Reinhardt cardinals in inner models

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July 29, 2021

1 Introduction

A cardinal is *Reinhardt* if it is the critical point of an elementary embedding from the universe of sets to itself. Kunen [1] famously refuted the existence of Reinhardt cardinals using the Axiom of Choice (AC). It is a longstanding open problem whether Reinhardt cardinals are consistent if AC is dropped.

Noah Schweber [2] introduced the notion of a *uniformly supercompact cardinal*, a cardinal that is the critical point of an elementary embedding $j : V \to M$ such that $M^\alpha \subseteq M$ for all ordinals α. He posed the question of whether such a cardinal must be Reinhardt, and he also asked about the consistency strength of uniformly supercompact cardinals. Both questions remain open, but this note makes some progress on the matter.

Say a cardinal is *weakly Reinhardt* if it is the critical point of an elementary embedding $j : V \to M$ such that $j \restriction P(\alpha) \in M$ for all ordinals α. This condition is equivalent to requiring that $P(P(\alpha)) \subseteq M$ for all ordinals α. It seems to be weaker than demanding that $M^{P(\alpha)} \subseteq M$ for all ordinals α.

Theorem 2.2. *If there is a proper class of weakly Reinhardt cardinals, then there is an inner model with a proper class of Reinhardt cardinals.*

As a corollary, we obtain a consistency strength lower bound for a large cardinal that looks a bit more like Schweber's: for lack of a better term, say κ is *ultrafilter Reinhardt* if it is the critical point of an elementary embedding $j : V \to M$ such that for all ordinals α, $M^\alpha \subseteq M$ and $\beta(\alpha) \subseteq M$. Here $\beta(X)$ denotes the set of ultrafilters on X.

Proposition 2.3. *If κ is ultrafilter Reinhardt, then κ is weakly Reinhardt.*

Finally, our methods show that Reinhardt cardinals are compatible with choice principles that naively one might expect them to refute:

Corollary 2.6. *If the existence of a proper class of Reinhardt cardinals is consistent, then it is consistent with the dual Kinna-Wagner principle: every nonempty set is the surjective image of the powerset of an ordinal.*

1.1 Preliminaries

Our background theory is von Neumman-Bernays-Gödel (NBG) set theory without AC. Even though we work without AC, for us a *cardinal* is an ordinal number that is not in bijection with any smaller ordinal. Of course, if AC fails, there are sets whose cardinality cannot be identified with a cardinal in this sense. Still, for any set Y, one can define the *Hartogs number of* Y, denoted by $\aleph(Y)$, as the least cardinal κ such that there is no injection from κ to Y.

2 The inner model N_ν

If ν is a cardinal and X is a set, $\beta_\nu(X)$ denotes the set of ν-complete ultrafilters on X. In the context of choiceless large cardinal axioms, sufficiently complete ultrafilters on ordinals can often be treated as "idealized ordinals." The following lemma is a simple example of this phenomenon, although the pattern runs quite a bit deeper than this.

Lemma 2.1. *If there is a weakly Reinhardt cardinal, then for all sufficiently large cardinals ν, for any ordinal α, $\beta_\nu(\alpha)$ can be wellordered.*

Proof. Assume not. Let $j : V \to M$ witness that κ is weakly Reinhardt. By transfinite recursion, define a sequence of ordinals δ_ξ for $\xi \in \text{Ord}$, taking suprema at limit ordinals and, at successor stages, setting $\delta_{\xi+1}$ equal to the least ordinal $\alpha > \aleph(\beta(\delta_\xi))$ such that $\beta_{\delta_\xi}(\alpha)$ cannot be wellordered. Let $\epsilon_\xi = (\delta_\xi)^M$. Then $j(\delta_\kappa) = \epsilon_{j(\kappa)} > \epsilon_{\kappa+1}$. For each $\gamma < \epsilon_{j(\kappa)+1}$, let D_γ be the ultrafilter on $\delta_{\kappa+1}$ derived from j using γ, so $D_\gamma = \{A \subseteq \delta_{\kappa+1} : \gamma \in j(A)\}$.

Note that the function $\mathcal{D}(\gamma) = D_\gamma$ is simply definable from $j \restriction P(\delta_{\kappa+1})$, and so $d \in M$. For any $W \in \beta_{\epsilon_{j(\kappa)}}(\epsilon_{j(\kappa)+1})$, $\mathcal{D}$ is constant on a set in W because W is $\epsilon_{j(\kappa)}$-complete and $\text{ran}(\mathcal{D})$ has cardinality less than $\epsilon_{j(\kappa)}$. Indeed, $|\text{ran}(\mathcal{D})| < \aleph^M(\beta(\delta_\kappa))$, the Hartogs number of $\beta(\delta_\kappa)$ as computed in M, since $\text{ran}(\mathcal{D})$ is a wellorderable subset of $\beta(\delta_\kappa)$ in M. Moreover, $\aleph^M(\beta(\delta_\kappa)) \leq \aleph^M(\beta(\epsilon_\kappa))$ since $\epsilon_\kappa = \sup j[\delta_\kappa] \geq \delta_\kappa$, and $\aleph^M(\beta(\epsilon_\kappa)) < \epsilon_{\kappa+1} < \epsilon_{j(\kappa)}$ by the definition of the ordinals δ_ξ and the elementarity of j.

