The Ultrapower Axiom

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Chapter 1

Introduction

Gödel, in his 1947 paper "What is Cantor's continuum problem," was the first to suggest that even those questions that cannot be answered using the commonly accepted ZFC axioms of set theory might be resolved in a principled way by adopting axioms that "assert the existence of still further iterations" of the powerset operation. Though the strong principles central to this monograph are admittedly wild extrapolations of Gödel's early intuitions, this remains the driving idea behind large cardinal axioms. Such axioms have been remarkably successful in settling classical problems left open by the ZFC axioms, but many problems, including Cantor's continuum problem, remain unsolvable under any of the known large cardinal hypotheses. Results of Lévy-Solovay [1] and others suggest that these problems cannot be solved using any large cardinal hypothesis that will ever be formulated.

Are there further principles which in conjunction with large cardinal axioms resolve all set theoretic questions? To answer this question, set theorists have sought to construct *canonical* models of set theory, free from the ambiguity inherent in the concept of set. The simplest example of such a model is Gödel's constructible universe L, the smallest model of Zermelo-Fraenkel set theory that contains every ordinal number. One sense in which L is canonical is that seemingly every question about its internal structure can be answered. For example, Gödel proved that L satisfies the Continuum Hypothesis. In contrast, many of the most basic properties of the universe of all sets V, the maximum model of set theory, cannot be determined in any commonly accepted axiomatic system.

To what extent does L provide a good approximation to the universe of sets? On the one hand, the principle that every set belongs to L (or in other words, $V = L$) cannot be refuted using the ZFC axioms, since L itself is a model of the theory $\text{ZFC} + V = L$. If $V = L$, then L approximates V *very* well. On the other hand, the model L fails to satisfy relatively weak large cardinal axioms. If one takes the stance that these large cardinal axioms are true in the universe of sets, one must conclude that $V \neq L$. Moreover, it follows from large cardinal axioms that L constitutes only a tiny fragment of the universe of sets. For example, assuming large cardinal axioms, the set of real numbers that lie in L

is countable.

Are there canonical models generalizing L that yield better approximations to V? A whole subfield of set theory known as inner model theory is devoted to answering this question. It turns out that there is a hierarchy of canonical models beyond L, satisfying stronger and stronger large cardinal axioms. The program of building such models has met striking success, reaching large cardinal axioms as strong as a Woodin limit of Woodin cardinals. Based on the pattern that has emerged so far, it seems plausible that every large cardinal axiom has a corresponding canonical model.

At present, however, a vast expanse of large cardinal axioms are not known to admit canonical models. A key target problem for inner model theory is the construction of a canonical model with a supercompact cardinal. Work of Woodin suggests that the solution to this problem alone will yield an *ultimate canonical model* that inherits essentially all large cardinals present in the universe. There is therefore hope that the goal of constructing inner models for all large cardinal axioms might be achieved in a single stroke. If this is possible, the resulting model would be of enormous set theoretic interest, since it would closely approximate the universe of sets and yet admit an analysis that is as detailed as that of Gödel's L.

This monograph investigates whether there can be a canonical model with a supercompact cardinal. To do this, we develop an abstract approach to inner model theory. This is accomplished by introducing a combinatorial principle called the Ultrapower Axiom, which is expected to hold in all canonical models. If one could show that the Ultrapower Axiom is inconsistent with a supercompact cardinal, one would arguably have to conclude that there can be no canonical model with a supercompact cardinal.

Supplemented with large cardinal axioms, the Ultrapower Axiom turns out to have surprisingly strong and coherent consequences for the structure of the upper reaches of the universe of sets, particularly above the first supercompact cardinal. These consequences are entirely consistent with what one would expect to hold in a canonical model, yet are proved by methods that are completely different from the usual techniques of inner model theory. The coherence of this theory provides compelling evidence that the Ultrapower Axiom is consistent with a supercompact cardinal. If this is the case, it seems that the only possible explanation is that the canonical model for a supercompact cardinal does indeed exist. Optimistically, studying the consequences of the Ultrapower Axiom will shed light on how this model should be constructed.

1.0.1 Outline

We now describe the main results of this monograph.

Chapter 2. In this introductory chapter, we introduce UA in the context of the problem of the linearity of the Mitchell order on normal ultrafilters. We show first that UA holds in all canonical inner models, a result that is philosophically central to this monograph. More precisely, we prove that UA is a consequence of Woodin's Weak Comparison principle:

Theorem 2.3.10. *Assume that* $V = \text{HOD}$ *and there is a* Σ_2*-correct worldly cardinal. If Weak Comparison holds, then the Ultrapower Axiom holds.*

We then show that UA implies the linearity of the Mitchell order:

Theorem 2.3.11 (UA)**.** *The Mitchell order is linear.*

Two applications of this result to longstanding problems of Solovay-Reinhardt-Kanamori [2] are explained in the introduction to Chapter 2.

Chapter 3. This chapter introduces the Ketonen order, a generalization of the Mitchell order to all countably complete ultrafilters on ordinals. The restriction of this order to weakly normal ultrafilters was originally introduced by Ketonen. The first proof of the wellfoundedness of the more general order is due to the author:

Theorem 3.3.8. *The Ketonen order is wellfounded.*

The main theorem of this chapter explains the fundamental role of the Ketonen order in applications of the Ultrapower Axiom:

Theorem 3.5.1. *The Ultrapower Axiom is equivalent to the linearity of the Ketonen order.*

In addition, we explain the relationship between the Ketonen order and various well-known orders like the Rudin-Keisler order and the Mitchell order.

Chapter 4. The topic of this chapter is the generalized Mitchell order, which is defined in exactly the same way as the usual Mitchell order on normal ultrafilters but removing the requirement that the ultrafilters involved be normal. This order is not linear (assuming there is a measurable cardinal), and in fact it is quite pathological when considered on ultrafilters in general. The two main results of this chapter generalize the linearity of the Mitchell order to nice classes of ultrafilters:

Theorem 4.3.29 (UA)**.** *The generalized Mitchell order is linear on Dodd sound ultrafilters.*

Dodd soundness is a generalization of normality that was first isolated in the context of inner model theory by Steel [3]. A uniform ultrafilter U on a cardinal λ is Dodd sound if the map $h : P(\lambda) \to M_U$ defined by $h(X) = j_U(X) \cap [\text{id}]_U$ belongs to M_U. The concept is discussed at great length in Section 4.3.

A better-known generalization of normality is the concept of a normal fine ultrafilter (Definition 4.4.7), introduced by Solovay, and underpinning the theory of supercompact cardinals. The second result of this chapter generalizes the linearity of the Mitchell order to this class of ultrafilters:

Theorem 4.4.2 (UA). *Suppose λ is a cardinal such that $2^{<\lambda} = \lambda$. Then the generalized Mitchell order is linear on normal fine ultrafilters on $P_{\text{bd}}(\lambda)$.*

Here $P_{\text{bd}}(\lambda)$ denotes the set of bounded subsets of λ. The theorem will be reproved later on (Theorem 7.5.39) without cardinal arithmetic hypotheses by a much more involved argument.

CHAPTER 5. We turn to another fundamental order on ultrafilters, the Rudin-Frolík order. The structure of the Rudin-Frolík order on countably complete ultrafilters is intimately related to the Ultrapower Axiom. For example, we point out the following simple connection:

Corollary 5.2.9. *The Ultrapower Axiom holds if and only if the Rudin-Frolík order is directed on countably complete ultrafilters.*

On the other hand, it is well-known that the Rudin-Frolík order is not directed on ultrafilters on ω.

The chapter is devoted to deriving deeper structural features of the Rudin-Frolík order from UA. The most interesting one is that it is locally finite:

Theorem 5.4.25 (UA). *A countably complete ultrafilter has at most finitely many predecessors in the Rudin-Frolík order up to isomorphism.*

Given the finiteness of the Rudin-Frolík order, it turns out to be possible to represent every ultrafilter as a finite iterated ultrapower consisting of *irreducible ultrafilters*, ultrafilters whose ultrapowers cannot be factored as an iterated ultrapower (Theorem 5.3.13). We apply this to analyze ultrafilters on the least measurable cardinal under UA:

Theorem 5.3.18 (UA). *Every countably complete ultrafilter on the least measurable cardinal κ is isomorphic to U^n where U is the unique normal ultrafilter on κ and n is a natural number.*

This generalizes a classic theorem of Kunen [4].

CHAPTER 6. This chapter exposits two inner model principles that follow abstractly from UA in the presence of a supercompact cardinal:

Theorem 6.2.8 (UA). *Assume there is a supercompact cardinal. Then V is a generic extension of* HOD.

Thus UA almost implies $V = \text{HOD}$. This is best possible in the sense that it is consistent that UA holds and V is a *nontrivial* generic extension of HOD.

The main result of the chapter is that UA implies the Generalized Continuum Hypothesis:

Theorem 6.3.26 (UA). *Suppose κ is supercompact. Then for all $\lambda \geq \kappa$, $2^\lambda = \lambda^+$.*

Thus UA almost implies the GCH. This is best possible in the sense that it is consistent that UA holds but CH fails.

Chapter 7. The final two chapters of this monograph are devoted to the analysis of strongly compact and supercompact cardinals under UA. In the first of these chapters, we investigate the structure of the least strongly compact cardinal, introducing the minimal ultrafilters $\mathscr{K}_\lambda$, and proving that they witness its supercompactness:

Theorem 7.4.23 (UA)**.** *The least strongly compact cardinal is supercompact.*

Chapter 8. This final chapter extends the UA analysis of the first supercompact cardinal enacted in Chapter 7 to all supercompact cardinals.

