), so $\mathrm{crit}(k) \geq \kappa_n(j)$ for all n, so $\mathrm{crit}(k) \geq \kappa_\omega(j) \geq \lambda$, contradiction.

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Consistency of choiceless cardinals, I

▶ Are the choiceless cardinals consistent?
  ▶ Not clear that there is a conception of the universe of sets for which the choiceless cardinals are *true*.
  ▶ Seems to be impossible to develop an inner model theory for choiceless cardinals.
    ▶ In fact, results of Woodin suggest that if inner model theory reaches supercompact cardinals, then at least some of the choiceless large cardinals are inconsistent.
  ▶ Until recently, little interesting structure theory.
▶ Rest of the talk focuses on results suggesting some of the choiceless cardinals are consistent.

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Periodicity in the cumulative hierarchy

▶ Assuming choiceless large cardinal axioms, the cumulative hierarchy exhibits a periodicity of order 2.

▶ In other words, the structure of V_ϵ for even ordinals ϵ (i.e., $\epsilon = \lambda + 2n$ for some limit ordinal λ) is quite different from that of the levels $V_{\epsilon+1}$.

    ▶ The most basic differences involve the structure of embeddings $j : V_\epsilon \to V_\epsilon$.

    ▶ More interestingly, there are purely structural dissimilarities making no reference to elementary embeddings.

▶ Periodicity is essentially unprecedented in set theory outside of determinacy theory.

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The definability of rank-to-rank embeddings

- ▶ Suzuki: no elementary embedding from V to V is definable (from parameters) over V.
- ▶ Suppose λ is a limit ordinal.
  - ▶ Schlutzenberg: no $j : V_\lambda \to V_\lambda$ is definable over V_λ.
  - ▶ Folklore: every elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ is definable over $V_{\lambda+1}$ from $i = j \restriction V_\lambda$: $j(A) = \bigcup_{\alpha < \lambda} i(A \cap V_\alpha)$.
  - ▶ Schlutzenberg asked: suppose $j : V_{\lambda+2} \to V_{\lambda+2}$. Can j be definable over $V_{\lambda+2}$? What about embeddings of $V_{\lambda+3}$?

Theorem (Goldberg, Schlutzenberg independently)

Suppose ϵ is an even ordinal.

- ▶ *No elementary embedding from V_ϵ to V_ϵ is definable over V_ϵ.*
- ▶ *Every elementary embedding from $V_{\epsilon+1}$ to $V_{\epsilon+1}$ is definable over $V_{\epsilon+1}$.*

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The theta sequence

Definition

For any ordinal α, θ_α denotes the supremum of all ordinals η such that for some $\xi < \alpha$, there is a surjection from V_ξ to η.

- For example, $\theta_\omega = \omega$, $\theta_{\omega+1} = \omega_1$.
- Assuming AC, $\theta_{\omega+2} = \mathfrak{c}^+$, and more generally $\theta_{\omega+\alpha} = \sup_{\xi < \alpha} \beth_\xi^+$.
- Assuming the Axiom of Determinacy, however, $\theta_{\omega+2}$, usually denoted by Θ, is a strong limit cardinal (for all $\beta < \theta_{\omega+2}$, there is no surjection from $P(\beta)$ to $\theta_{\omega+2}$).
- Still assuming AD, $\theta_{\omega+3} = \Theta^+$. The value of $\theta_{\omega+4}$ is independent of AD.

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The theta conjecture

Conjecture

Suppose ϵ is even and there is an elementary embedding $j : V_{\epsilon+2} \to V_{\epsilon+2}$. Then θ_ϵ is a strong limit cardinal and $\theta_{\epsilon+1} = \theta_\epsilon^+$.

- ▶ If ϵ is a limit ordinal, then θ_ϵ is a strong limit cardinal and $\theta_{\epsilon+1} = \theta_\epsilon^+$. The conjecture says that this generalizes to all sufficiently large ordinals. Also generalizes that Θ is a strong limit and $\theta_{\omega+3} = \Theta^+$ under AD.
    - ▶ ZFC large cardinals do not say nearly as much about the behavior of the continuum function at successor ordinals.
    - ▶ In some sense, the conjecture can be read to say that choiceless large cardinals asymptotically solve the continuum problem.
- ▶ Next few slides provide evidence for this conjecture beyond raw numerology.

