

# Comparison principles and very large cardinals

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# Independence

Impetus for most research in set theory: almost every problem is unsolvable (from the ZFC axioms)

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Two potential responses:

1. Study the structure of independence
  - ▶ What follows from what?
  - ▶ Consistency hierarchy
2. Search for “missing axioms” that resolve the independent propositions

## Gödel's Program

The known independent statements in arithmetic are not *absolutely* undecidable

- ▶ Each is decided in a well-justified higher order system
- ▶ For example, the Gödel sentence for first-order arithmetic is decided in second-order arithmetic, etc.

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- ▶ (1965-1990) Hundreds of problems in classical descriptive set theory settled in this way
- ▶ (1967) Lévy-Solovay Theorem: large cardinals *cannot* settle the Continuum Hypothesis

## Elementary embeddings

Modern large cardinal axioms are formulated in terms of elementary embeddings  $j : V \rightarrow M$  from the universe of sets  $V$  to a wellfounded model  $M$

- ▶ The large cardinal here is the *critical point* of  $j$ , the first cardinal  $\kappa$  such that  $j(\kappa)$  is not isomorphic to  $\kappa$
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Some examples: measurable, supercompact

## Large cardinals and ultrafilters

Key tool for understanding elementary embeddings:  
correspondence between embeddings and ultrafilters

- ▶ Given an ultrafilter  $U$ , one can form the ultrapower of the universe of sets, denoted  $j_U : V \rightarrow M_U$
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Combinatorial properties of ultrafilters correspond to model theoretic properties of the associated ultrapower

- ▶ An ultrafilter is countably complete if and only if its ultrapower is wellfounded
- ▶  $\kappa$  is measurable iff there is a  $\kappa$ -complete ultrafilter on  $\kappa$
- ▶  $\kappa$  is *strongly compact* if every  $\kappa$ -complete filter extends to a  $\kappa$ -complete ultrafilter

## Independence in the large cardinal hierarchy

Structure of large cardinals is itself susceptible to the independence phenomenon:

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  - ▶ Open: are strongly compacts and supercompacts equiconsistent?

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Dream:

1. Understand the structure of elementary embeddings of the universe of sets
2. Use this to answer classical problems in set theory

## Category of ultrapowers

If  $M$  is a model of ZFC, an *internal ultrapower embedding* of  $M$  is an elementary embedding of  $M$  isomorphic to  $(j_U)^M$  for some  $U \in M$  such that  $M \models U$  is an ultrafilter.

**Ult** denotes the category of internal ultrapowers:

- ▶ Objects: wellfounded ultrapowers of the universe
- ▶ Morphisms: internal ultrapower embeddings

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- ▶ Does **Ult** have pushouts?

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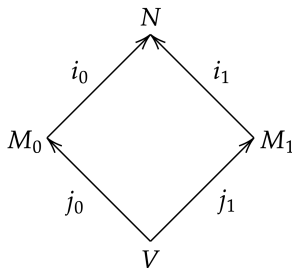
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Many questions about the structure of **Ult** are independent of ZFC:

- ▶ Nontriviality requires large cardinals
- ▶ Does **Ult** have pushouts?
- ▶ Is **Ult** locally finite?

# The Ultrapower Axiom

- ▶ The *Ultrapower Axiom* (UA) is the statement that **Ult** has the Amalgamation Property.



## Why UA? Part I

UA is motivated by inner model theory

- ▶ The goal of inner model theory is to construct and analyze canonical models of ZFC satisfying large cardinal axioms
  - ▶ Construction/analysis/canonicity flow from the fundamental *Comparison Lemma*
  - ▶ The Comparison Lemma implies UA holds in all the canonical models that have been constructed to date
  - ▶ Conversely, inner model theory is the only known way of building models of UA

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Open questions: Which large cardinal axioms have canonical models? Which large cardinal axioms are consistent with UA?

## Basic structural consequences

UA answers a lot of structural questions about **Ult**. For example, under UA, **Ult** is a lattice... Part of this:

### Theorem (UA)

**Ult** is a partial order.

Fancy way of saying that for any wellfounded ultrapower  $M$ , there is a unique ultrapower embedding  $j : V \rightarrow M$ .