Theorem 8.3.10 (UA)**.** *A cardinal κ is strongly compact if and only if it is supercompact or a measurable limit of supercompact cardinals.*

This is proved as a corollary of the main result of this chapter, the *Irreducibility Theorem* (Theorem 8.2.19, Corollary 8.2.21), relating supercompactness and irreducibility (that is, Rudin-Frolík minimality) under UA. The Irreducibility Theorem allows us to analyze various other large cardinals using UA. For example, we consider huge cardinals (Theorem 8.4.5) and rank-into-rank cardinals (Theorem 8.4.13).

Chapter 2

The Linearity of the Mitchell Order

2.1 Introduction

2.1.1 Normal ultrafilters and the Mitchell order

Normal ultrafilters[1] are among the most basic objects of study in modern large cardinal theory, and yet despite their apparent simplicity, and despite the past six decades of remarkable progress in the theory of large cardinals, even the class of normal ultrafilters remains in many ways mysterious, its underlying structure inextricably bound up with some of the deepest and most difficult problems in set theory. Take, for example, the following questions, posed by Solovay-Reinhardt-Kanamori [2] in the 1970s:

Question 2.1.1. *Assume κ is 2^κ-supercompact.*[2] *Must there be more than one normal ultrafilter on κ concentrating on nonmeasurable cardinals?*[3]

Question 2.1.2. *Assume κ is strongly compact.*[4] *Must κ carry more than one normal ultrafilter?*

These questions turn out to be merely the most concrete instances of a sequence of more and more general structural questions in the theory of large cardinals. Let us start down this path by stating a conjecture that would answer both questions at once:

Conjecture 2.1.3. *It is consistent with all large cardinal axioms that every measurable cardinal carries a unique normal ultrafilter concentrating on non-measurable cardinals.*

[1] See Definition 2.2.36.

[2] See Definition 4.2.15

[3] See Definition 2.2.27. A theorem of Kunen states that every measurable cardinal carries at least one normal ultrafilter concentrating on nonmeasurable cardinals.

[4] See Definition 7.2.4.

This would obviously answer Question 2.1.1 negatively, but what bearing does it have on Question 2.1.2? Assume every measurable cardinal carries a unique normal ultrafilter concentrating on nonmeasurable cardinals. Consider the least strongly compact cardinal κ that is a limit of strongly compact cardinals. By a theorem of Menas [5] (proved here as Theorem 8.1.1), the set of measurable cardinals below κ is nonstationary, and in particular, every normal ultrafilter on κ concentrates on nonmeasurable cardinals. Since we assumed there is only one such ultrafilter, κ is a strongly compact cardinal that carries a unique normal ultrafilter. Conjecture 2.1.3 thus supplies a negative answer to Question 2.1.2 as well.[5]

Why would someone make Conjecture 2.1.3? To answer this question, we must consider the broader question of the structure of the Mitchell order under large cardinal hypotheses. Recall that if U and W are normal ultrafilters, the *Mitchell order* is defined by setting $U \lhd W$ if U belongs to the ultrapower of the universe by W. It is not hard to see that a normal ultrafilter U on a cardinal κ concentrates on nonmeasurable cardinals if and only if U is a minimal element in the Mitchell order on normal ultrafilters on κ. The following conjecture therefore generalizes Conjecture 2.1.3:

Conjecture 2.1.4. *It is consistent with all large cardinal axioms that the Mitchell order is linear.*

The most general technique for proving consistency results in set theory, namely forcing, seems to be powerless in the face of Conjecture 2.1.4. To force the linearity of the Mitchell order, one would in particular have to force that the least measurable cardinal carries a unique normal ultrafilter, but even this much more basic problem remains open. So how could one possibly resolve Conjecture 2.1.4?

Kunen [4] famously did prove that it is consistent for the least measurable cardinal to carry a unique normal ultrafilter, not by forcing but instead by building an inner model. In fact, he showed that if U is a normal ultrafilter on a cardinal κ, then in the inner model $L[U]$, κ is the unique measurable cardinal and $U \cap L[U]$ is the unique normal ultrafilter on κ. In an attempt to generalize Kunen's results, Mitchell [6] isolated the Mitchell order and used it to guide the construction of generalizations of $L[U]$ that can have many measurable cardinals. Mitchell's proof of the linearity of the Mitchell order in these models proceeds as follows:

- Consider the model $M = L[\langle U_\alpha : \alpha < \gamma\rangle]$ built from a coherent sequence $\langle U_\alpha : \alpha < \gamma\rangle$ of normal ultrafilters.[6]
  - As a consequence of coherence, the sequence $\langle U_\alpha \cap M : \alpha < \gamma\rangle$ is linearly ordered by the Mitchell order in M.

[5]In recent work, Gitik-Kaplan show that if κ is supercompact and carries a unique normal ultrafilter concentrating on nonmeasurable cardinals, then in a generic extension, κ carries a unique normal ultrafilter and remains strongly compact.

[6]Coherence is a key technical definition that includes the assumption that $\langle U_\alpha : \alpha < \gamma\rangle$ is increasing in the Mitchell order.

- Show that every normal ultrafilter of M belongs to $\{U_\alpha \cap M : \alpha < \gamma\}$.

In the decades since Mitchell's result, inner model theory has ascended much farther into the large cardinal hierarchy. Combining the results of many researchers (especially Neeman [7] and Schlutzenberg [8]), the following is the best partial result towards Conjecture 2.1.4 to date:

Theorem. *If it is consistent that there is a Woodin limit of Woodin cardinals, then it is consistent that there is a Woodin limit of Woodin cardinals and the Mitchell order is linear.*

The linearity proof, due to Schlutzenberg, is much harder, but the argument still roughly follows Mitchell's:

- Consider the model $M = L[\langle E_\alpha : \alpha < \gamma\rangle]$ built from a coherent sequence of extenders $\langle E_\alpha : \alpha < \gamma\rangle$.
  - By the definition of a coherent sequence of extenders, $\langle E_\alpha : \alpha < \gamma\rangle$ is linearly ordered by the Mitchell order in M.
- Show that every normal ultrafilter of M belongs to $\{E_\alpha : \alpha < \gamma\}$.

By now, it may appear that Conjecture 2.1.4 itself is subsumed by the far more important (but far less precise) Inner Model Problem:

Conjecture 2.1.5. *Every large cardinal axiom holds in a canonical inner model.*

The relationship between Conjecture 2.1.4 and Conjecture 2.1.5 is actually not as straightforward as one might expect. It is open whether inner model theory can be extended to the level of supercompact cardinals, but if this is possible, then the models must be significantly different from the current models: so different, in fact, that it is not clear that Mitchell and Schlutzenberg's arguments still apply.

For example, take the Woodin and Neeman-Steel models with long extenders, which are canonical inner models designed to accommodate large cardinals at the finite levels of supercompactness. It is not known whether the constructions actually succeed, but the following conjecture is plausible:[7]

Conjecture 2.1.6. *If for all $n < \omega$, there is a cardinal κ that is κ^{+n}-supercompact, then for all $n < \omega$, there is an iterable Woodin model with a cardinal κ that is κ^{+n}-supercompact.*

Given the pattern described above, one might expect to generalize Mitchell and Schlutzenberg's results to the Woodin models, and therefore obtain for any $n < \omega$, the consistency of the linearity of the Mitchell order with a cardinal κ that is κ^{+n}-supercompact. But there is a catch: the proofs of these theorems cannot generalize verbatim to this level.

[7] This is proved in [9] under an iterability hypothesis.

Proposition 2.1.7. *If $L[\langle E_\alpha : \alpha < \gamma\rangle]$ is an iterable Woodin model satisfying that κ is κ^{++}-supercompact, then in $L[\langle E_\alpha : \alpha < \gamma\rangle]$, there is a normal ultrafilter on κ that does not belong to $\{E_\alpha : \alpha < \gamma\}$.*

The same result applies to the Neeman-Steel models at this level. Therefore Mitchell's proof of the consistency of the linearity of the Mitchell order does not extend to the level of a cardinal κ that is κ^{++}-supercompact.

2.1.2 The Ultrapower Axiom

The problem of generalizing the linearity of the Mitchell order to canonical inner models at the finite levels of supercompactness was the impetus for the work that led to this monograph. Our initial discovery was a new argument that shows that *any* canonical inner model built by the methodology of modern inner model theory must satisfy that the Mitchell order is linear. The argument is extremely simple and relies on a single fundamental property of the known canonical inner models: the *Comparison Lemma*. The Comparison Lemma roughly states that any two canonical inner models at the same large cardinal level can be embedded into a common model. The inner model constructions are perhaps best viewed as an attempt to build models that satisfy the Comparison Lemma yet accommodate large cardinals.

Upon further reflection, it became clear that this argument relied solely on a a single simple consequence of the Comparison Lemma, which could be distilled into an abstract combinatorial principle. This principle is called the *Ultrapower Axiom* (UA). (The definition appears in Section 2.3.4.) The Comparison Lemma implies that UA holds in all known canonical inner models. Since the Comparison Lemma is fundamental to the methodology of inner model theory, it seems likely that UA will hold in any canonical inner model that will ever be built.

Our theorem on the linearity of the Mitchell order now reads:

Theorem 2.3.11. *Assume the Ultrapower Axiom. Then the Mitchell order is linear.*

Granting our contention that UA holds in every canonical inner model, we have reduced Conjecture 2.1.4 to the Inner Model Problem (for example, Conjecture 2.1.5). Moreover, we can state a perfectly precise test question that seems to capture the essence of the Inner Model Problem:

Conjecture 2.1.8. *The Ultrapower Axiom is consistent with an extendible cardinal.*

It is our expectation that neither this conjecture nor even Conjecture 2.1.4 can be proved without first solving the Inner Model Problem itself. What sets Conjecture 2.1.8 apart from similar test questions like Conjecture 2.1.4 is that UA turns has a host of structural consequences in the theory of large cardinals. By studying UA, one can hope to glean insight into the inner models that have not yet been built, or perhaps to refute their existence by refuting UA from a large cardinal hypothesis. The latter has not happened. Instead a remarkable theory of large cardinals under UA has emerged which in our opinion provides evidence for Conjecture 2.1.8 and hence for the existence of inner models for very large cardinals.

