Embeddings of HOD

Cumulative hierarchy

- $V_0 = \emptyset$
- $V_{\alpha+1} = P(V_\alpha)$
- $V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$ (limit γ)

Universe of sets: $V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$

Problem. We don't know the axioms.

Question Should we add $V=L$ as an axiom?

Constructible hierarchy

- $L_0 = \emptyset$
- $L_{\alpha+1} = \mathrm{def}(L_\alpha)$
- $L_\gamma = \bigcup_{\alpha<\gamma} L_\alpha$ (limit γ)

Constructible universe $L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$

Principle "$V=L$": Every set lies in some L_α.

$L \models ZFC + V=L$

Definition. A cardinal κ is measurable if there's a κ-additive $\mu: P(\kappa) \longrightarrow \{0,1\}$.

[Measurables are inaccessible]

Scott's theorem. If there is a measurable cardinal, then $V \neq L$.

Jensen covering lemma.

Exactly one of the following holds:

① L has the Jensen covering property: for any uncountable set $A \subseteq L$, there is a $B \in L$ s.t. $A \subseteq B$ & $|A| = |B|$.

② There is an elementary embedding $j: L \longrightarrow L$

Canonical inner models for large cardinals

$M, N \models ZF$, $j : M \to N$ elementary, $\mathrm{Ord}^M, \mathrm{Ord}^N \leq \mathrm{Ord}$

$$\mathrm{crit}(j) = \min\{\alpha : j(\alpha) \neq \alpha\}$$

If $\exists\, j : L \to L$ w $\mathrm{crit}(j) = \kappa$, there's a minimum j_κ

$$\begin{tikzcd} L \arrow[r, "j"] \arrow[d, "j_\kappa"'] & L \\ L \arrow[ur, dashed] & \end{tikzcd}$$

Relativized L-hierarchy:

$L_0[A] = \emptyset$

$L_{\alpha+1}[A] = \mathrm{def}(L_\alpha[A], A \cap L_\alpha[A])$

$L_\gamma[A] = \bigcup_{\alpha<\gamma} L_\alpha[A]$

$L[A] = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha[A]$ $A \cap L[A] \in L[A]$

$L[j_{\kappa_0}] = L[j_{\kappa_1}]$

Coverings for $L[j_\kappa]$. $? : L[j_\kappa] \to L[j_\kappa]$.

Embeddings of V.

Scott: κ is mble iff $\kappa = \mathrm{crit}(j)$ for $j : V \to M$ where $\mathrm{Ord}^M = \mathrm{Ord}$.

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_\mu"'] & M \\ \mathrm{Ult}(V,\mu) \arrow[ur, dashed] & \end{tikzcd}$$

If $\mu_0, \mu_1 : P(\kappa) \to \{0,1\}$
$L[\mu_0] = L[\mu_1] \models GCH$

$L[\mu]$ has a weaker covering lemma.

Q1 Which large cardinals have canonical models?
- So far: Woodin cardinals, Woodin limits of Woodins
- Open: - strongly compact
  - supercompact

Q2 Is there an unconditional covering lemma?

Ordinal definability

Suppose $M \models ZF$.

$\mathrm{def}(M) \cap M \notin \mathrm{def}(M)$

- If $\mathrm{def}(M) \cap M \subseteq A \in \mathrm{def}(M)$...
  A contains all ordinals

- $OD^M = \mathrm{def}(M, Ord) \cap M$
- $OD^M \in \mathrm{def}(M)$:
- $x \in OD \iff x \in OD^{V_\alpha}$ for some α

HOD: class of x s.t. if $x_0 \in x_1 \in \cdots \in x_n = x$, then $x_0 \in OD$.

Principle "$V = HOD$": Every set is OD.

- HOD is not canonical
- Known large cardinals compatible w/ $V = HOD$

The HOD dichotomy

Jensen dichotomy.

One of the following holds:

1. every singular cardinal λ is singular in L and $\lambda^{+L} = \lambda^+$
2. every cardinal is inaccessible in L

HOD dichotomy (Woodin)

For extendible κ, either

1. every singular cardinal $\lambda \geq \kappa$, λ is singular in HOD and $\lambda^{+HOD} = \lambda^+$
2. every regular cardinal $\lambda \geq \kappa$ is is measurable in HOD

More general dichotomy.

$\alpha_n \in C \quad \sup_{n<\omega} \alpha_n \in C$

- κ is <u>strongly compact</u> if any κ-additive measure $\mu : \Lambda \to \{0,1\}$ on $\Lambda \subseteq P(X)$ extends to a κ-additive measure on $P(X)$

- $C \subseteq \delta$ is <u>ω-club</u> if unbounded & closed under countable suprema
- $A \in \mathcal{C}_{\delta,\omega}$ if A contains ω-club

- N is ω-club amenable if for all $\delta \in \mathrm{Ord}^N$, $\mathcal{C}_{\delta,\omega} \cap N \in N$.

<u>Theorem.</u> If κ is strongly compact and $N \models ZFC$ is ω-club amenable, $\mathrm{Ord}^N = \mathrm{Ord}$ either

1. For all singular $\lambda \geq \kappa$, λ is singular in N and $\lambda^{+N} = \lambda^+$
2. All suff large regular cardinals are mble in N.

Embeddings of HOD

Key difference between dichotomies:

- no large cardinal axiom can refute $V = HOD$
- no structure theory known (possible) in 2 of HOD dichotomy

<u>HOD conjecture</u> (Woodin). If there is a supercompact, 1 of HOD dichotomy holds.