## The Rudin-Frolík order

For ultrapowers  $M_0, M_1 \in \mathbf{Ult}$ , set  $M_0 \leq_{\text{RF}} M_1$  if there is an internal ultrapower embedding from  $M_0$  to  $M_1$ . [Under UA,  $M_0 \leq_{\text{RF}} M_1$  if and only if  $M_1 \subseteq M_0$ .]

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### Theorem (UA)

*An ultrapower has only finitely many  $\leq_{\text{RF}}$ -predecessors.*

A nontrivial ultrapower  $M$  is *irreducible* if its only  $\leq_{\text{RF}}$ -predecessors are  $V$  and  $M$ . The theorem reduces analysis of all ultrapowers to analysis of irreducible ultrapowers and their finite iterations.



## The linearity phenomenon, I

- ▶ Fundamental empirical fact: large cardinal hierarchy is linearly ordered by consistency strength
- ▶ Analogous to the semi-linearity of the hierarchy of definability in descriptive set theory:

### Theorem (LCA)

If  $A, B \subseteq 2^\omega$  are definable (e.g., Borel, projective, in  $L(\mathbb{R})$ , universally Baire), either  $A \leq_L B$  or  $B \leq_L \neg A$ .

$\leq_L$  denotes *Lipschitz reducibility* on subsets of  $2^\omega$ :

- ▶  $f : 2^\omega \rightarrow 2^\omega$  is *Lipschitz* if  $f(x) \upharpoonright n$  depends only on  $x \upharpoonright n$
- ▶ For  $A, B \subseteq 2^\omega$ ,  $f$  *reduces*  $A$  to  $B$  if  $A = f^{-1}[B]$

## The linearity phenomenon, II

Lipschitz reducibility generalizes to subsets  $A, B \subseteq 2^\kappa$  for an arbitrary ordinal  $\kappa$ :

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### Theorem (UA)

*For any ordinal  $\kappa$ , the set of countably complete ultrafilters on  $\kappa$  is wellordered by the Lipschitz order.*

## The Ketonen order

The *Ketonen order* is a structural refinement of  $\leq_L$ :

- ▶  $f : 2^\kappa \rightarrow 2^\kappa$  is *countably complete* if it is a countably complete endomorphism of the Boolean algebra  $2^\kappa$
- ▶  $U \leq_k W$  if there is a countably complete Lipschitz reduction from  $U$  to  $W$

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### Theorem

*The following are equivalent:*

- ▶ *For all ordinals  $\kappa$ , the set of countably complete ultrafilters on  $\kappa$  is wellordered by the Ketonen order.*
- ▶ *The Ultrapower Axiom holds.*

## Ordinal definability

A set is *ordinal definable from  $x$*  if it is first-order definable in  $V$  from  $x$  and an ordinal

The linearity of the Ketonen order implies that every countably complete ultrafilter on an ordinal is ordinal definable.

### Theorem (UA)

*If  $\kappa$  is strongly compact, then there is a set  $x \subseteq \kappa$  such that every set is ordinal definable from  $x$ .*

The proof works by coding sets into  $\kappa$ -complete ultrafilters

Note: UA + LCA does not imply every set is ordinal definable

## The Generalized Continuum Hypothesis

The idea of coding sets into ultrafilters gives various applications of UA + strongly compact cardinals to classical set theoretic questions. For example:

### Theorem (UA)

*Let  $\kappa$  be the least strongly compact cardinal. Then for all  $\lambda \geq \kappa$ ,  $2^\lambda = \lambda^+$ .*

One can also prove combinatorial principles like  $\diamond$  above  $\kappa$

Note: UA + LCA does not imply the Continuum Hypothesis

## Tarski's Question

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In fact, UA gives an analysis of arbitrary strongly compact cardinals...

