

Consistency beyond the Kunen inconsistency

Gabriel Goldberg*
UC Berkeley

June 16, 2026

Abstract

These notes discuss the problem of securing the consistency of large cardinal hypotheses at the level of the Kunen inconsistency. Motivated by the theory of Berkeley cardinals, we introduce *cover exacting cardinals*, a relativized variation on Aguilera–Bagaria–Lücke’s exacting cardinals. We show that ZFC plus a cover exacting cardinal is consistent relative to I_2 , but a cover exacting cardinal *cannot exist* above an extendible cardinal. We then draw some conclusions about the HOD conjecture. This is joint work with Doug Blue.

1 Consistency of large cardinals

Why do we think set theoretic hypotheses are consistent? Typically because we have proved them consistent from large cardinal hypotheses. But why do we think large cardinal hypotheses are consistent? This becomes a pressing question at the level of the Kunen inconsistency theorem [11].

Theorem 1.1 (Kunen). *For any ordinal λ , there is no nontrivial elementary embedding from $V_{\lambda+2}$ to itself.*

Thus the elementary embedding framework for large cardinal hypotheses seems to come to an abrupt halt. In this context, Solovay–Reinhardt–Kanamori [13] introduced the following *rank-to-rank* principles:

I_1 : There is a nontrivial elementary embedding from $V_{\lambda+1}$ to itself.

I_2 : There is a nontrivial elementary embedding $j : V \rightarrow M$ such that $V_{\kappa_\omega(j)} \subseteq M$.

I_3 : There is a nontrivial elementary embedding from V_λ to itself.

Here $\kappa_\omega(j)$ is the supremum of the critical sequence of j :

$$\kappa_\omega(j) = \sup\{\kappa, j(\kappa), j^2(\kappa), j^3(\kappa), \dots\}$$

where $\kappa = \text{crit}(j)$. Kunen’s theorem implies that if $j : V_\alpha \rightarrow V_\alpha$, then $\alpha \leq \kappa_\omega(j) + 1$. In particular the rank-to-rank cardinals λ all have cofinality ω .

*This research was supported by the National Science Foundation under Grant No. 2401789.

The past half century of research has led to a fairly strong conviction that these hypotheses are consistent, but in 1978, Solovay–Reinhardt–Kanamori cautioned: “It seems likely that I_1 , I_2 , and I_3 are all inconsistent since they appear to differ from the proposition proved inconsistent by Kunen only in an inessential technical way.”

The HOD conjecture [15] posits that the infinitary combinatorics leading to the Kunen inconsistency theorem can be carried out “effectively,” which would rule out various natural attempts to extend the large cardinal hierarchy beyond rank-to-rank.

Definition 1.2 (HOD conjecture). It is provable assuming a supercompact cardinal that there is a class well-ordering of $[\text{Ord}]^\omega$ that is definable over V from parameters.

This conjecture rules out extending the large cardinal hierarchy past the Kunen inconsistency by dropping the Axiom of Choice:

Theorem 1.3 (Woodin, ZF). *Assume the HOD conjecture and assume there is a proper class of extendible cardinals. Then there is no elementary embedding from V to itself.*

A cardinal κ is *extendible* if for all $\alpha > \kappa$, for some $\alpha' > \alpha$, there is an elementary embedding $j : V_\alpha \rightarrow V_{\alpha'}$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

The HOD conjecture also rules out the “sharp of HOD.”

Theorem 1.4. *Assume the HOD conjecture. If there is a strongly compact cardinal, then there is no nontrivial elementary embedding from HOD to itself.*

Theorem 1.3 actually follows from Theorem 1.4 combined with Woodin’s theorem that in ZF, if there is a proper class of extendible cardinals, then there is a definable homogeneous class forcing extension of ZFC with a proper class of extendible cardinals. If one applies this to a model with a $j : V \rightarrow V$, one winds up with a model of ZFC in which there is a $j : \text{HOD} \rightarrow \text{HOD}$. Since the HOD conjecture rules out the latter by Theorem 1.4, it also rules out the former.

In particular, the HOD conjecture eliminates several natural paths along which one might try to extend the large cardinal hierarchy, both in ZF and in ZFC.

Roughly, the reason the HOD conjecture rules out these hypotheses is the *HOD dichotomy theorem*:

Theorem 1.5 (Woodin). *Suppose κ is extendible. Then exactly one of the following holds:*

- (1) *Every set $X \subseteq \text{HOD}$ with $|X| \geq \kappa$ is covered by some $Y \in \text{HOD}$ with $|Y| = |X|$.*
- (2) *Every regular cardinal greater than or equal to κ is measurable in HOD.*

As a consequence of this theorem, in the context of an extendible cardinal κ , the HOD conjecture can be formulated in terms of:

- Definable wellorderings of $[\text{Ord}]^\omega$.
- Covering properties of HOD.
- Ordinal definable stationary partitions.

In particular, in the context of extendible cardinals, the HOD conjecture is equivalent to the conjecture that (2) of Theorem 1.5 never occurs; i.e., (1) is provable.

We briefly sketch the equivalence between the HOD conjecture as stated above and Woodin’s original conjecture. Woodin formulates the HOD conjecture in terms of the *HOD*

hypothesis, which is the following set-theoretic principle: for a proper class of regular cardinals δ , there is an ordinal definable partition of $S_\omega^\delta = \{\alpha < \delta : \text{cf}(\alpha) = \omega\}$ into γ -many stationary sets where $(2^\gamma)^{\text{HOD}} \geq \delta$. Woodin’s original HOD conjecture states that ZFC + a supercompact cardinal proves the HOD hypothesis.