If negation of HOD conj is consistent, might be very strong [large cardinals inconsistent w/ AC]

Suggests: develop structure on side 2 of HOD dichotomy

On embeddings from HOD to HOD

Q. (Woodin) if there is an elementary $j: \mathrm{HOD} \to \mathrm{HOD}$, must the HOD conj fail?

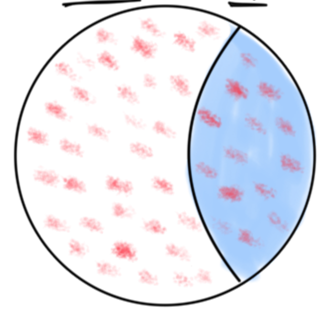
<u>Theorem</u>. Suppose there is a strongly compact cardinal and an elementary $j: \mathrm{HOD} \to \mathrm{HOD}$, then every suff large regular is measurable in HOD.

- Woodin had shown: if κ is extendible & covering holds for HOD, no $j: \mathrm{HOD} \to \mathrm{HOD}$ $\mathrm{crit}(j) \geq \kappa$
- Simulate by <u>replacing</u> $j: \mathrm{HOD} \to \mathrm{HOD}$ with $j: N \to \mathrm{HOD}$

Jonsson cardinals and HOD

<u>Definition</u>. λ is <u>definably</u> ω-Jonsson if $\forall f: [\lambda]^{\omega} \to \lambda$ $f \in \mathrm{OD}$ $\exists H \subsetneq \lambda$ s.t. $|H| = \lambda$ and H is closed under f.

$N \prec M$



$S \subseteq M$
$|S| = \lambda$
$|S \cap N| = \lambda$

<u>Proposition</u> TFAE:

- there is an e.e $L \to L$
- all cardinals λ are constructibly Jónsson ?

<u>Theorem</u> (Erdös-Hajnal)
No cardinal is ω-Jonsson.

<u>Theorem</u> (Strong compact) TFAE:

- HOD conjecture fails
- suff large cardinals are definably Jonsson
- suff large regular cardinals are definably ω-Jonsson
- proper class regulars are definably ω-Jonsson

are definably ω-Jonsson

Embeddings into HOD.

Suppose λ is definably ω-Jonsson. Get

$$j : N \longrightarrow HOD \cap V_\lambda$$

s.t. $j[\mathrm{Ord}^N]$ is ω-club. I.e., j is cofinal and ω-continuous: $j(\lim_{n\to\omega}\alpha_n) = \lim_{n\to\omega} j(\alpha_n)$

Theorem If κ is strongly compact, TFAE

1. HOD conj fails
2. For arbitrarily large regular λ, there is a $j : N \longrightarrow HOD \cap V_\lambda$ that's cofinal & ω-continuous
3. For suff large regular λ, there is a $j : N \longrightarrow HOD \cap V_\lambda$ s.t. $\mathrm{Ord}^N = \lambda$ & N is ω-club amenable.

Large cardinals beyond choice.

Theorem (Erdös-Hajnal). There are no ω-Jonsson cardinals.

Cor (Kunen). There is no elementary embedding $j : V \longrightarrow V$. $\lambda > \mathrm{crit}(j)$

Proof. If $j(\lambda) = \lambda$, λ is ω-Jonsson: fix $f : [\lambda]^\omega \longrightarrow \lambda$. Then $\boxed{j[\lambda]}$ is closed under $j(f)$. By elementarity some $H \subseteq \lambda$ of ordertype λ is closed under f.

$$L(P(\mathrm{Ord})) \overset{?}{\neq} V$$

Berkeley hypotheses

- $\mathcal{C}$ – Berkely hypothesis: for all suff large structures $\mathcal{M} \in \mathcal{C}$, there is a $j : \mathcal{M} \to \mathcal{M}$

- λ is weakly Lowenheim-Skolem if for all $\eta < \lambda$ and $N \supseteq V_\lambda$, there is $M \in V_\lambda$, a $j : M \to N$ s.t. $\text{crit}(j) > \eta$.

Woodin + Usuba: Assume ZF + proper class of weak Lowenheim-Skolems

- Then there is a class generic model of ZFC.
- Forcing preserves $\boxed{\text{HOD-Berkeley hypothesis}}$

The rank Berkeley hypothesis

Rank Berkeley =

=

=

Theorem (ZF+DC+RBH) Suppose α is an ordinal, λ limit of weak LS cardinals

① $\Lambda \subseteq P(\alpha)$, λ-additive measure $\mu : \Lambda \to \{0,1\}$ w/

② Every λ-additive $\mu : P(\alpha) \cap \text{HOD} \to \{0,1\}$

Theorem (ZFC) If κ is extendible, TFAE:

Theorem [illegible]

1. HOD conjecture
2. Every κ-additive $\mu : P(\alpha) \cap HOD \to \{0,1\}$ belongs to HOD.

Uniqueness of elementary embeddings.

Thm Suppose $j_0, j_1 : V \to M$ are elementary. $j_0 \restriction HOD = j_1 \restriction HOD$

Thm If κ is extendible, TFAE:

1. HOD conjecture
2. For all inaccessible $\delta > \kappa$, all suff large α, & all $j_0, j_1 : V_\alpha \to M$ s.t. $j_0(\delta) = j_1(\delta)$ & $\sup j_0[\delta] = \sup j_1[\delta]$, then $j_0 \restriction HOD \cap V_\kappa = j_1 \restriction HOD \cap V_\kappa$

THANKS!