The Ultrapower Axiom from Determinacy

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Abstract

We prove that the Ultrapower Axiom follows from determinacy hypotheses, revealing a new aspect of the connection between the Axiom of Determinacy and the theory of canonical models of large cardinal hypotheses.

1 Introduction

The Ultrapower Axiom (UA) is a set-theoretic principle that governs the structure of measures on transfinite cardinals, stratifying them into a well-ordered complexity hierarchy. Although UA is independent of the ZFC axioms, it is true in all known canonical models of ZFC. The purpose of this paper is to establish UA in a completely different context:

Theorem 4.13. *Assume* $\mathrm{ZF} + \mathrm{AD}^+$*. Then* UA *holds below* Θ*.*

The theorem demonstrates a structural resemblance between canonical models of ZFC and models of the Axiom of Determinacy (AD), even though the former satisfy very strong forms of the Axiom of Choice (AC) while the latter exhibit spectacular failures of it. Moreover, the result offers a unified explanation of the linearity and definability properties of measures long observed in both classes of models, and it provides a new technique for transferring inner model–theoretic arguments to the determinacy setting. These developments suggest that UA is not merely an artifact of the construction of canonical models but rather a general feature of large cardinals as they arise in nature.

AD was proposed by Mycielski–Steinhaus [24] as an alternative to AC; it asserts the existence of a winning strategy for one player or the other in any infinite two-player game of perfect information with a countable state-space. Woodin's AD^+ [19, 30] is an extension of AD that holds in all known models of AD but admits a more tractable global structure theory. It is an open problem whether these two principles are equivalent [2, Problem 14]. The cardinal Θ is the least ordinal that is not the range of a function on $\mathbb{R}$. AD^+ tells us essentially nothing about the structure of cardinals above Θ, and in particular, whether UA holds *above* Θ is independent of $\mathrm{ZF} + \mathrm{AD}^+$. On the other hand, we will prove:

Corollary 4.20. *Assume* $\mathrm{ZFC} + \mathrm{AD}^{L(\mathbb{R})}$*. Then* $L(\mathbb{R})$ *satisfies* UA*.*

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Theorem 4.13 has various applications to the global theory of measures under determinacy hypotheses, a major topic of study due to its intimate connection with the cardinal structure of determinacy models [11]. Among these applications, the most basic is the linearity of the *Mitchell order*, the ordering of normal measures given by $U \lhd W$ if U belongs to the ultrapower of the universe by W. In all canonical models of ZFC, the Mitchell order is linear [4, 22]. In the context of AD, work of Jackson [10] implies the linearity of the Mitchell order for cardinals smaller than the least weakly Mahlo cardinal, but yields no information about the global behavior of this order. Theorem 4.13 implies:

Corollary 4.21. *Assume* $\text{ZF} + \text{AD}^+$. *Then for any cardinal* $\kappa < \Theta$*, the Mitchell order well-orders the normal measures on* κ*.*

1.1 Background

Ultrapowers and UA. The proof of Scott's theorem [27] rests on a basic insight that transformed the theory of large cardinals: if one forms the ultrapower of the universe of sets V using a countably complete ultrafilter (or *measure*), the resulting structure is isomorphic to a submodel of the universe. A model obtained this way is called an *ultrapower* of the universe. Such ultrapowers are the basic objects of study in modern large cardinal theory.

The Ultrapower Axiom (UA) is an abstract set theoretic principle governing the structure of ultrapowers. Roughly speaking, it states that within the intersection of any two ultrapowers of the universe, one can always find a further ultrapower.

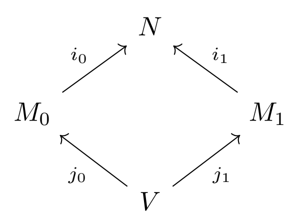


Figure 1: *The Ultrapower Axiom:* within any two ultrapowers M_0 and M_1 lies a further ultrapower N that is itself an ultrapower of M_0 and M_1. Moreover, the ultrapower embeddings commute.

Canonical models and the inner model problem. UA is motivated by the search for *canonical models of ZFC* satisfying large cardinal hypotheses. Though the term "canonical" is not precisely defined, set theorists have isolated a hierarchy of canonical models, starting with Gödel's constructible universe, capturing a certain initial segment of the large cardinal hierarchy. The longstanding *inner model problem* asks whether every large cardinal hypothesis has a canonical model. The question remains open for various important large cardinal notions, such as strong compactness and supercompactness.

All of the known canonical models satisfy UA, and the basic methodology of inner model theory can only produce models of UA. Thus, we can formulate a precise version of the inner model problem: is UA consistent with all large cardinal hypotheses?

UA and large cardinals. UA is most interesting in the context of powerful large cardinal hypotheses like strong compactness and supercompactness, for which canonical models are not yet known to exist. These hypotheses supply a wide array of measures and ultrapowers to test against the rigid structure imposed by UA. UA enables us to define a global well-ordering of the class of all measures on ordinals, called the *Ketonen order* [17, 6]. This means that each measure on an ordinal is classified by a single ordinal invariant (its rank in the Ketonen order), a striking contrast to the usual intuition of ultrafilters as wild, unclassifiable objects. The tension between the large number of measures and the rigid structure they carry yields information about the universe of sets. For example, under UA, the generalized continuum hypothesis holds above the least strongly compact cardinal [7].

Measures and determinacy. There is a second branch of set theory in which measures and their ultrapowers play a fundamental role: the study of the Axiom of Determinacy (AD). Under AD, the familiar filter extension lemma (that is, every filter extends to an ultrafilter) is false, and instead, there are no nonprincipal ultrafilters on ω at all. This implies that every ultrafilter is a measure, but it is still not obvious that there are any measures.

In the late 1960s, Solovay [14] discovered the canonical example of a measure under AD: the closed unbounded filter on ω_1. More generally, it turns out that measures are ubiquitous under AD: Kunen later showed that although the filter extension lemma fails, every *countably complete* filter on an ordinal below Θ extends to a measure. At the same time, he also established a key *rigidity property* of measures on ordinals: he defined a global well-ordering of the set of all measures on ordinals below Θ, showing that such measures are classified by ordinal invariants. This structural parallel raises the question of whether UA follows from AD [5, Question 3.6.17].

Measures below $\aleph_\omega$. Given the complexity of measures under AD, it seemed unlikely that the answer to this question would be positive. Seeking to find a counterexample, Jackson and the author began testing whether various pairs of measures on $\aleph_\omega$ satisfied the axiom. Since the measures in this region were already classified by Kunen in the 1970s [29], it was natural to expect this to be a straightforward task. However, no counterexample presented itself, but instead, a complex combinatorial structure emerged.

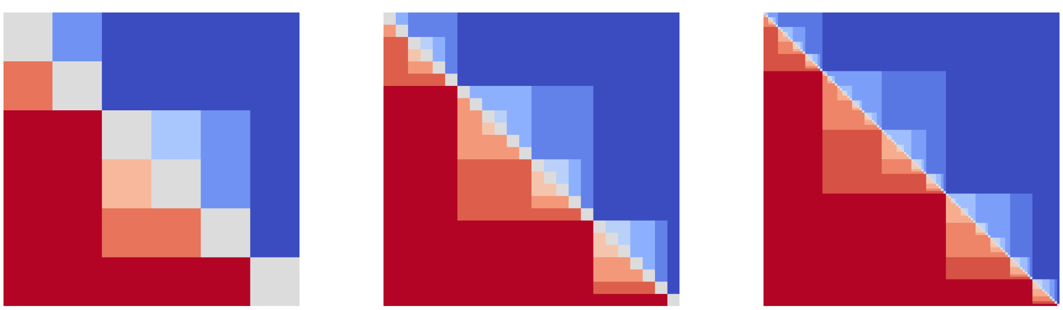


Figure 2: *Heatmaps of the Ketonen ultrametric on measures on* $\aleph_4, \aleph_5$, *and* $\aleph_6$*:* each point (x, y) on the grid represents a pair of measures on $\aleph_n$, colored based on their signed distance in the metric; the brighter the color, the greater the distance; reds indicate that $x <_{\Bbbk} y$, and blues that $x >_{\Bbbk} y$.

Jackson and the author ultimately proved UA below $\aleph_\omega$, but extending their analysis to all projective ordinals seems to be a formidable task.

1.2 A sketch of the proof.

Our proof of UA from determinacy closely follows the proof that UA is true in all canonical models of ZFC [8, Theorem 2.3.10], simulating Woodin's weak comparison axiom [35, Definition 5.29] using the Kechris coding theorem for measures and a generic ultrapower construction due to Woodin.

The proof divides into three steps: (1) showing that all elementary embeddings from countable transitive models into V are close, (2) proving a realization lemma for ultrapowers of such countable transitive models, and (3) combining these ideas to obtain close comparisons of any pair of ultrapowers.