2.1.3 Outline of Chapter 2

We now briefly outline the contents of the rest of this chapter.

Section 2.2. This section contains preliminary definitions, most of which are either standard or self-explanatory. The topics we cover are ultrapowers, close embeddings, uniform ultrafilters, and normal ultrafilters.

Section 2.3. This section contains proofs of the linearity of the Mitchell order and motivation for the Ultrapower Axiom. We begin in Section 2.3.1 by introducing and motivating Woodin's Weak Comparison axiom. Then in Section 2.3.2, we give our original argument for the linearity of the Mitchell order under Weak Comparison (Theorem 2.3.4). In Section 2.3.3, we abstract from this argument the Ultrapower Axiom, the central principle in this monograph and prove UA from Weak Comparison (Theorem 2.3.10). This proof is incomplete in the sense that several technical lemmas are deferred until the end of the chapter. In Section 2.3.4, we give the proof of the linearity of the Mitchell order from UA (Theorem 2.3.11), which is actually a simplification of the proof in Section 2.3.2. We also prove a sort of converse: UA restricted to normal ultrafilters is equivalent to the linearity of the Mitchell order. Finally, in Section 2.3.5, we prove the technical lemmas we had set aside in Section 2.3.3.

2.2 Preliminary definitions

2.2.1 Ultrapowers

We briefly put down our conventions on ultrapowers. If U is an ultrafilter, we denote by

$$j_U : V \to M_U$$

the ultrapower of the universe by U.

Recall that if κ is a cardinal, an ultrafilter U is *κ-complete* if it is closed under intersections of size less than κ. The *completeness of U* is the largest cardinal κ such that U is κ-complete.

We use the term *"countably complete"* as a synonym for ω_1-complete. The class of countably complete ultrafilters is the main topic of study in this monograph, serving to separate the wheat from the chaff:

Lemma 2.2.1. *An ultrafilter U is countably complete if and only if M_U is wellfounded.* $\square$

If M_U is wellfounded, or equivalently if U is countably complete, our convention is that M_U denotes the unique transitive class isomorphic to the ultrapower of the universe by U. More generally, we will often take the liberty of identifying wellfounded structures with their transitive collapses.

Higher degrees of completeness are related to the *critical point* of an elementary embedding:

Definition 2.2.2. If N is a transitive set and $j : N \to M$ is an elementary embedding, the *critical point of* j, denoted $\text{crit}(j)$, is the least ordinal κ such that $j(\kappa) \neq \kappa$.

We do not assume M is transitive here, but we do identify the wellfounded part of M with its transitive collapse.[8]

Lemma 2.2.3 (Scott)**.** *For any ultrafilter U, the completeness of U is equal to* $\text{crit}(j_U)$*.* $\square$

The *relativized ultrapower construction* will be a key tool in this monograph. If N is a transitive model of ZFC and $X \in N$, an *N-ultrafilter on X* is a set $U \subseteq P(X) \cap N$ such that $(N, U) \vDash U$ is an ultrafilter. Equivalently, U is an ultrafilter on the Boolean algebra $P(X) \cap N$. One can form the ultrapower of N by U, denoted

$$j_U^N : N \to M_U^N$$

using a modified ultrapower construction that uses only functions that belong to N. For any function $f \in N$ that is defined U-almost everywhere, we denote by $[f]_U^N$ the point in M_U^N represented by f. Since the point $[\text{id}]_U^N$ comes up so often, we introduce special notation for it:

Definition 2.2.4. If U is an N-ultrafilter, id_U^N denotes the point $[\text{id}]_U^N$.

We will drop the superscript N when it is convenient and unambiguous.

The notion of completeness makes sense for N-ultrafilters: if κ is an N-cardinal, an N-ultrafilter U is *N-κ-complete* if for all $\sigma \subseteq U$ with $\sigma \in N$ and $|\sigma|^N < \kappa$, $\bigcap \sigma \in U$. The *N-completeness of U* is the largest N-cardinal κ such that U is κ-complete. Lemma 2.2.3 generalizes:

Lemma 2.2.5. *If N is a transitive model and U is an N-ultrafilter, then the N-completeness of U is equal to* $\text{crit}(j_U^N)$*.*

Lemma 2.2.1, however, does not generalize to N-countably complete ultrafilters. (Here N-countable completeness is synonymous with N-$(\omega_1)^N$-completeness.) While this is often a significant issue in set theory, it will never cause problems in this monograph, because in our applications, N is closed under countable sequences.

Lemma 2.2.6. *If N is a transitive model that is closed under countable sequences and U is an N-ultrafilter, then M_U^N is wellfounded if and only if U is N-countably complete.* $\square$

[8]Without this convention, it would not really make sense to ask whether or not $j(\alpha)$ is equal to α. One could instead define $\text{crit}(j)$ to be the least ordinal κ such that $j(\kappa) \neq j[\kappa]$.

$$\begin{tikzcd}
 & M & a \\
N \arrow[ur, "j"] \arrow[r, "j^N_U"'] & M^N_U \arrow[u, "k"'] & a^N_U \arrow[u, mapsto]
\end{tikzcd}$$

Figure 2.1: The factor embedding associated to a derived ultrapower.

The derived ultrafilter construction allow us to extract combinatorial content from elementary embeddings:

Definition 2.2.7. Suppose N and M are transitive models of ZFC and $j : N \to M$ is an elementary embedding. Suppose $X \in N$ and $a \in j(X)$. The *N-ultrafilter on X derived from j using a* is the N-ultrafilter $\{A \in P(X) \cap N : a \in j(A)\}$.

What is the relationship between an elementary embedding and the ultrapowers associated to its derived ultrafilters? The answer is contained in the following lemma:

Lemma 2.2.8. *Suppose N and M are transitive models of* ZFC *and $j : N \to M$ is an elementary embedding. Suppose $X \in N$ and $a \in j(X)$. Let U be the N-ultrafilter on X derived from j using a. Then there is a unique embedding $k : M^N_U \to M$ such that $k \circ j^N_U = j$ and $k(\mathrm{id}_U) = a$.* □

We refer to the embedding k as the *factor embedding* associated to the derived ultrafilter U.

Often we will wish to discuss an ultrapower embedding without the need to choose any particular ultrafilter giving rise to it, so we introduce the following terminology:

Definition 2.2.9. If N and M are transitive models of ZFC, an elementary embedding $j : N \to M$ is an *ultrapower embedding* if there is an N-ultrafilter U such that $M = M^N_U$ and $j = j^N_U$.

Definition 2.2.10. If N is a transitive model of ZFC, a *countably complete ultrafilter of N* is a point in N that, in N, is a countably complete ultrafilter.

An N-ultrafilter U is a countably complete ultrafilter of N if and only if its corresponding ultrapower $j : N \to M$ is wellfounded and definable over N.

Definition 2.2.11. An ultrapower embedding $j : N \to M$ is an *internal ultrapower embedding* of N if there is a countably complete ultrafilter U of N such that $j = j^N_U$.

An important point is that for our purposes, when we speak of ultrapower embeddings, we only mean ultrapower embeddings between wellfounded models. For example, if U is an ultrafilter on ω, then the embedding $j_U : V \to M_U$ does not count as an ultrapower embedding.

There is a characterization of ultrapower embeddings that does not refer to ultrafilters at all.

Definition 2.2.12. Suppose N and M are transitive models of ZFC. An elementary embedding $j : N \to M$ is *cofinal* if for all $a \in M$, there is some $X \in N$ such that $a \in j(X)$.

Equivalently, j is cofinal if $j[\text{Ord} \cap N]$ is cofinal in $\text{Ord} \cap M$.

Definition 2.2.13. Suppose N and M are transitive set models of ZFC.[9] An elementary embedding $j : N \to M$ is a *weak ultrapower embedding* if there is some $a \in M$ such that every element of M is definable in M from parameters in $j[N] \cup \{a\}$.

Lemma 2.2.14. *An elementary embedding between two transitive set models of* ZFC *is an ultrapower embedding if and only if it is a cofinal weak ultrapower embedding.* □

The following notation will be extremely important in our analysis of elementary embeddings:

Definition 2.2.15. Suppose N and M are transitive models of ZFC. Suppose $j : N \to M$ is a cofinal elementary embedding and S is a subclass of M. Then the *hull of S in M over $j[N]$* is the class $H^M(j[N] \cup S) = \{j(f)(x_1, \ldots, x_n) : x_1, \ldots, x_n \in S\}$.

The fundamental theorem about these hulls, which we use repeatedly and implicitly, is the following:

Lemma 2.2.16. *Suppose N and M are transitive models of* ZFC*. Suppose $j : N \to M$ is a cofinal elementary embedding and S is a subclass of M. Then the hull of S in M over $j[N]$ is the minimum elementary substructure of M containing $j[N] \cup S$.* □

For more on hulls, see [10, Lemma 1.1.18]. (Larson's lemma should use a stronger theory than ZFC − Replacement. Σ_2-Replacement suffices.) Using hulls, we can give a metamathematically unproblematic model theoretic characterization of ultrapower embeddings between transitive models that are not assumed to be sets:

Lemma 2.2.17. *Suppose N and M are transitive models of* ZFC*. A cofinal elementary embedding $j : N \to M$ is an ultrapower embedding if and only if $M = H^M(j[N] \cup \{a\})$ for some $a \in M$.* □

[9] For metamathematical reasons (namely, the undefinability of definability), whether j is a weak ultrapower embedding is not in general a first-order property of $(V, N, M, j, \in)$ when M is a proper class.

