

Usuba's extendible cardinal

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Abstract

Answering a question of Usuba [5], we show that an extendible cardinal can be preserved by a set forcing that is not a small forcing.

1 Introduction

This note addresses the following question of Usuba [5] by showing that its answer is independent of ZFC:

Question 1.1. Suppose W is a set forcing extension of V and κ is a cardinal that is extendible in W . Must there exist a partial order $\mathbb{P} \in V$ of cardinality less than κ and a V -generic filter G on \mathbb{P} such that $W = V[G]$?

Usuba's question is motivated by his celebrated theorem on the relationship between extendible cardinals and forcing. Recall that if $M \subseteq N$ are models of ZFC, then M is a *ground of N* if N is a set forcing extension of M . Surprisingly, the grounds of a model of ZFC can be defined uniformly over that model, enabling one to study the structure of the grounds of the universe of set theory itself, which are usually simply called *grounds*.

Theorem (Usuba [5]). *If there is an extendible cardinal, the universe of sets has a minimum ground.*

Another theorem of Usuba shows that the intersection of all grounds of V is in general a model of ZFC, which is called the *mantle* and denoted by \mathbb{M} . If there is a minimum ground, then it is of course equal to the mantle. At heart, Question 1.1 asks about the distance between the universe of sets and the mantle:

Question. Suppose κ is an extendible cardinal. Must there be a partial order $\mathbb{P} \in V_\kappa \cap \mathbb{M}$ and an \mathbb{M} -generic filter $G \subseteq \mathbb{P}$ such that $V = \mathbb{M}[G]$?

If no such partial order exists, then Usuba's Theorem fails in V_κ where κ is the least extendible cardinal. Since one would expect V_κ to satisfy every large cardinal axiom short of extendibility, the consistency of a negative answer to this question would argue that the large cardinal hypothesis of Usuba's Theorem cannot be weakened.

The *Ground Axiom*, introduced by Hamkins-Reitz, states that $V = \mathbb{M}$. A consequence of Usuba's Theorem is that if there is an extendible cardinal, the mantle satisfies the Ground Axiom. The *Local Ground Axiom* states that the Ground Axiom holds in V_λ whenever λ is a Beth fixed point. It is not hard to force the Local Ground Axiom by a class Easton iteration that preserves extendible cardinals.

Proposition 2.5. *Assume that the mantle satisfies the Local Ground Axiom. Then the answer to Question 1.1 is yes.*

More interestingly, it is also consistent that the answer to Question 1.1 is no. A cardinal λ is *indestructibly Σ_n -correct* if V_λ is a Σ_n -elementary substructure of any λ -directed closed forcing extension of V . Every Σ_1 -correct cardinal is indestructibly Σ_1 -correct; every Laver indestructible supercompact cardinal is indestructibly Σ_2 -correct; and by a theorem of Bagaria-Hamkins-Tsaprounis-Usuba [2], there cannot be an indestructibly Σ_3 -correct cardinal.

Theorem 3.1. *Assume there is a proper class of indestructibly Σ_2 -correct cardinals. Then the answer to Question 1.1 is no.*

Given a model with an extendible cardinal and a proper class of supercompact cardinals, one can force to obtain a model satisfying the hypothesis of Theorem 3.1 using the global Laver preparation, as described in [1, Section 1]. Proposition 3.4 shows that if one just wants indestructible Σ_2 -correctness, the supercompact cardinals are overkill: it suffices to start with a model with an extendible cardinal and a proper class of inaccessible Σ_2 -correct cardinals. It seems likely that the inaccessible Σ_2 -correct cardinals can be dispensed with as well, but the author does not see how to do this.

2 The Local Ground Axiom

2.1 Extendibility over the mantle

In this section, we show that it is consistent that the answer to Question 1.1 is yes. This is straightforward and follows Usuba's original, but the proof suggests the route to proving the much more interesting negative consistency result by imposing two fundamental constraints on the structure of a counterexample.