The HOD dichotomy theorem implies this is equivalent to the HOD conjecture as we have stated it:

Proposition 1.6. *If there is a strongly compact cardinal, then the HOD hypothesis holds if and only if there is a class well-ordering of $[\text{Ord}]^\omega$ that is definable over V from parameters.*

Proof. Let κ be strongly compact.

In one direction, if the HOD hypothesis holds, [7, Proposition 2.3] shows that HOD has the κ -cover property. This yields a definable injection from $P_\kappa(\text{Ord})$ into $\text{HOD} \times P_\kappa(\kappa)$: send $s \in P_\kappa(\text{Ord})$ to (σ, \bar{s}) where $\sigma \in \text{HOD}$ is an OD-minimal set of ordinals of size $< \kappa$ covering s and \bar{s} is the image of s under the transitive collapse of σ . We can definably well-order $\text{HOD} \times P_\kappa(\kappa)$ using a well-order of $P_\kappa(\kappa)$ as a parameter. This pulls back to a definable well-order of $P_\kappa(\text{Ord})$.

Conversely, suppose there is a class well-ordering of $[\text{Ord}]^\omega$ that is definable over V from a parameter p . Then $[\text{Ord}]^\omega \subseteq \text{HOD}_p$. Since HOD_p is a generic extension of HOD, there is some γ such that if $\text{cf}(\alpha) = \omega$ then $\text{cf}^{\text{HOD}}(\alpha) < \gamma$. We finally show that for every regular $\delta > \gamma$, there is an ordinal definable partition of S_ω^δ into γ -many stationary sets where $(2^\gamma)^{\text{HOD}} \geq \delta$. Suppose not. By an argument due to Woodin [16, Lemma 3.37], there is an ordinal definable stationary set $S \subseteq S_\omega^\delta$ with the following property. Let F be the restriction of the club filter to S , which is generated by sets of the form $C \cap S$ where C is club in δ . Then $U = F \cap \text{HOD}$ is an ultrafilter in HOD. Since every $\alpha \in S_\omega^\delta$ has $\text{cf}^{\text{HOD}}(\alpha) < \gamma$, in HOD, U is a normal ultrafilter concentrated on singular cardinals. It’s a theorem of ZFC that normal measures concentrate on regular cardinals, so we have a contradiction. \square

The HOD dichotomy is reminiscent of Jensen’s covering lemma:

Theorem 1.7 (Jensen). *Exactly one of the following holds:*

- (1) *Every set $X \subseteq L$ with $|X| \geq \aleph_1$ is covered by some $Y \in L$ with $|Y| = |X|$.*
- (2) *Every cardinal greater than or equal to \aleph_1 is weakly compact in L .*

No set theorist would conjecture that (1) of Theorem 1.7 is provable (i.e., that the existence of $0^\#$ is inconsistent). How close is the analogy really? Is trying to prove the HOD conjecture as futile as trying to prove $0^\#$ does not exist? For now, let me just point out that there is much more evidence for the consistency of $0^\#$ than for the consistency of a Reinhardt cardinal.

Aguilera–Bagaria–Lücke [2] introduced a new hypothesis beyond rank-to-rank, again incompatible with the HOD conjecture:

Definition 1.8. A cardinal λ is *exacting* if for all $\alpha > \lambda$, there exists $X \preceq V_\alpha$ containing $V_\lambda \cup \{\lambda\}$ and an elementary $j : X \rightarrow V_\alpha$ with $\text{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Aguilera–Bagaria–Lücke show that if the theory ZFC + “there is an exacting cardinal above an extendible cardinal” is consistent, then the HOD conjecture is false. This follows from two key observations, which will be central below.

Observation 1. (1) *If λ is exacting and $Y \subseteq V_\lambda$ is OD, there is a $j : (V_\lambda, Y) \rightarrow (V_\lambda, Y)$.*

(2) If $Y \subseteq \lambda$ is cofinal with $\text{ot}(Y) < \lambda$, there is no $j : (V_\lambda, Y) \rightarrow (V_\lambda, Y)$.

Proof. For (1), take α such that Y is definable without parameters in V_α , and apply the definition of an exacting cardinal.

For (2), note that such an embedding j would have $\text{crit}(j) \geq \text{ot}(Y)$, and hence $j \upharpoonright Y$ is the identity. This contradicts that the fixed points of j are bounded below λ , since $\lambda = \kappa_\omega(j)$ by Kunen’s theorem. \square

These facts combine to show that any exacting cardinal λ is regular in HOD. On the other hand, $\text{cf}(\lambda) = \omega$ since λ is a rank-to-rank cardinal. Therefore assuming the HOD conjecture, since we are in case (1) of Theorem 1.5, there can be no exacting cardinals above an extendible cardinal.

The core idea behind the HOD conjecture is that certain attempts to define strong large cardinals may lead to inconsistency for unexpectedly deep reasons. Therefore we prefer a more neutral formulation of Aguilera–Bagaria–Lücke’s result. Namely, at least one of the following holds:

- ZFC + “there is an exacting cardinal above an extendible cardinal” is inconsistent.
- The HOD conjecture is false.

How can we figure out which?

Here is an attempt to argue for the consistency of an exacting cardinal above an extendible, paraphrasing Aguilera–Bagaria–Lücke:

1. Exacting and extendible cardinals are natural large cardinal hypotheses.
2. Each of these hypotheses is consistent with ZFC relative to I_2 .
3. Therefore an exacting cardinal above an extendible cardinal should be consistent.