Suppose $U \in \beta_{\delta_\kappa}(\delta_{\kappa+1})$, and we will show that for $j(U)$-almost all γ, $D_\gamma = U$. Let D be the unique ultrafilter on $\delta_{\kappa+1}$ such that $D_\gamma = D$ for $j(U)$-almost all $\gamma < \epsilon_{j(\kappa)+1}$. If $A \in U$, then for all $\gamma \in j(A)$, $A \in D_\gamma$, and hence for $j(U)$-almost all γ, $A \in D_\gamma$. It follows that $A \in D$. This proves $U \subseteq D$, and so $U = D$. Therefore $\mathcal{D}$ is a surjection from the ordinal $\epsilon_{j(\kappa)+1}$ to $\beta_{\delta_\kappa}(\delta_{\kappa+1})$, which contradicts that $\beta_{\delta_\kappa}(\delta_{\kappa+1})$ cannot be wellordered. $\square$

Let us now define the inner model in which weakly Reinhardt cardinals become Reinhardt. Suppose ν is a cardinal. Let $\beta_\nu(\text{Ord}) = \bigcup_{\alpha\in\text{Ord}} \beta_\nu(\alpha)$ denote the class of ν-complete ultrafilters on ordinals. For any class C, we denote the class of all subsets of C by $P(C)$. Finally, let

$$N_\nu = L(P(\beta_\nu(\text{Ord})))$$

Granting that sufficiently complete ultrafilters on ordinals are idealized ordinals, the models N_ν are the corresponding idealizations of the inner model $L(P(\text{Ord}))$.

Theorem 2.2. *If there is a proper class of weakly Reinhardt cardinals, then for all sufficiently large cardinals ν, N_ν contains a proper class of Reinhardt cardinals.*

Proof. Let ν be a cardinal large enough that for all ordinals α, $\beta_\nu(\alpha)$ can be wellordered. Let $N = N_\nu$. We claim that if $\kappa > \nu$ is weakly Reinhardt, then κ is Reinhardt in N. To see this, let $j : V \to M$ witness that κ is weakly Reinhardt. We will show that $j(N) = N$ and $j \restriction X \in N$ for all $X \in N$. Hence $j \restriction N$ is an amenable class of N and in N, $j \restriction N$ is an elementary embedding from the universe to itself. Letting $\mathcal{C}$ denote the collection of classes amenable to N, it follows that $(N, \mathcal{C})$ is a model of NBG with a proper class of Reinhardt cardinals.

We first show that $j(N) = N$, or in other words, that N is correctly computed by M. (Here we use that $j(\nu) = \nu$ since $\nu < \kappa$.) The closure properties of M guarantee that all ultrafilters on ordinals are in M, and the elementarity of j implies that for all α, $\beta_\nu(\alpha)$ is wellorderable in M. Finally, since M is closed under wellordered sequences, $P(\beta_\nu(\alpha))$ is contained in M. This implies that N is correctly computed by M.

Finally, we show that for any $X \in N$, $j \restriction X \in N$. For this, it suffices to show that for any ordinal α, $j \restriction P(\beta_\nu(\alpha))$ is in N. Since $\beta_\nu(\alpha)$ is wellorderable in N, it suffices to show that $j \restriction P(\delta)$ belongs to N where $\delta = |\beta_\nu(\alpha)|^N$. Then letting $f : P(\beta_\nu(\alpha)) \to P(\delta)$ be a bijection in N,

$$j \restriction P(\beta_\nu(\alpha)) = j(f)^{-1} \circ (j \restriction P(\delta)) \circ f$$

and $j(f) \in N$ since $N = j(N)$ by the previous paragraph. But $j \restriction P(\delta) \in N$ because it is encoded by the extender $E = \langle D_\gamma : \gamma < j(\delta)\rangle$ where D_γ is the ultrafilter on δ derived from j using γ: indeed, if $A \subseteq \delta$, then $j(A) = \{\gamma < j(\delta) : A \in D_\gamma\}$. Since E is a wellordered sequence of ν-complete ultrafilters, $E \in N$. $\square$

We now show that ultrafilter Reinhardt cardinals are weakly Reinhardt, so the same consistency strength lower bound applies to them.