UA from weak comparison. Let us briefly sketch the proof that a model W of ZFC plus weak comparison satisfies UA. We use the following consequence of weak comparison: there is a countable transitive model M elementarily embedded into W with the property that any two ultrapowers N_0 and N_1 of M admit a *close comparison*, a pair of *close embeddings* from N_0 and N_1 into a single transitive model P. Roughly speaking, an elementary embedding $j : M \to N$ is *close* if every finitely generated fragment of j is definable over M; more precisely, every measure derived from j belongs to M. Given such a close comparison, we can form a finitely generated hull inside of P to obtain an ultrapower of M that — by the closeness of the comparison — lies within both N_0 and N_1. This proves UA for M. Since M is elementarily equivalent to W, UA holds in W.

All embeddings are close. The proof of our main theorem simulates the weak comparison argument for UA in the determinacy context. So assume that V is a model of AD. The key to simulating weak comparison in V is Kechris's *coding theorem for measures* [15], which roughly implies that all embeddings are close (Corollary 4.3). More precisely, if M is a countable transitive model, and $j : M \to V$ is an elementary embedding, then j is close to M. This phenomenon cannot occur under AC because any such $j : M \to V$ has critical point ω_1^M, so that closeness implies M satisfies "ω_1 is measurable."

Jensen's realization lemma. Therefore fix a countable transitive model M that is elementarily embedded into V. Using that all embeddings are close, we can construct a close comparison of two ultrapowers N_0 and N_1 of M simply by finding elementary embeddings from N_0 and N_1 back into V. In the context of ZFC, it is a well-known fact that this is always possible. This result, known as Jensen's *realization lemma* [31, Lemma 3.7], underpins many of the iterability proofs in inner model theory. In our context, however, the realization lemma is *false*: the ultrapowers we are considering do not satisfy Łoś's theorem, and so they are not elementarily equivalent to V, let alone elementarily embeddable into it.

Woodin's generic ultrapower construction. To recover elementarity, we make use of a forcing technique due to Woodin (Section 4.2), which allows us to replace the ultrapowers N_0 and N_1 with *generic ultrapowers* N_0^* and N_1^* satisfying Łoś's theorem. These generic ultrapowers are sufficiently similar to the original ones that it suffices for UA to construct a close comparison of them instead. We prove a realization lemma (Lemma 4.10) for the generic ultrapowers, using Steel's construction of a winning strategy for Nonempty in a certain Galvin–Jech–Magidor precipitous game associated to the generic ultrapowers (Theorem 4.9). The realization lemma yields the desired close comparison, and UA follows as in the proof from weak comparison.

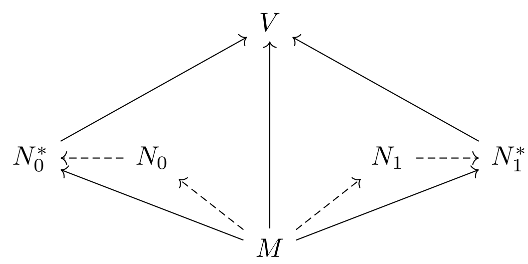


Figure 3: *Replacing ultrapowers with generic ultrapowers.* Dashed arrows denote non-elementary embeddings; solid arrows denote fully elementary embeddings.

Beyond the Suslin cardinals. Because the Kechris coding theorem requires the determinacy of certain games on reals, the outline above only leads to a proof of UA below the supremum of the Suslin cardinals (or assuming real determinacy, Corollary 4.12). To reach all the way up to Θ, we must (1) prove UA at the largest Suslin cardinal, if there is one, and (2) propagate UA to all ordinals between the largest Suslin cardinal and Θ. (1) is accomplished in Theorem 4.11, heavily using the fact that UA is equivalent to the linearity of the Ketonen order (Theorem 3.8). (2) is the subject of Section 4.4, and requires aspects of the proof of Woodin's theorem that Θ is Woodin in $\text{HOD}^{L(\mathbb{R})}$ [18].

2 Preliminaries

In this section, we sketch the basic constructions in the choiceless theory of ultrapowers of the universe of sets, attempting to keep the notation as close as possible to [8]. The basic challenge is that Łoś's theorem fails. For this reason, ultrapowers are no longer extensional structures, and therefore they cannot be identified with their transitive collapse.

2.1 Ultrapowers without choice

Measures. For any set X, a *measure on* X is a countably complete ultrafilter on X. (This terminology is justified by Corollary 4.23 below.) The set of measures on X is denoted by $\upsilon(X)$. If U is an ultrafilter on X and $A \subseteq X$, $\forall^*_U x\, A(x)$ means that $A \in U$.

The ultrapower construction. If M is a model of ZF and U is an M-ultrafilter, we denote by M_U the *uncollapsed* ultrapower of M by U using only functions in M. Let $j^M_U : M \to M_U$ denote the induced embedding. (We will typically drop the superscript when there is no risk of confusion.)

Since we do not assume M is a model of AC, Łoś's theorem may fail for this ultrapower. In particular, although in our applications M_U will typically be well-founded, it will generally not be extensional, which is why we do not take its transitive collapse.

Ultrapowers and ultrapower embeddings. A map $j : M \to N$ is an *ultrapower embedding of* M if for some M-ultrafilter U, there is an isomorphism $k : M_U \to N$ satisfying $k \circ j_U = j$. An *ultrapower of* M is a pair (N, j), where $j : M \to N$ is an ultrapower embedding.

2.2 Iterated ultrapowers

We fix throughout this section an ultrapower (N, j) of M.

Interpreted classes. If $C \subseteq M$, we define the *interpretation of* C *in* N by

$$(C)^N = \big\{a \in N : \exists A \in M \,\big(A \subseteq C \wedge N \vDash a \in j(A)\big)\big\}$$

If $N = M_U$, we have $(C)^N = \{[f]_U : \text{ran}(f) \subseteq C\}$. We use similar notation for subclasses of M^n, operations on M, etc. We usually write C^N instead of $(C)^N$.

The iterated ultrapower construction. For any $Y \in N$, an N*-ultrafilter on* Y is an ultrafilter on the Boolean algebra $P^N(Y)$ where $P : M \to M$ is the powerset operation of M. Let $\mathcal{F} \subseteq M$ denote the class of functions in M. The *ultrapower of* N *by* W, denoted by N_W, consists of equivalence classes of $f \in \mathcal{F}^N$ with $\text{dom}^N(f) = Y$ under the equivalence relation defined by $f =_W g$ if and only if $(\{y : f(y) = g(y)\})^N \in W$. The membership relation of N is defined in the usual way, as is the ultrapower embedding $j^N_W : N \to N_W$.

Note that N_W depends not only on W and N, but also on the ultrapower embedding $j : M \to N$.

Iterated ultrafilters. If $(N, j) = (M_U, j_U)$, we will denote N_W by $M_{U,W}$ instead of $(M_U)_W$. Assume U lies on X and W lies on $Y = [x \mapsto Y_x]_U$. In this case, N_W is isomorphic to the ultrapower of M by the M-ultrafilter

$$[U, W] = \left\{ A \subseteq \bigsqcup_{x \in X} Y_x : [x \mapsto A_x]_U \in W \right\}$$

where $\bigsqcup_{x\in X} Y_x = \bigcup_{x \in X}\{x\} \times Y_x$ and $A_x = \{y \in Y_x : (x, y) \in A\}$. More precisely, $(N_W, j_W \circ j_U) \cong (M_{[U,W]}, j_{[U,W]})$. Identifying these two models, j^N_W becomes the factor map from M_U to $M_{[U,W]}$ induced by the projection of $[U, W]$ to U.

Iterated ultrapowers and ultrapower embeddings. A map $i : N \to P$ is an *ultrapower embedding of* (N, j) if for some N-ultrafilter W, there is an isomorphism $k : N_W \to P$ satisfying $i = k \circ j^N_W$. In this case, we say the pair (P, i) is an *ultrapower of* (N, j). When there is no possibility of confusion, we will sometimes leave the embedding j implicit, and refer simply to ultrapowers and ultrapower embeddings of N, although formally this is not well-defined.

If (N, j) is an ultrapower of M and (P, i) is an ultrapower of (N, j), then in view of the iterated ultrafilter construction above, $(P, i \circ j)$ is an ultrapower of M. This observation allows us to define iterated ultrapowers of any finite length.

Internal measures and ultrapowers. We say an N-ultrafilter W on Y is an *internal measure of* N if there is some $w \in \upsilon^N(Y)$ such that $W = \{a \in N : N \vDash a \in w\}$. We will often abuse notation by confusing the point w with the internal measure W; thus we will speak of the ultrapower of N by w. Note however that there may be distinct points in N corresponding to the same internal measure (since N need not be extensional).

We say $i : N \to P$ is an *internal ultrapower embedding of* N, or (P, i) is an *internal ultrapower of* N, if there is an internal measure W of N and an isomorphism $k : N_W \to P$ such that $k \circ j^N_W = i$.