According to Aguilera–Bagaria–Lücke, we should either accept this argument or accept that two separately consistent large cardinal hypotheses can be jointly inconsistent. Our results show that if one stretches the usage of the term large cardinal hypothesis far enough, this kind of joint inconsistency actually occurs.

Motivated by the theory of Berkeley cardinals, we introduce a generalization of the notion of an exacting cardinal called a *cover exacting cardinal*. In Theorem 3.5, we show that the existence of a cover exacting cardinal is consistent with ZFC (relative to I_2). The argument above should generalize to a cover exacting cardinal if it works at all. But in Theorem 3.6, we prove that there cannot be a cover exacting cardinal above an extendible cardinal.

2 Some choiceless cardinals

Aguilera–Bagaria–Lücke motivate exacting cardinals as a variant of Schlutzenberg’s rank Berkeley cardinals.

Definition 2.1 (ZF). A cardinal λ is *rank Berkeley* if for all $\alpha > \lambda > \beta$, there is an elementary $j : V_\alpha \rightarrow V_\alpha$ with $\beta < \text{crit}(j) < \lambda$.

Similarly, δ is *Berkeley* if for all $\alpha > \delta > \beta$ and all $Y \subseteq V_\alpha$, there is an elementary $j : (V_\alpha, Y) \rightarrow (V_\alpha, Y)$ with $\beta < \text{crit}(j) < \delta$.

These hypotheses are closely related to Reinhardt’s ill-fated axiom:

Theorem 2.2 (Schlutzenberg, ZF). *If $j : V \rightarrow V$, $\kappa_\omega(j)$ is rank Berkeley.* □

Theorem 2.3 (Bagaria–Koellner–Woodin, ZF). *If δ is a Berkeley cardinal, then there is an inaccessible cardinal $\kappa < \delta$ and a nontrivial elementary $j : V_\kappa \rightarrow V_\kappa$.* □

Remark 2.4. Part of the reason Berkeley cardinals are considered so ridiculous is that they are so easy to refute in ZFC. For example, suppose δ is Berkeley. Assume towards a contradiction that $<$ is a linear order of $[\delta]^\omega / \simeq$ where $s \simeq t$ if $|s \triangle t| < \aleph_0$. Fix

$$j : (V_{\delta+2}, <) \rightarrow (V_{\delta+2}, <)$$

Let $s_{\text{even}} = \langle \kappa, j^2(\kappa), j^4(\kappa), \dots \rangle$ and $s_{\text{odd}} = \langle j(\kappa), j^3(\kappa), j^5(\kappa), \dots \rangle$ be the even and odd points of the critical sequence of j . Then $j(s_{\text{even}}) = s_{\text{odd}}$ and $j(s_{\text{odd}}) \simeq s_{\text{even}}$. This implies $[s_{\text{even}}] < [s_{\text{odd}}]$ if and only if $[s_{\text{odd}}] < [s_{\text{even}}]$, which is a contradiction.

Question 2.5 (Asperó). Assume that every set can be linearly ordered. Can there be a rank Berkeley cardinal?

It is natural to try to strengthen the notion of an exacting cardinal in the same way that Berkeley cardinals strengthen rank Berkeley cardinals; that is, by relativizing to predicates.

Definition 2.6. Given $Y \subseteq V_\alpha$, a λ -*exacting embedding relative to* (V_α, Y) is an elementary

$$j : (X, X \cap Y) \rightarrow (V_\alpha, Y)$$

where $X \preceq (V_\alpha, Y)$, $V_\lambda \cup \{\lambda\} \subseteq X$, $\text{crit}(j) < \lambda$, and $j(\lambda) = \lambda$.

If we didn't know about the Kunen inconsistency, we might define a cardinal λ to be *relativized exacting* if for all $\alpha > \lambda$, for all $Y \subseteq V_\alpha$, there is a λ -exacting embedding relative to (V_α, Y) . This seems about as “intrinsically plausible” as exactingness, but it is immediate from Kunen's theorem that no such λ exists: take Y to be a cofinal ω -sequence in λ .

The problem here seems to be that a Berkeley cardinal δ is *never* witnessed by embeddings with $\kappa_\omega(j) = \delta$, while in ZFC *every* rank-to-rank embedding $j : V_\lambda \rightarrow V_\lambda$ has $\kappa_\omega(j) = \lambda$. This κ_ω -*collision* is what leads to the trivial inconsistency of relativized exacting cardinals with ZFC.

To find a nontrivial relativization of exacting, we first identify a weakening of a Berkeley cardinal that escapes the collision.

Definition 2.7 (ZF). A cardinal λ is V_γ -*coarsely Berkeley* if for all $\alpha > \lambda > \beta$ and $y \in V_\alpha$, there exist $Y \subseteq V_\alpha$ with $y \in Y \leq^* V_\gamma$ and $j : (V_\alpha, Y) \rightarrow (V_\alpha, Y)$ with $\beta < \text{crit}(j) < \lambda$.

The notation $A \leq^* B$ means that A is the surjective image of B . The idea is that although we may not be able to fix an arbitrary point $y \in V_\alpha$ by a self-embedding of V_α (which would imply λ is Berkeley), we can replace y with a small cloud of points containing it, and demand only that this cloud be preserved.

