Comparison principles and very large cardinals

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Independence

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Two potential responses:

1. Study the structure of independence
   - What follows from what?
   - Consistency hierarchy
2. Search for "missing axioms" that resolve the independent propositions

Gödel's Program

The known independent statements in arithmetic are not *absolutely* undecidable

- Each is decided in a well-justified higher order system
- For example, the Gödel sentence for first-order arithmetic is decided in second-order arithmetic, etc.

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- (1965-1990) Hundreds of problems in classical descriptive set theory settled in this way
- (1967) Lévy-Solovay Theorem: large cardinals *cannot* settle the Continuum Hypothesis

Elementary embeddings

Modern large cardinal axioms are formulated in terms of elementary embeddings $j : V \to M$ from the universe of sets V to a wellfounded model M

- The large cardinal here is the *critical point* of j, the first cardinal κ such that $j(\kappa)$ is not isomorphic to κ
- Closure properties of M yield reflection properties of κ

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Some examples: measurable, supercompact

Large cardinals and ultrafilters

Key tool for understanding elementary embeddings: correspondence between embeddings and ultrafilters

- Given an ultrafilter U, one can form the ultrapower of the universe of sets, denoted $j_U : V \to M_U$
- In the other direction, one can derive ultrafilters from an elementary embedding

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Combinatorial properties of ultrafilters correspond to model theoretic properties of the associated ultrapower

- An ultrafilter is countably complete if and only if its ultrapower is wellfounded
- κ is measurable iff there is a κ-complete ultrafilter on κ
- κ is *strongly compact* if every κ-complete filter extends to a κ-complete ultrafilter

Independence in the large cardinal hierarchy

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  - Open: are strongly compacts and supercompacts equiconsistent?

Independence in the large cardinal hierarchy

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  - Magidor Identity Crisis: the least strongly compact can be the least measurable or the least supercompact
  - Open: are strongly compacts and supercompacts equiconsistent?

Dream:

1. Understand the structure of elementary embeddings of the universe of sets
2. Use this to answer classical problems in set theory

Category of ultrapowers

If M is a model of ZFC, an *internal ultrapower embedding of M* is an elementary embedding of M isomorphic to $(j_U)^M$ for some $U \in M$ such that $M \vDash U$ is an ultrafilter.

Ult denotes the category of internal ultrapowers:

- Objects: wellfounded ultrapowers of the universe
- Morphisms: internal ultrapower embeddings

Many questions about the structure of **Ult** are independent of ZFC:

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Many questions about the structure of **Ult** are independent of ZFC:

- Nontriviality requires large cardinals
- Does **Ult** have pushouts?
- Is **Ult** locally finite?

The Ultrapower Axiom

- The *Ultrapower Axiom* (UA) is the statement that **Ult** has the Amalgamation Property.

$$\begin{tikzcd}
& N & \\
M_0 \arrow[ur, "i_0"] & & M_1 \arrow[ul, "i_1"'] \\
& V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}$$

Why UA? Part I

UA is motivated by inner model theory

- The goal of inner model theory is to construct and analyze canonical models of ZFC satisfying large cardinal axioms
  - Construction/analysis/canonicity flow from the fundamental *Comparison Lemma*
  - The Comparison Lemma implies UA holds in all the canonical models that have been constructed to date
  - Conversely, inner model theory is the only known way of building models of UA

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Open questions: Which large cardinal axioms have canonical models? Which large cardinal axioms are consistent with UA?

Basic structural consequences

UA answers a lot of structural questions about **Ult**. For example, under UA, **Ult** is a lattice... Part of this:

Theorem (UA)

Ult *is a partial order.*

Fancy way of saying that for any wellfounded ultrapower M, there is a unique ultrapower embedding $j : V \to M$.

The Rudin-Frolík order

For ultrapowers $M_0, M_1 \in \textbf{Ult}$, set $M_0 \leq_{\text{RF}} M_1$ if there is an internal ultrapower embedding from M_0 to M_1. [Under UA, $M_0 \leq_{\text{RF}} M_1$ if and only if $M_1 \subseteq M_0$.]

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Theorem (UA)

An ultrapower has only finitely many $\leq_{\mathrm{RF}}$-predecessors.

A nontrivial ultrapower M is *irreducible* if its only $\leq_{\mathrm{RF}}$-predecessors are V and M. The theorem reduces analysis of all ultrapowers to analysis of irreducible ultrapowers and there finite iterations.

The linearity phenomenon, I

- Fundamental empircal fact: large cardinal hierarchy is linearly ordered by consistency strength
- Analogous to the semi-linearity of the hierarchy of definability in descriptive set theory:

Theorem (LCA)

If $A, B \subseteq 2^\omega$ are definable (e.g., Borel, projective, in $L(\mathbb{R})$, universally Baire), either $A \leq_L B$ or $B \leq_L \neg A$.

$\leq_L$ denotes *Lipschitz reducibility* on subsets of 2^ω:

- $f : 2^\omega \to 2^\omega$ is *Lipschitz* if $f(x) \restriction n$ depends only on $x \restriction n$
- For $A, B \subseteq 2^\omega$, f *reduces* A to B if $A = f^{-1}[B]$

The linearity phenomenon, II

Lipschitz reducibility generalizes to subsets $A, B \subseteq 2^\kappa$ for an arbitrary ordinal κ:

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We can identify ultrafilters on κ, which are subsets of $P(\kappa)$, with the corresponding subsets of 2^κ

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We can identify ultrafilters on κ, which are subsets of $P(\kappa)$, with the corresponding subsets of 2^κ

Theorem (UA)

For any ordinal κ, the set of countably complete ultrafilters on κ is wellordered by the Lipschitz order.