2.3 Łoś classes and the transitive collapse

Ordinals and sets of ordinals. If (N, j) is an ultrapower of M and Ord^N is well-founded, we will identify Ord^N with its collapse. We will furthermore identify $P(\mathrm{Ord})^N$ with its collapse, noting that $P(\mathrm{Ord})^N$ is provably extensional in ZF. This means that

$$j_U \restriction P(\mathrm{Ord})^M : P(\mathrm{Ord})^M \to P(\mathrm{Ord})^N$$

maps sets of ordinals to sets of ordinals.

Ultrapowers of Łoś classes. The desire to take transitive collapses leads to an alternate approach to ultrapowers of the universe by measures on ordinals, essentially laid out in Kechris [14]. Let us say that a transitive class H is a *Łoś class* if

$$H = \bigcup\{L[S] : S \in P(\mathrm{Ord}) \cap H\}$$

Then for any H-ultrafilter U, one can form the ultrapower $j_U : H \to H_U$ using only functions in H. This ultrapower satisfies Łoś's theorem for Δ_0-formulas, and moreover the range of j_U is cofinal in H_U, and so $j_U : H \to H_U$ is Σ_1-elementary. As a consequence, H_U is extensional, and if well-founded, its transitive collapse is again a Łoś class.

The hereditarily well-orderable sets. The class HWO of *hereditarily well-orderable sets* is the maximal Łoś class:

$$\mathrm{HWO} = \bigcup\{L[S] : S \in P(\mathrm{Ord})\}$$

The previous paragraph shows that (assuming DC) one can form iterated ultrapowers of HWO by measures on ordinals without leaving the category of Łoś classes and cofinal Σ_1-elementary embeddings.

Under $\mathrm{AD}^+ + V = L(P(\mathbb{R}))$, if U is a measure on an ordinal, then $\mathrm{HWO}_U = \mathrm{HWO}^{V_U}$. This is because every function from an ordinal to HWO belongs to HWO by Woodin's theorem [1, Theorem 1.4] that for every set X, either X is well-orderable or there is an injection from $\mathbb{R}$ to X. We will not make use of this observation in this paper.

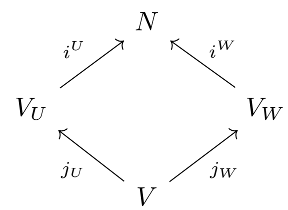
3 The Ultrapower Axiom

In this section, we formulate UA in the choiceless context, and show that some of its most basic theory can be carried out without using AC. In particular, we prove that UA is equivalent to the linearity of the Ketonen order (Theorem 3.8) and that UA implies that the Mitchell order is a well-order (Theorem 3.14).

3.1 UA in ZF

The Ultrapower Axiom is defined in almost the same way in ZF as it is assuming AC; the only difference is that we restrict to measures on ordinals rather than considering measures on arbitrary sets.

Definition 3.1. UA_κ states that for any $U, W \in \upsilon(\kappa)$, there exist internal ultrapower embeddings $i^U : V_U \to N$ and $i^W : V_W \to N$ such that the following diagram commutes:



The *Ultrapower Axiom* (UA) states that UA_κ is true for all ordinals κ. We say *UA holds below* κ if UA_λ holds for all $\lambda < \kappa$.

Note that UA_κ implies that UA holds below κ.

We note that many other natural choiceless formulations of UA for measures on ordinals are equivalent to this one.

Definition 3.2. A *combinatorial comparison* of $U, W \in \upsilon(\kappa)$ is a pair $\langle W_\alpha \rangle_{\alpha<\kappa}, \langle U_\beta \rangle_{\beta<\kappa}$ of sequences of measures on κ such that for any binary relation R on κ,

$$\forall^*_U \alpha\, \forall^*_{W_\alpha} \beta\, R(\alpha, \beta) \iff \forall^*_W \beta\, \forall^*_{U_\beta} \alpha\, R(\alpha, \beta)$$

Proposition 3.3. *For any ordinal κ, the following are equivalent:*

(1) UA_κ *holds.*

(2) For any $U, W \in \upsilon(\kappa)$, there exist $W_* \in j_U(\upsilon(\kappa))$ and $U_* \in j_W(\upsilon(\kappa))$ such that $(\mathcal{H}_U)_{W_*} = (\mathcal{H}_W)_{U_*}$ and $j^{\mathcal{H}_U}_{W_*} \circ j^{\mathcal{H}}_U = j^{\mathcal{H}_W}_{U_*} \circ j^{\mathcal{H}}_W$ where $\mathcal{H} = \mathrm{HWO}$.

(3) For any $U, W \in \upsilon(\kappa)$, there exist $W_* \in j_U(\upsilon(\kappa))$ and $U_* \in j_W(\upsilon(\kappa))$ such that

$$j_{W_*} \circ j_U \restriction P(\kappa) = j_{U_*} \circ j_W \restriction P(\kappa)$$

(4) Every pair of ultrafilters in $\upsilon(\kappa)$ admits a combinatorial comparison.

The proof of the proposition also shows that in the definition of UA, the internal ultrapower embeddings can be assumed to arise from internal measures on the images of κ in the respective ultrapowers.

3.2 The Ketonen order

In this section, we develop a key tool in the theory of UA: the Ketonen order. The main theorem proved here is that the linearity of the Ketonen order is *equivalent* to UA. This fact will be essential to the proof of Theorem 4.13. One of the main results of [6] is that the same statement is a consequence of ZFC. Here we must get by in ZF + DC, so we give a combinatorial proof that does not mention ultrapowers at all.

Definition 3.4. If $U, W \in \upsilon(\kappa)$, $U <_{\Bbbk} W$ if there exists $\langle U_\beta \rangle_{\beta<\kappa} \in {}^\kappa\upsilon(\kappa)$ such that for all $A \subseteq \kappa$, $A \in U$ if and only if $\forall^*_W \beta\, (A \cap \beta \in U_\beta)$.

Another way to define the Ketonen order is in terms of ultrafilter limits. If W is an ultrafilter on Y and $\langle U_i\rangle_{i\in Y}$ are ultrafilters on X, then

$$W\text{-}\lim_{i\in Y} U_i = \{A\subseteq X : \forall^*_W i\,(A\in U_i)\}$$

Then for $U, W \in \upsilon(\kappa)$, $U <_{\Bbbk} W$ if and only if $U = W\text{-}\lim_{\beta<\kappa} U_\beta$ for some $\langle U_\beta\rangle_{\beta<\kappa} \in {}^\kappa\upsilon(\kappa)$ such that $\forall^*_W\beta(\beta\in U_\beta)$.

Suppose $D\in\upsilon(\kappa)$ and $\beta\in D$ for some $\beta<\kappa$. Then D is essentially a measure on β, namely $D\cap P(\beta)$, and therefore the Ketonen order could also be formulated in terms of limits of measures U_β on β. For a similar reason, $(\upsilon(\beta), <_{\Bbbk})$ is canonically isomorphic to an initial segment of $(\upsilon(\kappa), <_{\Bbbk})$, and so we can think of the Ketonen order as a global partial order of the class of measures on ordinals.

There is an important ultrapower-theoretic characterization of the Ketonen order:

Lemma 3.5. *For $U, W\in\upsilon(\kappa)$, the following are equivalent:*

- *$U <_{\Bbbk} W$.*
- *For some ultrapower embedding $i^U : V_U \to N$ and internal ultrapower embedding $i^W : V_W\to N$, we have $i^U\circ j_U = i^W\circ j_W$ and $i^U([\mathrm{id}]_U) < i^W([\mathrm{id}]_W)$.*
- *There is an internal measure U_* of V_W such that $[\mathrm{id}]_W\in U_*$ and $j_W[U]\subseteq U_*$.*

Corollary 3.6. *Under* UA_κ*, the Ketonen order is total on $\upsilon(\kappa)$.*

Many of the key properties of the Ketonen order can be established without using AC (or UA):

Theorem 3.7 ([8, Theorem 3.3.8], DC)**.** *The Ketonen order is a strict well-founded partial order.*

In fact, only well-foundedness requires DC. Even this use of DC can be removed if one redefines $\upsilon(\kappa)$ to be the set of measures on κ with well-founded ultrapowers [9]. However, Theorem 3.7 suffices for our applications here.

The following is the main theorem of this section:

Theorem 3.8 (DC)**.** *For any ordinal κ,* UA_κ *is equivalent to the linearity of the Ketonen order on $\upsilon(\kappa)$.*

The proof requires some additional terminology.

- If $(X,\prec)$ is a well-order, the *Ketonen order on $\upsilon(X)$ relative to $\prec$* is defined by $U\prec_{\Bbbk} W$ if $U = W\text{-}\lim_{y\in X} U_y$ for some $\langle U_y\rangle_{y\in X}\in {}^X\upsilon(X)$ such that $\forall^*_W y\,\forall^*_{U_y} x\,(x\prec y)$.
- A well-order $\prec$ of $\kappa\times\kappa$ is a *product* if $\alpha_0\le\alpha_1$ and $\beta_0\le\beta_1$ implies $(\alpha_0,\beta_0)\preceq(\alpha_1,\beta_1)$.
- For $U, W\in\upsilon(\kappa)$, $U\times W$ denotes the filter on $\kappa\times\kappa$ generated by sets of the form $A\times B$ where $A\in U$ and $B\in W$.

