The Ultrapower Axiom and the GCH

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1 Introduction

In the mid 1960s, Cohen [1] and Lévy-Solovay [2] proved theorems that devastated Gödel's program to solve the Continuum Problem. Gödel [3] had conjectured that the problem was unsolvable on the basis of the commonly accepted axioms of mathematics, the Zermelo-Fraenkel system ZFC, but he suggested that it might still be solved by supplementing ZFC with large cardinal axioms. His conjecture that the problem is unsolvable in ZFC turned out to be correct, as Cohen famously proved, but his optimistic suggestion that it could still be resolved by large cardinal axioms did not: Lévy-Solovay soon extended Cohen's techniques to show that the Continuum Problem is unsolvable from essentially all large cardinal axioms.

Further results (variants of Easton's Theorem) show that large cardinal axioms impose almost no constraints on the behavior of the continuum function $\kappa \mapsto 2^\kappa$ at regular cardinals.[1] It therefore came as a surprise when Solovay showed that the opposite is true in the case of singular cardinals: if κ is a strongly compact cardinal, then $2^\lambda = \lambda^+$ for all singular strong limit cardinals $\lambda \geq \kappa$.

The main theorem of this paper generalizes Solovay's Theorem to regular cardinals. By the independence results cited above, this cannot be achieved in any extension of ZFC by large cardinal axioms. Instead, we use a further principle called the Ultrapower Axiom (UA).

Theorem 3.27 (UA)**.** *If κ is a strongly compact cardinal, then $2^\lambda = \lambda^+$ for all cardinals $\lambda \geq \kappa$.*

The Ultrapower Axiom is a structure principle for large cardinals that is motivated by the theory of canonical inner models. A single feature unifies all the known canonical inner model constructions, from Gödel's constructible universe L to the canonical inner model $L[U]$ with one measurable cardinal to the Mitchell-Steel extender models $L[\mathbb{E}]$ at the level of Woodin cardinals: all of these models are built as the limit of approximating structures that satisfy a central *Comparison Lemma*. In L, the Comparison Lemma reduces to Gödel's Condensation Lemma, while in $L[U]$, it is the statement that any two structures $L_\alpha[U]$ and $L_\beta[W]$ can be aligned by iterating the measures U and W. The Comparison Lemma for the models $L[\mathbb{E}]$ states that any two mice can be aligned by iteration trees. The Ultrapower Axiom is

[1] Scott notably showed that if κ is measurable, then either $2^\kappa = \kappa^+$ or κ is a limit of cardinals ν such that $2^\nu > \nu^+$.

an abstract formulation of the Comparison Lemma which can be stated as a combinatorial principle involving no mention of fine structure or even constructibility:

Ultrapower Axiom. Suppose U_0 and U_1 are countably complete ultrafilters. Then there is a countably complete ultrafilter W_0 of M_{U_0} and a countably complete ultrafilter W_1 of M_{U_1} such that $(M_{W_0})^{M_{U_0}} = (M_{W_1})^{M_{U_1}}$ and $j_{W_0}^{M_{U_0}} \circ j_{U_0} = j_{W_1}^{M_{U_1}} \circ j_{U_1}$.

UA holds in all known canonical inner models, and given the pivotal role of the Comparison Lemma in inner model theory, we conjecture that UA will hold in any canonical inner model that will ever be constructed. This is not a precise statement, but it does motivate a precise conjecture. A central initiative in modern set theory is to construct a canonical inner model with a supercompact cardinal. This vaguely posed Inner Model Problem should be solvable if and only if the following conjecture is true:

Conjecture. *The Ultrapower Axiom is consistent with a supercompact cardinal.*

In the presence of a supercompact cardinal, the structure of ultrapowers becomes so rich that the Ultrapower Axiom can be used to prove theorems not only about these ultrapowers but also about the asymptotic structure of the universe itself. We believe these theorems predict the structure of the canonical inner model with a supercompact cardinal, but perhaps they simply form the first steps of a proof that this inner model does not exist. If so, this is all the more reason to investigate the theory of supercompact cardinals under UA.

Our approach to proving GCH above a strongly compact cardinal is somewhat indirect. We start by proving the theorem from a supercompact cardinal:

Theorem 3.26 (UA). *If κ is a supercompact cardinal, then $2^\lambda = \lambda^+$ for all $\lambda \geq \kappa$.*

One of the first questions that arose after the proof of this theorem was whether a strongly compact cardinal suffices. Theorem 3.27 above answers this question positively, but it turns out that this is not really the right question. The more fundamental result is that assuming UA, the least strongly compact cardinal *is* supercompact. (This is one of the main theorems of the author's thesis [4].) Thus we obtain Theorem 3.27 as a consequence of Theorem 3.26.

We now state some results subsidiary to our main theorem Theorem 3.27. First, we have a local result:

Theorem 3.34 (UA). *If $\kappa \leq \lambda$ are cardinals and κ is 2^λ-strongly compact, then $2^\lambda = \lambda^+$.*

This will be proved as a consequence of the following theorem, which uses what looks like a very different cardinal hypothesis:

Theorem 3.33 (UA). *Suppose λ is a cardinal and λ^{++} carries a countably complete uniform ultrafilter. Then $2^\lambda = \lambda^+$.*

Finally, we will show in Section 3.6 that UA implies combinatorial principles stronger than GCH:

Theorem 3.35 (UA). *Suppose κ is δ^{++}-strongly compact and $\text{cf}(\delta) \geq \kappa$. Then $\diamondsuit(S^{\delta^{++}}_{\delta^+})$ holds, where $S^{\delta^{++}}_{\delta^+} = \{\alpha < \delta^{++} : \text{cf}(\alpha) = \delta^+\}$.*

2 Preliminaries

2.1 Ultrafilters and elementary embeddings

In this subsection, we briefly review some basic concepts in the general theory of ultrafilters.

Definition 2.1. An ultrafilter U is *uniform* if all the sets in U have the same cardinality.

Definition 2.2. If U is an ultrafilter, the *size of* U is the cardinal $\lambda_U = \min\{|A| : A \in U\}$.

Thus an ultrafilter U on a set X is uniform if $|X| = \lambda_U$. Notice that for any ultrafilter U, there is a set $A \in U$ such that $U \cap P(A)$ is a uniform ultrafilter on A. (Any set $A \in U$ of cardinality λ_U suffices.) Moreover, U is equivalent to a uniform ultrafilter on the cardinal λ_U itself in the following sense:

Definition 2.3. A pair of ultrafilters U and W are *Rudin-Keisler equivalent* if there exist sets $A \in U$ and $B \in W$ and a bijection $f : A \to B$ such that for all $S \subseteq A$, $S \in U$ if and only if $f[S] \in W$.

In other words, $U \cap P(A)$ and $W \cap P(B)$ are the same up to a relabeling of their underlying sets. When considering Rudin-Keisler equivalence invariant properties of ultrafilters, we can restrict consideration to uniform ultrafilters on cardinals:

Lemma 2.4. *Every ultrafilter U is Rudin-Keisler equivalent to a uniform ultrafilter on λ_U.* □

The following constructions are the key to many of the applications of model theory to the theory of ultrafilters:

- The *ultrapower construction* associates to each ultrafilter an elementary embedding of the universe of sets V called its *ultrapower*.
- The *derived ultrafilter construction* associates to each elementary embedding of the universe of sets a system of ultrafilters called its *derived ultrafilters*.

Our notation for ultrapowers is fairly standard in set theory. Suppose U is an ultrafilter on a set X. We denote the ultrapower of the universe of sets by U by:

$$j_U : V \to M_U$$

If $f : X \to V$ is a function, then $[f]_U$ denotes the element of M_U represented by f. If M_U is well-founded, we identify M_U with the transitive inner model to which it is isomorphic.

The following fact explains why we are typically only interested in ultrafilters up to Rudin-Keisler equivalence:

Proposition 2.5. *Two ultrafilters U and W are Rudin-Keisler equivalent if and only if there is an isomorphism $k : M_U \to M_W$ such that $k \circ j_U = j_W$.* □

In other words, two ultrafilters are Rudin-Keisler equivalent if and only if their ultrapower embeddings are isomorphic. Since isomorphisms of transitive structures are equalities, two countably complete ultrafilters are Rudin-Keisler equivalent if and only if their ultrapower embeddings are *equal*.

As for derived ultrafilters, we use the following terminology:

Definition 2.6. Suppose $j : V \to M$ is an elementary embedding, X is a set, and $a \in j(X)$. The *ultrafilter on X derived from j using a* is the ultrafilter $\{A \subseteq X : a \in j(A)\}$.

