Usuba's theorem is optimal

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Abstract

This paper answers a question of Usuba [18], establishing the optimality of the large cardinal assumption of his remarkable theorem that if there is an extendible cardinal, there is a minimum inner model from which the universe of sets can be obtained as a forcing extension.

1 Introduction

This paper studies the relationship between large cardinals and forcing from two perspectives. The first perspective, the classical one initiated by Lévy–Solovay [12], concerns the question: what impact does forcing have on large cardinals? The second perspective turns the first on its head: what influence do large cardinals exert on the structure of forcing? This second perspective can be traced back as far as the Martin-Solovay Σ^1_3-absoluteness theorem [13], but our approach is more closely aligned with Usuba's [18].

Lévy and Solovay proved that measurable cardinals are neither created nor destroyed by small forcings: that is, if κ is a cardinal and $\mathbb{P}$ is a partial order of cardinality less than κ, then κ is measurable if and only if κ is measurable in the forcing extension obtained by adjoining a generic subset of $\mathbb{P}$ to the universe of sets. This result has been generalized to all of the standard large cardinal properties [10, Theorem 21.2]. The interesting question that remains is the interaction between large cardinals and large forcings; that is, forcings with no cardinality constraints. Of course, it is easy to destroy a large cardinal by a large forcing: one can make it countable. A more subtle question is whether large cardinals can be preserved by forcings of unrestricted cardinality. This turns out to be possible in many situations, and often such preservation results are important aspects of consistency proofs; for example, the proof of the consistency of the failure of the singular cardinals hypothesis requires an analysis of the preservation of measurability under certain large forcings.

The work of Usuba, mentioned above, belongs to the field of set theoretic geology, a subject introduced by Hamkins and Reitz [14]. Their idea was to study not only the forcing extensions of the universe of sets, but also the ways in which the universe of sets itself can be represented as a forcing extension of some inner model. An inner model M is said to be a *ground* if there is a partial order $\mathbb{P} \in M$ and an M-generic filter $G \subseteq \mathbb{P}$ such that

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$M[G] = V$. Laver and Woodin [11] proved independently that every ground is (uniformly) definable from parameters over the universe of sets. This renders the existence of nontrivial grounds a question in first-order set theory. The statement that the universe of sets has no nontrivial grounds is the Hamkins-Reitz *Ground Axiom* [14].

On the one hand, the Ground Axiom holds in canonical inner models of set theory such as L and $L[U]$, but on the other hand, it is consistent that it fails; indeed, the Ground Axiom becomes false after one performs any nontrivial forcing. Since large cardinals are preserved by small forcings, no large cardinal hypothesis can imply the Ground Axiom. Surprisingly, however, assuming large cardinal hypotheses, Usuba established a bound on the number of grounds; thus large cardinals imply a weak version of the Ground Axiom, which asserts not that there are no nontrivial grounds whatsoever but instead that there are not too many. Specifically, Usuba's seminal result shows that if κ is an extendible cardinal, there are at most $2^{2^{(\kappa^+)}}$ grounds. Despite the perhaps coarse appearance of this bound (especially considering that κ itself is already enormous enough), the crucial conclusion is that there is only a *set* of grounds, not a proper class.

After proving this theorem, Usuba posed the question: if κ is extendible and M is a ground, is there a partial order $\mathbb{P} \in M$ of cardinality less than κ and a generic filter $G \subseteq \mathbb{P}$ such that $V = M[G]$? In this case, M is said to be a *κ-ground* of V. (Usuba's proof shows that there must be such a partial order $\mathbb{P}$ with cardinality at most $2^{(\kappa^+)}$.) From the more classical perspective on forcing, the question can be reformulated: if κ is extendible in a forcing extension W of V, must V be a κ-ground of W?

A negative answer to Usuba's question would demonstrate that the large cardinal hypothesis of Usuba's theorem is optimal. To see why, consider the situation where there is an extendible cardinal κ and a ground M such that for all partial orders $\mathbb{P} \in M \cap V_\kappa$ and all M-generic filters $G \subseteq \mathbb{P}$, $V \neq M[G]$. Then V_κ is a model of ZFC containing a proper class of grounds. (This is more fully explained in Corollary 3.2, but it follows from Usuba's theorem [18].) Since κ is extendible, V_κ satisfies all large cardinal hypotheses short of extendibility. It follows that an extendible cardinal is the weakest large cardinal hypothesis that proves that there is just a set of grounds.

The first result of this paper shows that it is consistent that Usuba's question has a positive answer, in that no large forcing preserves an extendible cardinal. This uses a strengthening of Reitz's Ground Axiom called the *Local Ground Axiom*, which states that the Ground Axiom holds in V_λ whenever λ is a Beth fixed point. Reitz's proof that one can class force the Ground Axiom in fact shows that one can class force the Local Ground Axiom; in particular, the Local Ground Axiom is consistent.

Theorem 2.4. *Assume the Local Ground Axiom. Suppose W is a forcing extension of V and κ is extendible in W. Then there is a partial order $\mathbb{P} \in V$ of cardinality less than κ and a V-generic filter $G \subseteq \mathbb{P}$ such that $W = V[G]$.*

The key consequence of the Local Ground Axiom that is used in this theorem is that it implies the local definability of the mantle, the inner model formed by intersecting all grounds of V. Building on this, we connect Usuba's question to the quantifier complexity of the *mantle*:

Theorem 2.5. *Suppose κ is extendible. Then the following are equivalent:*

(1) The mantle is a κ-ground.

(2) The mantle is Δ_2-definable from an ordinal less than κ.

(3) The mantle is Π_2-definable from parameters in V_κ.

The main result of this paper, however, is that it is consistent that the answer to Usuba's question is no. This requires a new argument for lifting extendibility to forcing extensions, and Theorem 2.4 shows that any such argument requires a preliminary class forcing to ensure that the Local Ground Axiom fails in the mantle.

The solution is related to a question posed by Woodin in the context of Solovay's Σ_2-potentialism. If a is a set and φ is a Σ_2-formula with one free variable, then $\varphi(a)$ is a *potential formula* if for every cardinal κ, there is a forcing extension W such that $(V_\kappa)^W = V_\kappa$ and W satisfies $\varphi(a)$. The Σ_2-*Potentialist Principle* states that every potential formula is true. Woodin posed the problem: is the Σ_2-Potentialist Principle consistent?[1] The author and Eyal Kaplan [6] have recently proved that the answer to Woodin's question is positive assuming the consistency of a proper class of supercompact cardinals by combining the techniques of this paper with ideas from Gitik's theory of Prikry-type forcings.

Our main result here, which predates Goldberg–Kaplan's work, hinges on a weaker principle that we will prove is consistent. If a is a set and φ is a Σ_2-formula with one free variable, then $\varphi(a)$ is a *strongly potential formula* if for every cardinal κ, there is a κ-directed closed forcing extension W such that W satisfies $\varphi(a)$. We will use the following *Weak Σ_2-Potentialist Principle*: every strongly potential formula is true.

The Weak Σ_2-Potentialist Principle implies that the Ground Axiom holds but the Local Ground Axiom fails. Using this principle, we get a negative answer to Usuba's question:

Theorem 3.1. *Assume the Weak Σ_2-Potentialist Principle. Then for any extendible cardinal κ, there is a forcing extension W such that κ is extendible in W and $W \neq V[X]$ for any $X \subseteq V$ in W with $|X| < \kappa$.*

Finally, we prove that the Weak Σ_2-Potentialist Principle is consistent:

Theorem 3.7. *There is a class forcing extension in which the Weak Σ_2-Potentialist Principle is true.*

This shows that the Weak Σ_2-Potentialist Principle is consistent relative to ZFC, but for our purposes it is important to show that it is consistent with extendible cardinals.

Proposition 3.9. *The forcing of Theorem 3.7 preserves extendible cardinals.*

1.1 Preliminaries

A natural question in the theory of forcing is whether the universe of sets V is always a definable subclass of its generic extensions. Of course, one must allow parameters, since it is consistent that there is a real number that is ordinal definable in V but not in one of its generic extensions; see [21, Example 3.1], which is due to McAloon. Moreover there are various counterexamples in class forcing [7], so it makes sense to restrict attention to set forcing.