Theorem 3.9. *Suppose $U, W\in\upsilon(\kappa)$ and $\langle W_\alpha\rangle_{\alpha<\kappa}\in{}^\kappa\upsilon(\kappa)$ is Ketonen minimal modulo U such that $W = U\text{-}\lim_{\alpha<\kappa} W_\alpha$. Then $D = \{R\subseteq\kappa\times\kappa : \forall^*_U\alpha\,\forall^*_{W_\alpha}\beta\, R(\alpha,\beta)\}$ is Ketonen minimal among all extensions of $U\times W$ relative to any product well-order $\prec$ of $\kappa\times\kappa$.*

Proof. Assume towards a contradiction that E extends $U \times W$ and $E \prec_{\Bbbk} D$. Fix measures $E_{\alpha,\beta}$ on $\kappa \times \kappa$ such that $E = D\text{-}\lim_{\alpha,\beta} E_{\alpha,\beta}$ and

$$\forall^*_D(\alpha,\beta)\, \forall^*_{E_{\alpha,\beta}}(\alpha_0,\beta_0)\, ((\alpha_0,\beta_0) \prec (\alpha,\beta)) \tag{1}$$

Since $\prec$ is a product order, we will contradict (1) by proving

$$\forall^*_D(\alpha,\beta)\, \forall^*_{E_{\alpha,\beta}}(\alpha_0,\beta_0)\, (\alpha_0 \geq \alpha) \tag{2}$$
$$\forall^*_D(\alpha,\beta)\, \forall^*_{E_{\alpha,\beta}}(\alpha_0,\beta_0)\, (\beta_0 \geq \beta) \tag{3}$$

For (2), note that

$$\begin{aligned} A \in U &\iff \forall^*_E(\alpha_0,\beta_0)\, (\alpha_0 \in A) \\ &\iff \forall^*_D(\alpha,\beta)\, \forall^*_{E_{\alpha,\beta}}(\alpha_0,\beta_0)\, (\alpha_0 \in A) \\ &\iff \forall^*_U\alpha\, \forall^*_{W_\alpha}\beta\, \forall^*_{E_{\alpha,\beta}}(\alpha_0,\beta_0)\, (\alpha_0 \in A) \end{aligned}$$

Therefore letting

$$U^*_\alpha = \{A \subseteq \kappa : \forall^*_{W_\alpha}\beta\, \forall^*_{E_{\alpha,\beta}}(\alpha_0,\beta_0)\, (\alpha_0 \in A)\}$$

we have $U = U\text{-}\lim_{\alpha<\kappa} U^*_\alpha$. Since $U \not<_{\Bbbk} U$, we must have $\forall^*_U\alpha\, \forall^*_{U^*_\alpha}\alpha_0\, (\alpha_0 \geq \alpha)$. From the definition of U^*_α, this yields $\forall^*_U\alpha\, \forall^*_{W_\alpha}\beta\, \forall^*_{E_{\alpha,\beta}}(\alpha_0,\beta_0)\, (\alpha_0 \geq \alpha)$ which is equivalent to (2) by the definition of D.

Finally, assume towards a contradiction that (3) fails. Note that

$$\begin{aligned} B \in W &\iff \forall^*_E(\alpha_0,\beta_0)\, (\beta_0 \in B) \\ &\iff \forall^*_D(\alpha,\beta)\, \forall^*_{E_{\alpha,\beta}}(\alpha_0,\beta_0)\, (\beta_0 \in B) \\ &\iff \forall^*_U\alpha\, \forall^*_{W_\alpha}\beta\, \forall^*_{E_{\alpha,\beta}}(\alpha_0,\beta_0)\, (\beta_0 \in B) \end{aligned} \tag{4}$$

For $\alpha < \kappa$, let

$$W^*_\alpha = \{B \subseteq \kappa : \forall^*_{W_\alpha}\beta\, \forall^*_{E_{\alpha,\beta}}(\alpha_0,\beta_0)\, (\beta_0 \in B)\}$$

Then

$$W = U\text{-}\lim_{\alpha<\kappa} W^*_\alpha \tag{5}$$

by (4). Moreover, for U-almost all $\alpha < \kappa$, letting $E^*_\beta = \{B \subseteq \kappa : \forall^*_{E_{\alpha,\beta}}(\alpha_0,\beta_0)\, (\beta_0 \in B)\}$, we have $\forall^*_{W_\alpha}\beta\, \forall^*_{E^*_\beta}\beta_0\, (\beta_0 < \beta)$ by the failure of (3), and

$$W^*_\alpha = W_\alpha\text{-}\lim_{\beta<\kappa} E^*_\beta$$

It follows that $W^*_\alpha <_{\Bbbk} W_\alpha$, and given (5), this contradicts the minimality of $\langle W_\alpha \rangle_{\alpha<\kappa}$. This proves (3). $\square$

Proof of Theorem 3.8. We may assume by induction that UA_γ holds for all ordinals $\gamma < \kappa$. Therefore, we can reduce to the case that κ is an infinite cardinal: otherwise, we have $\text{UA}_{|\kappa|}$, which implies UA_κ. Fix $U, W \in \upsilon(\kappa)$. Suppose $\langle W_\alpha \rangle_{\alpha<\kappa} \in {}^\kappa\upsilon(\kappa)$ is Ketonen minimal modulo U such that $W = U\text{-}\lim_{\alpha<\kappa} W_\alpha$ and $\langle U_\beta \rangle_{\beta<\kappa}$ is Ketonen minimal modulo W such that $U = W\text{-}\lim_{\alpha<\kappa} U_\alpha$. We claim that the pair $\langle W_\alpha \rangle_{\alpha<\kappa}, \langle U_\beta \rangle_{\beta<\kappa}$ is a combinatorial comparison of U, W.

Let

$$D = \{R \subseteq \kappa \times \kappa : \forall^*_U \alpha\, \forall^*_{W_\alpha} \beta\, R(\alpha, \beta)\}$$
$$E = \{R \subseteq \kappa \times \kappa : \forall^*_W \beta\, \forall^*_{U_\beta} \alpha\, R(\alpha, \beta)\}$$

To prove our claim, it suffices to show $D = E$. Let $\prec^0$ be the *Gödel order* on κ^2, defined by ordering first by maxima and then lexicographically. Let $\prec^1$ be defined instead by ordering first by maxima and then *reverse* lexicographically.

Since κ is a cardinal, $\text{ot}(\prec^0) = \kappa$. Therefore the linearity of the Ketonen order on $\upsilon(\kappa)$ implies that of the Ketonen order on $\upsilon(\kappa \times \kappa)$ relative to $\prec^0$. Since $\prec^0$ and $\prec^1$ are product orders, we can apply Theorem 3.9 twice to show $D \not<^0_{\Bbbk} E$ and $E \not<^0_{\Bbbk} D$. (For the latter, let E' and D' be the pushforwards of E and D under the coordinate flipping map $(\alpha, \beta) \mapsto (\beta, \alpha)$. We apply Theorem 3.9 to E' and D' to obtain $E' \not<^1_{\Bbbk} D'$; as a consequence, $E \not<^0_{\Bbbk} D$.) It follows that $D = E$, as desired. ☐

3.3 The Mitchell order

The Mitchell order is defined on normal measures $U, W \in \upsilon(\kappa)$ by $U \lhd W$ if $U \in V_W$. Since V_W is an uncollapsed ultrapower, the statement $U \in V_W$ formally means that there is some $u \in V_W$ such that $U = \{a \in V_W : V_W \vDash a \in u\}$.

The linearity of the Mitchell order is an immediate consequence of the linearity of the Ketonen order:

Proposition 3.10. *If W is normal and $U <_{\Bbbk} W$, then $U \lhd W$.*

Proof. Suppose $\langle U_\alpha \rangle_{\alpha<\kappa}$ is such that $\forall^*_W U_\alpha \in \upsilon(\alpha)$ and for $A \subseteq \kappa$, $A \in U$ if and only if $\forall^*_W \alpha\, A \cap \alpha \in U_\alpha$. Then $[\alpha \mapsto U_\alpha]_W = U$. ☐

The well-foundedness of the Mitchell order requires some more work. If in the definition of the Mitchell order we had demanded not only that $U \in V_W$ but also that U is an internal measure of V_W,[1] then the converse of Proposition 3.10 would be provable, and so well-foundedness would follow from Theorem 3.7. However, we will work with the weaker definition and show that it is equivalent to the stronger one under UA.