Finally, we establish our notation for critical points of elementary embeddings:

Definition 2.7. Suppose M and N are transitive models of ZFC and $j : M \to N$ is an elementary embedding. Then the *critical point of j*, denoted $\textsc{crt}(j)$, is the least ordinal α such that $j(\alpha) \neq \alpha$ (if there is such an ordinal).

2.2 M-ultrafilters and sums

Suppose M is a transitive model of ZFC (possibly a proper class), $X \in M$ is a set, and $U \subseteq P^M(X)$. Even if U does not belong to M, one can still ask whether U is an ultrafilter relative to M.

Definition 2.8. Suppose M is a transitive model of ZFC, $X \in M$, and κ is an M-cardinal. A set $U \subseteq P^M(X)$ is an *M-ultrafilter on X* if it is an ultrafilter on the Boolean algebra $P^M(X)$.

If U is an M-ultrafilter, one can form the ultrapower of M by U. The construction is exactly like the usual ultrapower construction except one uses only functions that belong to M. The key point is that Łoś's Theorem does not require that $U \in M$. The ultrapower of M by U is denoted by:

$$j^M_U : M \to M^M_U$$

If $f : X \to M$ is a function with $f \in M$, we let $[f]^M_U$ be the point in M^M_U represented by f. If M^M_U is well-founded, we identify M^M_U with its transitive collapse.

Definition 2.9. Suppose M is a transitive model of ZFC, U is an M-ultrafilter, and κ is an M-cardinal. Then U is *κ-complete* if for any $\sigma \subseteq U$ with $\sigma \in M$ and $|\sigma|^M < \kappa$, $\bigcap \sigma \in U$.

We need a slight generalization of the classical notion of a sum of ultrafilters. Suppose X and Y are sets. If A is a subset of $X \times Y$, and x is a point in X, we denote by A_x the *section of A at x*, defined by $A_x = \{y \in Y : (x, y) \in A\}$. If U is an ultrafilter on X and $\langle W_x : x \in X\rangle$ is a sequence of ultrafilters on Y, then classically, the *U-sum of $\langle W_x : x \in X\rangle$* is the ultrafilter defined by

$$U\text{-}\textstyle\sum_{x\in X} W_x = \{A \subseteq X \times Y : \{x \in X : A_x \in W_x\} \in U\}$$

The generalization we need is the following:

Definition 2.10. Suppose X and Y are sets. Suppose U is a countably complete ultrafilter on X and W is an M_U-ultrafilter on $j_U(Y)$. Then the *U-sum of W* is the ultrafilter on $X \times Y$ defined by $U\text{-}\sum W = \{A \subseteq X \times Y : [\langle A_x : x \in X\rangle]_U \in W\}$.

The relationship between these two types of sums is quite straightforward. If U is an ultrafilter on X and $\langle W_x : x \in X\rangle$ is a sequence of ultrafilters on Y, then $W = [\langle W_x : x \in X\rangle]_U$ is an M_U-ultrafilter on $j_U(Y)$ and

$$U\text{-}\sum_{x\in X} W_x = U\text{-}\sum W$$

The reader who prefers elementary embeddings to ultrafilter combinatorics will prefer the following characterization of a sum:

Proposition 2.11. *Suppose U is a countably complete ultrafilter on X and W is an M_U-ultrafilter on $j_U(Y)$. Then $U\text{-}\sum W$ is the ultrafilter on $X \times Y$ derived from $j_W^{M_U} \circ j_U$ using $(j_W^{M_U}([\mathrm{id}]_U), [\mathrm{id}]_W^{M_U})$. Hence $j_{U\text{-}\sum W} = j_W^{M_U} \circ j_U$.* □

Thus U-sums provide a canonical way of representing a two-step iterated ultrapower as the ultrapower by a single ultrafilter. Proposition 2.11 has the following immediate consequence:

Lemma 2.12. *Suppose κ is an uncountable cardinal, U is a κ-complete ultrafilter on X and W is a κ-complete M_U-ultrafilter on $j_U(Y)$. Then $U\text{-}\sum W$ is a κ-complete ultrafilter on $X \times Y$.* □

We use $U\text{-}\sum W$ only as a device to code U and W by another countably complete ultrafilter:

Lemma 2.13. *Suppose U is a countably complete ultrafilter on X. Suppose Y is a transitive set and W is a countably complete M_U-ultrafilter on $j_U(Y)$. Suppose N is an inner model such that $U\text{-}\sum W$ belongs to N. Then U and W belong to N.*

Proof. Since $U\text{-}\sum W$ belongs to N, $P(X \times Y)$ belongs to N. In particular, $P(X)$ belongs to N. Since $U = \{A \subseteq X : A \times Y \in U\text{-}\sum W\}$, it follows that $U \in N$.

The fact that $P(X \times Y)$ belongs to N implies that $P(Y)^X$ belongs to N as well. (Since we identify functions with their graphs, $P(Y)^X$ is formally a subset of $P(X \times Y)$.) Since $U \in N$ by the previous paragraph, $P(Y)^X/U$ is in N. In general:

$$P(Y)^X/U \cong j_U(P(Y))$$

Since Y is transitive, $j_U(P(Y))$ is transitive. Hence $j_U(P(Y))$ is the transitive collapse of $P(Y)^X/U$. Thus $j_U(P(Y)) \in N$, and moreover the function $f \mapsto [f]_U$ belongs to N.

By the definition of $U\text{-}\sum W$, we have

$$W = \{[f]_U : f \in P(Y)^X \text{ and } \{(x, y) : x \in X \text{ and } y \in f(x)\} \in U\text{-}\sum W\}$$

Since $f \mapsto [f]_U$, $P(Y)^X$, and $U\text{-}\sum W$ belong to N, so does W. □

2.3 Supercompact cardinals

Definition 2.14. If $\kappa \leq \lambda$ are cardinals, then κ is λ-*supercompact* if there is an elementary embedding $j : V \to M$ such that the following hold:

- M is an inner model that is closed under λ-sequences.
- $\textsc{crt}(j) = \kappa$ and $j(\kappa) > \lambda$.

Such an elementary embedding j is said to *witness that κ is λ-supercompact.*

We now describe the combinatorial formulation of supercompactness. Recall that if $\kappa \leq \lambda$ are cardinals then $P_\kappa(\lambda) = \{\sigma \subseteq \lambda : |\sigma| < \kappa\}$. Recall also that the *diagonal intersection* of a sequence $\langle A_\alpha : \alpha < \lambda \rangle$ of subsets of $P_\kappa(\lambda)$ is the set $\triangle_{\alpha<\lambda} A_\alpha = \{\sigma \in P_\kappa(\lambda) : \sigma \in \bigcap_{\alpha \in \sigma} A_\alpha\}$.

Definition 2.15. Suppose $\kappa \leq \lambda$ are cardinals. An ultrafilter $\mathcal{U}$ on $P_\kappa(\lambda)$ is:

- *fine* if for all $\tau \in P_\kappa(\lambda)$, $\{\sigma \in P_\kappa(\lambda) : \tau \subseteq \sigma\} \in \mathcal{U}$.
- *normal* if it is closed under diagonal intersections.

Any normal fine ultrafilter on $P_\kappa(\lambda)$ is automatically κ-complete because of the definition of fineness we have chosen.

Definition 2.16. Suppose $\kappa \leq \lambda$ are cardinals. Then $\mathcal{N}(\kappa, \lambda)$ denotes the set of normal fine ultrafilters on $P_\kappa(\lambda)$.

We sometimes refer to the elements of $\mathcal{N}(\kappa, \lambda)$ as *supercompactness measures* because of the following theorem:

Lemma 2.17. *A cardinal κ is λ-supercompact if and only if there is a normal fine ultrafilter $\mathcal{U}$ on $P_\kappa(\lambda)$. In fact:*

- *If $j : V \to M$ witnesses that κ is λ-supercompact, then the ultrafilter on $P_\kappa(\lambda)$ derived from j using $j[\lambda]$ is a normal fine ultrafilter on $P_\kappa(\lambda)$.*
- *If $\mathcal{U}$ is a normal fine ultrafilter on $P_\kappa(\lambda)$, then the ultrapower embedding $j_\mathcal{U} : V \to M_\mathcal{U}$ witnesses that κ is λ-supercompact.* □

We will use the local version of Solovay's Theorem [5] that SCH holds above a strongly compact cardinal:

Theorem 2.18 (Solovay)**.** *Suppose $\kappa \leq \lambda$ are cardinals and κ is λ-supercompact.*

(1) If $\mathrm{cf}(\lambda) < \kappa$, then $\lambda^{<\kappa} = \lambda^+$.

(2) If $\mathrm{cf}(\lambda) \geq \kappa$ then $\lambda^{<\kappa} = \lambda$. □

Solovay also proved this theorem for strong compactness, but we will only need the weaker supercompact version here, and in any case we have not defined strong compactness.