Our next theorem shows that this notion avoids the κ_ω -collision:

Theorem 2.8 (ZF). *If δ is a Berkeley cardinal, then there exist $\lambda < \delta < \gamma$ such that λ is V_γ -coarsely Berkeley via embeddings with $\kappa_\omega(j) = \lambda$.*

For the proof, we need a basic lemma from the theory of rank Berkeley cardinals [6].

Lemma 2.9 (Well-ordered collection lemma, ZF). *Suppose $V_\gamma \preceq_{\Sigma_2} V$ and there is a rank Berkeley cardinal below γ . Suppose $\delta < \gamma$ and $\langle Z_\xi : \xi < \delta \rangle$ is a sequence of nonempty classes. Then there is a set $X \leq^* V_\gamma$ such that $Z_\xi \cap X \neq \emptyset$ for all $\xi < \delta$.*

Sketch. A key point is that γ is *almost supercompact*: for any $\alpha > \gamma > \xi$ and $x \in V_\alpha$, there is an ordinal $\bar{\gamma} < \bar{\alpha} < \gamma$ and an elementary $\pi : V_{\bar{\alpha}} \rightarrow V_\alpha$ with $\pi(\bar{\gamma}) = \gamma$, $\pi(\xi) = \xi$, and $x \in \text{ran}(\pi)$. This is an analog of Magidor’s characterization of supercompact cardinals, except that we cannot control the critical point; instead we can only ensure a single ordinal is fixed. We omit the proof that γ is almost supercompact, which appears in [6, Theorem 6.3].

The main idea is to use the almost supercompactness of γ to prove the well-ordered collection lemma by induction. Suppose the lemma holds for all $\bar{\delta} < \delta$. Clearly we may assume δ is a limit ordinal, or we are done. Fix β large enough that $Z_\xi \cap V_\beta \neq \emptyset$ for all $\xi < \delta$.

Fix $\alpha > \beta$, and using almost supercompactness, find $\bar{\gamma} < \bar{\alpha} < \gamma$ and $\pi : V_{\bar{\alpha}} \rightarrow V_\alpha$ with $\pi(\bar{\gamma}) = \gamma$, $\pi(\delta) = \delta$, and $\langle Z_\xi \cap V_\beta \rangle_{\xi < \delta} \in \text{ran}(\pi)$. By elementarity and our induction hypothesis, for each $\bar{\delta} \in \text{ran}(\pi) \cap \delta$, there is a set $Y \leq^* V_\gamma$ in $\text{ran}(\pi)$ witnessing the well-ordered collection lemma for all $\langle Z_\xi \cap V_\beta \rangle_{\xi < \bar{\delta}}$. Note that $\text{ran}(\pi) \cap \delta$ is cofinal in δ since $\pi(\delta) = \delta$. Therefore

$$X = \bigcup \{Y \in \text{ran}(\pi) : Y \leq^* V_\gamma\}$$

witnesses the well-ordered collection lemma for δ . To see that $X \leq^* V_\gamma$, note that X is the surjective image of $\text{ran}(\pi) \times V_\gamma$ and $\text{ran}(\pi) \leq^* V_\gamma$. \square

There are many applications of the well-ordered collection lemma (unrelated to our current goal):

Theorem 2.10 (ZF). *Under the hypotheses of Lemma 2.9:*

- *Either γ or γ^+ is measurable;*
- *The club filter on any regular cardinal above γ is γ -complete and atomic.*
- *Every γ -complete filter on an ordinal extends to a γ -complete ultrafilter.* \square

As a warm-up, we prove the following consequence of well-ordered collection:

Theorem 2.11 (Goldberg–Mizuri, ZF). *Assume there is a nontrivial elementary embedding from any sufficiently large transitive set to itself. Then there is a Berkeley cardinal.*

Sketch. The natural approach uses AC_ω . Fix η such that for any transitive set $M \geq^* \eta$, there is a nontrivial elementary $j : M \rightarrow M$. Assume towards a contradiction that there are no Berkeley cardinals. By Bagaria–Koellner–Woodin [3], there are no “proto-Berkeley” cardinals. In other words, for any ordinal δ , there is a transitive set $M \supseteq \delta$ that admits no elementary $j : M \rightarrow M$ with $\text{crit}(j) < \delta$. To get a contradiction, choose a sequence $\vec{M} = \langle (\delta_n, M_n) : n < \omega \rangle$ such that there is no elementary $j : M_n \rightarrow M_n$ with $\text{crit}(j) < \delta_n$ and $\text{rank}(M_n) = \delta_{n+1}$. Let $\alpha = \sup_{n < \omega} \delta_n$, and using the hypothesis, find a nontrivial $j : (V_\alpha, \vec{M}) \rightarrow (V_\alpha, \vec{M})$. This easily yields a contradiction.

To avoid AC_ω , one uses the well-ordered collection lemma. The argument of the previous paragraph implies that there is a rank Berkeley cardinal, and above this cardinal we can fix some γ such that $V_\gamma \preceq_{\Sigma_2} V$. We claim that for any $\xi > \gamma$ of countable cofinality, there is

some $\delta_\xi < \xi$ such that for any $Y \subseteq V_\xi$, there is an elementary $j : (V_\xi, Y) \rightarrow (V_\xi, Y)$ with $\text{crit}(j) < \delta_\xi$. To see this, assume not, fix $\langle \alpha_n : n < \omega \rangle$ cofinal in ξ , and using the well-ordered collection lemma, find a set $X \leq^* V_\gamma$ such that for each $n < \omega$, there is some $Y \in X$ such that every $j : (V_\xi, Y) \rightarrow (V_\xi, Y)$ has $\text{crit}(j) > \alpha_n$. Let $f : V_\gamma \rightarrow X$ be a surjection, and fix $j : (V_\xi, f) \rightarrow (V_\xi, f)$ with $\text{crit}(j) > \gamma$. Then $j \upharpoonright X = \text{id}$, so taking n such that $\text{crit}(j) < \alpha_n$ yields a contradiction.