Given this characterization of ultrapower embeddings, the closure of ultrapower embeddings under compositions as well as certain kinds of factorizations becomes evident:

Lemma 2.2.18. *If $j : N \to P$ is an elementary embedding and $j = k \circ i$ where $N \xrightarrow{i} M \xrightarrow{k} P$ are ultrapower embeddings, then j is an ultrapower embedding.* $\square$

Lemma 2.2.19. *If $j : N \to P$ is an ultrapower embedding and $j = k \circ i$ where $N \xrightarrow{i} M \xrightarrow{k} P$ are elementary embeddings, then k is an ultrapower embedding.* $\square$

The combinatorial proofs of these facts (using ultrafilters) are not hard, but the ultrafilter-free characterization of ultrapowers makes them *transparent.*

2.2.2 Close embeddings

The property of being an internal ultrapower embedding is a very stringent requirement. *Closeness* is a natural weakening that originated in inner model theory:

Definition 2.2.20. Suppose N and M are transitive models of ZFC and $j : N \to M$ is an elementary embedding. Then j is *close* to N if j is cofinal and for all $X \in N$ and $a \in j(X)$, the N-ultrafilter on X derived from j using a belongs to N.

Close embeddings have a very natural model theoretic definition that makes no reference to ultrafilters:

Lemma 2.2.21. *Suppose N and P are transitive models of* ZFC *and $j : N \to P$ is an elementary embedding. Then the following are equivalent:*

(1) j is close to N.

(2) For any $a \in P$, j factors as $N \xrightarrow{i} M \xrightarrow{k} P$ where $i : N \to M$ is an internal ultrapower embedding, $k : M \to P$ is an elementary embedding, and $a \in k[M]$.

(3) For any set $A \in P$, the inverse image $j^{-1}[A]$ belongs to N.

Proof. (1) implies (2): Immediate from the factor embedding construction Lemma 2.2.8.

(2) implies (3): Fix $A \in P$, and we will show $j^{-1}[A] \in N$. Factor j as $N \xrightarrow{i} M \xrightarrow{k} P$ where $i : N \to M$ is an internal ultrapower embedding, $k : M \to N$ is an elementary embedding, and $A \in k[M]$. Fix $B \in M$ such that $k(B) = A$. Now $i^{-1}[B] \in N$ since i is an internal ultrapower embedding of N. We finish by showing $i^{-1}[B] = j^{-1}[A]$. First, by the elementarity of k, $B = k^{-1}[A]$. Therefore $i^{-1}[B] = i^{-1}[k^{-1}[A]] = (k \circ i)^{-1}[A] = j^{-1}[A]$.

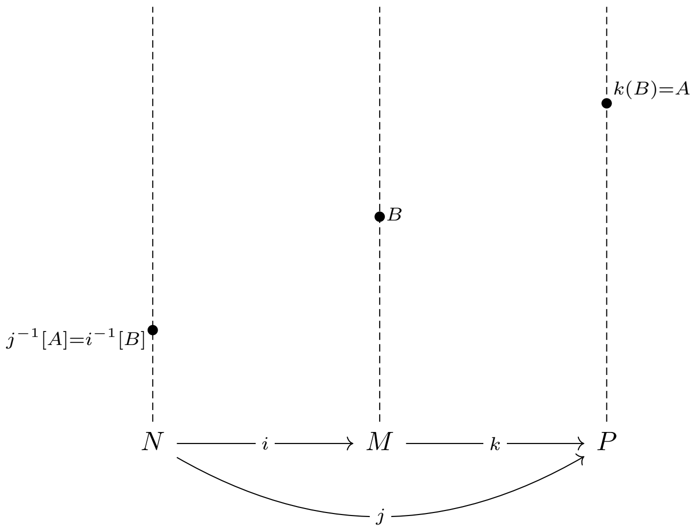

Figure 2.2: The proof that Lemma 2.2.21 (2) implies (3)

(3) implies (1): We first show that j is cofinal. Assume not, towards a contradiction. Then there is an ordinal $\alpha \in P$ that lies above all ordinals in the range of j. Therefore $j^{-1}[\alpha] = \mathrm{Ord} \cap N \notin N$, which is a contradiction.

Finally, fix $X \in N$ and $a \in P$ with $a \in j(X)$. We must show that the N-ultrafilter on X derived from j using a belongs to N. Let $\mathrm{p}_a^{j(X)}$ denote the principal N-ultrafilter on $j(X)$ concentrated at a. Then the N-ultrafilter on X derived from j using a is precisely $j^{-1}[\mathrm{p}_a^{j(X)}]$, which belongs to N by assumption. $\square$

Most texts on inner model theory define *close extenders* rather than close embeddings, so we briefly describe the relationship between these two concepts. If N is a transitive model of ZFC and E is an N-extender of length λ, then E is *close to N* if $E_a \in M$ for all $a \in [\lambda]^{<\omega}$.

Lemma 2.2.22. *An N-extender E is close to N if and only if the elementary embedding j_E^N is close to N.* $\square$

The fact that the comparison process gives rise to close embeddings is less well-known than the fact that all extenders applied in a normal iteration tree are close, which for example is proved in [11]. Given that each of the individual extenders that are applied are close, the following fact shows that all the embeddings between models of ZFC in a normal iteration tree are close:

Lemma 2.2.23. *(1) If $N \xrightarrow{i} M \xrightarrow{k} P$ are close embeddings, then the composition $k \circ i$ is close to N.*

(2) Suppose $\mathscr{D} = \{M_p, j_{pq} : p \leq q \in I\}$ is a directed system of transitive models of ZFC *and elementary embeddings. Suppose $p \in I$ is an index such that for all $q \geq p$ in I, $j_{pq} : M_p \to M_q$ is close to M_p. Let N be the direct limit of $\mathscr{D}$, and assume N is transitive. Then the direct limit embedding $j_{p\infty} : M_p \to N$ is close to M_p.*

Proof. (1) is immediate from Lemma 2.2.21 (3), and (2) is clear from Lemma 2.2.21 (2). □

An often useful trivial fact about close embeddings is that their right-factors are close:

Lemma 2.2.24. *If $j : N \to P$ is a close embedding and $j = k \circ i$ where $N \xrightarrow{i} M \xrightarrow{k} P$ are elementary embeddings, then i is close to N.* □

Another fact which is almost tautological is that an ultrapower embedding is internal if and only if it is close:

Lemma 2.2.25. *If $j : N \to M$ is an ultrapower embedding, then j is an internal ultrapower embedding of N if and only if j is close to N.* □

2.2.3 Uniform ultrafilters

One of the most basic notions from ultrafilter theory is that of a uniform ultrafilter:

Definition 2.2.26. An ultrafilter U is *uniform* if every set in U has the same cardinality. If U is an ultrafilter, the *size* of U, denoted λ_U, is the least cardinality of a set in U.

The cardinals λ_U for U a countably complete ultrafilter will become very important in Chapter 7.

Definition 2.2.27. A cardinal κ is *measurable* if it carries a κ-complete uniform ultrafilter.

Returning to uniformity, note that U is a uniform ultrafilter on X if and only if it extends the Fréchet filter on X, the collection F_X of all $A \subseteq X$ such that $|X \setminus A| < |X|$. It will be important to distinguish between the notion of a uniform ultrafilter and the similar but distinct notion of a *fine* ultrafilter on an ordinal, defined in Section 3.2.1. These notions are often confused in the literature.

Rudin-Keisler equivalence is the notion of isomorphism for ultrafilters.

$$\begin{tikzcd} & M_0 \arrow[dd, two heads, "k"] \\ N \arrow[ur, "j_0"] \arrow[dr, "j_1"'] & \\ & M_1 \end{tikzcd}$$

Figure 2.3: An isomorphism of embeddings, $(M_0, j_0) \cong (M_1, j_1)$

Definition 2.2.28. Suppose U and W are ultrafilters. We say U and W are *Rudin-Keisler equivalent*, and write $U \equiv_{\mathrm{RK}} W$, if there exist $X \in U$, $Y \in W$, and a bijection $f : X \to Y$ such that for all $A \subseteq X$, $A \in U$ if and only if $f[A] \in W$.

Rudin-Keisler equivalence is a special case of the measure-theoretic notion of almost isomorphic measure spaces. It can also be formulated as a model theoretic property:

Definition 2.2.29. Suppose $j_0 : N \to M_0$ and $j_1 : N \to M_1$ are elementary embeddings. We write $(M_0, j_0) \cong (M_1, j_1)$ to denote that there is an isomorphism $k : M_0 \to M_1$ such that $k \circ j_0 = j_1$.

The following lemma (due to Rudin-Keisler) is explained in Section 3.4.2:

Lemma 2.2.30. *If U and W are ultrafilters, then U and W are Rudin-Keisler equivalent if and only if $(M_U, j_U) \cong (M_W, j_W)$.* $\square$

For countably complete ultrafilters, there is a much simpler model theoretic characterization of Rudin-Keisler equivalence (so we will not really need the notation from Definition 2.2.29):

Corollary 2.2.31. *If U and W are countably complete ultrafilters, then U and W are Rudin-Keisler equivalent if and only if $j_U = j_W$.*

Proof. Since M_U and M_W are transitive, the only possible isomorphism between M_U and M_W is the identity. Hence $(M_U, j_U) \cong (M_W, j_W)$ if and only if $j_U = j_W$. $\square$

Notice that if $U \equiv_{\mathrm{RK}} W$ then $\lambda_U = \lambda_W$. Since we are mostly interested in ultrapower embeddings and not ultrafilters themselves, the following lemma lets us focus our attention on uniform ultrafilters that lie on cardinals:

Lemma 2.2.32. *Any ultrafilter U is Rudin-Keisler equivalent to a uniform ultrafilter on λ_U.*

Proof. Fix $X \in U$ such that $|X| = \lambda_U$. Let $f : X \to \lambda_U$ be a bijection. Let $W = \{A \subseteq \lambda_U : f^{-1}[A] \in U\}$. Then $U \equiv_{\text{RK}} W$. Moreover W is uniform since $\lambda_W = \lambda_U$. $\square$

The generalization of uniformity to relativized case is straightforward:

Definition 2.2.33. Suppose M is a transitive model of ZFC and U is an M-ultrafilter. Then the *size* of U is the M-cardinal $\lambda_U = \min\{|X|^M : X \in U\}$.