If $M \subseteq N$ are models of ZFC, then M is a κ -ground of N if there is a partial order $\mathbb{P} \in M$ of M -cardinality less than κ carrying a M -generic filter G such that $N = M[G]$. The κ -mantle is the intersection of all κ -grounds of V . We will need Usuba's Theorem in the following form:

Theorem 2.1 (Usuba). *If κ is extendible, then the mantle is equal to the κ -mantle.* \square

If M is an inner model of ZFC, a cardinal κ is *extendible over M* if for all ordinals $\lambda \geq \kappa$, for some $\lambda' > \lambda$, there is an elementary embedding $j : V_{\lambda+1} \rightarrow V_{\lambda'+1}$ such that $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $j \upharpoonright V_\lambda^M$ belongs to M , and $j(V_\lambda^M) = V_{\lambda'}^M$. The following proof is based on Usuba's original proof of his theorem using hyperhuge cardinals [4].

Proposition 2.2. *An extendible cardinal κ is extendible over a ground W if and only if W is a κ -ground.*

Proof. Let $\mathbb{Q} \in W$ be a partial order carrying a W -generic filter H such that $V = W[H]$.

Suppose λ is a Beth fixed point larger than the rank of \mathbb{Q} . Let $j : V_{\lambda+1} \rightarrow V_{\lambda'+1}$ be an elementary embedding such that $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $j \upharpoonright W_\lambda$ belongs to W , and $j(V_\lambda^W) = V_{\lambda'}^W$. Since $V_{\lambda'}^W[H] = V_{\lambda'}$, $V_{\lambda'}$ satisfies that there is a partial order $\mathbb{P} \in W_{j(\kappa)}$ carrying a $V_{\lambda'}^W$ -generic filter G such that $V_{\lambda'}^W[G] = V_{\lambda'}$. Therefore V_λ satisfies that there is a partial order $\mathbb{P} \in V_\kappa^W$ carrying a V_λ^W -generic filter G such that $V_\lambda^W[G] = V_\lambda$.

By the pigeonhole principle, there is a partial order $\mathbb{P} \in V_\kappa^W$ carrying a filter G such that for a proper class of Beth fixed points λ , G is V_λ^W -generic and $V_\lambda = V_\lambda^W[G]$. It follows that G is W -generic and $V = W[G]$, which proves the proposition. \square

Corollary 2.3. *If κ is the least extendible cardinal, the mantle is a κ -ground if and only if κ is extendible over \mathbb{M} .* \square

The *Local Ground Axiom* states that the Ground Axiom holds in V_λ whenever λ is a Beth fixed point.

Proposition 2.4. *If the mantle satisfies the Local Ground Axiom and κ is an extendible cardinal, then κ is extendible over the mantle.*

Proof. We claim that for all Beth fixed points $\lambda > \kappa$, $V_\lambda^{\mathbb{M}} = \mathbb{M}^{V_\lambda}$. By Usuba's Theorem [5], $\mathbb{M}^{V_\lambda} \subseteq \mathbb{M}$. Conversely, if W is a ground of V_λ , then since \mathbb{M} and W are grounds of V_λ , they have a common ground N , and since $V_\lambda^{\mathbb{M}}$ satisfies the Ground Axiom, $N = V_\lambda^{\mathbb{M}}$, and so $V_\lambda^{\mathbb{M}} \subseteq W$.