2.4 The Mitchell order

The following theorem (which in essence appears as Proposition 1.14 in [6]) expresses a fundamental fact about the relationship between an ultrafilter and its ultrapower:

Theorem 2.19. *A nonprincipal countably complete ultrafilter never belongs to its own ultrapower.* $\square$

This motivates the definition of the *Mitchell order*:

Definition 2.20. Suppose U and W are countably complete ultrafilters. The *Mitchell order* is defined by setting $U \lhd W$ if $U \in M_W$.

The following folklore fact (which is, for example, a consequence of [7]) places Theorem 2.19 in a far more general context:

Theorem 2.21. *The Mitchell order is well-founded on nonprincipal ultrafilters.* $\square$

As nice as this generalization is, the relation $U \lhd W$ suffers from a strong dependence on the choice of the underlying set X of U: before one can have $U \lhd W$, one must first have $X \in M_W$ and moreover $P(X) \subseteq M_W$. It is therefore convenient to restrict the Mitchell order to a class of ultrafilters on which it is invariant under Rudin-Keisler equivalence. Recall that the hereditary cardinality of a set is the cardinality of its transitive closure.

Definition 2.22. An ultrafilter U on a set X is *hereditarily uniform* if X has hereditary cardinality λ_U. If λ is a cardinal, then $\mathscr{U}(\lambda)$ denotes the set of countably complete hereditarily uniform ultrafilters U such that $\lambda_U < \lambda$.

Every uniform ultrafilter on a cardinal is hereditarily uniform, and therefore every ultrafilter is Rudin-Keisler equivalent to a hereditarily uniform ultrafilter.

Lemma 2.23. *Restricted to the class of hereditarily uniform countably complete ultrafilters, the Mitchell order is invariant under Rudin-Keisler equivalence.*

The proof relies on a general fact about hereditary cardinality.

Definition 2.24. For any cardinal λ, $H(\lambda)$ denotes the collection of sets of hereditary cardinality less than λ.

Lemma 2.25. *Suppose X is a set and M is an inner model of* ZFC *such that $P(X) \subseteq M$. Then $H(|X|^+) \subseteq M$ and every partial function from X to $H(|X|^+)$ belongs to M.* $\square$

Proof of Lemma 2.23. Suppose U is a countably complete ultrafilter on X and W is a countably complete ultrafilter such that $U \lhd W$. We will show that any countably complete hereditarily uniform ultrafilter U' that is Rudin-Keisler equivalent to U satisfies $U' \lhd W$. (Since whether $U \lhd W$ depends only on M_W, it is clear that the Mitchell order only depends on the Rudin-Keisler equivalence class of W.)

Since $U \lhd W$, $P(X) \subseteq M_W$ and therefore $H(|X|^+) \subseteq M_W$ by Lemma 2.25. Note that $|X| \geq \lambda_U = \lambda_{U'}$. Since U' is hereditarily uniform, it follows that the underlying set Y of U' has hereditary cardinality $\lambda_{U'}$, and hence $Y \in H(|X|^+)$. Since U and U' are Rudin-Keisler equivalent, there is a partial function $f : X \to Y$ such that $U' = \{S \subseteq Y : f^{-1}[S] \in U\}$. Since f is a partial function from X to $H(|X|^+)$, $f \in M_W$ by Lemma 2.25. Therefore $U' \in M_W$. In other words, $U' \lhd W$ as desired. $\square$

2.5 The Ultrapower Axiom

In this section we restate the Ultrapower Axiom and survey a few consequences of the Ultrapower Axiom that are relevant to the rest of this paper.

Definition 2.26. Suppose M and N are transitive models of ZFC. An elementary embedding $j : M \to N$ is an *internal ultrapower embedding of* M if there is a countably complete ultrafilter U of M such that $N = (M_U)^M$ and $j = (j_U)^M$.

Definition 2.27. Suppose P, M_0, M_1, and N are transitive models of ZFC.

- We write $(k_0, k_1) : (M_0, M_1) \to N$ to denote that $k_0 : M_0 \to N$ and $k_1 : M_1 \to N$ are elementary embeddings.
- Suppose $j_0 : P \to M_0$ and $j_1 : P \to M_1$ are elementary embeddings. Then a pair $(k_0, k_1) : (M_0, M_1) \to N$ is a *comparison* of (j_0, j_1) if $k_0 \circ j_0 = k_1 \circ j_1$.
- The pair (k_0, k_1) is an *internal ultrapower comparison* of (j_0, j_1) if moreover k_0 and k_1 are internal ultrapower embeddings of M_0 and M_1 respectively.

Ultrapower Axiom. Every pair of ultrapower embeddings of the universe of sets has an internal ultrapower comparison.

The earliest consequence of the Ultrapower Axiom was linearity of the Mitchell order on normal ultrafilters. The main theorem of [8] is that under a cardinal arithmetic hypothesis, the Ultrapower Axiom implies the linearity of the Mitchell order not only on normal ultrafilters but also on supercompactness measures:

Theorem 2.28 (UA)**.** *Suppose $\kappa \leq \lambda$ are cardinals and $2^{<\lambda} = \lambda$. Then the Mitchell order is linear on $\mathcal{N}(\kappa, \lambda)$.*

We will use this result a few times in this paper. Combining Theorem 2.18 and Theorem 3.28, one can *prove* the cardinal arithmetic hypothesis $2^{<\lambda} = \lambda$ in many cases assuming that $\mathcal{N}(\kappa, \lambda)$ is nonempty. If $\mathcal{N}(\kappa, \lambda)$ is empty, then it is of course linearly ordered by the Mitchell order. Thus one obtains the linearity of the Mitchell order from UA alone in these cases. The results of this paper, however, leave open the following key case: does UA imply that the Mitchell order is linear on $\mathcal{N}(\kappa, \kappa^+)$? The issue is that we seem to need κ to be κ^{++}-supercompact to prove $2^\kappa = \kappa^+$, while the nonemptiness of $\mathcal{N}(\kappa, \kappa^+)$ only yields that κ is κ^+-supercompact. Still, a positive answer can be established by an indirect argument which appears in the author's thesis [4]:

Theorem 2.29 (UA)**.** *Suppose $\kappa \leq \lambda$ are cardinals. Then the Mitchell order is linear on $\mathcal{N}(\kappa, \lambda)$.*

We cannot use this theorem here because the proof leans heavily on the results of the current paper.

Let us finally make some remarks on the equivalence of strong compactness and supercompactness under UA. The author first proved the GCH above a supercompact cardinal from UA, then generalized it to GCH above a strongly compact cardinal. Shortly after this discovery, however, came a realization that made this further proof obsolete:

Theorem 2.30 (UA)**.** *The least strongly compact cardinal is supercompact.* □

In fact, any successor strongly compact cardinal is supercompact. (The least strongly compact limit of strongly compact cardinals is never supercompact by a theorem of Menas [9].) There is actually a level-by-level equivalence between the notions of strong compactness and supercompactness, spelled out in [4]. One consequence of this is the following theorem:

Theorem 2.31 (UA)**.** *Suppose λ is a cardinal such that λ^+ carries a countably complete uniform ultrafilter. Then some cardinal $\kappa \leq \lambda$ is λ^+-supercompact.* □

3 Cardinal arithmetic and the Mitchell order

3.1 The number of supercompactness measures

The original motivation for this work was a remarkable observation due to Solovay [6] which shows that the linearity of the Mitchell order implies instances of GCH.

Under sufficient large cardinal assumptions, Solovay showed that $P_\kappa(\lambda)$ carries the maximum possible number of supercompactness measures. Note that if $\eta = \lambda^{<\kappa} = |P_\kappa(\lambda)|$, the cardinality of $\mathcal{N}(\kappa, \lambda)$ is bounded by 2^{2^η}, since $\mathcal{N}(\kappa, \lambda)$ is contained in the double powerset of $P_\kappa(\lambda)$. Solovay showed that this bound is achieved:

Theorem 3.1 (Solovay)**.** *Suppose $\kappa \leq \lambda$ are cardinals. Let $\eta = \lambda^{<\kappa}$, and assume κ is 2^η-supercompact. Then $|\mathcal{N}(\kappa, \lambda)| = 2^{2^\eta}$.* □

As a corollary, Solovay proved instances of GCH from the linearity of the Mitchell order.