Proposition 2.3. *If κ is ultrafilter Reinhardt, then κ is weakly Reinhardt.*

Proof. Suppose $j : V \to M$ is elementary and for all ordinals α, $M^\alpha \subseteq M$ and $\beta(\alpha) \subseteq M$. We claim that for all ordinals δ, $j \restriction P(\delta) \in M$. Consider the extender $E = \langle D_\gamma : \gamma < j(\delta)\rangle$ given by letting $D_\gamma = \{A \subseteq \delta : \gamma \in j(A)\}$ be the ultrafilter derived from j using γ. Then $E \in M$, and hence $j \restriction P(\delta) \in M$, since for $A \in P(\delta)$, $j(A) = \{\gamma < j(\delta) : A \in D_\gamma\}$. $\square$

Finally, observe that the inner models considered here show that Reinhardt cardinals are compatible with global choice-like principles.

Proposition 2.4. *If there is a proper class of Reinhardt cardinals, then there is an inner model with a proper class of Reinhardt cardinals in which every set is constructible from a wellordered sequence of ultrafilters on ordinals.* $\square$

Here a set X is *constructible from* Y if $X \in L(Z)$ where Z is the transitive closure of $Y \cup \{Y\}$.

For any class C and any ordinal β, we define the iterated operation $P^\beta(C)$ by recursion: let $P^0(C) = C$, let $P^{\alpha+1}(C) = P(P^\alpha(C))$, and let $P^\gamma(C) = \bigcup_{\alpha<\gamma} P^\alpha(C)$ for limits γ.

Corollary 2.5. *If the existence of a proper class of Reinhardt cardinals is consistent, then it is consistent with $V = L(P^2(\mathrm{Ord}))$.*

Proof. For any cardinal λ, a λ-sequence $\langle S_\alpha : \alpha < \lambda\rangle$ of subsets of $P(\lambda)$ can be coded by a single subset of $P(\lambda \times \lambda)$; namely, $\{\{\alpha\} \times A : A \in S_\alpha\}$. So if every set is constructible from a wellordered sequence of ultrafilters on ordinals, then $V = L(P^2(\mathrm{Ord}))$. $\square$

Suppose α is an ordinal. The *Kinna-Wagner principle* KW_α asserts that every set has the same cardinality as a subset of $P^\alpha(\mathrm{Ord})$. The dual Kinna-Wagner principle, asserting that every set is the surjective image of a subset of $P^\alpha(\mathrm{Ord})$, does not seem to have been given a name, so let us call it KW^*_α here. For context, KW_α implies KW^*_α which implies $\mathrm{KW}_{\alpha+1}$. Also KW^*_α implies $V = L(P^\alpha(\mathrm{Ord}))$ if α is a limit ordinal, and KW^*_α implies $V = L(P^{\alpha+1}(\mathrm{Ord}))$ if α is a successor.

The following corollary of Proposition 2.3 therefore strengthens Corollary 2.5:

Corollary 2.6. *If the existence of a proper class of Reinhardt cardinals is consistent, then it is consistent with the dual Kinna-Wagner principle: every nonempty set is the surjective image of the powerset of an ordinal.*

Proof. This follows from the fact that every set is constructible from a wellordered sequence of ultrafilters, noting that if $\vec{U}$ is a wellordered sequence of ultrafilters, then $L(\vec{U}) = L(P(\delta))[\vec{U}]$ where $\delta = \text{rank}(\vec{U})$, and so for every $X \in L(\vec{U})$, for some ordinal α, there is a surjection from $P(\delta) \times \alpha$ onto X that is constructible from $\vec{U}$. □

More advanced techniques yield the following theorem, whose proof is omitted:

Theorem 2.7. *If there is a Reinhardt cardinal, then for a closed unbounded class of cardinals ν, there is a Reinhardt cardinal in $N_\nu(V_{\nu+1})$.* □

We mention this because it is conceivable that the existence of a proper class of Reinhardts is inconsistent, while the existence of a single Reinhardt is not. Here $N_\nu(V_{\nu+1})$ is the smallest inner model N such that $P(\beta_\nu(\text{Ord})) \cup V_{\nu+1} \subseteq N$. We also note that the proofs here easily generalize to show that if there is a proper class of Berkeley cardinals, then for sufficiently large ν, there is a proper class of Berkeley cardinals in N_ν.

3 Questions

Let us list some variants of Schweber's original questions that seem natural given the results of this note.

Question 3.1. Is the existence of a Reinhardt cardinal equiconsistent with the existence of a weakly Reinhardt cardinal?

Question 3.2. Is the existence of a Reinhardt cardinal compatible with $V = L(P(\text{Ord}))$?

In the context of NBG, a cardinal κ is Ord-*supercompact* if for all ordinals α, there is an elementary embedding $j : V \to M$ such that $j(\kappa) > \alpha$ and $M^\alpha \subseteq M$.

Question 3.3. Is NBG plus the existence of a proper class of Ord-supercompact cardinals equiconsistent with ZFC plus a proper class of supercompact cardinals?

References

[1] Kenneth Kunen. Elementary embeddings and infinitary combinatorics. *J. Symbolic Logic*, 36:407–413, 1971.

[2] Noah Schweber. Supercompact and Reinhardt cardinals without choice. MathOverflow.