To finish, one uses a pressing down argument to find a fixed ordinal δ such that for a proper class of ξ , for all $Y \subseteq V_\xi$, there is an elementary $j : (V_\xi, Y) \rightarrow (V_\xi, Y)$ with $\text{crit}(j) < \delta$. It is immediate that δ is a proto-Berkeley cardinal, and this is a contradiction. \square

Proof of Theorem 2.8. Fix $\gamma > \delta$ such that $V_\gamma \preceq_{\Sigma_2} V$. Assume towards a contradiction that no $\lambda < \delta$ is V_γ -coarsely Berkeley via embeddings with $\kappa_\omega(j) = \lambda$.

For each $\xi < \delta$, let Z_ξ be the nonempty class of y such that for some ordinal $\alpha > \max\{\text{rank}(y), \xi\}$, there is no $Y \subseteq V_\alpha$ with $y \in Y \leq^* V_\gamma$ admitting an elementary embedding $j : (V_\alpha, Y) \rightarrow (V_\alpha, Y)$ with $\kappa_\omega(j) = \xi$. If $y \in Z_\xi$, let $\alpha_{y,\xi}$ denote the least α witnessing this.

By the well-ordered collection lemma, fix a set $X \leq^* V_\gamma$ such that $Z_\xi \cap X \neq \emptyset$ for all $\xi < \delta$. Let

$$\beta = \sup\{\alpha_{y,\xi} + 1 : \xi < \delta \text{ and } y \in Z_\xi \cap X\}$$

Since δ is Berkeley, there is an elementary embedding $i : (V_\beta, X) \rightarrow (V_\beta, X)$ with $\text{crit}(i) < \delta$, $i(\delta) = \delta$, and $i(\gamma) = \gamma$. We can also ensure that $\kappa_\omega(i) < \delta$: if δ has uncountable cofinality, this is automatic; otherwise, we can demand that i fix a cofinal ω -sequence in δ .

Let $\lambda = \kappa_\omega(i)$ and

$$\alpha = \min\{\alpha_{y,\lambda} : y \in Z_\lambda \cap X\}$$

so that $i(\alpha) = \alpha$. Fix $y \in Z_\lambda \cap X$ with $\alpha_{y,\lambda} = \alpha$. Set $Y = X \cap V_\alpha$. Then $y \in Y \leq^* V_\gamma$, but i restricts to an elementary embedding $j : (V_\alpha, Y) \rightarrow (V_\alpha, Y)$ with $\kappa_\omega(j) = \lambda$, contradicting the fact that $y \in Z_\lambda$ and $\alpha = \alpha_{y,\lambda}$. \square

2.1 Aside: homogeneously Berkeley cardinals

In the 1990s, Woodin introduced Berkeley cardinals and challenged the UC Berkeley set theory seminar to refute them in ZF. Although we now have a better understanding of their consequences and nothing close to a contradiction, these cardinals still look a little bit decadent.

Here is a hypothesis we might have a better chance of refuting this week.

Definition 2.12 (ZF). A cardinal δ is *homogeneously Berkeley* if for all transitive sets $M \supseteq P(\delta)$ and all $\beta < \delta$, there is some $A \in [\delta]^\delta$ such that for all $B \in [A]^\delta$, there is an elementary $j : M \rightarrow M$ with $\beta < \text{crit}(j) < \delta$ and $j(A) = B$.

It is unclear whether this should count as a large cardinal hypothesis, particularly given its consequences for small sets:

Proposition 2.13 (ZF + DC). *Suppose δ is homogeneously Berkeley.*

- For all $\gamma < \delta$, $\delta \rightarrow (\delta)_\gamma^\delta$.
- ω_1 is measurable.

The main idea in the second item comes from an old result of Woodin; it basically uses the existence of strong partition cardinals past the continuum. The key new question here is whether the strong partition property can be witnessed by elementary embeddings.

Problem. Refute homogeneously Berkeley cardinals in ZF.

3 Cover exacting cardinals

Theorem 2.8 leads to the central definition of these notes.

Definition 3.1. A cardinal λ is γ -cover exacting if for all $\alpha > \lambda$ and $y \in V_\alpha$, there exist $Y \in [V_\alpha]^{\leq \gamma}$ with $y \in Y$ and a λ -exacting embedding $j : (X, X \cap Y) \rightarrow (V_\alpha, Y)$.

If $\gamma < \lambda$, then λ is not γ -cover exacting by the argument that refutes relativized exacting cardinals. On the other hand, most of the arguments that an exacting cardinal is a natural large cardinal hypothesis extend to a cardinal λ that is γ -cover exacting for some $\gamma \geq \lambda$.

- The notion naturally strengthens exactingness, avoiding the Kunen inconsistency.
- γ -cover exacting cardinals arise from V_γ -coarsely Berkeley cardinals in the same way that exacting cardinals arise from rank Berkeley cardinals.
- There are various equivalent formulations; e.g., in terms of structural reflection or using any of the alternate formulations of exacting, but also see Proposition 3.4.
- Cover exacting cardinals cannot be created or destroyed by small forcing (roughly).