The following theorem, proved independently by Laver and Woodin [11, 19] settles the definability question positively:

Theorem 1.1 (Ground Model Definability Theorem)**.** *If $\mathbb{P}$ is a partial order of cardinality κ and $G \subseteq \mathbb{P}$ is V-generic, then V is Δ_2-definable over $V[G]$ from $P(\kappa) \cap V$.*

[1] In fact, he asked about the parameter-free version of the principle.

Theorem 1.1 is instrumental in ensuring that the statements of the theorems to follow are in fact expressible in first-order set theory.

The now-standard proof of Theorem 1.1, which is due to Hamkins [4], proceeds by establishing a more general statement. If $M \subseteq N$ are models of ZFC, then M has the *κ-cover property* in N if every subset of M of cardinality less than κ in N is included in a set in M of cardinality less than κ in M (or equivalently in N). If $X \in M$ and $A \subseteq X$, then A is *κ-approximated by M* if for all $\sigma \subseteq X$ with $\sigma \in M$ and $|\sigma|^M < \kappa$, $A \cap \sigma \in M$. We say M has the *κ-approximation property* in N if every $A \in N$ that is κ-approximated by M is an element of M.

Theorem 1.2 (Hamkins)**.** *If M is an inner model that has the δ-approximation and cover properties in N, then M is Δ_2-definable over N from $H(\delta^+) \cap M$.*

If $\mathbb{P}$ is a partial order of cardinality κ and $G \subseteq \mathbb{P}$ is V-generic, then V has the κ^+-cover and approximation properties in $V[G]$ and $H(\kappa^{++}) \cap V$ is definable in $V[G]$ from $P(\kappa) \cap V$, so Theorem 1.1 follows from Theorem 1.2 in the case $\delta = \kappa^+$.[2]

The following theorem of Usuba [17] proves the (Downwards Directedness of Grounds) DDG conjecture of Fuchs-Hamkins-Reitz [4], establishing that the grounds have a more intricate structure than one might at first expect:

Theorem 1.3 (Downwards Directedness of Grounds)**.** *If $\langle M_i \rangle_{i \in I}$ is a set of grounds, then there is a ground N contained in $\bigcap_{i \in I} M_i$.*

We will actually make use of a sharper version of this theorem that follows from Usuba's proof:

Theorem 1.4 (Usuba)**.** *If κ is a cardinal and $\langle M_i : i < \kappa \rangle$ is a family of κ-cc grounds, then $\bigcap_{i<\kappa} M_i$ is a κ^+-cc ground.*

The *mantle*, denoted by $\mathbb{M}$, is the intersection of all the grounds. The DDG theorem has the following consequence for the mantle:

Corollary 1.5 (Usuba)**.** *The mantle is an inner model of* ZFC*.*

The mantle is the largest definable inner model of ZFC that is invariant under forcing. There is little one can prove about the structure of this model, however:

Theorem 1.6 (Fuchs-Hamkins-Reitz [4])**.** *Every model of* ZFC *is the mantle of another model of* ZFC*.*

Corollary 1.7. *For any sentence φ in the language of set theory,* ZFC *proves that $\mathbb{M} \vDash \varphi$ if and only if* ZFC *proves φ.*

Under large cardinal axioms, Usuba showed that more can be said about the structure of the mantle. We will actually need a stronger form of Usuba's theorem that also appears in [18]. If κ is a cardinal, then an inner model M is a *κ-ground* if there is a partial order $\mathbb{P} \in M$ of cardinality less than κ such that $V = M[G]$ for some M-generic filter $G \subseteq \mathbb{P}$. The *κ-mantle*, denoted by $\mathbb{M}_\kappa$, is the intersection of all κ-grounds.

[2] To define $H(\kappa^{++}) \cap V$ from $P(\kappa) \cap V$ in $V[G]$, first note that $H(\kappa^+)^V$ is definable from $P(\kappa) \cap V$ (since the former is in fact interpretable in the latter). Then we use that $H(\kappa^+)^V = H(\kappa^+) \cap V$ since V and $V[G]$ agree about κ^+. One can now define $P(\kappa^+) \cap V$ from $H(\kappa^+) \cap V$ in $V[G]$, because by the κ^+-approximation property, a set $A \in P(\kappa^+) \cap V[G]$ belongs to $P(\kappa^+) \cap V$ if and only if $A \cap \alpha \in H(\kappa^+) \cap V$ for all $\alpha < \kappa^+$. Finally, one can define $H(\kappa^{++}) \cap V$ from $P(\kappa^+) \cap V$ as before.

Theorem 1.8 (Usuba [18])**.** *If there is an extendible cardinal, then the mantle is a ground. In fact, the mantle is equal to the intersection of all κ-grounds.*

Corollary 1.9. *If there is an extendible cardinal, then there is a set of grounds. More precisely, there is a Σ_2-formula φ with two free variables and a set I such that for any ground M, there is some $i \in I$ such that $M = \{x : \varphi(i, x)\}$.*

Proof. Applying Theorem 1.8, let $\mathbb{B}$ be a complete Boolean algebra in the mantle such that for some $\mathbb{M}$-generic $G \subseteq \mathbb{B}$, $V = \mathbb{M}[G]$. By the intermediate model theorem ([10, Lemma 15.43]), every ground N of V is of the form $V[G \cap \mathbb{A}]$ where $\mathbb{A} \subseteq \mathbb{B}$ is a complete subalgebra of $\mathbb{B}$ in $\mathbb{M}$. This easily yields the conclusion of the corollary. $\square$

Corollary 1.10. *If there is an extendible cardinal, then the mantle satisfies the Ground Axiom.*

2 The Local Ground Axiom

2.1 Extendibility over the mantle

Recall Usuba's question: if κ is an extendible in an outer model W of V, must V be a κ-ground of W? In this section, we show that it is consistent that the answer to Usuba's question is yes, and furthermore that the answer is yes in every set forcing extension. This is straightforward, but the proof suggests the route to proving the much more interesting negative consistency result by imposing two fundamental constraints on the structure of a counterexample (Section 2.2).

If M is an inner model of ZFC, a cardinal κ is *extendible over* M if for all ordinals $\lambda \geq \kappa$, for some $\lambda' > \lambda$ and some elementary embedding $j : V_{\lambda+1} \to V_{\lambda'+1}$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$, $j \restriction V^M_\lambda$ belongs to M, and $j(V^M_\lambda) = V^M_{\lambda'}$. This seems to be a natural generalization to extendibility of Woodin's concept of a weak extender model for supercompactness [20]. A ground may be a weak extender model for the supercompactness of κ without being a κ-ground, but the following proposition, based on Usuba's original proof of Theorem 1.8 under the hypothesis of a hyperhuge cardinal [17], shows that the same cannot be said for this generalization.

Proposition 2.1. *An extendible cardinal κ is extendible over a ground M if and only if M is a κ-ground.*

Proof. The reverse direction is clear: if κ is extendible and M is a κ-ground, then κ is extendible over M by a generalization of the Lévy–Solovay theorem; for example, this follows from Hamkins's theorem that extensions with the approximation and cover properties have no new large cardinals [9].

Assume instead that κ is extendible over a ground M, and we will show that M is a κ-ground. Let $\mathbb{Q} \in M$ be a partial order carrying a M-generic filter H such that $V = M[H]$.

Suppose λ is a Beth fixed point larger than the rank of $\mathbb{Q}$. Let $j : V_{\lambda+1} \to V_{\lambda'+1}$ be an elementary embedding such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $j \restriction V^M_\lambda$ belongs to M, and $j(V^M_\lambda) = V^M_{\lambda'}$. Since $V^M_{\lambda'}[H] = V_{\lambda'}$, $V_{\lambda'}$ satisfies that there is a partial order $\mathbb{P} \in V^M_{j(\kappa)}$ carrying a $V^M_{\lambda'}$-generic filter G such that $V^M_{\lambda'}[G] = V_{\lambda'}$. Therefore V_λ satisfies that there is a partial order $\mathbb{P} \in V^M_\kappa$ carrying a V^M_λ-generic filter G such that $V^M_\lambda[G] = V_\lambda$.