Though it will not be relevant to us, we note that if $U \notin M$, the wellfoundedness of M is required to ensure that λ_U is well-defined.

The ultrapower associated to an ultrafilter U is in a certain sense trivial above λ_U, which will be very useful in later chapters:

Lemma 2.2.34. *Suppose U is an M-ultrafilter. Then for all M-regular cardinals $\delta > \lambda_U$, $j_U(\delta) = \sup j_U[\delta]$.*

Proof. We may assume the underlying set of U is λ_U by the relativized version of Lemma 2.2.32. Then fix $a < j_U(\delta)$. We must find $\alpha < \delta$ such that $a < j_U(\alpha)$. For this, fix a function $f : \lambda_U \to \delta$ in M be such that $a = [f]^M_U$. Since δ is M-regular, f is bounded below δ, so let $\alpha < \delta$ be such that $f[\lambda_U] \subseteq \alpha$. Then $a = [f]^M_U < j_U(\alpha)$ by Łoś's Theorem, as desired. $\square$

2.2.4 The Mitchell order on normal ultrafilters

Definition 2.2.35. Suppose $\langle X_\alpha : \alpha < \delta\rangle$ is a sequence of subsets of δ. The *diagonal intersection* of $\langle X_\alpha : \alpha < \delta\rangle$ is the set

$$\triangle_{\alpha<\delta} X_\alpha = \{\alpha < \delta : \alpha \in \textstyle\bigcap_{\beta<\alpha} X_\beta\}$$

Definition 2.2.36. A uniform ultrafilter on an infinite cardinal κ is *normal* if it is closed under diagonal intersections.

Lemma 2.2.37. *Suppose U is an ultrafilter on an ordinal κ. The following are equivalent:*

(1) U is normal.

(2) U is κ-complete and $\text{id}_U = \kappa$. $\square$

The Mitchell order was introduced by Mitchell in [6].

Definition 2.2.38. Suppose U and W are normal ultrafilters. The *Mitchell order* is defined by setting $U \lhd W$ if $U \in M_W$.

This definition makes sense by our convention that the ultrapower of the universe by a countably complete ultrafilter is taken to be transitive.

Lemma 2.2.39. *The Mitchell order is a wellfounded partial order.* $\square$

Actually, the interested reader will find several generalizations of this fact scattered throughout this monograph. For example, Theorem 3.3.8 and Theorem 4.2.45 come to mind.

Definition 2.2.40. If U is a normal ultrafilter on a cardinal κ, then $o(U)$ denotes the rank of U in the restriction of the Mitchell order to normal ultrafilters on κ. For any ordinal κ, $o(\kappa)$ denotes the rank of the restriction of the Mitchell order to normal ultrafilters on κ.

2.3 The linearity of the Mitchell order

Our original proof of the linearity of the Mitchell order did not use the Ultrapower Axiom as a hypothesis. Instead, it used a principle called *Weak Comparison* that was introduced by Woodin [12] in his work on the Inner Model Problem for supercompact cardinals.

Weak Comparison is directly motivated by the Comparison Lemma of inner model theory, and it is immediately clear that Weak Comparison holds in all known canonical inner models. On the other hand, although the Ultrapower Axiom is a more elegant principle than Weak Comparison, the fact that the Ultrapower Axiom holds in all known canonical inner models is not nearly as obvious. But our proof of the linearity of the Mitchell order from Weak Comparison actually shows that the Ultrapower Axiom follows from Weak Comparison, and this is how the Ultrapower Axiom was originally isolated.

In this section, we will introduce Weak Comparison and then prove that Weak Comparison implies the linearity of the Mitchell order. We then isolate the Ultrapower Axiom by remarking that this proof breaks naturally into two implications: first, that Weak Comparison implies the Ultrapower Axiom, and second, that the Ultrapower Axiom implies the linearity of the Mitchell order. The reader who does not want to learn about Weak Comparison can skip ahead to Section 2.3.4. We emphasize, however, that the fact that Weak Comparison implies UA is central to the motivation for this work.

2.3.1 Weak Comparison

Stating Weak Comparison requires a number of definitions. The following notational convention will make many of our arguments easier to read:

Definition 2.3.1. Suppose N_0, N_1, P are transitive models of ZFC. We write

$$(j_0, j_1) : (N_0, N_1) \to P$$

to mean that $j_0 : N_0 \to P$ and $j_1 : N_1 \to P$ are elementary embeddings.

Weak Comparison is a comparison principle for a class of structures. The next two definitions specify this class.

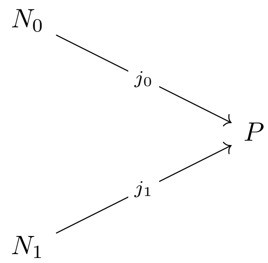

Figure 2.4: A pair of embeddings $(j_0, j_1) : (N_0, N_1) \to P$

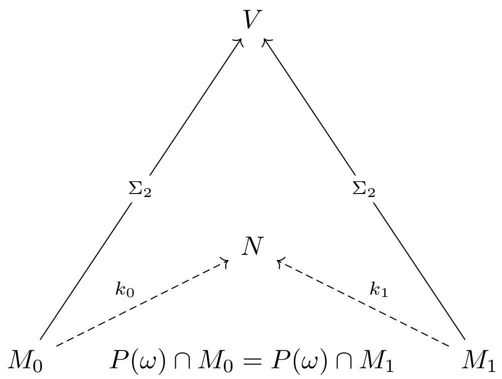

Figure 2.5: Weak Comparison

Definition 2.3.2. Suppose M is a model of ZFC. Then M is *finitely generated* if there is some $a \in M$ such that every point in M is definable in M using a as a parameter.

Definition 2.3.3. Suppose M is a transitive set that satisfies ZFC. Then M is a Σ_2-*hull* if there is a Σ_2-elementary embedding $\pi : M \to V$.

We can now state Weak Comparison:

Weak Comparison. If M_0 and M_1 are finitely generated Σ_2-hulls such that $P(\omega) \cap M_0 = P(\omega) \cap M_1$, there are close embeddings $(k_0, k_1) : (M_0, M_1) \to N$.

We conclude this section by sketching Woodin's argument that Weak Comparison holds in all known canonical inner models. Assume that V itself is a canonical inner model, so that there is some sort of Comparison Lemma for countable sufficiently elementary substructures of V. Assume M_0 and M_1 are finitely generated Σ_2-hulls. We will show that there are close embeddings $(k_0, k_1) : (M_0, M_1) \to N$.

The fact that M_0 and M_1 are countable Σ_2-hulls implies that the Comparison Lemma applies to them. The comparison process therefore produces transitive structures N_0 and N_1 such that one of the following holds:

Case 1. $N_0 = N_1$ *and there are close embeddings* $(k_0, k_1) : (M_0, M_1) \to N_0$.

Case 2. $N_0 \in N_1$, $P(\omega) \cap N_1 \subseteq M_1$, *and there is a close embedding* $k_0 : M_0 \to N_0$.

Case 3. $N_1 \in N_0$, $P(\omega) \cap N_0 \subseteq M_0$, *and there is a close embedding* $k_1 : M_1 \to N_1$.

Case 1 is the result of "coiteration," while in Case 2 and Case 3, one of the models has "outiterated" the other. To obtain weak comparison for the pair M_0 and M_1, it suffices to show that Case 1 holds. To do this we show that Case 2 and Case 3 cannot hold.

Assume towards a contradiction that Case 2 holds. Since M_0 is finitely generated, there is some $a \in M_0$ such that every point in M_0 is definable in M_0 from a. Therefore $k_0[M_0]$ is equal to the set of points in N_0 that are definable in N_0 from $k_0(a)$. Since $N_0 \in N_1$, it follows that $k_0[M_0] \in N_1$ and $k_0[M_0]$ is countable in N_1. Therefore its transitive collapse, namely M_0, is in N_1 and is countable in N_1. Let $x \in P(\omega) \cap N_1$ code M_0 in the sense that any transitive model H of ZFC with $x \in H$ has $M_0 \in H$. Then $x \in P(\omega) \cap N_1 \subseteq P(\omega) \cap M_1 = P(\omega) \cap M_0$. It follows that $x \in M_0$. Since x codes M_0, this implies $M_0 \in M_0$, which is a contradiction.

A similar argument shows that Case 3 does not hold. Therefore Case 1 must hold.

This argument actually shows that a slight strengthening of Weak Comparison is true in all known canonical inner models:

Weak Comparison (Strong Version)**.** If M_0 and M_1 are finitely generated Σ_2-hulls, either $M_0 \in M_1$, $M_1 \in M_0$, or there are close embeddings $(k_0, k_1) : (M_0, M_1) \to N$.

The strong version of Weak Comparison implies the Continuum Hypothesis.[10] It is not clear if it has any other advantages.

2.3.2 Weak Comparison and the Mitchell order

Now, a confession: we cannot actually prove the linearity of the Mitchell order from Weak Comparison. Instead, we will need some auxiliary hypotheses:

Theorem 2.3.4. *Assume that* $V = \text{HOD}$ *and there is a* Σ_2*-correct worldly cardinal. Assume Weak Comparison holds. Then the Mitchell order is linear.*

---

[10] Here one must assume in addition to the strong version of Weak Comparison that $V = \text{HOD}$ and there is a Σ_2-correct worldly cardinal. In fact, these hypotheses are necessary for all our consequences of Weak Comparison.