Now suppose λ is a Beth fixed point, and let $j : V_{\lambda+1} \rightarrow V_{\lambda'+1}$ be an elementary embedding such that $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. We must show that $j \upharpoonright V_\lambda^{\mathbb{M}}$ belongs to \mathbb{M} and $j(V_\lambda^{\mathbb{M}}) = V_{\lambda'}^{\mathbb{M}}$. That $j \upharpoonright V_\lambda^{\mathbb{M}}$ belongs to \mathbb{M} is immediate by Usuba's Theorem and the absoluteness of extendibility to κ -grounds. On the other hand, clearly λ' is a Beth fixed point, and so

$$j(V_\lambda^{\mathbb{M}}) = j(\mathbb{M}^{V_\lambda}) = \mathbb{M}^{V_{\lambda'}} = V_{\lambda'}^{\mathbb{M}} \quad \square$$

Proposition 2.5. *If the mantle satisfies the Local Ground Axiom, then the answer to Question 1.1 is yes.*

Proof. Suppose N is a set forcing extension and κ is extendible in N . Then applying Proposition 2.4 in N , κ is extendible over the mantle, and so by Corollary 2.3, the mantle is a κ -ground of N . Since $\mathbb{M} \subseteq V \subseteq N$, the Intermediate Model Theorem implies that V is a κ -ground of N . \square

2.2 Two constraints

Proposition 2.5 highlights two key constraints that guide the way to a counterexample to Usuba's question. The first constraint is that by Proposition 2.5, it is consistent that Question 1.1 has a positive answer in all forcing extensions. This suggests that in order to find a counterexample, one should start with a preparatory class forcing.

The second constraint is more subtle. By Corollary 2.3, if one is to preserve an extendible cardinal by a set forcing that is not a small forcing, this preservation cannot be proved using the standard lifting arguments for extendible cardinals. The reason is that if a cardinal κ is shown to be extendible in a forcing extension N of a model M using these lifting arguments, then N satisfies that κ is extendible over M , and hence M is a κ -ground of N .

Since it is hard to see how to preserve extendible cardinals without a lifting argument, answering Question 1.1 seems to require inventing a novel forcing notion along with an entirely new preservation argument for extendible cardinals, a task that lies above the author's paygrade. The solution instead is simply to *reformulate* extendibility in terms of normal fine ultrafilters (Lemma 3.2) and then to employ the standard lifting arguments from the theory of supercompactness to show that, thanks to our preliminary preparatory forcing, this reformulation is preserved by a forcing notion that is about as far from novel as one can get: an Easton product of Cohen forcings.

3 Indestructible correctness

3.1 The main theorem

In this section, we show that under indestructibility hypotheses, the answer to Question 1.1 is no. A cardinal λ is Σ_n -correct if V_λ is a Σ_n -elementary substructure of V . A cardinal λ is *indestructibly* Σ_n -correct if V_λ is a Σ_n -elementary substructure of V in any λ -directed closed forcing extension.

Theorem 3.1. *Assume there is a proper class of indestructibly Σ_2 -correct cardinals. Then the answer to Question 1.1 is no.*

Proposition 3.4 shows that the hypothesis of Theorem 3.1 can be class forced starting with a model with a proper class of inaccessible Σ_2 -correct cardinals.

The key to the proof of Theorem 3.1 is the following characterization of extendibility, observed independently and earlier by Bagaria. For any cardinal λ , let $T_{\kappa,\lambda}$ (resp. $T_{\kappa,\lambda}^*$) denote the set of $\sigma \in P_\kappa(\lambda)$ such that the ordertype of σ is Σ_2 -correct (resp. indestructibly Σ_2 -correct) in V_κ . In most cases of interest, κ itself will be Σ_2 -correct, in which case a cardinal is Σ_2 -correct in V_κ if and only if it is truly Σ_2 -correct. Similarly, if κ is Σ_2 -correct, then a cardinal is indestructibly Σ_2 -correct in V_κ if and only if it is truly indestructibly Σ_2 -correct.