Theorem 3.2 (Solovay)**.** *Suppose $\kappa \leq \lambda$ are cardinals, $\mathrm{cf}(\lambda) \geq \kappa$, and $|\mathcal{N}(\kappa, \lambda)| = 2^{2^\lambda}$. Assume the Mitchell order is linear on $\mathcal{N}(\kappa, \lambda)$. Then $2^{2^\lambda} = (2^\lambda)^+$.*

Proof. Since $(\mathcal{N}(\kappa, \lambda), \lhd)$ is a wellorder,

$$2^{2^\lambda} = |\mathcal{N}(\kappa, \lambda)| \leq \mathrm{ot}(\mathcal{N}(\kappa, \lambda), \lhd)$$

It therefore suffices to show that $\mathrm{ot}(\mathcal{N}(\kappa, \lambda), \lhd) \leq (2^\lambda)^+$. To accomplish this, we show that any $\mathcal{U} \in \mathcal{N}(\kappa, \lambda)$ has at most 2^λ-many predecessors in $(\mathcal{N}(\kappa, \lambda), \lhd)$. Note that the set of predecessors of $\mathcal{U}$ in $(\mathcal{N}(\kappa, \lambda), \lhd)$ is equal to $\mathcal{N}(\kappa, \lambda) \cap M_\mathcal{U}$. But

$$\mathcal{N}(\kappa, \lambda) \cap M_\mathcal{U} \subseteq j_\mathcal{U}(V_\kappa) = (V_\kappa)^{P_\kappa(\lambda)}/\mathcal{U}$$

so $|\mathcal{N}(\kappa, \lambda) \cap M_\mathcal{U}| \leq |(V_\kappa)^{P_\kappa(\lambda)}| = \kappa^\lambda = 2^\lambda$. This calculation uses that $|P_\kappa(\lambda)| = \lambda$, which is a consequence of Theorem 2.18. □

Thus under UA, if κ is 2^κ-supercompact, then GCH holds at 2^κ. More generally, we have the following consequence of UA:

Corollary 3.3 (UA)**.** *Suppose $\kappa \leq \lambda$ are cardinals, $\mathrm{cf}(\lambda) \geq \kappa$, and $2^{<\lambda} = \lambda$. If κ is 2^λ-supercompact, then $2^{2^\lambda} = (2^\lambda)^+$.*

Proof. Since $2^{<\lambda} = \lambda$, Theorem 2.28 implies that the Mitchell order is linear on $\mathcal{N}(\kappa, \lambda)$. By Theorem 3.1, $|\mathcal{N}(\kappa, \lambda)| = 2^{2^\lambda}$. We can therefore apply Theorem 3.2. $\square$

As a corollary, we obtain a result that strongly suggests that UA plus a supercompact cardinal implies the eventual GCH:

Corollary 3.4 (UA)**.** *Suppose κ is supercompact. Let $\lambda \geq \kappa$ be a strong limit singular cardinal with* $\text{cf}(\lambda) \geq \kappa$*. Then for all $n \leq \omega$, $2^{(\lambda^{+n})} = \lambda^{+n+1}$.*

Proof. We first claim that for all $n < \omega$, $2^{(\lambda^{+n})} = \lambda^{+n+1}$. The proof is by induction. For the base case $n = 0$, we have $2^\lambda = \lambda^+$ by Solovay's Theorem [5] since λ is a singular strong limit cardinal above a supercompact cardinal. Now suppose that the claim is true for $n \leq k$, and we will show it is true when $n = k + 1$. By our induction hypothesis (or if $k = 0$, by the fact that λ is a strong limit cardinal), $2^{<\lambda^{+k}} = \lambda^{+k}$, Corollary 3.3 implies

$$2^{(\lambda^{+k+1})} = 2^{2^{(\lambda^{+k})}} = (2^{(\lambda^{+k})})^+ = \lambda^{+k+2}$$

The final equality follows from our induction hypothesis that $2^{(\lambda^{+k})} = \lambda^{+k+1}$.

To finish, we show that $2^{(\lambda^{+\omega})} = \lambda^{+\omega+1}$. The previous paragraph implies that $\lambda^{+\omega}$ is a singular strong limit cardinal. Thus $2^{(\lambda^{+\omega})} = \lambda^{+\omega+1}$ by Solovay's Theorem. $\square$

The proof breaks down when one tries to show that $2^{\lambda^{+\omega+1}} = \lambda^{+\omega+2}$. Moreover, the argument yields no insight into the value of 2^κ itself. To handle these cases, we must take a closer look at the proof of Theorem 3.1.

3.2 The Local Capturing Property

Habič-Honzík [10] define a generalization of the Mitchell order, extracted from the proof of Theorem 3.1, that describes the relationship between ultrafilters and powersets:

Definition 3.5. Suppose $\mathscr{U}$ is a set of countably complete ultrafilters and λ is a cardinal. The *Local Capturing Property*, denoted $\text{LCP}(\lambda, \mathscr{U})$, states that every subset of λ belongs to the ultrapower of the universe by an ultrafilter in $\mathscr{U}$.

The proof of Theorem 3.1 shows that the Local Capturing Property holds for supercompactness measures:

Theorem 3.6 (Solovay)**.** *Suppose $\kappa \leq \gamma$ are cardinals and $j : V \to M$ is an elementary embedding witnessing that κ is γ-supercompact. Let $\mathcal{U}$ be the ultrafilter on $P_\kappa(\gamma)$ derived from j using $j[\gamma]$.*

- *If $\mathcal{U} \in M$, then* $\text{LCP}(2^\gamma, \mathcal{N}(\kappa, \gamma))$ *holds in M.*
- *Therefore if $\mathcal{U} \in M$, $\lambda \leq 2^\gamma$, and $P(\lambda) \subseteq M$, then* $\text{LCP}(\lambda, \mathcal{N}(\kappa, \gamma))$ *holds.* $\square$

We will consider the statement $\text{LCP}(\lambda, \mathscr{U}(\delta))$. This is equivalent to the statement that for any $A \subseteq \lambda$, there is some U with $\lambda_U < \delta$ such that $A \in M_U$.

Since $\mathcal{N}(\kappa, \gamma) \subseteq \mathscr{U}(\eta)$ where $\eta = (\gamma^{<\kappa})^+$, we have the following implication:

Proposition 3.7. $\mathrm{LCP}(\lambda, \mathscr{N}(\kappa, \gamma))$ *implies* $\mathrm{LCP}(\lambda, \mathscr{U}(\eta))$ *where* $\eta = (\gamma^{<\kappa})^+$. $\square$

It will be convenient to use the following self-improvement of $\mathrm{LCP}(\lambda, \mathscr{U}(\delta))$:

Lemma 3.8. *Suppose δ and λ are cardinals such that $\mathrm{LCP}(\lambda, \mathscr{U}(\delta))$ holds. Then for any $A \subseteq \lambda$, there is a cardinal $\gamma < \delta$ and a countably complete uniform ultrafilter D on γ such that $A \in j_D(P(\gamma))$.*

The proof requires a corollary of the Kunen Inconsistency Theorem [11]:

Theorem 3.9 (Kunen)**.** *Suppose $j : V \to M$ is a nontrivial elementary embedding and $\iota \geq \mathrm{CRT}(j)$ is a fixed point of j such that $P(\alpha) \subseteq M$ for all $\alpha < \iota$. Then $P_{\aleph_1}(\iota) \nsubseteq M$.* $\square$

Lemma 3.10. *Suppose D is a nonprincipal countably complete ultrafilter and η is a cardinal such that such that $P(\alpha) \subseteq M_D$ for all ordinals $\alpha < \eta$. Then there is a strong limit cardinal $\iota \leq \lambda_D$ such that $j_D(\iota) > \eta$.*

Proof. Let $\lambda = \lambda_D$ Since $P(2^\lambda) \nsubseteq M_D$ by Theorem 2.19, $\eta \leq 2^\lambda$. Let ι be the supremum of all measurable cardinals $\kappa \leq \lambda$. Thus $\mathrm{CRT}(j_D) \leq \iota$. We claim that $j_D(\iota) > \eta$. Assume towards a contradiction that $j_D(\iota) \leq \eta$. Since $P(\alpha) \subseteq M_D$ for all $\alpha < \eta$, $j_D(\iota)$ really is a limit of measurable cardinals. In particular, $j_D(\iota)$ is a strong limit cardinal, and so since $j_D(\iota) \leq \eta \leq 2^\lambda$, we must have $j_D(\iota) \leq \lambda$. Since ι is the supremum of all measurable cardinals $\kappa \leq \lambda$ and $j_D(\iota)$ is a limit of measurable cardinals, $j_D(\iota) \leq \iota$. Since $\iota \geq \mathrm{CRT}(j)$, Theorem 3.9 implies $P_{\aleph_1}(\iota) \nsubseteq M_D$. This contradicts the fact that the ultrapower of the universe by a countably complete ultrafilter is closed under countable sequences. $\square$

Proof of Lemma 3.8. Let $\eta \leq \lambda$ be the least cardinal such that $2^\eta > \lambda$. Fix a sequence $\langle X_\alpha : \alpha < \lambda \rangle$ of distinct subsets of η. Fix a set $B \subseteq \lambda$ such that $P(\alpha) \subseteq L[B]$ for all $\alpha < \eta$. Suppose $D \in \mathscr{U}(\delta)$ has the property that $\langle X_\alpha : \alpha < \lambda \rangle$ and B belong to M_D. We claim $j_D(\lambda_D) \geq \lambda$. Since $B \in M_D$, $P(\alpha) \subseteq M_D$ for all $\alpha < \eta$. It follows from Lemma 3.10 that there is an strong limit cardinal $\kappa \leq \lambda_D$ such that $j_D(\kappa) > \eta$. Since $j_D(\kappa)$ is a strong limit in M_D,

$$j_D(\kappa) > (2^\eta)^{M_D} \geq \lambda$$

The final inequality follows from the fact that $\langle X_\alpha : \alpha < \lambda \rangle \in M_D$ is an injection from λ into $P(\eta) \cap M_D$ that belongs to M_D. Thus $j_D(\lambda_D) \geq j_D(\kappa) \geq \lambda$, as desired.