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The even thetas are large

Theorem

Suppose ϵ is even and there is an elementary embedding $j : V_{\epsilon+2} \to V_{\epsilon+2}$. Then for all $\eta < (\theta_{\epsilon+1})^{+\kappa_\omega(j)}$, there is no surjection from $P(\eta)$ to $\theta_{\epsilon+2}$.

▶ For example, there is no surjection from the powerset of $(\theta_{\epsilon+1})^+$ to $\theta_{\epsilon+2}$, which is already a nontrivial result.

▶ The result derives from a strengthening of the undefinability result, one form of which is as follows:

Theorem

Suppose $j : V \to V$ and $\epsilon > \mathrm{crit}(j)$ is an even ordinal such that $j(\epsilon) = \epsilon$. Then $j \upharpoonright \theta_\epsilon$ is not ordinal definable from parameters in V_ϵ.

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The odd thetas are small

Theorem

Suppose ϵ is an even ordinal and there is an elementary embedding $j : V_{\epsilon+3} \to V_{\epsilon+3}$. Then there are fewer than $\mathrm{crit}(j)$ many regular cardinals between $\theta_{\epsilon+2}$ and $\theta_{\epsilon+3}$.

- ▶ One would like to conclude that $\theta_{\epsilon+3} < (\theta_{\epsilon+2})^{+\mathrm{crit}(j)}$. The problem is that one cannot prove that successor cardinals are regular in ZF.
- ▶ It seems likely that some successor cardinals *must* be singular under choiceless large cardinal assumptions.

Question

If λ is a definably Berkeley cardinal, is there a singular successor cardinal above λ? Can every cardinal above λ be singular?

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Weak choice assumptions

- ▶ If λ is a cardinal, λ-*DC* asserts that any partial order in which every chain of size less than λ has an upper bound contains a maximal element or a chain of ordertype λ.

- ▶ If X and Y are sets, (X, Y)-*Collection* asserts that every relation $R \subseteq X \times Y$ has a subrelation $S \subseteq R$ with the same domain such that S is the surjective image of X.

Theorem

Suppose λ is definably Berkeley, λ-DC holds, $\epsilon \geq \lambda$ is even, and $(V_{\epsilon+1}, V_{\epsilon+2})$-Collection holds.

- ▶ $\theta_{\epsilon+2}$ *is a strong limit cardinal.*

- ▶ *Moreover, if $\eta < \theta_{\epsilon+2}$, there is a surjection from $V_{\epsilon+1}$ to $P(\eta)$.*

- ▶ $\theta_{\epsilon+2}$ *is a limit of regular cardinals.*

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Measurable cardinals from Reinhardt cardinals

The structure of ultrafilters on ordinals in the context of choiceless large cardinal axioms turns out to be very complex and interesting, and in many ways analogous to the structure of ultrafilters under the Axiom of Determinacy.

Theorem (Solovay)

Assume the Axiom of Determinacy. Then ω_1 is a measurable cardinal. In fact, the ω-club filter on ω_1 is a countably complete ultrafilter.

Theorem (Woodin)

Suppose $j : V_{\lambda+2} \to V_{\lambda+2}$ and λ-DC holds. Then λ^+ is a measurable cardinal. In fact, the ω-club filter restricted to a stationary set is a λ^+-complete ultrafilter.

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Wadge's Lemma

Definition

Suppose δ is an ordinal.

▶ A function $h : P(\delta) \to P(\delta)$ is *Lipschitz* if for all $A \subseteq \delta$ and $\alpha \leq \delta$, $h(A) \cap \alpha = h(A \cap \alpha)$.

▶ If $X, Y \subseteq P(\delta)$, then $X \leq_L Y$ if there is a Lipschitz function $h : P(\delta) \to P(\delta)$ such that $h^{-1}[Y] = X$.

Theorem (Wadge)

Assume the Axiom of Determinacy. If X and Y are subsets of $P(\omega)$, either $X \leq_L Y$ or $Y \leq_L P(\omega) \setminus X$.

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The Ketonen order

The Ketonen order is the Lipschitz order "in the category of countably complete Boolean algebras":

Definition

Suppose δ is an ordinal and U and W are countably complete ultrafilters on δ. Then $U \leq_{\Bbbk} W$ if there is a Lipschitz countably complete homomorphism $h : P(\delta) \to P(\delta)$ such that $h^{-1}[W] = U$.

▶ In ZF + DC, one can prove that the Lipschitz order is a wellfounded partial order.