Lemma 3.2. *A cardinal κ is extendible if and only if for arbitrarily large cardinals λ , there is a normal fine κ -complete ultrafilter on $T_{\kappa,\lambda}$.*

Proof. For the forwards direction, suppose λ is Σ_2 -correct and there is an elementary embedding $j : V_{\lambda+1} \rightarrow V_{\lambda'+1}$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. We will show that there is a normal fine κ -complete ultrafilter on $T_{\kappa,\lambda}$. Since λ is Σ_2 -correct, λ is Σ_2 -correct in $V_{j(\kappa)}$, and so $j[\lambda] \in T_{j(\kappa),j(\lambda)}$. It follows that there is a normal fine κ -complete ultrafilter on $T_{\kappa,\lambda}$; namely, the ultrafilter derived from j using $j[\lambda]$, or in symbols, $\{A \subseteq T_{\kappa,\lambda} : j[\lambda] \in j(A)\}$.

Now we show that if there is a normal fine κ -complete ultrafilter \mathcal{U} on $T_{\kappa,\lambda}$, then κ is γ -extendible for all $\gamma < \lambda$. In particular, this implies the reverse direction of the lemma. Let $j : V \rightarrow M$ be the ultrapower embedding associated to \mathcal{U} .

We first show that λ is Σ_2 -correct in $V_{j(\lambda)}^M$. By Łoś's theorem, since $T_{\kappa,\lambda} \in \mathcal{U}$ and $[\text{id}]_{\mathcal{U}} = j[\lambda]$, clearly $\lambda = \text{ot}(j[\lambda])$ is Σ_2 -correct in $V_{j(\kappa)}^M$. But κ is Σ_2 -correct in V_λ since \mathcal{U} witnesses that κ is λ -supercompact. Therefore by the elementarity of j , $j(\kappa)$ is Σ_2 -correct in $V_{j(\lambda)}^M$. Now $V_\lambda \preceq_{\Sigma_2} V_{j(\kappa)}^M \preceq_{\Sigma_2} V_{j(\lambda)}^M$, which implies that λ is Σ_2 -correct in $V_{j(\lambda)}^M$.

Note that $j \upharpoonright V_\lambda$ belongs to $V_{j(\lambda)}^M$, and in $V_{j(\lambda)}^M$, $j \upharpoonright V_\lambda$ is an elementary embedding from V_λ to $V_{\lambda'}$ where $\lambda' = \sup j[\lambda]$. Therefore $V_{j(\lambda)}^M$ satisfies that κ is γ -extendible for all $\gamma < \lambda$. Since λ is Σ_2 -correct in $V_{j(\lambda)}^M$, V_λ satisfies that κ is γ -extendible for all $\gamma < \lambda$, and since the γ -extendibility of κ is expressed by a Σ_2 -formula, this is upwards absolute to V . Hence κ is γ -extendible for all $\gamma < \lambda$, as claimed. \square

The proof of Lemma 3.2 yields:

Lemma 3.3. *If κ is extendible and λ is indestructibly Σ_2 -correct, then there is a normal fine κ -complete ultrafilter on $T_{\kappa,\lambda}^*$.* \square

Given this, we turn to the proof of Theorem 3.1

Proof of Theorem 3.1. For each cardinal δ , let λ_δ denote the least indestructibly Σ_2 -correct cardinal λ such that there is no normal fine δ -complete ultrafilter on $T_{\kappa,\lambda}^*$ if there is such a cardinal. Let $\mathbb{P}_\delta = \text{Add}(\lambda_\delta^+, 1)$, and let \mathbb{P} be the Easton product of the partial orders \mathbb{P}_δ for $\delta < \kappa$.

Let $G \subseteq \mathbb{P}$ be a V -generic filter. Clearly there is no partial order $\bar{\mathbb{P}} \in V_\kappa$ carrying a V -generic filter $\bar{G} \in V[G]$ such that $V[G] = V[\bar{G}]$. To show that the answer to Question 1.1 is no, it therefore suffices to show that κ is extendible in $V[G]$. To prove this, we will verify the criterion of Lemma 3.2.