Now suppose $A \subseteq \lambda$, and we will find a countably complete uniform ultrafilter D on a cardinal $\gamma < \delta$ such that $A \in j_D(P(\gamma))$. By $\mathrm{LCP}(\lambda, \mathscr{U}(\delta))$, there is a countably complete uniform ultrafilter $D \in \mathscr{U}(\delta)$ such that $\langle X_\alpha : \alpha < \lambda \rangle$, B, and A belong to M_D. Let $\gamma = \lambda_D$. We may assume without loss of generality that γ is the underlying set of D. Since $\langle X_\alpha : \alpha < \lambda \rangle$ and B belong to M_D, $j_D(\gamma) \geq \lambda$ by the previous paragraph. Thus $A \in P(\lambda) \cap M_D \subseteq j_D(P(\gamma))$. $\square$

3.3 λ-Mitchell ultrafilters

The key concept in our proof of GCH is that of a λ-Mitchell ultrafilter:

Definition 3.11. Suppose λ is a cardinal. A countably complete ultrafilter U is λ-*Mitchell* if every countably complete uniform ultrafilter on a cardinal less than λ belongs to M_U.

Lemma 3.12. *Suppose λ is a cardinal and U is a λ-Mitchell ultrafilter. Then $\mathscr{U}(\lambda) \subseteq M_U$.*

Proof. This is immediate from Lemma 2.23, which asserts the invariance of the Mitchell order on hereditarily uniform ultrafilters under Rudin-Keisler equivalence. □

Assuming $2^{<\lambda} = \lambda$, any countably complete ultrafilter U such that $P(\lambda) \subseteq M_U$ is λ-Mitchell. Under UA, we can get away without the cardinal arithmetic hypothesis:

Theorem (UA)**.** *Suppose λ is a cardinal and U is a countably complete ultrafilter such that M_U is closed under λ-sequences. Then U is λ-Mitchell.*

This theorem, proved as Theorem 3.16 below, will be the engine for our results on cardinal arithmetic under UA. In this subsection, let us show how we will use it:

Theorem 3.13. *Suppose $\kappa \leq \gamma$ are cardinals with $\mathrm{cf}(\gamma) \geq \kappa$. Assume the following hold:*

- *There is a γ^+-Mitchell ultrafilter on a set of cardinality γ^+*
- *There is a γ^{++}-Mitchell ultrafilter on a set of cardinality γ^{++}.*
- *There is an elementary embedding $j : V \to M$ with the following properties:*
  - *j witnesses that κ is γ-supercompact.*
  - *The normal fine ultrafilter on $P_\kappa(\gamma)$ derived from j using $j[\gamma]$ belongs to M.*
  - *$P(\gamma^{++}) \subseteq M$.*

Then $2^\gamma = \gamma^+$.

Theorem 3.16 below implies that all the conditions of Theorem 3.13 follow under UA from the assumption that κ is γ^{++}-supercompact. This immediately yields GCH above a supercompact (Theorem 3.26) and more.

The proof of Theorem 3.13 requires one or two interesting lemmas which are motivated by a theorem of Cummings [12], which states that it is is consistent that there is a normal ultrafilter U on a cardinal κ with the property that $P(\kappa^+) \subseteq M_U$, or in other words (abusing notation slightly), $\mathrm{LCP}(\kappa^+, U)$. Since $P(2^\kappa)$ is never contained in M_U, $\mathrm{LCP}(\kappa^+, U)$ implies $2^\kappa > \kappa^+$. The key lemma on the way to Theorem 3.13 implies that the existence of a κ^+-Mitchell ultrafilter on a set of size κ^+ refutes $\mathrm{LCP}(\kappa^+, U)$ for all ultrafilters U on κ:

Lemma 3.14. *Suppose there is a nonprincipal λ-Mitchell ultrafilter on a set of cardinality λ. Suppose U is a countably complete ultrafilter such that $P(\lambda) \subseteq M_U$. Then $\lambda_U \geq \lambda$.*

Proof. We may assume without loss of generality that the underlying set of U is the cardinal λ_U, which we denote by γ. Assume towards a contradiction that $\gamma < \lambda$. Since $P(\lambda) \subseteq M_U$, we must have $j_U(\gamma) > \lambda$ by Lemma 3.10. Let W be a λ-Mitchell ultrafilter on λ. Let Z be the M_U-ultrafilter on $j_U(\gamma)$ projecting to W: in other words,

$$Z = \{A \subseteq j_U(\gamma) : A \in M_U \text{ and } A \cap \lambda \in W\}$$

Consider the ultrafilter $U\text{-}\sum Z$ on $\gamma \times \gamma$. (See Section 2.2 for the explanation of this notation.) By Lemma 2.12, $U\text{-}\sum Z$ is a countably complete ultrafilter, and it is easy to see that $U\text{-}\sum Z$

is hereditarily uniform with $\lambda_{U\text{-}\sum Z} = \gamma$. Thus $U\text{-}\sum Z \in \mathscr{U}(\lambda)$. Since W is λ-Mitchell, Lemma 3.12 implies that $U\text{-}\sum Z \lhd W$. In other words, $U\text{-}\sum Z \in M_W$. But $U\text{-}\sum Z$ codes Z (Lemma 2.13), so $Z \in M_W$. Hence $W \in M_W$: indeed $W = \{A \cap \lambda : A \in Z\}$. No countably complete nonprincipal ultrafilter belongs to its own ultrapower, so this is a contradiction. □

We also need the following lemma:

Lemma 3.15. *Suppose δ and λ are cardinals. Assume* $\text{LCP}(\lambda, \mathscr{U}(\delta))$ *holds. Then for any δ-Mitchell ultrafilter U, $P(\lambda) \subseteq M_U$.*

Proof. Fix $A \subseteq \lambda$, and we will show $A \in M_U$. By Lemma 3.8, there is a countably complete uniform ultrafilter D on a cardinal $\gamma < \delta$ such that $A \in j_D(P(\gamma))$. Since U is δ-Mitchell, $D \lhd U$. Therefore in particular $P(\gamma) \in M_U$, so $j_D(P(\gamma)) = P(\gamma)^\gamma/D \in M_U$. Since $A \in j_D(P(\gamma))$, it follows that $A \in M_U$. □

This yields the proof of Theorem 3.13:

Proof of Theorem 3.13. Assume towards a contradiction that $2^\gamma > \gamma^+$. Then Theorem 3.6 combined with the fact that $\gamma^{++} \leq 2^\gamma$ yields $\text{LCP}(\gamma^{++}, \mathscr{N}(\kappa, \gamma))$. By Theorem 2.18, $\gamma^{<\kappa} = \gamma$. Therefore $\mathscr{N}(\kappa, \gamma) \subseteq \mathscr{U}(\gamma^+)$, so $\text{LCP}(\gamma^{++}, \mathscr{N}(\kappa, \gamma))$ implies $\text{LCP}(\gamma^{++}, \mathscr{U}(\gamma^+))$.

Now let U be a γ^+-Mitchell ultrafilter on γ^+. By Lemma 3.15, since $\text{LCP}(\gamma^{++}, \mathscr{U}(\gamma^+))$ holds, $P(\gamma^{++}) \subseteq M_U$. This contradicts Lemma 3.14: since γ^{++} carries a γ^{++}-Mitchell ultrafilter, no countably complete ultrafilter D on γ^+ can satisfy $P(\gamma^{++}) \subseteq M_D$. □

3.4 λ-Mitchell ultrafilters from UA

The main theorem of this section shows that assuming the Ultrapower Axiom, every λ-supercompact ultrafilter is λ-Mitchell.