Exacting cardinals are undeniably *simpler* than cover exacting cardinals, but it is unclear they are any more “natural.”

We begin with some alternate characterizations of cover exacting cardinals.

Definition 3.2. If Y is a set, a cardinal λ is *exacting relative to Y* if for any $\alpha \geq \text{rank}(Y)$, there is a λ -exacting embedding relative to (V_α, Y) .

Lemma 3.3. *If λ is exacting relative to Y and Y' is ordinal definable from Y , then λ is exacting relative to Y' .*

Proof. Let Y' be OD_Y -least such that the lemma fails. Then for all sufficiently large α , Y' is definable from Y in V_α without parameters. But then any λ -exacting embedding relative to (V_α, Y) is a λ -exacting embedding relative to (V_α, Y') , contradiction. \square

Proposition 3.4. *The following are equivalent:*

- (1) λ is γ -cover exacting.
- (2) Every set belongs to a set of size at most γ relative to which λ is exacting.
- (3) For all α , λ is exacting relative to a stationary set of $\sigma \in P_{\gamma^+}(V_\alpha)$.
- (4) Every set X is covered by a set of size $\max\{|X|, \gamma\}$ relative to which λ is exacting.
- (5) For all α , λ is exacting relative to a prewellorder of V_α with levels of size $\leq \gamma$.

Proof. Assume (1), and let us show (2). We first show that if α is a limit ordinal and there is a λ -exacting embedding $j : (X, X \cap Y) \rightarrow (V_\alpha, Y)$ where $Y \in V_\alpha$, then for all $\beta < \alpha$ above $\text{rank}(Y)$, there is a λ -exacting embedding relative to (V_β, Y) . Suppose not. Then the least $\beta < \alpha$ for which this fails is definable in (V_α, Y) , and so by elementarity $\beta \in X$ and

$j(\beta) = \beta$. Then $j \upharpoonright (X \cap V_\beta)$ is a λ -exacting embedding relative to (V_β, Y) , contradicting the definition of β .

Now fix y and let us find some Y of size at most γ relative to which λ is exacting. For each limit ordinal α of cofinality greater than γ , there is some $Y \in P_{\gamma^+}(V_\alpha)$ containing y such that there is a λ -exacting embedding relative to (V_α, Y) . Note that such a Y belongs to V_α since $\text{cf}(\alpha) > \gamma$, so Fodor's lemma implies there is a single Y that works for proper class of limit ordinals α . By the previous paragraph, this suffices.

Now assume (2), and let us show (3). Let λ be γ -cover exacting. Suppose $C \subseteq P_{\gamma^+}(V_\alpha)$ is closed unbounded. We will find $\sigma \in C$ relative to which λ is exacting.

Fix $f : V_\alpha^{<\omega} \rightarrow V_\alpha$ such that C contains every $\sigma \in P_{\gamma^+}(V_\alpha)$ with $f[\sigma^{<\omega}] \subseteq \sigma$ and $\sigma \cap \gamma^+ \in \gamma^+$. Take $Y \subseteq V_{\alpha+\omega}$ such that $f \in Y$ and λ is Y -exacting.

Let σ be the intersection of all sets $\tau \subseteq V_\alpha$ such that:

- τ is closed under every $g : V_\alpha^{<\omega} \rightarrow V_\alpha$ in Y .
- $\tau \cap \gamma^+ \in \gamma^+$.

Then $\sigma \in C$ and σ is ordinal definable from Y , so λ is exacting relative to σ .

The other implications are straightforward. \square

We show the existence of cover exacting cardinals is consistent relative to I_2 .

Theorem 3.5 (Blue–Goldberg). *Assuming I_2 is consistent, it is consistent that there is a cardinal λ that is λ -cover exacting.*

Sketch. Let κ be the critical point of an I_2 -embedding $j : V \rightarrow M$, and let D be the normal ultrafilter derived from j ; we argue that after Prikry forcing with D , κ is κ -cover exacting.

Let $j_{0\omega} : V \rightarrow M_\omega$ be the ω -th iterate of j , and let $g = \langle \kappa_n : n < \omega \rangle$ be the critical sequence. Let $\lambda = \sup(g) = \kappa_\omega(j) = j_{0\omega}(\kappa)$. It suffices to show that in $M_\omega[g]$, λ is λ -cover exacting.

First, every $y \in M_\omega[g]$ belongs to a set Y of size at most λ that is ordinal definable in $M_\omega[g]$ from parameters in M_ω . To see this, take $\dot{y} \in M_\omega$ such that $y = (\dot{y})_g$. Let $Y = \{(\dot{y})_h : M_\omega[h] = M_\omega[g]\}$; here h ranges over $j_{0\omega}(D)$ -Prikry generics that construct the whole extension. Each such h has the property that $|g \triangle h| < \omega$, so there are at most λ many.

To verify λ is λ -cover exacting in $M_\omega[g]$, by Proposition 3.4 and the remark preceding it, it suffices to show that for all $Y \in M_\omega$, λ is exacting relative to Y in $M_\omega[g]$. Suppose towards a contradiction that there is some $Y \in M_\omega$ relative to which λ is not exacting. Then there is such a $Y \in \text{ran}(j_{0\omega})$. In particular, $j(Y) = Y$. Fix $\alpha > \lambda$ least such that $M_\omega[g]$ satisfies that there is no λ -exacting embedding relative to $(V_\alpha^{M_\omega[g]}, \bar{Y})$ where $\bar{Y} = Y \cap V_\alpha$. Note that $j(\alpha) = \alpha$ since α is definable in $M_\omega[g]$ from Y .