Note that the class of indestructibly Σ_2 -correct cardinals is closed. Therefore there is a proper class of indestructibly Σ_2 -correct *singular* cardinals λ , which have the property that $2^\lambda = \lambda^+$ by Solovay's Theorem [3] that the Singular Cardinals Hypothesis holds above a supercompact cardinal. Suppose $\lambda > \kappa$ is an indestructibly Σ_2 -correct cardinal such that $2^\lambda = \lambda^+$. We claim that $V[G]$ satisfies that there is a normal fine κ -complete ultrafilter on $(T_{\kappa,\lambda})^{V[G]}$.

Working in V , let \mathcal{U} be Mitchell minimal among all κ -complete normal fine ultrafilters on $T_{\kappa,\lambda}^*$. Such an ultrafilter exists by Lemma 3.3 and the wellfoundedness of the Mitchell order. We claim that $\lambda = (\lambda_\kappa)^{M_{\mathcal{U}}}$. To see this, note that since $V_\kappa^{M_{\mathcal{U}}} = V_\kappa$, for all $\gamma \leq \lambda$, $(T_{\kappa,\gamma}^*)^{M_{\mathcal{U}}} = T_{\kappa,\gamma}^*$. Moreover, since $V_\lambda \subseteq M_{\mathcal{U}}$ and λ is a strong limit cardinal, every ultrafilter on $T_{\kappa,\gamma}^*$ for $\gamma < \lambda$ belongs to $M_{\mathcal{U}}$. Therefore $(\lambda_\kappa)^{M_{\mathcal{U}}} \leq \lambda$. But there can be no normal fine ultrafilter \mathcal{W} on $T_{\kappa,\lambda}^*$ in $M_{\mathcal{U}}$: otherwise \mathcal{W} is a normal fine ultrafilter on $T_{\kappa,\lambda}^*$ in V since $P(T_{\kappa,\lambda}) \subseteq M_{\mathcal{U}}$, and this contradicts the Mitchell minimality of \mathcal{U} . It follows that $\lambda \geq (\lambda_\kappa)^{M_{\mathcal{U}}}$, which proves the claim.

Since $\lambda = (\lambda_\kappa)^{M_{\mathcal{U}}}$, the forcing $j_{\mathcal{U}}(\mathbb{P})$ is isomorphic to the product $\mathbb{P} \times \mathbb{Q}$ where $\mathbb{Q} = \prod_{\kappa \leq \delta < j_{\mathcal{U}}(\kappa)} \mathbb{P}_\delta$ is λ^+ -directed closed in $M_{\mathcal{U}}$. Since $M_{\mathcal{U}}$ is closed under λ -sequences, \mathbb{Q} really is λ^+ -directed closed. In addition, $|P^M(\mathbb{Q})|^M \leq (2^\kappa)^{M_{\mathcal{U}}} < j_{\mathcal{U}}(\lambda) < (2^\lambda)^+ = \lambda^{++}$. The final bound follows from the fact that $2^\lambda = \lambda^+$. Therefore in V , $|P^M(\mathbb{Q})| \leq \lambda^+$ and \mathbb{Q} is λ^+ -closed, and so one can build an M -generic filter $H \subseteq \mathbb{Q}$ with $H \in V$.

The closure of \mathbb{Q} implies that $M[H]$ contains no new dense subsets of \mathbb{P} , and so G is an $M[H]$ -generic filter on \mathbb{P} . By standard results on mutual genericity, this means that $G \times H$ is an M -generic filter on $\mathbb{P} \times \mathbb{Q}$. The cardinal λ is Σ_2 -correct in $M[H]$ by since λ is indestructibly Σ_2 -correct in M . Since G is $M[H]$ -generic for a forcing in $(V_\lambda)^{M[H]}$, λ is Σ_2 -correct in $M[H \times G]$.