Theorem 3.16 (UA)**.** *Suppose W is a countably complete ultrafilter such that M_W is closed under λ-sequences. Then W is λ-Mitchell.*

The first step in the proof is a straightforward fact about the relationship between supercompactness and the Mitchell order:

Proposition 3.17. *Suppose γ is an ordinal, U is a countably complete ultrafilter on γ, and W is a countably complete ultrafilter such that M_W is closed under γ-sequences. Then the following are equivalent:*

(1) $U \lhd W$.

(2) There is an internal ultrapower comparison $(k, h) : (M_U, M_W) \to N$ of (j_U, j_W) such that $k([\text{id}]_U) \in h(j_W[\gamma])$.

Proof. (1) implies (2): We will not need this direction so we only sketch the proof. One takes $k = j_U(j_W)$ and $h = (j_U)^{M_W}$. Since M_W is closed under γ-sequences, $(j_U)^{M_W} = j_U \restriction M_W$, so $j_U(j_W) \circ j_U = j_U \circ j_W = (j_U)^{M_W} \circ j_W$. Thus $(j_U(j_W), (j_U)^{M_W})$ is an internal ultrapower comparison of (j_U, j_W). Moreover, $j_U(j_W)([\text{id}]_U) \in j_U(j_W)[j_U(\gamma)] = (j_U)^{M_W}(j_W[\gamma])$.

(2) implies (1): We will show that U is definable over M_W, and hence $U \in M_W$. In fact, we will prove:

$$U = \{A \subseteq \gamma : k([\mathrm{id}]_U) \in h(j_W[A])\} \tag{1}$$

Since $j_W \restriction \gamma \in M_W$, the function on $P(\gamma)$ given by $A \mapsto j_W[A]$ belongs to M_W. Moreover, h is an internal ultrapower embedding of M_W, and so in particular, h is a definable subclass of M_W. Thus (1) implies that U is definable over M_W.

To finish, we prove (1). For any $A \subseteq \gamma$:

$$\begin{aligned} A \in U &\iff [\mathrm{id}]_U \in j_U(A) \\ &\iff k([\mathrm{id}]_U) \in k(j_U(A)) \\ &\iff k([\mathrm{id}]_U) \in h(j_W(A)) \\ &\iff k([\mathrm{id}]_U) \in h(j_W(A)) \cap h(j_W[\gamma]) \end{aligned}$$

For the final equivalence, we use that $k([\mathrm{id}]_U) \in h(j_W[\gamma])$. Note that

$$h(j_W(A)) \cap h(j_W[\gamma]) = h(j_W(A) \cap j_W[\gamma]) = h(j_W[A])$$

This yields (1). $\square$

Remark 3.18. In the context of Proposition 3.17, the statement that $k([\mathrm{id}]_U) \in h(j_W[\gamma])$ is actually equivalent to the a priori weaker statement that $k([\mathrm{id}]_U) \in h(j_W[\beta])$ for some ordinal $\beta \geq \gamma$ such that $j_W[\beta] \in M_W$.

To see this, suppose $k([\mathrm{id}]_U) \in h(j_W[\beta])$. Since $[\mathrm{id}]_U < j_U(\gamma)$,

$$k([\mathrm{id}]_U) < k(j_U(\gamma)) = h(j_W(\gamma))$$

Therefore $k([\mathrm{id}]_U) \in h(j_W[\beta]) \cap h(j_W(\gamma)) = h(j_W[\beta] \cap j_W(\gamma))$. Finally, $j_W[\beta] \cap j_W(\gamma) = j_W[\gamma]$, so we have $k([\mathrm{id}]_U) \in h(j_W[\gamma])$, as desired.

To prove Theorem 3.16, it now suffices to prove the following fact:

Lemma 3.19. *Suppose γ is an ordinal, U is a countably complete ultrafilter on γ, and W is a countably complete ultrafilter whose ultrapower M_W is closed under γ^+-sequences. If (k, h) is an internal ultrapower comparison of (j_U, j_W), then $k([\mathrm{id}]_U) \in h(j_W[\gamma])$.*

Proof of Theorem 3.16. By the invariance of the Mitchell order on hereditarily uniform ultrafilters under Rudin-Keisler equivalence (Lemma 2.23), it suffices to show that for any countably complete ultrafilter U on an ordinal $\gamma < \lambda$, $U \vartriangleleft W$. Fix such an ultrafilter U. By UA, there is an internal ultrapower comparison (k, h) of (j_U, j_W). By Lemma 3.19, this implies $U \vartriangleleft W$. $\square$

If one drops the assumption that k is an internal ultrapower embedding of M_U, then the conclusion of Lemma 3.19 that $k([\mathrm{id}]_U) \in h(j_W[\gamma])$ can easily fail. Thus the argument must make use of the fact that k is an internal ultrapower embedding.

The proof uses the following concepts which are essentially part of the theory of strongly compact cardinals:

Definition 3.20. Suppose $j : V \to M$ is an elementary embedding and λ is a cardinal. A set $A \subseteq j(\lambda)$ is a *cover of* $j[\lambda]$ if $j[\lambda] \subseteq A$. A cover of $j[\lambda]$ is *j-closed* if for any $f : \lambda \to \lambda$, $j(f)[A] \subseteq A$.

We need three general lemmas regarding closed covers. The first concerns the interaction of closed covers with compositions:

Lemma 3.21. *Suppose $V \xrightarrow{j} M \xrightarrow{k} N$ are elementary embeddings and λ is a cardinal.*

- *If B is a $k \circ j$-closed cover of $k \circ j[\lambda]$, then $k^{-1}[B]$ is a j-closed cover of $j[\lambda]$.*
- *If $A \in M$ is a j-closed cover of $j[\lambda]$, then $k(A)$ is a $k \circ j$-closed cover of $k \circ j[\lambda]$.* □

Ultrafilters on small sets cannot have small covers:

Lemma 3.22. *Suppose U is a countably complete ultrafilter and $\lambda > \lambda_U$ is a regular cardinal. Suppose $A \in M_U$ is a cover of $j_U[\lambda]$. Then $|A|^{M_U} = j_U(\lambda)$.*

Proof. Since $\lambda > \lambda_U$ is regular, $j_U(\lambda) = \sup j_U[\lambda]$. Thus $j_U[\lambda]$ is cofinal in $j_U(\lambda)$. It follows that A is cofinal in $j_U(\lambda)$ Since $j_U(\lambda)$ is a regular cardinal of M_U, $|A|^{M_U} = j_U(\lambda)$. □

Combined with Lemma 3.22, the following lemma shows that closed covers past λ_U are highly constrained:

Lemma 3.23. *Suppose U is a countably complete ultrafilter and $\lambda \geq \lambda_U$ is a cardinal. Then $j_U(\lambda)$ is the unique j_U-closed cover A of $j_U[\lambda]$ such that $A \in M_U$ and $|A|^{M_U} = j_U(\lambda)$.*

Proof. We may assume without loss of generality that U is a uniform ultrafilter on λ_U. Let A be a closed cover of j_U at λ such that $A \in M_U$ and $|A|^{M_U} = j_U(\lambda)$. Fix $f : \lambda_U \to \lambda$, and we will show that $[f]_U \in A$.

Since $A \in M_U$, there is a sequence $\langle A_\alpha : \alpha < \lambda_U\rangle$ of subsets of λ with $A = [\langle A_\alpha : \alpha < \lambda_U\rangle]_U$. Since $|A|^{M_U} = j_U(\lambda)$, we may assume by Łoś's Theorem that $|A_\alpha| = \lambda$ for all $\alpha < \lambda$. Therefore there is an injective function $g : \lambda_U \to \lambda$ such that $g(\alpha) \in A_\alpha$ for all $\alpha < \lambda_U$. Let $h : \lambda \to \lambda$ be a function such that $h \circ g = f$. (Such a function necessarily exists because g is injective.) Then

$$j_U(h)([g]_U) = j_U(h)(j_U(g)([\text{id}]_U)) = j_U(h \circ g)([\text{id}]_U) = j_U(f)([\text{id}]_U) = [f]_U$$

Since $[g]_U \in A$ and $j_U(h)[A] \subseteq A$, it follows that $[f]_U \in A$, as desired. □

Lemma 3.24. *Suppose U and W are countably complete ultrafilters. Suppose $\lambda > \lambda_U$ is a regular cardinal and $B \in M_W$ is a j_W-closed cover of $j_W[\lambda]$. Suppose $(k, h) : (M_U, M_W) \to N$ is an internal ultrapower comparison of (j_U, j_W). Then $k[j_U(\lambda)] \subseteq h(B)$.*

Proof. Since $B \in M_W$ is a j_W-closed cover of $j_W[\lambda]$, $h(B)$ is a $h \circ j_W$-closed cover of $h \circ j_W[\lambda]$ by Lemma 3.21. Since $h \circ j_W = k \circ j_U$, it follows that $h(B)$ is a $k \circ j_U$-closed cover of $k \circ j_U[\lambda]$. Therefore $k^{-1}[h(B)]$ is a j_U-closed cover of $j_U[\lambda]$ by Lemma 3.21.