By the pigeonhole principle, there is a partial order $\mathbb{P} \in V_\kappa^M$ carrying a filter G such that for a proper class of Beth fixed points λ, G is V_λ^M-generic and $V_\lambda = V_\lambda^M[G]$. It follows that G is M-generic and $V = M[G]$, which proves the proposition. $\square$

Corollary 2.2. *If κ is the least extendible cardinal, the mantle $\mathbb{M}$ is a κ-ground if and only if κ is extendible over $\mathbb{M}$.* $\square$

The *Local Ground Axiom* states that the Ground Axiom holds in V_λ whenever λ is a Beth fixed point.

Proposition 2.3. *If the mantle satisfies the Local Ground Axiom and κ is an extendible cardinal, then κ is extendible over the mantle.*

Proof. We claim that for all Beth fixed points $\lambda > \kappa$, $V_\lambda^{\mathbb{M}} = \mathbb{M}^{V_\lambda}$. By Usuba's theorem (Theorem 1.8), $\mathbb{M}^{V_\lambda} \subseteq \mathbb{M}$; this is because the mantle is the intersection of all κ-grounds, and $\mathbb{M}^{V_\lambda}$ is also contained in the intersection of all κ-grounds since $\lambda \geq \kappa$.

Conversely, if P is a ground of V_λ, then since $V_\lambda^{\mathbb{M}}$ and P are grounds of V_λ, they have a common ground N, and since $V_\lambda^{\mathbb{M}}$ satisfies the Ground Axiom, $N = V_\lambda^{\mathbb{M}}$, and so $V_\lambda^{\mathbb{M}} \subseteq P$.

Now suppose $\lambda > \kappa$ is a Beth fixed point, and let $j : V_{\lambda+1} \to V_{\lambda'+1}$ be an elementary embedding such that $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. We must show that $j \restriction V_\lambda^{\mathbb{M}}$ belongs to $\mathbb{M}$ and $j(V_\lambda^{\mathbb{M}}) = V_{\lambda'}^{\mathbb{M}}$. That $j \restriction V_\lambda^{\mathbb{M}}$ belongs to $\mathbb{M}$ is follows from Usuba's theorem and the Lévy–Solovay absoluteness of extendibility to κ-grounds. That is, by the Lévy–Solovay absoluteness of extendibility to κ-grounds (which again follows from Hamkins's theorem [9]), $j \restriction V_\lambda^{\mathbb{M}}$ belongs to every κ-ground. Since the mantle is the intersection of all κ-grounds, $j \restriction V_\lambda^{\mathbb{M}}$ belongs to $\mathbb{M}$.

On the other hand, clearly λ' is a Beth fixed point, and so

$$j(V_\lambda^{\mathbb{M}}) = j(\mathbb{M}^{V_\lambda}) = \mathbb{M}^{V_{\lambda'}} = V_{\lambda'}^{\mathbb{M}}$$

$\square$

Theorem 2.4. *Assume the mantle satisfies the Local Ground Axiom. Suppose W is a forcing extension of V and κ is extendible in W. Then there is a partial order $\mathbb{P} \in V$ of cardinality less than κ and a V-generic filter $G \subseteq \mathbb{P}$ such that $W = V[G]$.*

Proof. Applying Proposition 2.3 in W, κ is extendible over the mantle. Therefore by Corollary 2.2, the mantle is a κ-ground of W. Since $\mathbb{M} \subseteq V \subseteq W$, the intermediate model theorem [10, Lemma 15.43] implies that V is a κ-ground of W. $\square$

We now consider the connection between Usuba's theorem and the definability of the mantle. The mantle is Π_3-definable: $x \in \mathbb{M}$ if and only if there are arbitrarily large Beth fixed points λ such that $x \in \mathbb{M}^{V_\lambda}$. If κ is extendible, then Usuba's theorem proves that the mantle is Δ_2-definable using κ as a parameter, since it is equal to the κ-mantle. We now show that if the mantle is Π_2-definable from anything smaller than κ, then the mantle is a ground.

Theorem 2.5. *Suppose κ is extendible. Then the following are equivalent:*

(1) The mantle is a κ-ground.

(2) The mantle is Δ_2-definable from an ordinal less than κ.

(3) The mantle is Π_2-definable from parameters in V_κ.

Proof. To see that (1) implies (2), note that if the mantle is a κ-ground, then it is equal to the γ-mantle for some $\gamma < \kappa$. The γ-mantle is Δ_2-definable from γ by the ground model definability theorem [11]. The implication from (2) to (3) is trivial.

Finally, the proof that (3) implies (1) is a generalization of the proof of Theorem 2.4. Assume the mantle is Π_2-definable from some $p \in V_\kappa$. We will show that κ is extendible over the mantle and then appeal to Corollary 2.2 to conclude that the mantle is a κ-ground.

Fix a Π_2 formula φ such that

$$\mathbb{M} = \{a \in V : \varphi(a, p)\}$$

By the reflection theorem, there is a proper class of Beth fixed points $\lambda > \kappa$ such that

$$V_\lambda \cap \mathbb{M} = \mathbb{M}^{V_\lambda} = \{a \in V_\lambda : V_\lambda \vDash \varphi(a, p)\}$$

Let $j : V_\lambda \to V_{\lambda'}$ be an elementary embedding such that $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda'$. Then $j \restriction (V_\lambda \cap \mathbb{M}) \in \mathbb{M}$ since $\mathbb{M} = \mathbb{M}_\kappa$. (The details appear in the second-to-last paragraph of Proposition 2.3.)

By elementarity,

$$\mathbb{M}^{V_{\lambda'}} = \{a \in V_{\lambda'} : V_{\lambda'} \vDash \varphi(a, p)\} \supseteq \{a \in V_{\lambda'} : \varphi(a, p)\} = V_{\lambda'} \cap \mathbb{M}$$

since Π_2 formulas are downwards absolute to any Beth fixed point level of the cumulative hierarchy. Since $j(V_\lambda \cap \mathbb{M}) \in \mathbb{M}$, we must have $\mathbb{M}^{V_{\lambda'}} \subseteq V_{\lambda'} \cap \mathbb{M}$, and hence equality holds. Therefore j witnesses that κ is λ-extendible over $\mathbb{M}$. $\square$

It is consistent with ZFC that the mantle is not a ground and yet is Σ_1-definable, so the previous theorem requires a large cardinal hypothesis. (For example, if V is a proper class Easton product forcing extension of L, then $\mathbb{M} = L$, and L is Σ_1-definable.) In Corollary 3.10, we will show that it is consistent with an extendible cardinal that the mantle is Σ_2-definable but not a κ-ground, where κ is the least extendible; in fact, the model from Theorem 3.1 (our a negative answer to Usuba's question) will satisfy $\mathbb{M} = \text{HOD}$. So the definability hypotheses of Theorem 2.5 cannot be weakened.

Finally we note the connection between the definability of the mantle and the Local Ground Axiom. If the mantle satisfies the Local Ground Axiom, then the mantle is Δ_2-definable since it is equal to the union over all Beth fixed points λ of the mantle of V_λ. Conversely, if the mantle is Δ_2-definable and satisfies the Ground Axiom, then a version of the Local Ground Axiom must hold for the mantle. Namely, there is a Δ_2-definable (over V) proper class of cardinals λ such that $V_\lambda \cap \mathbb{M}$ satisfies the Ground Axiom. The Local Ground Axiom is just the special case in which this class is the class of Beth fixed points.

2.2 Two constraints

Theorem 2.4 highlights two key constraints that guide the way to a counterexample to Usuba's question. The first constraint is that by Theorem 2.4, it is consistent that Usuba's question has a positive answer in all forcing extensions. This suggests that to find a counterexample, one should start with a preparatory class forcing.