## Larger strongly compact cardinals

Not every strongly compact cardinal can be supercompact:

### Theorem (Menas)

*The first measurable limit of strongly compacts is strongly compact but not supercompact.*

In fact, any measurable limit of strongly compacts is strongly compact, but not necessarily supercompact. Under UA, this is the only obstruction to supercompactness:

### Theorem (UA)

*A cardinal is strongly compact if and only if it is supercompact or a measurable limit of supercompacts.*

## Irreducible ultrapowers revisited

The analysis of arbitrary strongly compact cardinals turns out to be closely related to irreducible ultrapowers and the Rudin-Frolík order

If  $M$  is an ultrapower of  $V$ , the *width* of  $M$  is the least cardinal  $\lambda$  such that  $M = M_U$  for some ultrafilter  $U$  on  $\lambda$

### Theorem (UA)

*Suppose  $M$  is an irreducible ultrapower of width  $\lambda$ . If  $\lambda$  is either a successor cardinal or a singular strong limit cardinal, then  $M$  is closed under  $\lambda$ -sequences.*

## How to be tall and strong

$\kappa$  is *tall* (resp. *strong*) if for all  $\lambda \geq \kappa$ , there is an embedding  $j : V \rightarrow M$  with critical point  $\kappa$  such that  $j(\kappa) > \lambda$  (resp.  $V_\lambda \subseteq M$ ) and  $M$  is closed under  $\kappa$ -sequences

Analog of Tarski's question: is the first tall cardinal larger than the first measurable cardinal?

- ▶ This is independent of ZFC: the first measurable could be tall and the first tall cardinal could be strong
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### Theorem

UA does not decide whether the first tall cardinal is strong.

## Extender embeddings, I

What happens if you generalize UA to a wider class of elementary embeddings?

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### Definition

An elementary embedding  $j : V \rightarrow M$  is  $\lambda$ -generated if there is a set  $A \subseteq M$  with  $|A| < \lambda$  such that every element of  $M$  is definable in  $M$  from parameters in  $A \cup \text{ran}(j)$ .

An ultrapower embedding is simply an  $\aleph_0$ -generated elementary embedding. An *extender embedding* is an elementary embedding that is generated by a set (i.e., is  $\lambda$ -generated for some  $\lambda$ ).

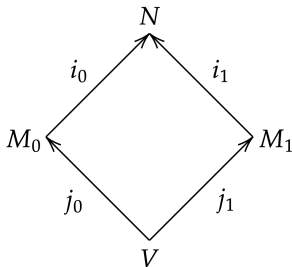


## UA for Extenders

Define a category **Ext**:

- ▶ Objects are extender ultrapowers of the universe
- ▶ Morphisms are internal extender embeddings

The *Extender Power Axiom* (EPA) asserts that **Ext** has the Amalgamation Property.



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UA is motivated by the inner model theoretic Comparison Lemma, but the Comparison Lemma doesn't justify anything like EPA...

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*If there is a Woodin cardinal, then there is a canonical inner model with a Woodin cardinal in which EPA is false.*

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Larger cardinals refute EPA outright:

### Theorem (with Woodin)

*If there is a supercompact cardinal, then EPA is false in  $V$ .*

## The Ground Axiom

Another approach to eliminating independence:

- ▶ Recall Cohen's proof that  $ZFC \not\vdash CH$ : fix a countable model  $M \models ZFC$  and build a forcing extension  $M[G]$  satisfying  $ZFC + \neg CH$

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- ▶ Could one decide CH by simply positing that  $V$  is not of the form  $M[G]$ ?

The *Ground Axiom* (GA) asserts that if  $M \subsetneq V$  is an inner model and  $G \in V$  is an  $M$ -generic set, then  $M[G] \subsetneq V$ .

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Under GA, can have  $V = M[G]$  where  $G$  is an  $M$ -generic class



## Combining axioms

UA seems to enforce a rigid structure above the first supercompact cardinal, while the known class forcings exhibiting independence from GA disturb the universe at arbitrarily large cardinals and therefore destroy UA

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**Theorem (UA + GA + a supercompact cardinal)**

*Every set is ordinal definable.*

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Theorem (UA + GA + a supercompact cardinal)

*Every set is ordinal definable.*

Conjecture

UA + GA + a supercompact cardinal implies CH.

Thanks!