▶ In the context of ZFC, the *Ultrapower Axiom* (UA) asserts that Wadge's Theorem holds for the Ketonen order, or in other words, the Ketonen order wellorders the set of countably complete ultrafilters on any ordinal δ.

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The Ultrapower Axiom and choiceless cardinals

▶ Only known way of producing models of UA with large cardinals is inner model theory.

    ▶ It is open whether UA is consistent with a supercompact cardinal.

▶ Under choiceless large cardinal axioms, UA is almost provable:

Theorem

Suppose λ is a definably Berkeley cardinal such that λ-DC holds. Then for any ordinals $\delta \leq \alpha$, the set of ultrafilters on δ of rank α in the Ketonen order has cardinality less than λ. Moreover, any set of Ketonen incomparable ultrafilters on δ has cardinality at most λ.

Idea: if $j : V_\alpha \to V_\alpha$, $\delta < \alpha$, and $j(\delta) = \delta$, then $j \restriction P(\delta)$ is a countably complete Lipschitz homomorphism from $P(\delta)$ to $P(\delta)$.

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Strong compactness

Theorem

Suppose λ is a definably Berkeley cardinal such that λ-DC holds. Then every λ^+-complete filter on an ordinal extends to a λ^+-complete ultrafilter.

Idea: Let F be a minimal filter in the Ketonen order on δ for which the theorem fails. Show that F is compatible with the *fixed point filter* G on δ, which is the λ^+-complete filter generated by sets of the form $\{\alpha < \delta : j(\alpha) = \alpha\}$ for sufficiently elementary embeddings j. Using a generalization of Woodin's argument that the ω-club filter restricts to an ultrafilter, show that every extension of G extends to a λ^+-complete ultrafilter. In particular, $F \cup G$ extends to a λ^+-complete ultrafilter.

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Ordinal definability of ultrafilters

Theorem (Kunen)

Assume the Axiom of Determinacy. Then every countably complete ultrafilter on an ordinal less than Θ is ordinal definable.

Theorem

Suppose λ is a definably Berkeley cardinal such that λ-DC holds. Then every λ^+-complete ultrafilter U on an ordinal δ is ordinal definable from a subset of δ. Moreover $U \cap \mathrm{HOD} \in \mathrm{HOD}$ and U belongs to an ordinal definable set of cardinality less than λ.

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The axiom I_0

The following large cardinal axioms are widely believed to be consistent with ZFC:

- ▶ $I_3(\lambda)$: there is a $j : V_\lambda \to V_\lambda$.
- ▶ $I_2(\lambda)$: there is a $j : V \to M$ such that $V_{\kappa_\omega(j)} \subseteq M$.
- ▶ $I_1(\lambda)$: there is a $j : V_{\lambda+1} \to V_{\lambda+1}$.
- ▶ $I_0(\lambda)$: there is a $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$.

The Axiom I_0 in particular has been developed in great detail and has some fairly deep consequences.

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Schlutzenberg's theorem

Theorem (Schlutzenberg)

The following are equiconsistent:

- ▶ *$ZFC + I_0(\lambda)$.*
- ▶ *There is a $j : V_{\lambda+2} \to V_{\lambda+2} + \lambda$-DC.*
- ▶ *There is a $j : V \to M$ such that M is closed under $\kappa_\omega(j)$-sequences and $\kappa_\omega(j)$-DC holds.*

Thus the simplest large cardinal axiom refuted by Kunen *is* consistent in the choiceless context relative to traditional large cardinal axioms.

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Consistency of choiceless cardinals, II

Theorem

If $\aleph_0$-DC holds and there is an elementary embedding from $V_{\lambda+3}$ to $V_{\lambda+3}$, then ZFC $+$ I_0 is consistent.

▶ Woodin defined a hierarchy of inner models beyond $L(V_{\lambda+1})$ and corresponding ZFC large cardinal axioms beyond I_0, analogous to the hierarchy of models of determinacy beyond $L(\mathbb{R})$ and the corresponding determinacy hypotheses.

▶ Plausible that the choiceless cardinals line up with levels of this hierarchy.

    ▶ Some evidence that the existence of an elementary embedding from $V_{\lambda+3}$ to $V_{\lambda+3}$ is equiconsistent with the analog of $AD_{\mathbb{R}}$.

▶ Opens the door to providing far more conclusive evidence of the consistency of the axioms in both hierarchies.

Thanks!