Let $X \preceq (V_\alpha^{M_\omega[g]}, \bar{Y})$ be any elementary substructure containing $V_\lambda \cup \{\lambda\}$ with $X \in M_\omega[g]$ and $|X|^{M_\omega[g]} \leq \lambda$. Note that externally, $j \upharpoonright X$ is a λ -exacting embedding relative to $(V_\alpha^{M_\omega[g]}, \bar{Y})$. Since $\text{cf}(\lambda) = \omega$ in $M_\omega[g]$ and $V_\lambda \subseteq M_\omega[g]$, an easy absoluteness argument shows that there is some λ -exacting embedding $i : (X, \bar{Y} \cap X) \rightarrow (V_\alpha^{M_\omega[g]}, \bar{Y})$ that belongs to $M_\omega[g]$. \square

The argument above is analogous to one from [1], and it has the benefit of showing that it is consistent to assume there is a cover exacting cardinal *below* an extendible (or below any reasonable large cardinal). We note that Aguilera–Bagaria–Lücke's other consistency

proofs for exacting and ultraexacting cardinals also produce models of cover exacting and cover ultraexacting cardinals.¹

We finally show that cover exacting cardinals cannot exist *above* extendible cardinals:

Theorem 3.6 (Blue–Goldberg). *If κ is extendible and $\lambda > \kappa$, then λ is not γ -cover exacting for any $\gamma \geq \lambda$.*

The proof of this result goes through a weak version of the HOD conjecture that is provable from an extendible cardinal. If γ is a regular cardinal, a set x is γ -club definable if for some $\alpha > \gamma$, for a club of $\sigma \in P_\gamma(V_\alpha)$, x is ordinal definable in (V_α, σ) ; x is ∞ -club definable if it is γ -club definable for all regular γ . Let N denote the class of hereditarily ∞ -club definable sets.

Theorem 3.7. *If κ is extendible, then N is a model of ZFC and N is a ground of V for a forcing of size less than $(2^{2^\kappa})^+$.*

The proof involves an auxiliary notion that traces back to the proof that the Ultrapower Axiom implies that V is a generic extension of HOD. If κ is a cardinal, then HCD_κ denotes $\text{HOD}_\mathcal{U}$ where \mathcal{U} is the class of κ -complete ultrafilters on ordinals; $\text{HCD} = \bigcap_{\delta \in \text{Ord}} \text{HCD}_\delta$. If $M \subseteq N$ are transitive models of ZFC, M is a *ground* of N if there is a poset $\mathbb{P} \in M$ and an M -generic filter $G \subseteq \mathbb{P}$ such that $N = M[G]$.

Theorem 3.8. *If κ is supercompact, then HCD_κ is a ground of V for a forcing of size less than $(2^{2^\kappa})^+$. If κ is extendible, then $\text{HCD}_\kappa = \text{HCD}$.*

Sketch. A complete proof appears in [8, Proposition 4.11, Theorem 4.15], we only sketch the result. We omit the proof that HCD_κ satisfies the Axiom of Choice.

To see HCD_κ is a ground of V , it suffices by Vopenka’s theorem to show that there is a set $A \subseteq \kappa$ such that V is in $(\text{HCD}_\kappa)_A$, by which we mean HCD_κ relative to A . This is proved by taking A such that $V_\kappa \subseteq L[A]$. Then for any supercompactness measure \mathcal{U} on $P_\kappa(\lambda)$, since $V_\lambda \subseteq \text{Ult}(V, \mathcal{U})$,

$$V_\lambda \subseteq j_\mathcal{U}(L[A]) = L[j_\mathcal{U}(A)] \subseteq j_\mathcal{U}((\text{HCD}_\kappa)_A) \subseteq (\text{HCD}_\kappa)_A$$

where the last inclusion uses that \mathcal{U} is interdefinable with a κ -complete ultrafilter on an ordinal.

The fact that $\text{HCD}_\kappa = \text{HCD}$ when κ is extendible follows from the proof of Usuba’s theorem [14] that if κ is extendible, there is a minimum ground. \square

Proof of Theorem 3.7. It suffices to show that $N = \text{HCD}$. Here, we will only show $\text{HCD} \subseteq N$, since that is all we need for the proof of Corollary 3.9. For this it suffices to show that any η -complete ultrafilter U on an ordinal δ is γ -club definable for all $\gamma < \eta$. Fix $\alpha > \delta$. For $\sigma \in P_\gamma(V_\alpha)$, let $\beta_\sigma = \min \bigcap (U \cap \sigma)$. Then if $U \in \sigma \preceq V_\alpha$, U is ordinal definable in (V_α, σ) as the unique measure $W \in \sigma$ such that $W \cap \sigma = p_{\beta_\sigma} \cap \sigma$, where p_{β_σ} is the principal ultrafilter on δ concentrated at β_σ . \square

Corollary 3.9. *If κ is extendible and $\lambda > \kappa$ is singular, then there is a club definable cofinal set $a \subseteq \lambda$ with $\text{ot}(a) < \lambda$.* \square

¹No cardinal λ can be λ -cover ultraexacting, but assuming $I_0(\lambda)$, if $G \subseteq \text{Add}(\lambda^+, 1)$ is generic, λ is λ^+ -cover ultraexacting in $L(V_{\lambda+1})[G]$.