The need for these auxiliary hypotheses is one of the quirks of Weak Comparison, and it is part of the reason we think the Ultrapower Axiom is a more elegant principle.

Here a cardinal κ is *worldly* if V_κ satisfies ZFC, and κ is Σ_2-*correct* if $V_\kappa \prec_{\Sigma_2} V$. This is a very weak large cardinal hypothesis. For example, if κ is inaccessible, then in V_κ, there is a Σ_2-correct worldly cardinal; indeed, Morse-Kelley set theory implies the existence of a Σ_2-correct worldly cardinal. If κ is a strong cardinal, then κ itself is a Σ_2-correct worldly cardinal. The hypothesis is motivated by the following lemma, which we defer to a later section:

Lemma 2.3.19. *The existence of a Σ_2-hull is equivalent to the existence of a Σ_2-correct worldly cardinal.*

If one wants to apply Weak Comparison at all, at the very least, one needs the existence of a Σ_2-hull, and therefore one needs a Σ_2-correct worldly cardinal. One also needs *finitely generated* models, and this is where we use the principle $V = \text{HOD}$:

Definition 2.3.5. Suppose M is a model of ZFC. Then M is *pointwise definable* if every point in M is definable in M without parameters.

Lemma 2.3.6. *Assume $V = \text{HOD}$. If there is a Σ_2-hull, then there is a pointwise definable Σ_2-hull.*

Proof. If H is a Σ_2-hull, then H satisfies $V = \text{HOD}$ since H satisfies every Π_3-sentence true in V. Since H is a model of ZFC + $V = \text{HOD}$, H has a Σ_2-definable Skolem function. Therefore the set X of elements of H that are definable in H forms an elementary substructure of H. The transitive collapse of X is a pointwise definable Σ_2-hull. $\square$

The principle $V = \text{HOD}$ arguably does not hold in all canonical inner models. (The standard counterexample is $L[M_1^\#]$.) The proof that the Mitchell order is linear, however, really does work in any inner model. The fact that we must assume $V = \text{HOD}$ is again a quirk of Weak Comparison.

The key technical lemma of Theorem 2.3.4 is the following closure property:

Lemma 2.3.17. *The set of finitely generated Σ_2-hulls is closed under internal ultrapowers.*

We defer the proof to Section 2.3.5. We now proceed to the proof of Theorem 2.3.4 granting the lemmas.

Proof of Theorem 2.3.4. Since there is a Σ_2-correct worldly cardinal and $V = \text{HOD}$, we can fix a pointwise definable Σ_2-hull H (by Lemma 2.3.6). It suffices to show that the Mitchell order is linear in H, since, being Π_2-expressible, this is absolute between H and V.

Suppose U_0 and U_1 are normal ultrafilters of H. We must show that in H, either $U_0 = U_1$, $U_0 \lhd U_1$, or $U_0 \rhd U_1$. We might as well assume that U_0 and U_1

are normal ultrafilters on the same cardinal κ, since otherwise it is immediate that $U_0 \lhd U_1$ or $U_0 \rhd U_1$.

Let $j_0 : H \to M_0$ be the ultrapower of H by U_0 and let $j_1 : H \to M_1$ be the ultrapower of H by U_1. By the closure of finitely generated Σ_2-hulls under internal ultrapowers (Lemma 2.3.17), M_0 and M_1 are finitely generated Σ_2-hulls. Since M_0 and M_1 are internal ultrapowers of H, $P(\omega) \cap M_0 = P(\omega) \cap M_1$. Therefore, by Weak Comparison there are close embeddings

$$(k_0, k_1) : (M_0, M_1) \to N$$

Since H is pointwise definable,

$$k_0 \circ j_0 = k_1 \circ j_1 \tag{2.1}$$

This is because $k_0 \circ j_0$ and $k_1 \circ j_1$ are both elementary embeddings from H to N, and therefore each must shift all parameter-free definable points in the same way.

The proof now splits into three cases.

Case 1. $k_0(\kappa) = k_1(\kappa)$.

Case 2. $k_0(\kappa) < k_1(\kappa)$.

Case 3. $k_0(\kappa) > k_1(\kappa)$.

In Case 1, we will show $U_0 = U_1$, in Case 2, we will show $U_0 \lhd U_1$, and in Case 3, we will show $U_0 \rhd U_1$. This will complete the proof.

Proof in Case 1. Suppose $A \subseteq \kappa$ and $A \in H$. We have

$$\begin{aligned} A \in U_0 &\iff \kappa \in j_0(A) \\ &\iff k_0(\kappa) \in k_0(j_0(A)) \\ &\iff k_0(\kappa) \in k_1(j_1(A)) \qquad (2.2) \\ &\iff k_1(\kappa) \in k_1(j_1(A)) \qquad (2.3) \\ &\iff \kappa \in j_1(A) \\ &\iff A \in U_1 \end{aligned}$$

To obtain (2.2), we use (2.1) above. To obtain (2.3), we use the case hypothesis that $k_0(\kappa) = k_1(\kappa)$. It follows that $U_0 = U_1$. □

Proof in Case 2. Suppose $A \subseteq \kappa$ and $A \in H$. We have

$$\begin{aligned} A \in U_0 &\iff \kappa \in j_0(A) \\ &\iff k_0(\kappa) \in k_0(j_0(A)) \\ &\iff k_0(\kappa) \in k_1(j_1(A)) \qquad (2.4) \\ &\iff k_0(\kappa) \in k_1(j_1(A)) \cap k_1(\kappa) \qquad (2.5) \\ &\iff k_0(\kappa) \in k_1(j_1(A) \cap \kappa) \\ &\iff k_0(\kappa) \in k_1(A) \qquad (2.6) \end{aligned}$$

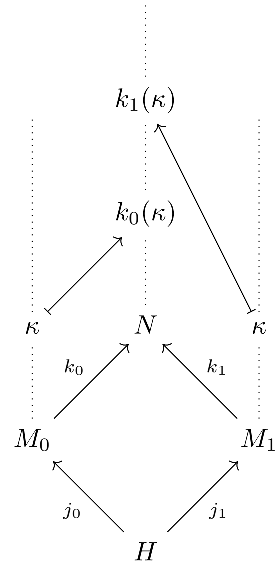

Figure 2.6: Case 2 of Theorem 2.3.4

To obtain (2.4), we use (2.1) above. To obtain (2.5), we use the case hypothesis that $k_0(\kappa) < k_1(\kappa)$. To prove (2.6), we use that U_1 is κ-complete, so $\text{crit}(j_1) = \kappa$ and hence $j_1(A) \cap \kappa = A$ for any $A \subseteq \kappa$.

It follows from this calculation that U_0 is the M_1-ultrafilter on κ derived from k_1 using $k_0(\kappa)$. (Here we use that $P(\kappa) \cap M_1 = P(\kappa) \cap H$.) Since k_1 is close to M_1, it follows that $U_0 \in M_1$. Since $M_1 = M^H_{U_1}$, this means that $U_0 \lhd U_1$ in H. $\square$

Proof in Case 3. The proof in this case is just like the proof in Case 2 but with U_0 and U_1 swapped. $\square$

This completes the proof of Theorem 2.3.4. $\square$

2.3.3 Weak Comparison and the Ultrapower Axiom

We now define the Ultrapower Axiom, which arises naturally from the proof of Theorem 2.3.4. Notice that the first half of this proof, which justifies our application of Weak Comparison to the ultrapowers M_0 and M_1, does not actually require that U_0 and U_1 are normal ultrafilters. Instead, it simply requires that they are countably complete.

In order to state UA succinctly, we make the following definitions.

Definition 2.3.7. Suppose N, M_0, M_1, P are transitive models of ZFC and $j_0 : N \to M_0$, $j_1 : N \to M_1$, and $(k_0, k_1) : (M_0, M_1) \to P$ are elementary embeddings.

$$\begin{tikzcd}
& P & \\
M_0 \arrow[ur, dashed] & & M_1 \arrow[ul, dashed] \\
& V \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Figure 2.7: The Ultrapower Axiom

- (k_0, k_1) is a *comparison* of (j_0, j_1) if $k_0 \circ j_0 = k_1 \circ j_1$.
- (k_0, k_1) is an *internal ultrapower comparison* if k_0 is an internal ultrapower embedding of M_0 and k_1 is an internal ultrapower embedding of M_1.
- (k_0, k_1) is a *close comparison* if k_0 is close to M_0 and k_1 is close to M_1.

Ultrapower Axiom. Every pair of ultrapower embeddings of the universe of sets has an internal ultrapower comparison.

The Ultrapower Axiom can be formulated in a first order way by quantifying over ultrafilters instead of ultrapowers.

On the face of it, the statement that every pair of ultrapowers has a comparison by internal ultrapowers looks *much* stronger than the conclusion of Weak Comparison, which only supplies close comparisons. But this is an illusion.

Lemma 2.3.8. *Suppose N, M_0, M_1 are transitive set models of* ZFC *and $j_0 : N \to M_0$ and $j_1 : N \to M_1$ are weak ultrapower embeddings. If (j_0, j_1) has a close comparison, then (j_0, j_1) has an internal ultrapower comparison.*

Proof. Suppose $(k_0, k_1) : (M_0, M_1) \to P$ is a comparison by close embeddings. Let $H \prec P$ be defined by

$$H = H^P(k_0[M_0] \cup k_1[M_1])$$

Let Q be the transitive collapse of H and let $h : Q \to P$ be the inverse of the transitive collapse embedding. Let $i_0 = h^{-1} \circ k_0$ and $i_1 = h^{-1} \circ k_1$. See Fig. 2.8.

Obviously $(i_0, i_1) : (M_0, M_1) \to Q$ is a comparison of (j_0, j_1) and

$$Q = H^Q(i_0[M_0] \cup i_1[M_1])$$

We need to show it is a comparison by internal ultrapowers, or in other words that i_0 is an internal ultrapower embedding of M_0 and i_1 is an internal ultrapower embedding of M_1.