Fix an ordinal κ. A function $f : P(\kappa) \to P(\kappa)$ is *Lipschitz* if for all $\alpha < \kappa$, $f(A) \cap \alpha$ depends only on $A \cap \alpha$; f is *super-Lipschitz* if $f(A) \cap (\alpha + 1)$ depends only on $A \cap \alpha$. For $X, Y \subseteq P(\kappa)$, X is *super-Lipschitz reducible to* Y if there is a super-Lipschitz function $f : P(\kappa) \to P(\kappa)$ such that $f^{-1}[Y] = X$. The super-Lipschitz order is defined by setting $X <_L Y$ if X is super-Lipschitz reducible both to Y and its complement.

Theorem 3.11 ([8, Theorem 3.4.30])**.** *The super-Lipschitz order is a strict partial order.*

We consider the super-Lipschitz order restricted to ultrafilters. Note that then the super-Lipschitz order is equal to the order of super-Lipschitz reducibility, because ultrafilters are Lipschitz self-dual.

Theorem 3.12. *Under* UA_κ*, for $U, W \in \upsilon(\kappa)$, $U <_{\Bbbk} W$ if and only if $U <_L W$.*

Proof. Since $<_L$ is a strict partial order and the Ketonen order is linear, it suffices to show that $U <_{\Bbbk} W$ implies $U <_L W$. But if $\langle U_\beta \rangle_{\beta<\kappa}$ witnesses $U <_{\Bbbk} W$, then the function $f(A) = \{\beta : A \cap \beta \in U_\beta\}$ witnesses U is super-Lipschitz reducible to W. ☐

[1] That is, $U = [\alpha \mapsto U_\alpha]_W$ where each U_α is a measure.

In particular, UA can be seen as a form of long game determinacy [6]; namely, the determinacy of the Lipschitz game associated to any two ultrafilters on κ.

The proof of Proposition 3.10 yields:

Proposition 3.13. *If $U \in \upsilon(\kappa)$ is normal and $X \subseteq P(\kappa)$, then $X \in V_U$ if and only if $X <_L U$. In particular, the super-Lipschitz and Mitchell orders are the same when restricted to normal measures on κ.*

An immediate corollary of Theorem 3.12 and Proposition 3.13, we obtain:

Theorem 3.14. *Under UA_κ, the Mitchell order coincides with the Ketonen order restricted to normal measures on κ, and in particular, it is a well-order.*

4 UA from determinacy

In this section, we prove our main theorems deriving UA from determinacy. We begin in Section 4.1 with an exposition of the relationship between the Kechris coding theorem for measures and the concept of a close embedding. Then in Section 4.2, we briefly exposit Woodin's generic ultrapower construction, the key to recovering Łoś's theorem in our proof of UA. In Section 4.3, we combine these ingredients to prove that UA holds at any Suslin cardinal. Following this, in Section 4.4, we propagate UA all the way to Θ under AD^+. Finally, Section 4.5 contains another application of the proof technique of this paper (Theorem 4.22), showing that under AD, every precipitous ideal on an ordinal below Θ is atomic.

4.1 Close embeddings and the Kechris coding theorem

If X is a set, the *cut-and-choose game* on X is a game in which two players, Cut and Choose, alternate playing subsets $X_n \subseteq X$ and numbers $i_n \in \{0,1\}$ respectively:

Cut:	X_0		X_1		X_2		$\cdots$
Choose:		i_0		i_1		i_2	$\cdots$

An infinite run of the game is a win for Cut if $\bigcap_{n<\omega}(X_n)_{i_n} \neq \emptyset$ where

$$(Y)_i = \begin{cases} Y & \text{if } i = 0 \\ X \setminus Y & \text{if } i = 1 \end{cases}$$

If $\pi : \mathbb{R} \to P(\gamma)$ is a surjection, *cut-and-choose game coded by* π is the variant of the cut-and-choose game on γ in which rather than playing the sets X_n themselves, Cut must play reals coding them according to π.

Theorem 4.1 (Kechris, AD)**.** *If there is a Suslin cardinal above γ, then there is some surjection $\pi : \mathbb{R} \to P(\gamma)$ such that Choose has a winning strategy in the cut-and-choose game coded by π.*

Proof. Fix $\pi : \mathbb{R} \to P(\gamma)$ so that the cut-and-choose game coded by π has Suslin and co-Suslin pay-off. (This is possible by the coding lemma, using the closure properties of the pointclass of sets that are Suslin and co-Suslin and the fact that this class contains a prewellorder of length γ; see [19, Theorem 6.18].) By Kechris's theorem that Suslin and co-Suslin real-integer games are determined [16], this game is determined. Moreover, Kechris [15] showed that under AD, Cut cannot have a winning strategy in the coded game; indeed, Cut cannot even win the original (uncoded) cut-and-choose game. This is because a winning strategy τ for Cut in the latter game well-orders the reals: if $f : 2^\omega \to P(\gamma)^\omega$ is the Lipschitz function induced by τ, then $a \mapsto \min \bigcap_{n<\omega} (f(a)_n)_{a(n)}$ is an injection from 2^ω into γ. $\square$

Definition 4.2. Suppose M and N are transitive sets and $\eta \in \text{Ord}^M$. A Σ_2-elementary embedding $j : M \to N$ is *close to* M *below* η if for all $\gamma < \eta$, for all $\xi < j(\gamma)$, the M-ultrafilter on γ derived from j using ξ belongs to M.

Corollary 4.3 (AD)**.** *If M is a countable transitive model, $j : M \to V$ is Σ_2-elementary, and M satisfies that κ is a Suslin cardinal, then j is close to M below κ.*

Proof. Fix $\gamma < \kappa$ and $\xi < j(\gamma)$, and we will show that the M-ultrafilter D on γ derived from j using ξ belongs to M. Working in M, let τ be a strategy for Choose in the cut-and-choose game coded by some surjection $\pi : \mathbb{R} \to P(\gamma)$. Outside of M, let $\langle x_n, i_n : n < m\rangle$ be a maximal run of the game consistent with τ such that, letting $X_n = \pi(x_n)$, the set $(X_n)_{i_n}$ belongs to D for all $n < m$. We claim that $m < \omega$: otherwise, the sequence $\langle x_n, i_n : n < \omega\rangle$ can be viewed as a run in the cut-and-choose game coded by $j(\pi)$ that is consistent with the winning strategy $j(\tau)$ for Choose; but $\bigcap_{n<\omega} (j(\pi)(x_n))_{i_n} = \bigcap_{n<\omega} j((X_n)_{i_n})$ contains ξ because each $(X_n)_{i_n}$ belongs to D. Thus $m < \omega$, but then

$$D = \{\pi(x) : \tau(x_0, i_0, \dots, x_{m-1}, i_{m-1}, x) = 1\} \in M. \qquad \square$$

4.2 Woodin's generic ultrapower construction

In this section, we describe a forcing technique due to Woodin for extending ultrapower embeddings of models of determinacy to elementary embeddings.

Suppose U is a measure on an ordinal κ and $\pi : \mathbb{R} \to \kappa$ is a surjection. The *pull-back ideal* associated to U is the ideal

$$I_{U,\pi} = \{A \subseteq \mathbb{R} : \pi[A] \notin U\}$$

The ideal depends to a certain extent on the choice of π, but when there is no chance of confusion, we will denote it simply by I_U.

An ideal I on $\mathbb{R}$ has the *uniformization property* if for all $R \subseteq \mathbb{R} \times \mathbb{R}$ with $\text{dom}(R) \in I^+$, there is a function $f \subseteq R$ with $\text{dom}(f) \in I^+$.

Theorem 4.4 (Woodin, AD^+)**.** *The pullback ideal I_U has the uniformization property.*

Proof. We will only apply the theorem in the special case that π is Suslin, by which we mean that the prewellorder $\preceq_\pi$ of $\mathbb{R}$ induced by π is Suslin. Then the result can be proved from AD alone. By the Moschovakis coding lemma [23], any relation $R \subseteq \mathbb{R} \times \mathbb{R}$ has a Suslin subrelation $S \subseteq R$ such that $\pi[\text{dom}(S)] = \pi[\text{dom}(R)]$. Therefore if $\text{dom}(R)$ belongs to I_U^+, so does $\text{dom}(S)$. Since S is Suslin, it can be uniformized on its domain, and this yields the special case of the theorem. We note that one can actually bound the complexity of the uniformizing function, which is essential in the general case: applying the coding lemma to the uniformizing function we have just obtained, there is in fact a function $g \subseteq R$ with $\text{dom}(g) \in I_U^+$ that is moreover *projective* in $\preceq_\pi$.

The general theorem follows from the special case by Woodin's $\mathbf{\Sigma}^2_1$-reflection theorem [30]: under AD^+, every true $\mathbf{\Sigma}^2_1$ statement about sets of reals is witnessed by a set that is Suslin and co-Suslin. Assume towards a contradiction that for some norm $\psi : \mathbb{R} \to \lambda$, there is a measure W on λ such that for some relation $R \subseteq \mathbb{R} \times \mathbb{R}$ with $\mathrm{dom}(R) \in I^+_{W,\psi}$, there is no function $g \subseteq R$ projective in $\preceq_\psi$ with $\mathrm{dom}(g) \in I^+_{W,\psi}$. Then by $\mathbf{\Sigma}^2_1$-reflection, there is a witness such that ψ is Suslin, and this contradicts the previous paragraph. □

An ideal I on $\mathbb{R}$ has the *collection property* if for any $R \subseteq \mathbb{R} \times V$ with $\mathrm{dom}(R) \in I^+$, there is a relation $S \subseteq R$ such that $\mathrm{dom}(S) \in I^+$ and $\mathrm{ran}(S)$ is the surjective image of $\mathbb{R}$.