Proof of Theorem 3.6. Assume towards a contradiction that λ is γ -cover exacting. Since $\lambda > \kappa$ is singular, by Corollary 3.9, there is a club definable cofinal set $a \subseteq \lambda$ with $\text{ot}(a) < \lambda$. Fix $\alpha > \lambda$ and $\delta \geq \gamma$ such that a is ordinal definable in (V_α, σ) for all σ belonging to a club $C \subseteq P_\delta(V_\alpha)$. Since λ is δ -cover exacting, there is some $\sigma \in C$ relative to which λ is exacting. It follows that λ is exacting relative to a , which is impossible. \square

The moral: it takes more to refute the HOD conjecture than to formulate a natural large cardinal hypothesis whose consistency implies the HOD conjecture is false. Even finding two independently consistent large cardinal hypotheses whose joint consistency refutes the HOD conjecture does not suffice. One needs a stronger argument for joint consistency.

Aguilera–Bagaria–Lücke give a second, more compelling argument for the consistency of exacting cardinals above extendible cardinals: they can be forced from choiceless large cardinal hypotheses. This relative consistency proof distinguishes them from notions like cover exacting cardinals; it is not clear what else does. The consistency of exacting cardinals above extendibles therefore seems to rest on the consistency of choiceless large cardinals.

The consistency of choiceless cardinals is not a settled question, not even in the sense that the consistency of a supercompact cardinal is settled. Similarly the HOD conjecture is not a settled question.

4 Against the HOD conjecture

Although we have argued that the HOD conjecture is not settled, the past decade has produced significant evidence against it. Here we present this evidence and sketch some scenarios in which one might become confident that the HOD conjecture is false.

4.1 Consistency of choiceless cardinals

When the HOD conjecture was posed, it was conjectured that the weakest of the choiceless cardinals, an elementary embedding from $V_{\lambda+2}$ to itself, was inconsistent. This intuition was overturned by Schlutzenberg [12]:

Theorem 4.1 (Schlutzenberg). *ZFC+ I_0 is equiconsistent with $\text{ZF}+\text{DC}_\lambda+j : V_{\lambda+2} \rightarrow V_{\lambda+2}$.*

This casts some doubt on the idea that choiceless large cardinals are that strong. Perhaps $j : V_{\lambda+3} \rightarrow V_{\lambda+3}$ is consistent relative to a reasonable ZFC-large cardinal. Maybe this pattern persists to rank Berkeley cardinal.

There is weaker but still considerable evidence for the consistency of large cardinals at the level of rank Berkeley cardinals based on the emergence of structure theory, especially the periodicity patterns discovered by Schlutzenberg–Goldberg [10], Goldberg [6] and Goldberg–Hou [9]. It is not known whether the consistency of rank Berkeley cardinals does not contradict the HOD conjecture. Berkeley cardinals do, but there is currently very little structural evidence for the consistency of Berkeley cardinals.

Eventually these hypotheses may be sufficiently well-understood that their consistency will be granted the same credence as that of supercompact cardinals today (or the Axiom of Determinacy in the 1970s). The same could happen for exacting cardinals above extendible cardinals, but at this point, these cardinals are still in their infancy. To provide independent evidence of the failure of the HOD conjecture, there needs to be some deeper interaction between exacting cardinals and extendibility, something that doesn't already follow from being in case (2) of the HOD dichotomy.

4.2 Strong HOD conjectures and the Nairian model

When the HOD conjecture was posed, it was conceivable that a strong form of the HOD conjecture is provable: ZFC proves there is a proper class of regular cardinals that are not ω -strongly measurable in HOD. This is partly because it seemed so hard to force this, given the difficulty of forcing near successors of singular cardinals.

The first cracks appeared with Ben-Neria–Hayut’s work [4], showing that all successors of regulars can be ω -strongly measurable in HOD from fairly weak assumptions. (Before this, it was unknown whether more than three cardinals could be ω -strongly measurable in HOD.) Very recently, Blue–Larson–Sargsyan [5] refuted this strengthened HOD conjecture outright by forcing over Nairian models:

Theorem 4.2 (Blue–Larson–Sargsyan). *Relative to a Woodin limit of Woodin cardinals, it is consistent that all regular cardinals are ω -strongly measurable in HOD.*

If there is a model of AD^+ whose full Nairian model $V_\theta \cap \text{HOD}_{\text{Ord}^\omega}$ contains a proper class of supercompact cardinals, then the HOD conjecture is false. So there is a speculative scenario for refuting the HOD conjecture, perhaps not even using extremely large cardinals.

5 Questions

Problem 5.1. Refute homogeneously Berkeley cardinals in ZF.

Problem 5.2. Can there be a γ -cover exacting cardinal above a strongly compact cardinal?

Definition 5.3. A cardinal λ is *weakly γ -cover exacting* if every set of ordinals of size at least γ is covered by a set of the same size relative to which λ is exacting.

Problem 5.4. Show that it is consistent that there is a weakly γ -cover exacting cardinal above an extendible cardinal.

A positive answer seems likely from choiceless large cardinals. In this case, how could we possibly sharpen our notion of large cardinal to distinguish between γ -cover exacting and weakly γ -cover exacting cardinals?

Problem 5.5 (ZF). Show that a V_γ -coarsely Berkeley cardinal has higher consistency strength than a rank Berkeley cardinal.

We conjecture that it is consistent relative to a Berkeley cardinal that the least rank Berkeley cardinal is V_γ -coarsely Berkeley for some γ . If this is right, the problem above is probably out of reach of current techniques.

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