The Ketonen order

The *Ketonen order* is a structural refinement of $\leq_L$:

- $f : 2^\kappa \to 2^\kappa$ is *countably complete* if it is a countably complete endomorphism of the Boolean algebra 2^κ
- $U \leq_k W$ if there is a countably complete Lipschitz reduction from U to W

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Theorem

The following are equivalent:

- *For all ordinals κ, the set of countably complete ultrafilters on κ is wellordered by the Ketonen order.*
- *The Ultrapower Axiom holds.*

Ordinal definability

A set is *ordinal definable from x* if it is first-order definable in V from x and an ordinal

The linearity of the Ketonen order implies that every countably complete ultrafilter on an ordinal is ordinal definable.

Theorem (UA)

If κ is strongly compact, then there is a set $x \subseteq \kappa$ such that every set is ordinal definable from x.

The proof works by coding sets into κ-complete ultrafilters

Note: UA + LCA does not imply every set is ordinal definable

The Generalized Continuum Hypothesis

The idea of coding sets into ultrafilters gives various applications of UA + strongly compact cardinals to classical set theoretic questions. For example:

Theorem (UA)

Let κ be the least strongly compact cardinal. Then for all $\lambda \geq \kappa$, $2^\lambda = \lambda^+$.

One can also prove combinatorial principles like $\diamondsuit$ above κ

Note: UA + LCA does not imply the Continuum Hypothesis

Tarski's Question

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This is settled by UA:

Theorem (UA)

The least strongly compact cardinal is supercompact.

In fact, UA gives an analysis of arbitrary strongly compact cardinals...

Larger strongly compact cardinals

Not every strongly compact cardinal can be supercompact:

Theorem (Menas)

The first measurable limit of strongly compacts is strongly compact but not supercompact.

In fact, any measurable limit of strongly compacts is strongly compact, but not necessarily supercompact. Under UA, this is the only obstruction to supercompactness:

Theorem (UA)

A cardinal is strongly compact if and only if it is supercompact or a measurable limit of supercompacts.

Irreducible ultrapowers revisited

The analysis of arbitrary strongly compact cardinals turns out to be closely related to irreducible ultrapowers and the Rudin-Frolík order

If M is an ultrapower of V, the *width* of M is the least cardinal λ such that $M = M_U$ for some ultrafilter U on λ

Theorem (UA)

Suppose M is an irreducible ultrapower of width λ. If λ is either a successor cardinal or a singular strong limit cardinal, then M is closed under λ-sequences.

How to be tall and strong

κ is *tall* (resp. *strong*) if for all $\lambda \geq \kappa$, there is an embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ (resp. $V_\lambda \subseteq M$) and M is closed under κ-sequences

Analog of Tarski's question: is the first tall cardinal larger than the first measurable cardinal?

- This is independent of ZFC: the first measurable could be tall and the first tall cardinal could be strong
- UA answers the modified Tarski question positively, but

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Theorem

UA *does not decide whether the first tall cardinal is strong.*

Extender embeddings, I

What happens if you generalize UA to a wider class of elementary embeddings?

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Definition

An elementary embedding $j : V \to M$ is *λ-generated* if there is a set $A \subseteq M$ with $|A| < \lambda$ such that every element of M is definable in M from parameters in $A \cup \mathrm{ran}(j)$.

An ultrapower embedding is simply an $\aleph_0$-generated elementary embedding. An *extender embedding* is an elementary embedding that is generated by a set (i.e., is λ-generated for some λ).

UA for Extenders

Define a category **Ext**:

- Objects are extender ultrapowers of the universe
- Morphisms are internal extender embeddings

The *Extender Power Axiom* (EPA) asserts that **Ext** has the Amalgamation Property.

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Theorem

If there is a Woodin cardinal, then there is a canonical inner model with a Woodin cardinal in which EPA *is false.*

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Theorem

If there is a Woodin cardinal, then there is a canonical inner model with a Woodin cardinal in which EPA *is false.*

Larger cardinals refute EPA outright:

Theorem (with Woodin)

If there is a supercompact cardinal, then EPA *is false in* V.

The Ground Axiom

Another approach to eliminating independence:

- Recall Cohen's proof that $\text{ZFC} \nvdash \text{CH}$: fix a countable model $M \vDash \text{ZFC}$ and build a forcing extension $M[G]$ satisfying $\text{ZFC} + \neg\text{CH}$

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- Could one decide CH by simply positing that V is not of the form $M[G]$?

The *Ground Axiom* (GA) asserts that if $M \subsetneq V$ is an inner model and $G \in V$ is an M-generic set, then $M[G] \subsetneq V$.

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The Ground Axiom does not decide CH.

Under GA, can have $V = M[G]$ where G is an M-generic class

Combining axioms

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Theorem (UA + GA + a supercompact cardinal)

Every set is ordinal definable.

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Theorem (UA + GA + a supercompact cardinal)

Every set is ordinal definable.

Conjecture

UA + GA + *a supercompact cardinal implies* CH.

Thanks!