Let $A = k^{-1}[h(B)]$. Since k is an internal ultrapower embedding of M_U, $A \in M_U$. Since $A \in M_U$ is a cover of $j_U[\lambda]$ and $\lambda_U < \lambda$, by Lemma 3.22, $|A|^{M_U} \geq j_U(\lambda)$. By Lemma 3.23, $A = j_U(\lambda)$. Thus $k^{-1}[h(B)] = j_U(\lambda)$, or in other words, $k[j_U(\lambda)] \subseteq h(B)$. □

As an immediate consequence, we have proved Lemma 3.19:

Proof of Lemma 3.19. Trivially, $j_W[\gamma^+]$ is a closed cover of $j_W[\gamma^+]$. Since $j_W[\gamma^+] \in M_W$, applying Lemma 3.24 with $\lambda = \gamma^+$ and $B = j_W[\gamma^+]$ yields that $k[j_U(\gamma^+)] \subseteq h(j_W[\gamma^+])$. In particular, $k([\mathrm{id}]_U) \in h(j_W[\gamma^+])$. By Remark 3.18, this is equivalent to the statement that $k([\mathrm{id}]_U) \in h(j_W[\gamma])$. By Proposition 3.17, we can conclude that $U \lhd W$. □

3.5 GCH from UA

Our main theorems follow at once from Theorem 2.18, Theorem 3.13, and Theorem 3.16.

Theorem 3.25 (UA). *Suppose $\kappa \leq \lambda$ are cardinals and κ is λ^+-supercompact. Then for any cardinal γ with $\kappa \leq \gamma < \lambda$, $2^\gamma = \gamma^+$.*

Proof. There are two cases. Suppose first that $\mathrm{cf}(\gamma) \geq \kappa$. We claim that the hypotheses of Theorem 3.13 are satisfied.

We first show that there is a γ^+-Mitchell ultrafilter on a set of cardinality γ^+. Let $\mathcal{U}$ be a normal fine ultrafilter on $P_\kappa(\gamma^+)$, which exists by Lemma 2.17 since κ is γ^+-supercompact. Note that $|P_\kappa(\gamma^+)| = \gamma^+$ by Theorem 2.18. By Lemma 2.17, $M_\mathcal{U}$ is closed under γ^+-sequences, so by Theorem 3.16, $\mathcal{U}$ is γ^+-Mitchell.

Let $\mathcal{W}$ be a normal fine ultrafilter on $P_\kappa(\gamma^{++})$. As in the previous paragraph, $\mathcal{W}$ is a γ^{++}-Mitchell ultrafilter on a set of cardinality γ^{++}.

Finally, consider the elementary embedding $j_\mathcal{W} : V \to M_\mathcal{W}$. Let $\mathcal{D}$ be the normal fine ultrafilter on $P_\kappa(\gamma)$ derived from $j_\mathcal{W}$ using $M_\mathcal{W}$. Then $\mathcal{D} \lhd \mathcal{W}$ since $\mathcal{W}$ is γ^{++}-Mitchell and $\mathcal{D} \in \mathscr{U}(\gamma^{++})$. In other words $\mathcal{D} \in M_\mathcal{W}$. By Lemma 2.17, $M_\mathcal{W}$ is closed under γ^{++}-sequences, and as a consequence $M_\mathcal{W}$ is closed under γ-sequences and $P(\gamma^{++}) \subseteq M_\mathcal{W}$.

This verifies that the hypotheses of Theorem 3.13 are satisfied with $j = j_\mathcal{W}$, so $2^\gamma = \gamma^+$.

This leaves us with the case that $\mathrm{cf}(\gamma) < \kappa$. Note that γ is a limit of regular cardinals, and by the previous case, GCH holds at all of them. In particular, $2^{<\gamma} = \gamma$. Thus $2^\gamma = (2^{<\gamma})^{\mathrm{cf}(\gamma)} \leq \gamma^{<\kappa} = \gamma^+$ by Theorem 2.18. □

The global implication was stated in the introduction:

Theorem 3.26. *If κ is a supercompact cardinal, then $2^\lambda = \lambda^+$ for all $\lambda \geq \kappa$.* □

Applying the supercompactness analysis of the author's thesis [4], we obtain the full generalization of Solovay's Theorem:

Theorem 3.27 (UA). *If κ is a strongly compact cardinal, then $2^\lambda = \lambda^+$ for all cardinals $\lambda \geq \kappa$.*

Proof. Without loss of generality, we may assume κ is the least strongly compact cardinal. By Theorem 2.30, κ is supercompact. By Theorem 3.26, $2^\lambda = \lambda^+$ for all $\lambda \geq \kappa$. □

One can actually prove two more local instances of GCH by incorporating the argument of Corollary 3.4:

Theorem 3.28 (UA). *Suppose κ is λ^{++}-supercompact and* $\mathrm{cf}(\lambda) \geq \kappa$. *Then for any cardinal γ such that $\kappa \leq \gamma \leq \lambda^{++}$, $2^\gamma = \gamma^+$.*

Proof. By Theorem 3.25, $2^\gamma = \gamma^+$ for any cardinal $\gamma \in [\kappa, \lambda]$. It therefore suffices to show that $2^{(\lambda^+)} = \lambda^{++}$ and $2^{(\lambda^{++})} = \lambda^{+++}$.

We begin by showing $2^{(\lambda^+)} = \lambda^{++}$. Since $2^{<\lambda} = \lambda$, Corollary 3.3 implies $2^{2^\lambda} = (2^\lambda)^+$. In other words, $2^{(\lambda^+)} = \lambda^{++}$, as desired.

We continue by showing $2^{(\lambda^{++})} = \lambda^{+++}$. Since $2^\lambda = \lambda^+$, Corollary 3.3 implies $2^{2^{(\lambda^+)}} = (2^{(\lambda^+)})^+$. Since $2^{(\lambda^+)} = \lambda^{++}$ by the previous paragraph, this yields $2^{(\lambda^{++})} = \lambda^{+++}$, as desired. □

Applying the results of [4], we can replace the supercompactness hypothesis here with one that appears much weaker.

Definition 3.29. A cardinal λ is Fréchet if it carries a countably complete uniform ultrafilter.

Theorem 3.30 (UA)**.** *Suppose λ is a cardinal.*

(1) If λ^+ is Fréchet, then $2^{<\lambda} = \lambda$.

(2) If λ is regular and λ^{++} is Fréchet, then $2^\gamma = \gamma^+$ for $\gamma = \lambda, \lambda^+$, or λ^{++}.

Proof. (1) is immediate from Theorem 2.31 and Theorem 3.25. (2) is immediate from Theorem 2.31 and Theorem 3.28. □

The following proposition is a consequence of the proof of Solovay's Theorem [5] that SCH holds above a strongly compact cardinal along with Silver's Theorem [13] on SCH at singular cardinals of uncountable cofinality:

Proposition 3.31. *Suppose λ is a singular strong limit cardinal.*

- *If λ has countable cofinality and λ^+ is Fréchet, then $2^\lambda = \lambda^+$.*
- *If λ has uncountable cofinality and all sufficiently large regular cardinals below λ are Fréchet, then $2^\lambda = \lambda^+$.* □

Corollary 3.32 (UA)**.** *Suppose λ is a singular cardinal.*

- *If λ^+ is Fréchet, then $2^\lambda = \lambda^+$.*
- *If λ has uncountable cofinality and all sufficiently large regular cardinals below λ are Fréchet, then $2^\lambda = \lambda^+$.* □

Combining this with Theorem 3.30, we obtain:

Theorem 3.33 (UA)**.** *Suppose λ is a cardinal and λ^{++} is Fréchet. Then $2^\lambda = \lambda^+$.* □

Notice that if λ^{++} is Fréchet, then by Theorem 3.30, $2^\lambda = \lambda^+$. Moreover by Theorem 2.31, there is some cardinal $\kappa \leq \lambda$ that is λ^+-supercompact, and hence 2^λ-supercompact. We can actually use Theorem 3.33 to prove this instance of GCH under the weaker assumption that some cardinal $\kappa \leq \lambda$ is 2^λ-strongly compact:

Theorem 3.34 (UA)**.** *If $\kappa \leq \lambda$ are cardinals and κ is 2^λ-strongly compact, then $2^\lambda = \lambda^+$.*

Proof. Assume towards a contradiction that $2^\lambda \neq \lambda^+$. Then $2^\lambda \geq \lambda^{++}$. Since κ is 2^λ-strongly compact, it follows that λ^{++} carries a κ-complete uniform ultrafilter. Thus $2^\lambda = \lambda^+$ by Theorem 3.30. This is a contradiction. □

3.6 $\diamondsuit$ on the critical cofinality

Our final result shows that UA implies instances of Jensen's $\diamondsuit$ Principle above a supercompact cardinal. Results of Shelah generalizing Jensen's Theorem that CH does not imply $\diamondsuit_{\omega_1}$ show that under GCH, $\diamondsuit(S^{\kappa^+}_{\kappa})$ may also fail for κ a regular uncountable cardinal. The following result therefore goes beyond Theorem 3.30:

Theorem 3.35 (UA)**.** *Suppose κ is δ^{++}-strongly compact and* $\text{cf}(\delta) \geq \kappa$. *Then $\diamondsuit(S^{\delta^{++}}_{\delta^+})$ holds, where $S^{\delta^{++}}_{\delta^+} = \{\alpha < \delta^{++} : \text{cf}(\alpha) = \delta^+\}$.*

For the proof, we need a theorem of Kunen.