The second constraint is more subtle. By Corollary 2.2, if one is to preserve an extendible cardinal by a set forcing that is not a small forcing, this preservation cannot be proved using the standard lifting arguments for extendible cardinals. The reason is that if a cardinal κ

is shown to be extendible in a forcing extension W of V using these lifting arguments, then W will satisfy that κ is extendible over V, and hence V is a κ-ground of W.

Since it is hard to see how to preserve extendible cardinals without a lifting argument, answering Usuba's question seems to require inventing a novel forcing notion along with an entirely new preservation argument for extendible cardinals. The solution instead is simply to reformulate extendibility in terms of normal fine ultrafilters (Lemma 3.4) and then to employ the standard lifting arguments from the theory of supercompactness to show that, thanks to our preliminary preparatory forcing, this reformulation is preserved by a forcing notion that is about as far from novel as one can get: an Easton product of Cohen forcings.

3 The Weak Σ_2-Potentialist Principle

3.1 The main theorem

In this section, we show that assuming the Weak Σ_2-Potentialist Principle, the answer to Usuba's question is no.

Recall from the introduction that if a is a set and φ is a Σ_2-formula with one free variable, then $\varphi(a)$ is a *strongly potential formula* if for every cardinal κ, there is a κ-directed closed forcing extension W such that W satisfies $\varphi(a)$. The *Weak Σ_2-Potentialist Principle* states that every strongly potential formula is true.

Theorem 3.1. *Assume the Weak Σ_2-Potentialist Principle. Then for any extendible cardinal κ, there is a forcing extension W such that κ is extendible in W and $W \neq V[X]$ for any $X \subseteq V$ in W with $|X| < \kappa$.*

This theorem shows that the large cardinal hypothesis of Usuba's theorem ([17] or Theorem 1.8 above) is optimal in the following sense.

Corollary 3.2. *If* ZFC *is consistent with the existence of an extendible cardinal, it is consistent with the existence of an extendible cardinal κ such the conclusion of Usuba's theorem fails in V_κ.*

If κ is extendible, then V_κ should satisfy every large cardinal hypothesis short of extendibility. For example, V_κ satisfies that there is a proper class of supercompact cardinals, a proper class of supercompact limits of supercompact cardinals, a proper class of supercompact cardinals that are ω-extendible, and so on. Therefore no large cardinal hypothesis short of extendibility implies the conclusion of Usuba's theorem.[3]

The Weak Σ_2-Potentialist Principle turns out to be related to an indestructibility hypothesis that is slightly easier to use. A cardinal λ is *Σ_n-correct* if V_λ is a Σ_n-elementary substructure of V. A cardinal λ is *indestructibly Σ_n-correct* if V_λ is a Σ_n-elementary substructure of V in any λ-directed closed forcing extension.

Proposition 3.3. *The following are equivalent:*

(1) The Weak Σ_2-Potentialist Principle holds.

(2) There is a proper class of indestructibly Σ_2-correct cardinals.

[3]Usuba observed that his theorem does not follow from a supercompact cardinal; see the remarks following the statement of [17, Theorem 1.4]. This was extended to a proper class of supercompact cardinals by [3, Theorem 3.8]. Corollary 3.2 is in a sense the ultimate extension of these results.

Proof. That (2) implies (1) is immediate. For the other direction, assume the Weak Σ_2-Potentialist Principle and let λ be a Σ_3-correct cardinal. We claim that λ is indestructibly Σ_2-correct.

To see this, fix $a \in V_\lambda$, and suppose that in some λ-directed closed forcing extension, the Σ_2-formula $\varphi(a)$ is true. We will show that $\varphi(a)$ holds in V_λ. Then in V, for all $\gamma < \lambda$, there is a γ-directed closed forcing extension in which $\varphi(a)$ holds, and since V_λ is Σ_2-correct, V_λ satisfies that there is a γ-directed closed forcing extension in which $\varphi(a)$ holds. Now V_λ satisfies that for all cardinals γ, there is a γ-directed closed forcing extension in which $\varphi(a)$ holds, and so since V_λ is Σ_3-correct, this is true; that is, V satisfies that for all cardinals γ, there is a γ-directed closed forcing extension in which $\varphi(a)$ holds. Applying the Weak Σ_2-Potentialist Principle, it follows that $\varphi(a)$ is true in V. Now since V_λ is Σ_2-correct, $\varphi(a)$ is true in V_λ, as desired. □

Theorem 3.7 shows that the existence of a proper class of indestructibly Σ_2-correct cardinals, and hence the hypothesis of Theorem 3.1, can be class forced over any model of ZFC while preserving all extendible cardinals. Before turning to this, however, let us prove Theorem 3.1.

The key to the proof of Theorem 3.1 is the following characterization of extendibility, observed independently and earlier by Bagaria [1]. For any cardinal λ, let $T_{\kappa,\lambda}$ (resp. $T^*_{\kappa,\lambda}$) denote the set of $\sigma \in P_\kappa(\lambda)$ such that the ordertype of σ is Σ_2-correct (resp. indestructibly Σ_2-correct) in V_κ. In most cases of interest, κ itself will be Σ_2-correct, in which case a cardinal is Σ_2-correct in V_κ if and only if it is truly Σ_2-correct. Similarly, if κ is Σ_2-correct, then a cardinal is indestructibly Σ_2-correct in V_κ if and only if it is truly indestructibly Σ_2-correct.

Lemma 3.4. *A cardinal κ is extendible if and only if for arbitrarily large cardinals λ, there is a normal fine κ-complete ultrafilter on $T_{\kappa,\lambda}$.*

To keep the paper self-contained, we recall the definition of a normal fine ultrafilter. A family of sets $\mathcal{C}$ is a *cover* of a set X if $\bigcup_{A\in\mathcal{C}} A = X$. An ultrafilter U on a cover $\mathcal{C}$ of X is *fine* if every set in U covers X and *normal* if every choice function on $\mathcal{C}$ is constant on a set in U.[4] We will mostly deal with the elementary embedding characterization of normality and fineness [5, Lemma 4.4.9]:

Lemma 3.5. *An ultrafilter U on a cover $\mathcal{C}$ of X is normal and fine if and only if there is an elementary embedding $j : V \to M$ such that U is the ultrafilter on $\mathcal{C}$ derived from j using $j[X]$, or in other words, $U = \{A \subseteq \mathcal{C} : j[X] \in j(A)\}$.* □

Proof of Lemma 3.4. For the forwards direction, suppose λ is Σ_2-correct and there is an elementary embedding $j : V_{\lambda+1} \to V_{\lambda'+1}$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. We will show that there is a normal fine κ-complete ultrafilter on $T_{\kappa,\lambda}$. Since λ is Σ_2-correct, λ is Σ_2-correct in $V_{j(\kappa)}$, and so $j[\lambda] \in T_{j(\kappa),j(\lambda)}$. It follows that there is a normal fine κ-complete ultrafilter on $T_{\kappa,\lambda}$; namely, the ultrafilter derived from j using $j[\lambda]$.

Now we show that if there is a normal fine κ-complete ultrafilter $\mathcal{U}$ on $T_{\kappa,\lambda}$, then κ is γ-extendible for all $\gamma < \lambda$. In particular, this implies the reverse direction of the lemma. Let $j : V \to M$ be the ultrapower embedding associated to $\mathcal{U}$.

[4] Note that if κ is a limit ordinal, then κ is a cover of itself; a fine ultrafilter on κ is just an ultrafilter that does not concentrate on a bounded subset of κ; and a normal fine ultrafilter on κ is just a normal ultrafilter on κ in the usual sense. Similarly, a normal fine κ-complete ultrafilter on $P_\kappa(\lambda)$ is a supercompactness measure, and a normal fine κ-complete ultrafilter on $[\lambda]^\kappa$ is a huge measure.