Proposition 4.5. *If I is an ideal on $\mathbb{R}$ with the uniformization and collection properties, and $G \subseteq I^+$ is V-generic, then the generic ultrapower $j_G : V \to V_G$ satisfies Łoś's theorem.*

Lemma 4.6 (AD^+)**.** *Assume $V = \mathrm{HOD}_{P(\mathbb{R})}$ and $j_U(\Theta) = \Theta$. Then I_U has the collection property.*

The hypotheses of this lemma are satisfied assuming $V = L(P(\mathbb{R}))$ and Θ is regular or $j_U(\mathrm{cf}(\Theta)) = \mathrm{cf}(\Theta)$; in any case, we will only apply the lemma under the assumption that $V = L(A, \mathbb{R})$ for some $A \subseteq \mathbb{R}$.

Finally, we need that the generic elementary embedding associated to the pull-back ideal I_U extends the ultrapower embedding associated to U:

Theorem 4.7 (Woodin)**.** *Let $G \subseteq I^+_U$ be V-generic. Then the factor map $k : V_U \to V_G$ defined by*

$$k([f]_U) = [f \circ \pi]_G$$

is an isomorphism from Ord^{V_U} to Ord^{V_G}. Thus k is an inclusion on $P(\mathrm{Ord})^{V_U}$ and a surjection onto OD^{V_G}.

Proof. Note that for any partial function $\tilde{f} : \mathbb{R} \to \mathrm{Ord}$, there is a function $f : \kappa \to \mathrm{Ord}$ such that $\pi[\{x : \tilde{f}(x) = f(\pi(x))\}] = \pi[\mathrm{dom}(\tilde{f})]$: set $f(\alpha) = \min\{\tilde{f}(x) : \pi(x) = \alpha\}$. □

Definition 4.8. An ideal I is *precipitous* if for any V-generic $G \subseteq I^+$, the ultrapower V_G is well-founded.

Theorem 4.7 shows that assuming DC, the ideal I_U is precipitous. Together with our determinacy assumptions, this suggests considering the associated Galvin–Jech–Magidor *precipitous game* [3].

In the *precipitous game* associated to an ideal I on X, two players, Empty and Nonempty, alternate to construct a decreasing sequence of sets $X_n \in I^+$:

Empty:	X_0		X_2		X_4		$\cdots$
Nonempty:		X_1		X_3		X_5	$\cdots$

A run of the game is a win for Empty if $\bigcap_{n<\omega} X_n = \emptyset$.

Theorem 4.9 (Steel, AD)**.** *Suppose I_U is the pullback ideal induced by a Suslin surjection $\pi : \mathbb{R} \to \kappa$, and let $\mathcal{G}$ denote the precipitous game associated to I_U. Let $H \subseteq \mathrm{Col}(\omega_1, \mathbb{R})$ be V-generic. Then in $V[H]$, there is a strategy for Nonempty in $\mathcal{G}$.*

Proof. Since π is Suslin, the coding lemma implies that in V there is a dense set $D \subseteq I_U^+$ that consists only of λ-Suslin sets for some fixed $\lambda < \Theta$. Therefore in $V[H]$, there is a well-order of both D and the trees T on $\omega \times \lambda$ projecting to elements of D. This is because in V, $P(\lambda)$ is the surjective image of $\mathbb{R}$. We use this well-order implicitly below to choose various objects.

Working in $V[H]$, we define a strategy for Nonempty in $\mathcal{G}$. At each round m, in addition to choosing the set X_{2m+1}, Nonempty will also choose certain auxiliary objects T_m, s_m, and $(t_k^m)_{k\leq m}$. These are required to satisfy the following recursive constraints:

- T_m is a tree on $\omega \times \lambda$ such that $p[T_m] = X_m$.
- For all $x \in X_{2m+1}$, $x \restriction m = s_m$.
- For $k < m$, for all $\alpha \in \pi[X_{2m+1}]$, $t_k^m(\alpha) \in \lambda^m$ extends $t_k^{m-1}(\alpha)$.
- For all $x \in X_{2m+1}$, for all $k \leq m$, there is some $f \in \lambda^\omega$ extending $t_k^m(\pi(x))$ such that $(x, f) \in [T_k]$.

Suppose it is the n-th round of the game, and it is Nonempty's turn to play. Thus sets $X_0, \ldots, X_{2n}$ have been played, and for $m < n$, the objects T_m, s_m, and $(t_k^m)_{k\leq m}$ have been chosen satisfying the bullets above. We must show it is possible for Nonempty to choose T_n, s_n and $(t_k^n)_{k\leq n}$.

Nonempty first arbitrarily selects a tree T_n on $\omega \times \lambda$ such that $p[T_n] \subseteq X_{2n}$.

Then, using the countable completeness of I_U, Nonempty chooses s_n such that

$$B = \{x \in X_{2n} : x \restriction n = s_n\} \in I_U^+$$

Finally, we define, for $\alpha < \kappa$, the sequences $t_k^n(\alpha) \in \lambda^n$. First, if $x \in X_{2n}$ and $k < n$, let $f_k(x)$ be the left-most branch of $T_k(x)$ extending $t_k^{n-1}(\pi(x))$. If $k = n$, let $f_n(x)$ be the leftmost branch of $T_n(x)$.

For $k \leq n$, let $\tilde{t}_k(x) = f_k(x) \restriction n$. By Theorem 4.7, we can choose $X_{2n+1} \subseteq B$ in I_U^+ such that for any $x, y \in X_{2n+1}$ with $\pi(x) = \pi(y)$, $\tilde{t}_k(x) = \tilde{t}_k(y)$. Finally, for $\alpha \in \pi[X_{2n+1}]$, let $t_k^n(\alpha)$ be the constant value of $\tilde{t}_k(x)$ for $x \in X_{2n+1} \cap \pi^{-1}[\{\alpha\}]$.

Suppose $\langle X_n \rangle_{n<\omega}$ is a run of $\mathcal{G}$ consistent with the strategy we have just defined. Let us show that $\bigcap_{n<\omega} X_n$ is nonempty. Let $T_n, s_n, (t_k^n)_{k\leq n}$ be the objects chosen by Nonempty at round n of the run. We claim that $x = \bigcup_{n<\omega} s_n$ belongs to $\bigcap_{n<\omega} X_n$. To see this, fix $\alpha \in \bigcap_{n<\omega} \pi[X_n]$, and for $k < \omega$, let $f_k = \bigcup_{n<\omega} t_k^n(\alpha)$. Since $(x, f_k) \in [T_k]$, $x \in X_{2k}$, and hence $x \in \bigcap_{n<\omega} X_n$, as desired. □

4.3 Comparing ultrapowers of determinacy models

Lemma 4.10 (AD)**.** *Assume $V = L(A, \mathbb{R})$ for some $A \subseteq \mathbb{R}$. Suppose M is a countable transitive model, $j : M \to V$ is Σ_2-elementary, and M satisfies that κ is a Suslin cardinal. Then for any internal measure U of M on κ, there is an M-generic filter $G \subseteq I_U^+$ and a Σ_2-elementary $i : M_G \to V$ such that $i \circ j_G = j$.*

Proof. Let $H \in V$ be M-generic for $\text{Col}(\omega_1, \mathbb{R})^M$. Since $\text{Col}(\omega_1, \mathbb{R})$ is countably closed, there is a condition $p \in \text{Col}(\omega_1, \mathbb{R})$ below $j[H]$, and letting $\tilde{H} \subseteq \text{Col}(\omega_1, \mathbb{R})$ be V-generic with $p \in \tilde{H}$, we can extend j to a Σ_2-elementary $j^+ : M[H] \to V[\tilde{H}]$.

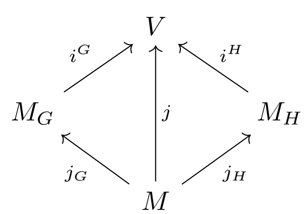
In $M[H]$, apply Steel's theorem (Theorem 4.9) to obtain a strategy τ for Nonempty in the precipitous game associated to I_U. Working in $V[\tilde{H}]$, let $\langle D_m \rangle_{m<\omega}$ enumerate the dense subsets of I_U^+ that belong to M. Build a run $\langle X_n : n < \omega\rangle$ of the precipitous game in which Nonempty plays according to τ and Empty plays so that $X_{2m} \in D_m$. Let G be the M-generic filter on I_U^+ generated by $\{X_n : n < \omega\}$.