Definition 3.36. Suppose λ is a regular uncountable cardinal and $S \subseteq \lambda$ is a stationary set. Suppose $\langle \mathcal{A}_\alpha : \alpha \in S\rangle$ is a sequence of sets with $\mathcal{A}_\alpha \subseteq P(\alpha)$ and $|\mathcal{A}_\alpha| \leq \alpha$ for all $\alpha < \lambda$. Then $\langle \mathcal{A}_\alpha : \alpha \in S\rangle$ is a $\diamondsuit^-(S)$-sequence if for all $X \subseteq \lambda$, $\{\alpha \in S : X \cap \alpha \in \mathcal{A}_\alpha\}$ is stationary.

Definition 3.37. $\diamondsuit^-(S)$ is the assertion that there is a $\diamondsuit^-(S)$-sequence.

Theorem 3.38 (Kunen, [14])**.** *Suppose λ is a regular uncountable cardinal and $S \subseteq \lambda$ is a stationary set. Then $\diamondsuit^-(S)$ is equivalent to $\diamondsuit(S)$.* $\square$

Proof of Theorem 3.35. By Theorem 3.30, GCH holds on the interval $[\kappa, \delta^{++}]$, and we will use this without further comment.

For each $\alpha < \delta^{++}$, let $\mathcal{U}_\alpha$ be the unique ultrafilter of rank α in the wellorder $(\mathcal{N}(\kappa,\delta), \lhd)$. (The linearity of the Mitchell order on normal fine ultrafilters on $P_\kappa(\delta)$ is a consequence of Theorem 2.28 which applies in this context since $2^{<\delta} = \delta$.) Let $\mathcal{A}_\alpha = P(\alpha) \cap M_{\mathcal{U}_\alpha}$. Note that $|\mathcal{A}_\alpha| \leq \kappa^\delta = \delta^+$. Let

$$\vec{\mathcal{A}} = \langle \mathcal{A}_\alpha : \alpha < \delta^{++}\rangle$$

Note that $\vec{\mathcal{A}}$ is definable in $H_{\delta^{++}}$ without parameters.

Claim 1. *$\vec{\mathcal{A}}$ is a $\diamondsuit^-(S^{\delta^{++}}_{\delta^+})$-sequence.*

Proof. Suppose towards a contradiction that $\vec{\mathcal{A}}$ is not a $\diamondsuit^-(S^{\delta^{++}}_{\delta^+})$-sequence. Let $\mathcal{W}$ be a normal fine ultrafilter on $P_\kappa(\delta^{++})$. Then in $M_\mathcal{W}$, $\vec{\mathcal{A}}$ is not a $\diamondsuit^-(S^{\delta^{++}}_{\delta^+})$-sequence. Let $\mathcal{U}$ be the normal fine ultrafilter on $P_\kappa(\delta)$ derived from $\mathcal{W}$ and let $k : M_\mathcal{U} \to M_\mathcal{W}$ be the factor embedding. Let $\gamma = \textsc{crt}(k) = \delta^{++M_\mathcal{U}}$.

Since $\vec{\mathcal{A}}$ is definable in $H_{\delta^{++}}$ without parameters, $\vec{\mathcal{A}} \in \text{ran}(k)$. Therefore $k^{-1}(\vec{\mathcal{A}}) = \vec{\mathcal{A}} \restriction \gamma$ is not a $\diamondsuit^-(S^\gamma_{\delta^+})$-sequence in $M_\mathcal{U}$. Fix a witness $A \in P(\gamma) \cap M_\mathcal{U}$ and a closed unbounded set $C \in P(\gamma) \cap M_\mathcal{U}$ such that for all $\alpha \in C \cap S^\gamma_{\delta^+}$, $A \cap \alpha \notin \mathcal{A}_\alpha$. By elementarity, for all $\alpha \in k(C) \cap S^{\delta^{++}}_{\delta^+}$, $k(A) \cap \alpha \notin \mathcal{A}_\alpha$. Since $M_\mathcal{U}$ is closed under δ-sequences, $\text{cf}(\gamma) = \delta^+$, and so in particular $k(A) \cap \gamma \notin \mathcal{A}_\gamma$. Since $\gamma = \textsc{crt}(k)$, this means $A \notin \mathcal{A}_\gamma$.

Note however that $\mathcal{U}$ has Mitchell rank $\delta^{++M_\mathcal{U}} = \gamma$, so $\mathcal{U} = \mathcal{U}_\gamma$. Therefore $\mathcal{A}_\gamma = P(\gamma) \cap M_\mathcal{U}$, so $A \in \mathcal{A}_\gamma$ by choice of A. This is a contradiction. $\square$

By Theorem 3.38, this completes the proof. $\square$

4 Questions

A number of problems remain open. Many are variants on the following problem:

Question 4.1 (UA)**.** Suppose κ is κ^+-supercompact. Must $2^\kappa = \kappa^+$?

One can ask further whether UA implies that GCH holds at any cardinal λ such that some $\kappa \leq \lambda$ is λ-supercompact. We do not know how to refute this, but we conjecture a negative answer:

Conjecture 4.2. *It is consistent with* UA *that* GCH *fails at a measurable cardinal.*

Friedman-Magidor forcing [15] establishes an approximation to this conjecture: Friedman-Magidor construct a model in which the least measurable cardinal κ carries a unique normal ultrafilter and yet $2^\kappa > \kappa^+$. Assuming without loss of generality that there is just one measurable cardinal, the Ultrapower Axiom is equivalent to the statement that κ carries a unique normal ultrafilter U and moreover every κ-complete ultrafilter on κ is Rudin-Keisler equivalent to U^n for some $n < \omega$. Thus to affirm Conjecture 4.2, one seems to have to modify Friedman-Magidor forcing to control the structure of all κ-complete ultrafilters, rather than just the normal ones.

Question 4.3 (UA)**.** Suppose κ is a strong cardinal. Can GCH fail at κ or above?

It remains to be seen whether further combinatorial principles follow from UA.

Question 4.4 (UA)**.** Suppose κ is supercompact. Does $\diamondsuit(S^{\kappa^+}_\kappa)$ hold? (Note that $\diamondsuit^+(\lambda)$ is false if κ is λ-supercompact.) Does $S^{\kappa^{++}}_{\geq\kappa}$ carry a partial square?

Finally, one might try to prove GCH from the linearity of the Mitchell order alone (assuming large cardinals), rather than the full Ultrapower Axiom. To state this question it is convenient to make the following definition:

Definition 4.5. Let $\mathcal{N} = \bigcup\{\mathcal{N}(\kappa,\lambda) : \kappa,\lambda \in \text{Card and cf}(\lambda) \geq \kappa\}$.

If $\text{cf}(\lambda) < \kappa$, then every ultrafilter in $\mathcal{N}(\kappa,\lambda)$ is Rudin-Keisler equivalent to an ultrafilter in $\mathcal{N}(\kappa,\lambda^+)$, which is why we have thrown the former type of ultrafilters out of $\mathcal{N}$.

Recall that UA implies that the Mitchell order is linear on $\mathcal{N}$ (Theorem 2.28). Notice also that certain instances of GCH at regular cardinals do not require the full strength of UA, and instead only use the linearity of the Mitchell order on $\mathcal{N}$ (Corollary 3.4).

Question 4.6. Assume there is a supercompact cardinal and the Mitchell order is linear on $\mathcal{N}$. Does GCH hold at all sufficiently large cardinals?

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