We first show that λ is Σ_2-correct in $V^M_{j(\lambda)}$. By Łoś's theorem, since $T_{\kappa,\lambda} \in \mathcal{U}$ and $[\mathrm{id}]_{\mathcal{U}} = j[\lambda]$, clearly $\lambda = \mathrm{ot}(j[\lambda])$ is Σ_2-correct in $V^M_{j(\kappa)}$. But κ is Σ_2-correct in V_λ since $\mathcal{U}$ witnesses that κ is λ-supercompact. Therefore by the elementarity of j, $j(\kappa)$ is Σ_2-correct in $V^M_{j(\lambda)}$. Now $V_\lambda \preceq_{\Sigma_2} V^M_{j(\kappa)} \preceq_{\Sigma_2} V^M_{j(\lambda)}$, which implies that λ is Σ_2-correct in $V^M_{j(\lambda)}$.

For all $\gamma < \lambda$, $j \restriction V_\gamma$ belongs to $V^M_{j(\lambda)}$, and in $V^M_{j(\lambda)}$, $j \restriction V_\gamma$ is an elementary embedding from V_γ to $V_{j(\gamma)}$. Therefore $V^M_{j(\lambda)}$ satisfies that κ is γ-extendible for all $\gamma < \lambda$. Since λ is Σ_2-correct in $V^M_{j(\lambda)}$, V_λ satisfies that κ is γ-extendible for all $\gamma < \lambda$, and since the γ-extendibility of κ is expressed by a Σ_2-formula, this is upwards absolute to V. Hence κ is is γ-extendible for all $\gamma < \lambda$, as claimed. □

The proof of Lemma 3.4 yields:

Lemma 3.6. *If κ is extendible and λ is indestructibly Σ_2-correct, then there is a normal fine κ-complete ultrafilter on $T^*_{\kappa,\lambda}$.* □

Given this, we turn to the proof of Theorem 3.1

Proof of Theorem 3.1. We begin by defining a partial function $f : \kappa \to \kappa$ with the following Laver-like property: for any indestructibly Σ_2-correct cardinal $\lambda \geq \kappa$, there is a normal fine κ-complete ultrafilter $\mathcal{U}$ on $T^*_{\kappa,\lambda}$ such that $j_{\mathcal{U}}(f)(\kappa) = \lambda$. The function f is defined by recursion: if $f \restriction \alpha$ has been defined, let $f(\alpha)$ be the least indestructibly Σ_2-correct cardinal $\gamma \geq \alpha$ such that there is no normal fine α-complete ultrafilter $\mathcal{W}$ on $T^*_{\alpha,\lambda}$ such that $j_{\mathcal{W}}(f \restriction \alpha)(\alpha) = \gamma$. If no such γ exists, then $f(\alpha)$ is left undefined. Note that if f is defined at α, then $f(\alpha) < \kappa$ because κ is Σ_3-correct.

Suppose towards a contradiction that f is not as desired, and let λ be the least indestructibly Σ_2-correct cardinal above κ such that there is no normal fine κ-complete ultrafilter $\mathcal{W}$ on $T^*_{\kappa,\lambda}$ such that $j_{\mathcal{W}}(f)(\kappa) = \lambda$. By Lemma 3.6, let $\mathcal{U}$ be *any* normal fine κ-complete ultrafilter on $T^*_{\kappa,\lambda}$, and we will show that $j_{\mathcal{U}}(f)(\kappa) = \lambda$, contrary to the definition of λ.

Let $j : V \to M$ be the ultrapower embedding associated to $\mathcal{U}$, and note that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $M^\lambda \subseteq M$, and λ is indestructibly Σ_2-correct in M. (To see that λ is indestructibly Σ_2-correct in M, note that λ is indestructibly Σ_2-correct in $V^M_{j(\kappa)}$ by Łoś's theorem and $V^M_{j(\kappa)} \preceq_{\Sigma_3} M$ by elementarity and the Σ_3-correctness of κ. Indestructible Σ_2-correctness is a Π_2 property, and therefore λ is indestructibly Σ_2-correct in M.)

Since $V_{\lambda+1} \subseteq M$, λ is the least indestructibly Σ_2-correct cardinal of M such that $\lambda \geq \kappa$ and there is no normal fine κ-complete ultrafilter $\mathcal{W}$ on $T^*_{\kappa,\lambda}$ such that $j_{\mathcal{W}}(f)(\kappa) = \lambda$. Since f is a function from κ to κ and κ is the critical point of j, $j(f) \restriction \kappa = f$. Therefore by elementarity and the definition of f, $j(f)(\kappa)$ is the least indestructibly Σ_2-correct cardinal γ of M such that $\gamma \geq \kappa$ and there is no normal fine κ-complete ultrafilter $\mathcal{W}$ on $T^*_{\kappa,\lambda}$ such that $j_{\mathcal{W}}(f)(\kappa) = \gamma$; in other words, $j(f)(\kappa) = \lambda$. Thus the ultrafilter $\mathcal{U}$ contradicts the definition of λ.

For $\alpha < \kappa$, define an increasing continuous sequence of ordinals γ_α by setting $\gamma_0 = 0$ and $\gamma_{\alpha+1} = f(\gamma_\alpha)^+$. For $\eta_0 \leq \eta_1 \leq \kappa$, let

$$\mathbb{P}_{\eta_0,\eta_1} = \prod_{\eta_0 \leq \alpha < \eta_1} \mathrm{Add}(\gamma_{\alpha+1}, 1)$$

be the Easton support product. Let $G \subseteq \mathbb{P}_{0,\kappa}$ be a V-generic filter, and note that for each $\eta_0 \leq \eta_1 \leq \kappa$,

$$G_{\eta_0,\eta_1} = \{p \restriction [\eta_0, \eta_1) : p \in G\}$$

is a V-generic filter on $\mathbb{P}_{\eta_0,\eta_1}$. By Easton's Lemma [10, Lemma 15.19], for any $X \subseteq V$ in $V[G]$ with $|X| < \kappa$, $X \in V[G_{0,\eta}]$ where $\eta < \kappa$ is least such that $\gamma_{\eta+1} > |X|$. Therefore $V[X] \neq V[G]$.

The result is proved by taking $W = V[G]$ as soon as we show that κ is extendible in $V[G]$. To prove this, we will verify the criterion of Lemma 3.4.

Note that the class of indestructibly Σ_2-correct cardinals is closed, and by our hypothesis it is unbounded (Proposition 3.3). Therefore there is a proper class of indestructibly Σ_2-correct *singular* cardinals $\lambda > \kappa$. Such a cardinal λ has the property that $2^\lambda = \lambda^+$ by Solovay's theorem [15] that the singular cardinals hypothesis holds above a supercompact cardinal. Suppose $\lambda > \kappa$ is an indestructibly Σ_2-correct cardinal such that $2^\lambda = \lambda^+$. We claim that $V[G]$ satisfies that there is a normal fine κ-complete ultrafilter on $(T_{\kappa,\lambda})^{V[G]}$.

Working in V, let $\mathcal{U}$ be a κ-complete normal fine ultrafilter on $T^*_{\kappa,\lambda}$ such that $j_\mathcal{U}(f)(\kappa) = \lambda$. Let $j : V \to M$ be the ultrapower embedding, so $j = j_\mathcal{U}$.

The forcing $j(\mathbb{P}_{0,\kappa})$ is isomorphic to the product $\mathbb{P}_{0,\kappa} \times (\mathbb{P}_{\kappa,j(\kappa)})^M$. Note that $\mathbb{Q} = (\mathbb{P}_{\kappa,j(\kappa)})^M$ is λ^+-directed closed in M since

$$M \vDash \mathbb{Q} = \prod_{\kappa \leq \alpha < j(\kappa)} \mathrm{Add}(\gamma_{\alpha+1}, 1)$$

and $\gamma_\alpha > \lambda$ for $\kappa \leq \alpha < j(\kappa)$. (Here we extend the sequence γ_α to values of α greater than or equal to κ by setting $\gamma_\alpha = j(\vec{\gamma})_\alpha$ where $\vec{\gamma} = \langle \gamma_\alpha \rangle_{\alpha<\kappa}$.)