Since $\langle j(X_m) : m < \omega\rangle$ is consistent with $j(\tau)$, $\bigcap_{m<\omega} j(X_m)$ is nonempty, so fix some $x \in \bigcap_{m<\omega} j(X_m)$. Note that G is the M-ultrafilter on $\mathbb{R}^M$ derived from j using x. Define $i : M_G \to V$ by $i([f]_G) = j(f)(x)$. The ultrapower M_G satisfies Łoś's theorem for Σ_2-formulas by Proposition 4.5, so i is a Σ_2-elementary embedding. □

Theorem 4.11 ($\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$). *If κ is a Suslin cardinal, then UA_κ holds.*

Proof. By the coding lemma, we may assume $V = L(A, \mathbb{R})$ for some $A \subseteq \mathbb{R}$. By DC, there is a countable transitive M and a Σ_2-elementary $j : M \to V$. It suffices to show that for any Suslin cardinal κ of M, M satisfies UA_κ. By Theorem 3.8, we only have to establish that in M, the Ketonen order on $\upsilon(\kappa)$ is linear. We will show by induction on $\delta \leq \kappa$ that in M, the Ketonen order on $\upsilon(\delta)$ is linear.

For this, fix internal measures U and W of M on δ. Applying Lemma 4.10, let G and H be generic filters for their respective pull-back ideals, for which we have the following commutative diagram of Σ_2-elementary embeddings:



Let $\xi_U = [\mathrm{id}]_U$ and $\xi_W = [\mathrm{id}]_W$. If $i^G(\xi_U) = i^H(\xi_W)$, it is easy to show that $U = W$. Therefore assume without loss of generality that $i^G(\xi_U) < i^H(\xi_W)$, and we will show that $U <_{\Bbbk} W$.

Let U_* be the M_H-ultrafilter on ξ_W derived from i^H using $i^G(\xi_U)$. By Corollary 4.3, i^H is close to M_H below $j_H(\kappa)$, and therefore $U_* \in M_H$. By our induction hypothesis, the Ketonen order on $\upsilon(\xi_W)$ is linear in M_H, and in particular, U_* is OD^{M_H}. Therefore by Theorem 4.7, U_* is in the range of the factor embedding $k : M_W \to M_H$. Fix $\langle U_\beta\rangle_{\beta<\delta}$ such that $U_* = k([\beta \mapsto U_\beta]_W)$. Note that by Łoś's theorem for H, for W-almost all β, $U_\beta \in \upsilon(\beta)$. Moreover, for $A \in P^M(\delta)$,

$$\begin{aligned} A \in U &\iff [\mathrm{id}]_U \in j_U(A) \\ &\iff i^G([\mathrm{id}]_U) \in i^G(j_U(A)) \\ &\iff i^G([\mathrm{id}]_U) \in i^H(j_W(A)) \\ &\iff j_W(A) \in U_* \\ &\iff \forall^*_W \beta\, (A \cap \beta \in U_\beta) \end{aligned}$$

so $\langle U_\beta\rangle_{\beta<\delta}$ witnesses $U <_{\Bbbk} W$, as desired. □

The equivalence of UA with the linearity of the Ketonen order (Theorem 3.8) seems essential to the proof of Theorem 4.11. If at the end of the proof one tries to obtain a combinatorial comparison directly, one will only succeed in proving that UA holds below κ.

The use of induction on $\delta \leq \kappa$ in Theorem 4.11 is only necessary to conclude that U_* is OD^{M_H}. We could instead have used Kunen's theorem on the ordinal definability of measures, but for aesthetic reasons, we chose to present Theorem 4.11 as a new proof of Kunen's theorem.

Corollary 4.12 ($\text{AD}_{\mathbb{R}}$)**.** UA *holds below* Θ.

Proof. This follows from Theorem 4.11 and Martin–Woodin's theorem [21] that under $\text{AD}_{\mathbb{R}}$, every set is Suslin. □

4.4 UA from AD^+

We now turn to our AD^+ theorem:

Theorem 4.13 (AD^+)**.** UA *holds below* Θ.

The most obvious approach would be to show that UA, or some strengthening of UA, is a Π^2_1-statement. Then one could appeal to Woodin's Σ^2_1-reflection theorem to conclude Theorem 4.13 from Theorem 4.11, as in Theorem 4.4. Unfortunately, it is not clear what Π^2_1-statement to consider. Therefore we take a different approach.

By Theorem 4.11, we can assume there is a largest Suslin cardinal: recall that Steel and Woodin [19, Theorem 11.20] proved that AD^+ implies the Suslin cardinals form a closed subset of Θ, so if there is no largest Suslin cardinal, then Θ is a limit of Suslin cardinals, and then UA holds below Θ by Theorem 4.11.

Throughout this section, we let κ denote the largest Suslin cardinal. We propagate Theorem 4.11 using the fact that under AD^+, κ is strong to Θ:

Theorem 4.14 (Woodin, AD^+)**.** *For all $\lambda < \Theta$, there is a normal measure D on κ such that $j_D(\kappa) > \lambda$ and $P(\lambda) \subseteq V_D$.*

Proof. This follows from the proof of Woodin's theorem that Θ is Woodin in $\text{HOD}^{L(\mathbb{R})}$ [18]. In that paper, the authors work in $L(\mathbb{R})$ and show that the largest Suslin cardinal δ^2_1 is $<\Theta$-strong in HOD. In fact, for all $\lambda < \Theta$, there is a normal measure D on δ^2_1 such that $j_D(\delta^2_1) > \lambda$ and $P(\lambda) \cap \text{HOD} \subseteq j_D(\text{HOD})$. The ultrafilter D witnessing this is strongly normal (see [18, Definition 4.11]), and this implies $P(\lambda) \subseteq V_D$ by the argument used in [18, p. 2015]. □

We will need a minor strengthening of Woodin's theorem concerning the *generalized Mitchell order*, which for the purposes of this section, we define as follows:

Definition 4.15. Suppose $\delta \leq \lambda$ are cardinals, $D \in \upsilon(\delta)$, and $U \in \upsilon(\lambda)$. Then $U \lhd D$ if $j_D(\delta) > \lambda$, $P(\lambda) \subseteq V_D$, and U is an internal measure of V_D.

Proposition 4.16. *For all $\lambda < \Theta$ and $U, W \in \upsilon(\lambda)$, there is a normal measure D on κ such that $U, W \lhd D$.*

Proof. The result is obtained by considering the ultrafilter $D = \mu_X$, in Koellner–Woodin's notation [18, p. 2009], where the parameter X is the tuple consisting of a prewellordering of length λ along with the measures U and W. □

We will also need the following lemmas on the uniqueness of ultrapower embeddings, the second of which is a consequence of the strictness of the Ketonen order:

Lemma 4.17. *If M is a transitive model of* ZF*, U is an M-ultrafilter and $i_0, i_1 : M_U \to N$ are ultrapower embeddings with $i_0 \circ j_U = i_1 \circ j_U$ and $i_0([\mathrm{id}]_U) = i_1([\mathrm{id}]_U)$, then $i_0 = i_1$.*

Lemma 4.18. *If U is a measure on an ordinal and $i_0, i_1 : V_U \to N$ are internal ultrapower embeddings with $i_0 \circ j_U = i_1 \circ j_U$, then $i_0 = i_1$.*

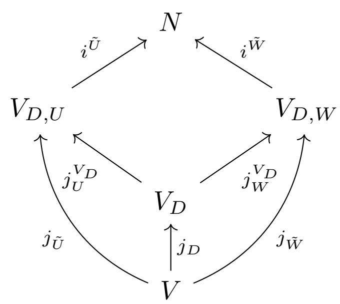
Proof. By Lemma 4.17, it suffices to show $i_0([\mathrm{id}]_U) = i_1([\mathrm{id}]_U)$. If $i_0([\mathrm{id}]_U) < i_1([\mathrm{id}]_U)$ or vice versa, then Lemma 3.5 implies $U <_{\Bbbk} U$, contrary to the strictness of the Ketonen order (Theorem 3.7). □

Proof of Theorem 4.13. Suppose $\lambda < \Theta$, and we will prove UA_λ. Fix $U, W \in \upsilon(\lambda)$; we will show that U and W are Ketonen comparable and conclude the theorem by Corollary 3.6. Let D be a normal measure on κ such that $U, W \lhd D$. Let

$$\tilde{U} = [D, U] = \{R \subseteq \kappa \times \kappa : \forall^*_D \xi \, \forall^*_{U_\xi} \alpha \, R(\xi, \alpha)\}$$

where $\langle U_\xi \rangle_{\xi<\kappa} \in {}^\kappa\upsilon(\kappa)$ represents some $U_* \in j_D(\upsilon(\kappa))$ such that $U = U_* \cap P(\lambda)$. Recall that $V_{\tilde{U}} = V_{D,U}$ and $j_{\tilde{U}} = j^{V_D}_U \circ j_D$. Similarly let $\tilde{W} = [D, W]$.