Finally, identifying $j_{\mathcal{U}}(\mathbb{P})$ with $\mathbb{P} \times \mathbb{Q}$ in the natural way, $j_{\mathcal{U}}[G] = G \times \{1\} \subseteq G \times H$, and so the embedding $j_{\mathcal{U}}$ lifts uniquely to an elementary embedding $j : V[G] \rightarrow M[G \times H]$ such that $j(G) = G \times H$. Since λ is Σ_2 -correct in $M[G \times H]$, the set $j[\lambda]$ belongs to $T_{j(\kappa),j(\lambda)}$ as computed in $M[G \times H]$. As a consequence, working in $V[G]$, the ultrafilter

$$\mathcal{D} = \{A \subseteq T_{\kappa,\lambda} : j[\lambda] \in j(A)\}$$

derived from j using $j[\lambda]$ is a normal fine κ -complete ultrafilter on $T_{\kappa,\lambda}$. □

3.2 Forcing indestructible correctness

In this section, we show that the hypotheses of Theorem 3.1 are consistent relative to a large cardinal hypothesis just past an extendible cardinal.

Proposition 3.4. *There is a class forcing \mathbb{Q} such that every inaccessible Σ_2 -correct cardinal of V is an inaccessible indestructibly Σ_2 -correct cardinal of $V^{\mathbb{Q}}$. Moreover this forcing preserves extendible cardinals.*

Proof. Suppose η is an ordinal, y is a set of rank at most η , and $\varphi(x)$ is a set-theoretic formula in one free variable. Let $F(\eta, \varphi, y)$ denote the least Beth fixed point $\beta \geq \text{rank}(y)$ for which there is an η -directed closed partial order $\mathbb{P} \in V_\beta$ such that $V_\beta^{\mathbb{P}} \models \varphi(y)$. For each ordinal η , let $f(\eta) = \sup_{\varphi, y \in V_\eta} F(\eta, \varphi, y)$. Let $\mathbb{P}(\eta)$ denote the lottery sum of all η -directed closed forcings in $V_{f(\eta)}$.

We define a class Easton iteration $\langle \mathbb{Q}_\alpha, \dot{\mathbb{P}}_\alpha : \alpha \in \text{Ord} \rangle$ and a continuous sequence of ordinals η_α by letting $\dot{\mathbb{P}}_\alpha$ be the canonical name for $\mathbb{P}(\eta_\alpha)$ as computed in $V^{\mathbb{Q}_\alpha}$ and letting $\eta_{\alpha+1}$ be the least Beth fixed point above $\text{rank}(\mathbb{P}(\eta_\alpha))$. Let \mathbb{Q} be the class direct limit of the iteration and let G be a V -generic filter on \mathbb{Q} . If κ is an inaccessible Σ_2 -correct cardinal, then κ is closed under f , and so $\mathbb{Q}^{V_\kappa} \cong \mathbb{Q}_\kappa$. In particular, \mathbb{Q}_κ is κ -cc, and so κ remains inaccessible in $V[G]$.

We will show that every Σ_2 -correct inaccessible cardinal κ of V becomes indestructibly Σ_2 -correct in $V[G]$. Assume that in $V[G]$, $\beta > \kappa$ is an ordinal and there is a κ -directed closed partial order $\mathbb{P} \in V[G]_\beta$ such that $V[G]_\beta^{\mathbb{P}} \models \varphi(y)$ where φ is a formula and $y \in V[G]_\kappa$. We claim that there is some $\bar{\beta} < \kappa$ such that $V[G]_{\bar{\beta}} \models \varphi(y)$. For ease of notation, and without loss of generality, we assume $y \in V_\kappa$. (More formally, we work in $V[G \upharpoonright \lambda]$ where $\lambda < \kappa$ is the hereditary cardinality of y .)