Since M is closed under λ-sequences, $\mathbb{Q}$ really is λ^+-directed closed. In addition, $|P^M(\mathbb{Q})|^M \leq j(2^\kappa) < j(\lambda) < (2^\lambda)^+ = \lambda^{++}$. The final bound follows from the fact that $2^\lambda = \lambda^+$. Therefore in V, $|P^M(\mathbb{Q})| \leq \lambda^+$ and $\mathbb{Q}$ is λ^+-closed, and so one can build an M-generic filter $H \subseteq \mathbb{Q}$ with $H \in V$.

The closure of $\mathbb{Q}$ implies that $M[H]$ contains no new dense subsets of $\mathbb{P}_{0,\kappa}$, and so G is an $M[H]$-generic filter on $\mathbb{P}_{0,\kappa}$. By standard results on mutual genericity, this means that $G \times H$ is an M-generic filter on $\mathbb{P}_{0,\kappa} \times \mathbb{Q}$. The cardinal λ is Σ_2-correct in $M[H]$ since λ is indestructibly Σ_2-correct in M. (Small forcing preserves Σ_2-correct cardinals by the usual Lévy–Solovay argument.) Since G is $M[H]$-generic for a forcing in $(V_\lambda)^{M[H]}$, λ is Σ_2-correct in $M[H \times G]$.

Finally, identifying $j(\mathbb{P}_{0,\kappa})$ with $\mathbb{P}_{0,\kappa} \times \mathbb{Q}$ in the natural way, $j[G] = G \times \{1\} \subseteq G \times H$, and so the embedding $j : V \to M$ lifts uniquely to an elementary embedding $j^* : V[G] \to M[G \times H]$ such that $j(G) = G \times H$. Since λ is Σ_2-correct in $M[G \times H]$, the set $j[\lambda]$ belongs to $T_{j(\kappa),j(\lambda)}$ as computed in $M[G \times H]$. As a consequence, working in $V[G]$, the ultrafilter

$$\mathcal{D} = \{A \subseteq T_{\kappa,\lambda} : j[\lambda] \in j(A)\}$$

derived from j using $j[\lambda]$ is a normal fine κ-complete ultrafilter on $T_{\kappa,\lambda}$. $\square$

3.2 Forcing the Weak Σ_2-Potentialist Principle

In this section, we show that the hypotheses of Theorem 3.1 are consistent relative to an extendible cardinal.

Theorem 3.7. *There is a class forcing $\mathbb{Q}$ such that the Weak Σ_2-Potentialist Principle holds in $V^\mathbb{Q}$.*

The idea of the proof is to build an iterated forcing that makes each strongly potential formula true one by one, in such a way that once a formula is made true, it remains true in the final extension. The only subtlety is that it is not clear that the strongly potential formulas of the final extension are strongly potential in the intermediate stages, so we will instead proceed by forcing formulas that are potential in a more local sense.

Turning to the details, we define an iterated forcing $\langle \mathbb{Q}_\alpha, \dot{\mathbb{P}}_\alpha : \alpha \in \text{Ord}\rangle$ by recursion as follows. The forcing will be an Easton support iteration, so all that must be specified is $\mathbb{Q}_0$ and the forcing $\dot{\mathbb{P}}_\alpha$ to be used at stage α. In order to use Hamkins's gap forcing theorem [8] later on (Theorem 3.8), we let $\mathbb{Q}_0$ be the Cohen forcing, $\text{Add}(\omega, 1)$. Now suppose $\mathbb{Q}_\alpha$ has been defined. Let $G_\alpha \subseteq \mathbb{Q}_\alpha$ be V-generic. Working in $V[G_\alpha]$, we will define a forcing $\mathbb{P}_\alpha$ for which $\dot{\mathbb{P}}_\alpha$ will be the canonical $\mathbb{Q}_\alpha$-name.

Let $\gamma = |\mathbb{Q}_\alpha|^+$. For each $y \in H(\gamma)^{V[G_\alpha]}$ and each Σ_2-formula φ, say that $\varphi(y)$ is an *α-potential formula* if in $V[G_\alpha]$, there is a γ-directed closed partial order that forces $\varphi(y)$. For each α-potential formula $\varphi(y)$ and each Beth fixed point $\lambda > \gamma$, let $S_{\varphi,y,\lambda}$ be the set of partial orders $\mathbb{P}$ in $V[G_\alpha]_\lambda$ such that the following hold in $V[G_\alpha]$:

- $\mathbb{P}$ is a γ-directed closed partial order.
- If $H \subseteq \mathbb{P}$ is $V[G_\alpha]$-generic, then $\varphi(y)$ is true in $V[G_\alpha][H]_\lambda$.

Let $S_{\varphi,y} = S_{\varphi,y,\lambda}$ for the least λ such that $S_{\varphi,y,\lambda} \neq \emptyset$. Let $\mathbb{P}_\alpha$ be the lottery sum of all forcings $\mathbb{P}$ such that $\mathbb{P} \in S_{\varphi,y}$ for some α-potential formula.

Let $\mathbb{Q}$ denote the class direct limit of the forcings $\mathbb{Q}_\alpha$, and let G be a V-generic filter on $\mathbb{Q}$. Let G_α be the restriction of G to $\mathbb{Q}_\alpha$. (The class forcing $\mathbb{Q}$ preserves ZFC for fairly standard reasons: for every cardinal λ, $\mathbb{Q}$ is isomorphic to the two-step iteration $\mathbb{Q}_\lambda * \dot{\mathbb{Q}}_{\lambda,\infty}$ where $\mathbb{Q}_\lambda$ is a set forcing and $\dot{\mathbb{Q}}_{\lambda,\infty}$ is λ-closed. This ensures that the powerset of λ exists in $V[G]$ and there is no unbounded function from λ to the ordinals definable over $V[G]$.) Let $\mathbb{P}_\alpha = (\dot{\mathbb{P}}_\alpha)_{G_\alpha}$, and let H_α be the $V[G_\alpha]$-generic filter on $\mathbb{P}_\alpha$ induced by G.

Proof of Theorem 3.7. Assume towards a contradiction that in $V[G]$, there is a strongly potential Σ_2-formula $\varphi(y)$ that is false.

Suppose α is an ordinal such that $y \in H(\gamma)^{V[G]}$ where $\gamma = |\mathbb{Q}_\alpha|^+$. We claim that $\varphi(y)$ is an α-potential formula, meaning that there is a γ-directed closed partial order in $V[G_\alpha]$ that forces $\varphi(y)$. Since $\varphi(y)$ is strongly potential in $V[G]$, there is a γ-directed closed partial order $\mathbb{S}$ in $V[G]$ such that $\mathbb{S}$ forces $\varphi(y)$. Let α' be large enough that $\mathbb{S} \in V[G_{\alpha'}]$ and $\mathbb{S}$ forces $\varphi(y)$ in $V[G_{\alpha'}]$. Let $\mathbb{Q}_{\alpha,\alpha'} \in V[G_\alpha]$ denote the factor forcing from $V[G_\alpha]$ to $V[G_{\alpha'}]$, so that G induces a $V[G_\alpha]$ generic filter $G_{\alpha,\alpha'}$ such that $V[G_{\alpha'}] = V[G_\alpha][G_{\alpha,\alpha'}]$. Then $\mathbb{Q}_{\alpha,\alpha'}$ is γ-directed closed in $V[G_\alpha]$. Let $\dot{\mathbb{S}}$ be a $\mathbb{Q}_{\alpha,\alpha'}$-name for $\mathbb{S}$ in $V[G_\alpha]$; so $\dot{\mathbb{S}}_{G_{\alpha,\alpha'}} = \mathbb{S}$ and $\mathbb{Q}_{\alpha,\alpha'}$ forces $\dot{\mathbb{S}}$ to be γ-closed. Then $\mathbb{Q}_{\alpha,\alpha'} * \dot{\mathbb{S}}$ is a γ-directed closed forcing in $V[G_\alpha]$ that forces $\varphi(y)$, so $\varphi(y)$ is an α-potential formula.