We first show that i_0 is an ultrapower embedding of M_0. Since $j_1 : N \to M_1$ is a weak ultrapower embedding, there is some $a \in M_1$ such that every element

$$\begin{tikzcd}
 & P & \\
 & Q \arrow[u, "h"] & \\
M_0 \arrow[uur, "k_0"] \arrow[ur, "i_0"] & & M_1 \arrow[uul, "k_1"'] \arrow[ul, "i_1"'] \\
 & N \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}$$

Figure 2.8: An internal ultrapower comparison from a close comparison

of M_1 is definable in M_1 from parameters in $j_1[N] \cup \{a\}$. It follows easily that $Q = H^Q(i_0[M_0] \cup \{i_1(a)\})$. Therefore i_0 is an ultrapower embedding by Lemma 2.2.17.

Next, we show that i_0 is an *internal* ultrapower embedding. Since $h \circ i_0 = k_0$ and k_0 is close, in fact, i_0 is close to M_0 (Lemma 2.2.24). Since i_0 is a close ultrapower embedding of M_0, in fact, i_0 is an internal ultrapower embedding of M_0 (Lemma 2.2.25).

A symmetric argument shows that i_1 is an internal ultrapower embedding of M_1, completing the proof. ☐

This yields a strengthening of Weak Comparison:

Theorem 2.3.9. *Assume Weak Comparison and* $V = \text{HOD}$. *Suppose* M_0 *and* M_1 *are finitely generated* Σ_2*-hulls such that* $P(\omega) \cap M_0 = P(\omega) \cap M_1$. *Then there are internal ultrapower embeddings* $(i_0, i_1) : (M_0, M_1) \to Q$.

Proof. Applying Weak Comparison, fix close embeddings $(k_0, k_1) : (M_0, M_1) \to P$.

Since M_0 and M_1 are Σ_2-hulls, they satisfy any Π_3 sentence true in V. Therefore they both satisfy $V = \text{HOD}$. Let $H_0 \prec M_0$ be the set of points that are definable without parameters in M_0. Let $H_1 \prec M_1$ be the set of points that are definable without parameters in M_1. Then $k_0[H_0] = k_1[H_1]$ is the set of points that are definable without parameters in P. It follows that $H_0 \cong H_1$. Let N be the common transitive collapse of H_0 and H_1, and let $j_0 : N \to M_0$ and $j_1 : N \to M_1$ be the inverses of the transitive collapse maps. Note that j_0 and j_1 are weak ultrapower embeddings, and since N is pointwise definable, $k_0 \circ j_0 = k_1 \circ j_1$.

The embeddings (j_0, j_1) therefore have a close comparison, namely (k_0, k_1). Being weak ultrapower embeddings, they have an internal ultrapower comparison by Lemma 2.3.8. □

Lemma 2.3.8 also yields a proof of the Ultrapower Axiom from the same hypotheses as Theorem 2.3.4:

Theorem 2.3.10. *Assume that $V = \text{HOD}$ and there is a Σ_2-correct worldly cardinal. If Weak Comparison holds, then the Ultrapower Axiom holds.*

Proof. Since there is a Σ_2-correct worldly cardinal and $V = \text{HOD}$, we can fix a pointwise definable Σ_2-hull H (by Lemma 2.3.6). Since UA is a Π_2-statement and $H \equiv_{\Pi_2} V$, it suffices to show that H satisfies UA.

Suppose $j_0 : H \to M_0$ and $j_1 : H \to M_1$ are internal ultrapower embeddings of H. We must show that H satisfies that (j_0, j_1) has an internal ultrapower comparison.

By Lemma 2.3.17, finitely generated Σ_2-hulls are closed under internal ultrapowers, and so M_0 and M_1 are finitely generated Σ_2-hulls. Moreover, since M_0 and M_1 are internal ultrapowers of H, $P(\omega) \cap M_0 = P(\omega) \cap H = P(\omega) \cap M_1$. Therefore by Theorem 2.3.9, there are internal ultrapower embeddings $(i_0, i_1) : (M_0, M_1) \to Q$. Moreover since H is finitely generated, $i_0 \circ j_0 = i_1 \circ j_1$. It follows that (i_0, i_1) is an internal ultrapower comparison of (j_0, j_1). This is absolute to H, and therefore H satisfies that (j_0, j_1) has an internal ultrapower comparison, as desired. □

2.3.4 The Ultrapower Axiom and the Mitchell order

In this subsection, we prove the linearity of the Mitchell order from the Ultrapower Axiom. We include this proof largely for the benefit of the reader who would prefer to skip over our discussions of Weak Comparison, since the proof is very similar to that of Theorem 2.3.4. The reader who has followed Theorem 2.3.4 will no doubt notice that both the statement and proof of Theorem 2.3.11 below are much simpler and more elegant than those of Theorem 2.3.4. It is a general pattern that UA is easier to use than Weak Comparison. In fact, almost every known consequence of Weak Comparison is a consequence of UA.

Theorem 2.3.11 (UA)**.** *The Mitchell order is linear.*

Proof. Suppose U_0 and U_1 are normal ultrafilters. We must show that either $U_0 = U_1$, $U_0 \lhd U_1$, or $U_0 \rhd U_1$. We may assume without loss of generality that U_0 and U_1 are normal ultrafilters on the same cardinal κ, since otherwise it is obvious that either $U_0 \lhd U_1$ or $U_0 \rhd U_1$.

Let $j_0 : V \to M_0$ be the ultrapower of the universe by U_0. Let $j_1 : V \to M_1$ be the ultrapower of the universe by U_1. Applying UA, there is an internal ultrapower comparison $(i_0, i_1) : (M_0, M_1) \to P$ of (j_0, j_1).

The proof now breaks into three cases.

Case 1. $i_0(\kappa) = i_1(\kappa)$.

Case 2. $i_0(\kappa) < i_1(\kappa)$.

Case 3. $i_0(\kappa) > i_1(\kappa)$.

In Case 1, we will prove $U_0 = U_1$. In Case 2, we will prove $U_0 \lhd U_1$. In Case 3, we will prove $U_0 \rhd U_1$.

Proof in Case 1. Suppose $A \subseteq \kappa$. Then

$$\begin{aligned} A \in U_0 &\iff \kappa \in j_0(A) \\ &\iff i_0(\kappa) \in i_0(j_0(A)) \\ &\iff i_0(\kappa) \in i_1(j_1(A)) && (2.7) \\ &\iff i_1(\kappa) \in i_1(j_1(A)) && (2.8) \\ &\iff \kappa \in j_1(A) \\ &\iff A \in U_1 \end{aligned}$$

To obtain (2.7), we use the fact that (i_0, i_1) is a comparison, and in particular that $i_1 \circ j_1 = i_0 \circ j_0$. To obtain (2.8), we use the case hypothesis that $i_0(\kappa) = i_1(\kappa)$. It follows that $U_0 = U_1$. $\square$

Proof in Case 2. Suppose $A \subseteq \kappa$. Then

$$\begin{aligned} A \in U_0 &\iff \kappa \in j_0(A) \\ &\iff i_0(\kappa) \in i_0(j_0(A)) \\ &\iff i_0(\kappa) \in i_1(j_1(A)) \\ &\iff i_0(\kappa) \in i_1(j_1(A)) \cap i_1(\kappa) && (2.9) \\ &\iff i_0(\kappa) \in i_1(j_1(A) \cap \kappa) \\ &\iff i_0(\kappa) \in i_1(A) && (2.10) \end{aligned}$$

To obtain (2.9), we use the case hypothesis that $i_0(\kappa) < i_1(\kappa)$. To obtain (2.10), we use that U_1 is κ-complete; therefore $\text{crit}(j_1) = \kappa$ so $j_1(A) \cap \kappa = A$ for any $A \subseteq \kappa$.

It follows that U_0 is the M_1-ultrafilter derived from i_1 using $i_0(\kappa)$. (Here we use that $P(\kappa) \subseteq M_1$.) Since i_1 is an internal ultrapower embedding of M_1, i_1 is definable over M_1, and therefore U_0 is definable over M_1 from i_1 and $i_0(\kappa)$. It follows that $U_0 \in M_1$. Since $M_1 = M_{U_1}$, this means $U_0 \lhd U_1$, as desired. $\square$

Proof in Case 3. The proof in this case is identical to the proof in Case 2 but with U_0 and U_1 swapped. $\square$

Thus no matter which of the cases hold, either $U_0 = U_1$, $U_0 \lhd U_1$, or $U_0 \rhd U_1$. This completes the proof. $\square$

There is a partial converse to Theorem 2.3.11 that helps explain the motivation for the proof of Theorem 2.3.11. To state this converse, we first define a restricted version of the Ultrapower Axiom for ultrapower embeddings coming from normal ultrafilters:

Definition 2.3.12. The *Normal Ultrapower Axiom* is the statement that any pair of ultrapower embeddings of the universe of sets associated to normal ultrafilters have a comparison by internal ultrapowers.

Proposition 2.3.13. *The Normal Ultrapower Axiom is equivalent to the linearity of the Mitchell order.*

Proof. The proof that the Normal Ultrapower Axiom implies the linearity of the Mitchell order is immediate from the proof of Theorem 2.3.11.