By Theorem 4.11, since $\tilde{U}$ and $\tilde{W}$ are measures on $\kappa \times \kappa$, there are internal ultrapower embeddings $i^{\tilde{U}} : V_{D,U} \to N$ and $i^{\tilde{W}} : V_{D,W} \to N$ witnessing UA_κ for $\tilde{U}$ and $\tilde{W}$:



The key observation, which follows from Lemma 4.18, is that the upper diamond commutes:

$$i^{\tilde{U}} \circ j^{V_D}_U = i^{\tilde{W}} \circ j^{V_D}_W \tag{6}$$

Assume without loss of generality that $i^{\tilde{U}}([\mathrm{id}]_U) < i^{\tilde{W}}([\mathrm{id}]_W)$, noting that if equality holds, then (6) easily implies $U = W$. Let U_* be the internal measure of $V_{D,W}$ on $j_W(\lambda)$ derived from $i^{\tilde{W}}$ using $i^{\tilde{U}}([\mathrm{id}]_U)$. Since $i^{\tilde{U}}([\mathrm{id}]_U) < i^{\tilde{W}}([\mathrm{id}]_W)$, $[\mathrm{id}]_W \in U_*$, and by (6), $j^{V_D}_W[U] \subseteq U_*$.

Since $P(\lambda) \subseteq V_D$, $j_W \restriction P(\lambda) = j^{V_D}_W \restriction P(\lambda)$, $j^{V_D}_W(P(\lambda)) = j_W(P(\lambda))$, and the set of internal measures of $V_{D,W}$ on $j_W(\lambda)$ is contained in the set of internal measures of V_W on $j_W(\lambda)$. Therefore U_* is an internal measure of V_W and $j_W[U] \subseteq U_*$. By Lemma 3.5, it follows that $U <_{\Bbbk} W$. □

In Theorem 4.13, it seems essential that we apply UA_κ in the form of Definition 3.1; that is, it does not suffice here to use the linearity of the Ketonen order on $\upsilon(\kappa)$. In view of the proof of Theorem 4.11, it seems that even to establish the linearity of the Ketonen order below Θ from AD^+, one needs to go through the proof of Theorem 3.8.

Corollary 4.19 (AD^+)**.** *If $V = L(P(\mathbb{R}))$ and Θ is regular or* $\mathrm{cf}(\Theta) = \omega$*, then* UA *holds.*

Proof. This hypothesis implies that every measure on an ordinal concentrates on a set of size less than Θ. □

Corollary 4.20. *Assume* $\mathrm{ZF} + \mathrm{AD}^{L(\mathbb{R})}$*. Then* $L(\mathbb{R})$ *satisfies* UA.

Proof. The Cabal has shown that if $L(\mathbb{R})$ satisfies AD, then it satisfies AD^+; see [19, Theorem 8.20]. The corollary can be proved without going through this, however: to run our proof, one just needs Martin–Steel's theorem [20] that in $L(\mathbb{R})$, every $\mathbf{\Sigma}^2_1$ set is Suslin and the proof of Woodin's theorem [18] that δ^2_1 is $<\Theta$-strong. □

Finally, we have our basic application:

Corollary 4.21 (AD^+)**.** *The Mitchell order well-orders the normal measures on any cardinal less than* Θ*.*

Proof. By Theorem 3.14 and Theorem 4.13. □

4.5 More negative results on precipitous ideals

A theme in inner model theory is that in canonical models of ZFC, all precipitous ideals arise from measurable cardinals [25, 26, 34]. In this section, we adapt the proof of Theorem 4.11 to obtain an analogous conclusion from determinacy.

Theorem 4.22 (AD^+)**.** *Suppose* $\kappa < \Theta$ *and* J *is a precipitous ideal on* κ*. Then* J *is atomic.*

As a corollary, we obtain a determinacy version of Ulam's analysis of real-valued measurable cardinals [33]; see [13, Theorem 10.1].

Corollary 4.23 (AD^+)**.** *Any total probability measure on a cardinal below* Θ *is a convex combination of countably many two-valued measures.*

Proof. Fix $\kappa < \Theta$ and a probability measure $\mu : P(\kappa) \to [0,1]$. By passing to $L(A,\mathbb{R})$ for some $A \subseteq \mathbb{R}$ such that $\kappa < \Theta^{L(A,\mathbb{R})}$ and $\mu \in L(A,\mathbb{R})$, we can assume DC holds. By a standard argument [13, p. 125], the ideal $J = \{S \subseteq \kappa : \mu(S) = 0\}$ is countably saturated. Using DC, it follows by an argument of Solovay [28] that J is precipitous, and hence J is atomic by Theorem 4.22. The saturation of J implies that J has only countably many atoms, and this yields the corollary. □

Definition 4.24. Suppose $\pi : X \to Y$ is a surjection and J is an ideal on Y. The *pullback of* J *through* π is the ideal

$$\pi^*(J) = \{A \subseteq X : \pi[A] \in J\}$$

We start by generalizing Woodin's analysis of pullback ideals.

Lemma 4.25. *If* G *is a* V*-generic filter on* $\pi^*(J)^+$*, then* $\pi_*(G)$ *is a* V*-generic filter on* J^+*. Moreover, the factor embedding* $k : V_{\pi_*(G)} \to V_G$ *is the identity on* $\mathrm{Ord}^{V_{\pi_*(G)}}$*.*

The first part is straightforward and the second part is similar to Theorem 4.7.

We will secure Łoś's theorem for our generic ultrapowers using Proposition 4.5 and the following lemma, whose proof is similar to Theorem 4.4 and Lemma 4.6:

Lemma 4.26 (AD^+)**.** *If* J *is an ideal on an ordinal* κ *and* $\pi : \mathbb{R} \to \kappa$ *is a surjection, then* $\pi^*(J)$ *has the uniformization property. If* $V = \mathrm{HOD}(P(\mathbb{R}))$ *and* $j_G(\Theta) = \Theta$ *for all* V*-generic* $G \subseteq J^+$*, then* $\pi^*(J)$ *has the collection property.*

Proof of Theorem 4.22. Again, we can assume that $V = L(A, \mathbb{R})$ for some $A \subseteq \mathbb{R}$. By restricting to a positive set, it suffices to show that J has an atom; we reduce to this special case only for simplicity of notation. More importantly, by Woodin's $\mathbf{\Sigma}^2_1$-reflection theorem, we may assume that there is a Suslin cardinal above κ.

Let $\pi : \mathbb{R} \to \kappa$ be a surjection and let $I = \pi^*(J)$ be the associated pullback ideal on $\mathbb{R}$. By Lemma 4.25, I is precipitous, and by Lemma 4.26 and $V = L(A, \mathbb{R})$, I has the uniformization and collection properties.

Let $G \subseteq I^+$ be a V-generic ultrafilter on $\mathbb{R}$, and let $H = \pi_*(G)$. We will show that $H \in V$. Since H is a V-generic filter on J^+, it follows that J has an atom.

Let $\tau : \mathbb{R}^{<\omega} \to \{0, 1\}$ be a strategy for Choose in the cut-and-choose game coded by some $\psi : \mathbb{R} \to P(\kappa)$. Working in a further collapse extension of $V[G]$, let $\langle x_k, e_k : k < n\rangle \subseteq V$ be a maximal run consistent with τ such that letting $X_k = \psi(x_k)$, $[\mathrm{id}]_H \in j_G((X_k)_{e_k})$ for all $k < n$. Such a run exists once we collapse everything to be countable, and we must have $n \leq \omega$, but in fact, we claim $n < \omega$.

Otherwise, in $V[G]$, $\langle x_k, e_k : k < \omega\rangle$ is a run of the cut-and-choose game coded by $j_G(\psi)$ that is consistent with $j_G(\tau)$ but such that $[\mathrm{id}]_H \in \bigcap_{k<\omega}(Y_k)_{e_k}$ for all $k < \omega$. Although $\langle x_k, e_k : k < \omega\rangle$ may not belong to V_G, this still violates that $j_G(\tau)$ is winning for Choose in V_G: in V_G, there is a rank function on the tree of attempts to construct a run of the game consistent with $j_G(\tau)$ such that $[\mathrm{id}]_H$ belongs to all the chosen subsets of $j_G(\kappa)$, and the existence of a branch through this tree contradicts the well-foundedness of V_G.

Therefore $n < \omega$, and it follows that $\langle x_k, e_k : k < n\rangle \in V$. Finally, as in the proof of Kechris's coding theorem (or Corollary 4.3), this implies that $H \in V$. $\square$

5 Conjectures

Kunen proved that under $\mathrm{AD} + \mathrm{DC}_\mathbb{R}$, every measure on an ordinal less than Θ is ordinal definable.

Conjecture 5.1. $\mathrm{AD} + \mathrm{DC}_\mathbb{R}$ *implies* UA *below* Θ.

Our next conjecture concerns the analog of Corollary 4.19 in the case that Θ is singular of uncountable cofinality:

Conjecture 5.2. $\mathrm{AD}^+ + V = L(P(\mathbb{R}))$ *implies* UA.

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