Fix an ordinal α large enough that $y \in H(\gamma)^{V[G]}$ where $\gamma = |\mathbb{Q}_\alpha|^+$. Note that $y \in V[G_\alpha]$ since $V[G_\alpha]^\gamma \cap V[G] \subseteq V[G_\alpha]$. Let $\dot{y}$ be a $\mathbb{Q}_\alpha$-name such that $(\dot{y})_{G_\alpha} = y$. Let $p \in G$ be a condition forcing that $\varphi(\dot{y})$ is strongly potential (identifying $\dot{y}$ with a $\mathbb{Q}$-name in the natural way). We claim that the set of conditions in $\mathbb{Q}$ forcing $\varphi(\dot{y})$ is dense below p. Let $q \leq p$ be any condition. Let $\beta \geq \alpha$ be large enough that $q \in \mathbb{Q}_\beta$. As a condition in $\mathbb{Q}_\beta$, q forces that $\varphi(\dot{y})$ is β-potential by the previous paragraph. Let $\gamma' = |\mathbb{Q}_{\beta+1}|^+$. Since q forces that $\varphi(\dot{y})$ is β-potential, there is a condition $r \in \mathbb{Q}_{\beta+1}$ extending q such that for some Beth fixed point $\lambda < \gamma'$, r forces that $V_\lambda^{\mathbb{Q}_{\beta+1}}$ satisfies $\varphi(\dot{y})$. Since $V^{\mathbb{Q}_{\beta+1}}$ and $V^{\mathbb{Q}}$ have the same γ'-sequences, when viewed as a condition in $\mathbb{Q}$, r still forces $\varphi(\dot{y})$ to hold in $V_\lambda^{\mathbb{Q}}$, and so r is a condition in $\mathbb{Q}$ forcing $\varphi(\dot{y})$.

Since the set of conditions in $\mathbb{Q}$ forcing $\varphi(\dot{y})$ is dense below p, there is some $q \in G$ forcing $\varphi(\dot{y})$. Therefore $V[G]$ satisfies $\varphi(y)$, contrary to our assumption. $\square$

Theorem 3.8. *For all ordinals λ, the following are equivalent:*

(1) λ is Σ_2-correct in V.

(2) λ is indestructibly Σ_2-correct in $V[G_\lambda]$.

(3) λ is indestructibly Σ_2-correct in $V[G_\alpha]$ for all $\alpha \geq \lambda$.

(4) λ is indestructibly Σ_2-correct in $V[G]$.

(5) λ is Σ_2-correct in $V[G]$.

Proof. The key observation is that if λ is a Σ_2-correct cardinal, then $\langle \mathbb{Q}_\alpha, \dot{\mathbb{P}}_\alpha : \alpha < \lambda \rangle$ is contained in V_λ. This is because $\dot{\mathbb{P}}_\alpha$ is Σ_2-definable using $H(\gamma)$ as a parameter, where $\gamma = |\mathbb{Q}_\alpha|^+$.

We first show that (1) implies (2). Fix a set $y \in V[G_\lambda]_\lambda$ and a Σ_2-formula φ such that $\varphi(y)$ can be forced by a λ-directed closed forcing in $V[G_\lambda]$. An argument similar to the proof of Theorem 3.7 shows that $\varphi(y)$ is α-potential for all sufficiently large $\alpha < \lambda$, and that the set of conditions in $\mathbb{Q}_\lambda$ forcing $\varphi(y)$ is dense below any condition that forces the existence of a λ-directed closed partial order $\mathbb{S}$ such that $\mathbb{S}$ forces $\varphi(y)$. It follows that $\varphi(y)$ is true in $V[G_\lambda]$.

To see that (2) implies (3), note that an indestructibly Σ_2-correct cardinal λ remains indestructibly Σ_2-correct in any λ-directed closed set forcing extension.

Now assume (3), and let us show (4). The issue is that while $V[G]$ is a λ-directed closed forcing extension of $V[G_\lambda]$, it is not a set forcing extension. But suppose that $y \in V[G]_\lambda$ and in $V[G]$, $\varphi(y)$ holds in a λ-directed closed forcing extension. Let η be a Beth fixed point of $V[G]$ that is large enough that $V[G]_\eta$ satisfies that $\varphi(y)$ holds in a λ-directed closed forcing extension. Let α be large enough that $V[G_\alpha]_\eta = V[G]_\eta$. Then $\varphi(y)$ holds in a λ-directed closed set forcing extension of $V[G_\alpha]_\eta$, and hence in a λ-directed closed set forcing extension of $V[G_\alpha]$. Since λ is indestructibly Σ_2-correct in $V[G_\alpha]$, $\varphi(y)$ holds in $V[G_\alpha]_\lambda = V[G]_\lambda$. This verifies that λ is indestructibly Σ_2-correct in $V[G]$.

(4) trivially implies (5). The proof that (5) implies (1) uses Hamkins's gap forcing theorem [8]. This theorem implies that V has the ω_1-approximation and cover properties in $V[G]$. (The reason is that the forcing factors as an atomless countable forcing (namely, Cohen forcing) followed by a countably strategically closed forcing. The gap forcing theorem implies that any forcing of this form has the ω_1-approximation and cover properties.)

Now assuming (5), $\lambda > \omega_1$ and so V is Δ_2-definable over $V[G]$ from a parameter $p \in V_\lambda$. This is because of the generalization of the ground model definability theorem to inner models with the approximation and cover properties (Theorem 1.2).

Let ψ_0 and ψ_1 be Σ_2 and Π_2-formulas respectively defining V from p in $V[G]$. Suppose that $y \in V_\lambda$ and φ is a Σ_2-formula such that there is some α such that $V_\alpha \vDash \varphi(y)$. Note that for some Beth fixed point $\gamma > \alpha$, there is a transitive set $N \in V[G]_\gamma$ such that for $i = 0, 1$,

$$N = \{x \in V[G]_\alpha : V[G]_\gamma \vDash \psi_i(x, p)\}$$

and $N \vDash \varphi(y)$. (Choose any sufficiently large Beth fixed point γ and let $N = V_\alpha$.) The statement "there is some level of the cumulative hierarchy containing a transitive set N defined by ψ_0 and ψ_1 from p and satisfying $\varphi(y)$" is Σ_2-expressible in terms of y and p.

Therefore this statement reflects into $V[G]_\lambda$. This implies that for some $\bar{\alpha} < \bar{\gamma} < \lambda$, there is a transitive set $\bar{N}$ such that $y \in \bar{N}$,

$$\bar{N} = \{x \in V[G]_{\bar{\alpha}} : V[G]_{\bar{\gamma}} \vDash \psi_i(x,p)\}$$

and $\bar{N} \vDash \varphi(y)$. It follows that $\bar{N} = V_{\bar{\alpha}}$, and hence $V_{\bar{\alpha}} \vDash \varphi(y)$. This proves that λ is Σ_2-reflecting in V. $\square$

The proofs of all of the previous theorems of this section can be carried out if we weaken directed closure everywhere to strategic closure. This yields a variant of the Σ_2-Potentialist Principle for *strategically potential formulas*, which are Σ_2-formulas (with parameters) that can be forced by arbitrarily strategically closed forcings. The restriction to directed closed forcing is necessary in the master condition arguments used to establish the preservation of large cardinals under the forcing $\mathbb{Q}$.

Proposition 3.9. *The forcing $\mathbb{Q}$ preserves extendible cardinals.*

Proof. The standard preservation arguments for extendible cardinals under Easton support iterations of increasingly directed closed forcings can be used to establish this proposition; for example, see [2]. For variety, however, we will show how the characterization of extendibility from Lemma 3.4 can be used to prove the preservation result using the standard lifting arguments for supercompact cardinals.