For any ordinal ξ , let D_ξ denote the class of conditions $p \in \mathbb{Q}$ that force $V_\xi^{\mathbb{Q}} \models \varphi(y)$, and let C_ξ be the class of conditions $p \in \mathbb{Q}$ such that for some $\dot{\mathbb{P}} \in V_\xi^{\mathbb{Q}}$, p forces $(V^{\mathbb{Q}})_{\dot{\mathbb{P}}} \models \varphi(y)$. We will show that for any $p \in C_\beta$, for some $\bar{\beta} < \kappa$, there is a condition $r \in D_{\bar{\beta}}$ compatible with p . It follows that $\bigcup_{\bar{\beta} < \kappa} D_{\bar{\beta}}$ is predense below any element of C_β . Since $V[G]_\beta^{\mathbb{P}} \models \varphi(y)$, there is a condition $q \in G \cap C_\beta$, and so since $\bigcup_{\bar{\beta} < \kappa} D_{\bar{\beta}}$ is predense below q , $G \cap \bigcup_{\bar{\beta} < \kappa} D_{\bar{\beta}}$ is nonempty. In other words, there is some $\bar{\beta} < \kappa$ such that $V[G]_{\bar{\beta}} \models \varphi(y)$.

Fix $p \in C_\beta$. Let $\bar{p} = p \upharpoonright \kappa \in \mathbb{Q}_\kappa$. Let $\gamma = \text{supp}(\bar{p})$ be the support of \bar{p} . Note that if H is a V -generic filter on \mathbb{Q} containing p , then in $V[H \upharpoonright \gamma]$, one can find an ordinal β and an η_γ -closed partial order $\mathbb{P} \in V[H \upharpoonright \gamma]_\beta$ such that $V[H \upharpoonright \gamma]_\beta^{\mathbb{P}} \models \varphi(y)$; namely, $\beta = \beta$ and $\mathbb{P} = \mathbb{Q}_{\gamma, \beta} * \dot{\mathbb{P}}$ where $\dot{\mathbb{P}}$ is a $\mathbb{Q}_{\gamma, \beta}$ -name for a κ -directed closed partial order \mathbb{P} in $V[H \upharpoonright \beta]$ such that $V[H]_\beta^{\mathbb{P}} \models \varphi(y)$. The least such ordinal $\bar{\beta}$ lies below $f(\eta_\gamma)$ as computed in $V[H \upharpoonright \gamma]$, and so since κ is Σ_2 -correct in $V[H \upharpoonright \gamma]$, $\bar{\beta} < \kappa$.

It follows from the definition of \mathbb{P}_γ that there is an extension $r \leq \bar{p}$ in $\mathbb{Q}_{\gamma+1}$ forcing that $V_{\bar{\beta}}^{\mathbb{Q}_{\gamma+1}} \models \varphi(y)$. By construction, $V_{\bar{\beta}}^{\mathbb{Q}_{\gamma+1}} = V_{\bar{\beta}}^{\mathbb{Q}}$, and so r forces that $V_{\bar{\beta}}^{\mathbb{Q}} \models \varphi(y)$, or in other words, $r \in D_{\bar{\beta}}$. Note that r and p are compatible since r extends $p \upharpoonright \kappa$ and $\text{supp}(r) < \kappa$.

Finally, we sketch a proof that the forcing \mathbb{Q} preserves extendible cardinals. Suppose κ is extendible and $\lambda > \kappa$ is a limit of Σ_2 -correct inaccessible cardinals. Also assume that $2^\lambda = \lambda^+$. (By Solovay's Theorem [3], there is a proper class of cardinals λ with these properties.) Let \mathcal{U} be a normal fine κ -complete ultrafilter on $T_{\kappa, \lambda}$. Using a master condition and the fact that $2^\lambda = \lambda^+$, in $V[G]$, $j_{\mathcal{U}} : V \rightarrow M_{\mathcal{U}}$ lifts to an elementary embedding $j : V[G] \rightarrow M_{\mathcal{U}}[H]$. We have shown that every inaccessible Σ_2 -correct cardinal is Σ_2 -correct in $V[G]$, and so by elementarity, λ remains a limit of Σ_2 -correct cardinals in $M[H]$. It follows that λ is Σ_2 -correct in $M[H]$, and therefore one can derive from j a normal fine κ -complete $V[G]$ -ultrafilter on $(T_{\kappa, \lambda})^{V[G]}$. \square

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