In V, let $\lambda \geq \kappa$ be a singular Σ_2-reflecting cardinal and let $\mathcal{U}$ be a normal fine ultrafilter on $T_{\kappa,\lambda}$, which exists by Lemma 3.4. Let $j : V \to M$ be the ultrapower embedding. By Solovay's theorem that SCH holds above a strongly compact cardinal [15], $2^\lambda = \lambda^+$, and so a standard master condition argument argument allows us to extend j to an elementary embedding $j^* : V[G_\lambda] \to M[G_\lambda][H]$ where $H \in V[G_\lambda]$ is an $M[G_\lambda]$-generic filter on $(\dot{\mathbb{Q}}_{\lambda,j(\lambda)})^M_{G_\lambda}$ The ultrafilter $\mathcal{U}^*$ derived from j using $j[\lambda]$ is a normal fine ultrafilter in $V[G_\lambda]$. By Theorem 3.8, λ remains Σ_2-correct in $M[G_\lambda][H]$, and so $j[\lambda] \in j^*((T_{\kappa,\lambda})^{V[G_\lambda]})$. Let $\mathcal{U}^*$ be the $V[G_\lambda]$-ultrafilter on $(T_{\kappa,\lambda})^{V[G_\lambda]}$ derived from j^* using $j[\lambda]$. Then $\mathcal{U}^*$ remains an ultrafilter in $V[G]$, and since $V[G]_\lambda = V[G_\lambda]_\lambda$, $(T_{\kappa,\lambda})^{V[G_\lambda]} = (T_{\kappa,\lambda})^{V[G]}$.

Therefore for every singular Σ_2-correct cardinal of V, there is in $V[G]$ a normal fine κ-complete ultrafilter on $(T_{\kappa,\lambda})^{V[G]}$. Since there are arbitrarily large singular Σ_2-correct cardinals in V, Lemma 3.4 implies that κ is extendible in $V[G]$. $\square$

The following corollary shows that Theorem 2.5 cannot be improved:

Corollary 3.10. *It is consistent that the mantle is Σ_2-definable without parameters but is not a κ-ground where κ is the least extendible cardinal.*

For the proof, we will need the following simple lemma which involves Reitz's Continuum Coding Axiom [14]:

Lemma 3.11. *The Weak Σ_2-Potentialist Principle implies the Continuum Coding Axiom.*

Proof. For any set of ordinals A, the statement that A is coded into the continuum function is a strongly potential Σ_2-formula, and therefore by the Weak Σ_2-Potentialist Principle, it is true. $\square$

Proof of Corollary 3.10. Let W be the model constructed in the proof of Theorem 3.1. We claim that $\mathbb{M}^W = \mathrm{HOD}^W$. Since V is a model of the Weak Σ_2-Potentialist Principle, V satisfies Reitz's Continuum Coding Axiom [14]. Therefore V satisfies the Ground Axiom, and hence V is the mantle of W. Moreover the Continuum Coding Axiom implies that $V \subseteq \mathrm{HOD}^W$. On the other hand, $\mathrm{HOD}^W \subseteq V$ since W is obtained from V via a forcing that is ordinal definable and homogeneous in V. Here it is important to note that the function f used in the proof of Theorem 3.1 is ordinal definable in V. $\square$

Finally, let us prove Corollary 3.2 on the optimality of Usuba's theorem.

Proof of Corollary 3.2. Let κ be extendible. By Theorem 3.7, we may assume the Weak Σ_2-Potentialist Principle. Let W be the forcing extension of Theorem 3.1. Let $W_\kappa = (V_\kappa)^W$. We will show that in W_κ, the conclusion of Usuba's theorem fails, meaning that the set of grounds of W_κ has cardinality κ.

Recall that the mantle is Π_3-definable; see the comments before Theorem 2.5. Also recall that extendible cardinals are Σ_3-correct [16, Proposition 23.10].

Assume towards a contradiction that the set of grounds of W_κ has cardinality less than κ. Then in particular, there are fewer than κ-many κ-grounds of W, since if N is a κ-ground of W, then $N \cap W_\kappa$ is a ground of W_κ. By Usuba's downward directed grounds theorem [17], the intersection $\mathbb{M}_\kappa$ of all κ-grounds is a κ-ground of W. But by Usuba's theorem Theorem 1.8, $\mathbb{M}_\kappa$ is equal to the mantle of W, and hence the mantle is a κ-ground of W. Since $\mathbb{M} \subseteq V \subseteq W$, the intermediate model theorem implies that V is a κ-ground of W, contrary to our choice of W. $\square$

4 Questions

This work leaves several variants of Usuba's question open.

Question 4.1. Suppose κ is extendible. Must the mantle be a κ^+-ground? Must the mantle be a ground for κ-cc forcing?

Usuba's proof of the Downwards Directed Grounds Hypothesis does show that the κ-mantle is a ground for κ^+-cc forcing, and therefore so is the mantle assuming κ is extendible.

Question 4.2. Suppose κ is extendible. Must κ be extendible in the mantle? Must κ be strongly compact in the mantle?

We close the paper with two results connected to this question.

Proposition 4.3. *If κ is extendible, then in the mantle, every κ^+-complete filter extends to a κ-complete ultrafilter.*

Proof. Note that if U is a κ-complete ultrafilter, then $U \cap \mathbb{M} \in \mathbb{M}$. This is because $U \cap N \in N$ for every κ-ground N by the Lévy–Solovay theorem [12], and $\mathbb{M}$ is equal to the intersection of all κ-grounds by Usuba's theorem.

Suppose F is a κ^+-complete filter in the mantle. Since V is a κ^+-cc extension of $\mathbb{M}$, F generates a κ^+-complete filter F' in V. Since κ is strongly compact in V, F' extends to a κ-complete ultrafilter U. But $U \cap \mathbb{M} \in \mathbb{M}$, so F extends to a κ-complete ultrafilter in $\mathbb{M}$. $\square$

Proposition 4.4. *If κ is extendible and the mantle is Σ_3-definable from parameters in V_κ, then κ is a strong cardinal in the mantle.*

Proof. Fix a set $p \in V_\kappa$ and a formula φ such that $x \in \mathbb{M}$ if and only if for all sufficiently large ordinals α, $V_\alpha \vDash \varphi(x,p)$. By the reflection theorem, there are arbitrarily large cardinals λ such that $\mathbb{M}^{V_\lambda} = \mathbb{M} \cap V_\lambda$ and $x \in V_\lambda \cap \mathbb{M}$ if and only if for all sufficiently large ordinals $\alpha < \lambda$, $V_\alpha \vDash \varphi(x,p)$.

Fix any such cardinal λ, and let ν be large enough such that for all $x \in V_\lambda \cap \mathbb{M}$, for all $\beta \geq \nu$, $V_\beta \vDash \varphi(x,p)$. Since κ is extendible, for some ordinal $\lambda' > \nu$ there is an elementary $j : V_\lambda \to V_{\lambda'}$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. Since the mantle is equal to the κ-mantle, $j \restriction (V_\lambda \cap \mathbb{M}) \in \mathbb{M}$. (Again, we are using the Lévy-Solovay theorem to get $j \restriction (V_\lambda \cap \mathbb{M})$ into every κ-ground; this version of Lévy-Solovay follows from Hamkins's results in [9].) Since $\lambda' \geq \nu$, for all $x \in V_\lambda \cap \mathbb{M}$, $V_{\lambda'} \vDash \varphi(x,p)$. Since $(V_{\lambda'}, p)$ is elementarily equivalent to (V_λ, p), for all $x \in V_{\lambda'}$ such that $V_{\lambda'} \vDash \varphi(x,p)$, $V_{\lambda'} \vDash x \in \mathbb{M}$. In particular, $V_\lambda \cap \mathbb{M} \subseteq \mathbb{M}^{V_{\lambda'}} = j(V_\lambda \cap \mathbb{M})$. Therefore $j \restriction (V_\lambda \cap \mathbb{M})$ witnesses that κ is λ-strong in $\mathbb{M}$. ☐

Our final questions are basic definability questions in set-theoretic geology:

Question 4.5. Can the mantle be Σ_3-definable but not Σ_2-definable? Can the mantle be Π_3-definable but not Σ_3-definable?

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