Conversely, assume the Mitchell order is linear. Suppose U_0 and U_1 are normal ultrafilters, and let $j_0 : V \to M_0$ and $j_1 : V \to M_1$ be their ultrapowers. We will show (j_0, j_1) has a comparison by internal ultrapowers. Assume without loss of generality that $U_0 \lhd U_1$. Let $i_0 : M_0 \to P_0$ be the ultrapower of M_0 by $j_0(U_1)$. Let $i_1 : M_1 \to P_1$ be the ultrapower of M_1 by U_0. Then i_0 and i_1 are internal ultrapowers of M_0 and M_1 respectively. Moreover[11] $i_0 = j_0(j_1)$ and $i_1 = j_0 \restriction M_1$, so

$$i_0 \circ j_0 = j_0(j_1) \circ j_0 = j_0 \circ j_1 = i_1 \circ j_1$$

It follows that (i_0, i_1) is a comparison of (j_0, j_1) by internal ultrapowers. $\square$

The proof of Proposition 2.3.13 is local in the sense that it shows that the comparability of two normal ultrafilters in the Mitchell order is equivalent to their comparability by internal ultrapowers. This is a special feature of the Mitchell order on normal ultrafilters. For the generalized Mitchell order (defined in Chapter 4), neither direction of this equivalence is provable. Motivated by this issue, we develop in Section 5.5 a variant of the generalized Mitchell order called the *internal relation*.

2.3.5 Technical lemmas related to Weak Comparison

In this section, we prove several lemmas cited in Section 2.3.2.

Lemma 2.3.14. *Suppose N is a finitely generated model of* ZFC *and U is an N-ultrafilter. Then M_U^N is finitely generated.*

Proof. Fix $b \in N$ such that every element of N is definable in N using b as a parameter. Obviously every element of $j_U[N]$ is definable in M_U^N using $j_U(b)$ as a parameter. But $M_U^N = \{j_U(f)(a_U) : f \in N\} = \{g(a_U) : g \in j_U[N]\}$. Therefore every element of M_U^N is definable using $j_U(b)$ and a_U as parameters. $\square$

[11] Suppose M, N, and P are transitive models of ZFC. Suppose $j : M \to N$ and $i : M \to P$ are elementary embeddings. Assume $j \restriction x \in M$ for all $x \in M$. Assume i is a cofinal embedding. Then $i(j) = \bigcup_{X \in M} i(j \restriction X)$.

The next lemma, standard in the case of fully elementary embeddings, is the key to our analysis of Σ_2-hulls:

Lemma 2.3.15. *Suppose $j : N \to M$ is a Σ_2-elementary embedding between transitive models of* ZFC. *Suppose $X \in N$, and $a \in j(X)$. Let U be the N-ultrafilter on X derived from j using a. Then there is a unique Σ_2-elementary embedding $k : M^N_U \to M$ such that $k \circ j^N_U = j$ and $k(a_U) = a$.*

Proof. Suppose $\varphi(v)$ is a Σ_2 formula, which we assume has just one free variable for ease of notation. Suppose f is a function on X in N. The statement "$S = \{x \in X : N \vDash \varphi(f(x))\}$" can be written as a Boolean combination of Σ_2 formulas in the variables S and f. It follows that

$$j(\{x \in X : N \vDash \varphi(f(x))\}) = \{x \in j(X) : M \vDash \varphi(j(f)(x))\}$$

For any function $f \in N$ defined U-almost everywhere, set

$$k([f]_U) = j(f)(a)$$

Fix a Σ_2 formula $\varphi(v)$. The following calculation shows that k is a well-defined Σ_2-elementary embedding from M^N_U to M:

$$\begin{aligned} M^N_U \vDash \varphi([f]_U) &\iff \{x \in X : N \vDash \varphi(f(x))\} \in U \\ &\iff M \vDash a \in j(\{x \in X : N \vDash \varphi(f(x))\}) \\ &\iff M \vDash a \in \{x \in j(X) : M \vDash \varphi(j(f)(x))\} \\ &\iff M \vDash \varphi(j(f)(a)) \\ &\iff M \vDash \varphi(k([f]_U)) \end{aligned}$$

□

Lemma 2.3.15 yields a Σ_2-elementary generalization of what is known as the Realizability Lemma [13, Theorem 10.74]:

Lemma 2.3.16. *Suppose N is a countable Σ_2-hull and $N \vDash U$ is a countably complete ultrafilter. Then M^N_U is a Σ_2-hull.*

Proof. Let $\pi : N \to V$ be a Σ_2-elementary embedding. Let $U' = \pi(U)$, so U' is a countably complete ultrafilter. Since $\pi[U] \subseteq U'$ is countable, fix some $a \in \bigcap \pi[U]$. Then U is the N-ultrafilter derived from π using a, and so by Lemma 2.3.15, there is a Σ_2-elementary embedding $k : M^N_U \to V$. It follows that M^N_U is a Σ_2-hull. □

Lemma 2.3.17. *The set of finitely generated Σ_2-hulls is closed under internal ultrapowers.*

Proof. Immediate from the conjunction of Lemma 2.3.14 and Lemma 2.3.16. □

Lemma 2.3.15 can also be used to prove the following fact, essentially using the extender ultrapower construction:

Lemma 2.3.18. *Suppose N is a set model of* ZFC *and $j : N \to M$ is a Σ_2-elementary embedding. Then j factors as a cofinal elementary embedding followed by a Σ_2-elementary end-extension.* □

Sketch. Let $H \subseteq M$ be the collection of $a \in M$ such that there is some $X \in N$ with $a \in j(X)$. It is easy to see that M is an end-extension of H.

For each $(a, X) \in H \times N$ with $a \in j(X)$, let $U_{a,X}$ be the N-ultrafilter on X derived from j using a; let $j_a : N \to M_a$ be the associated ultrapower embedding; and applying Lemma 2.3.15, let $k_a : M_a \to M$ be the associated Σ_2-elementary factor embedding. If $a \in k_b[M_b]$, then $k_a[M_a] \subseteq k_b[M_b]$ and so $k_{a,b} = k_b^{-1} \circ k_a$ is a cofinal Σ_2-elementary embedding between the models M_a and M_b, which satisfy ZFC. By Gaifman's theorem [2, Proposition 5.1(c)], $k_{a,b}$ is a fully elementary embedding. As a consequence, $k_a[M_a] \prec k_b[M_b]$.

Thus $\{k_a[M_a] : a \in H\}$ is a family of substructures of M, directed under the elementary substructure relation, with union H. By basic model theory, $k_a[M_a]$ is therefore an elementary substructure of H for all $a \in H$. In particular, taking $a = 0$, $k_0[M_0] = j[N] \prec H$. In other words, $j : N \to H$ is an elementary embedding. Moreover, j is cofinal by the definition of H.

On the other hand, since k_a is Σ_2-elementary for all $a \in H$, $\{k_a[M_a] : a \in H\}$ is a directed family of Σ_2-elementary substructures of M. Hence the union of this family, namely H, is a Σ_2-elementary substructure of M. This proves the lemma. □

Proposition 2.3.19. *There is a Σ_2-hull if and only if there is a Σ_2-correct worldly cardinal.*

Proof. Suppose N is a Σ_2-hull. Let $\pi : N \to V$ be a Σ_2-elementary embedding. By Lemma 2.3.15, π factors as a cofinal elementary embedding $\pi : N \to H$ followed by a Σ_2-elementary end-extension $H \prec_{\Sigma_2} V$. Since $H \prec_{\Sigma_2} V$, $H = V_\kappa$ for some cardinal κ. Since $\pi : N \to V_\kappa$ is fully elementary, V_κ satisfies ZFC. Thus κ is a Σ_2-correct worldly cardinal. □

Chapter 3

The Ketonen Order

3.1 Introduction

3.1.1 Ketonen's order

The central result of Chapter 2 is that the Mitchell order is linear in all known canonical inner models. In Section 2.3.3, we delved deeper into the first half of this proof, extracting from it a general inner model principle called the Ultrapower Axiom. A closer look at the second half of the proof also yields more information: it shows that the Ultrapower Axiom implies not only the linearity of the Mitchell order, but also the linearity of a much more general order on countably complete ultrafilters.

This order dates back to the early 1970s. A remarkable theorem of Ketonen [14] from this period states that if every regular cardinal $\lambda \geq \kappa$ carries a κ-complete uniform ultrafilter, then κ is strongly compact. Ketonen gave two proofs of this theorem. The first proceeds by a straightforward but somewhat opaque induction. The second is not as well-known, but is of much greater interest here. Ketonen introduces a wellfounded order on countably complete weakly normal ultrafilters and shows that certain minimal elements in this order witness the strong compactness of κ. (This result is included below as Theorem 7.2.15 since generalizations of the proof form a key component of our analysis of strong compactness and supercompactness under UA.)

Independently of Ketonen's work (and half a century late to the game), the author extracted from the proof of the linearity of the Mitchell order under UA (Theorem 2.3.11) a more general version of Ketonen's order, which we now call the *Ketonen order*. The Ketonen order is a wellfounded partial order on countably complete ultrafilters concentrating on ordinals. The key realization, which distinguishes our work from Ketonen's, is that the Ketonen order can be *linear*. In fact, the linearity of the Ketonen order is an immediate consequence of UA (Theorem 3.3.6). The main result of this chapter (Theorem 3.5.1) states that the linearity of the Ketonen order is equivalent to the Ultrapower Axiom. The Ketonen order will be one of our main tools in the investigation of the structure of countably complete ultrafilters under UA.

3.1.2 Outline of Chapter 3

Section 3.2. We introduce some more preliminary definitions that will be used throughout the rest of this monograph. Especially important are limits of ultrafilters, which we introduce both in the traditional ultrafilter theoretic sense and in a generalized setting in terms of inverse images.

Section 3.3. We introduce the Ketonen order, the main object of study of this chapter and a fundamental tool in the theory of the Ultrapower Axiom. In Section 3.3.1, we define the Ketonen order and give various alternate characterizations. The most important characterization is given by Lemma 3.3.4, which shows that the Ketonen order can be reformulated in terms of comparisons. This immediately leads to the observation that the Ketonen order is linear under the Ultrapower Axiom. In Section 3.3.2, we establish the basic order-theoretic properties of the Ketonen order: it is a preorder on the class of countably complete ultrafilters concentrating on ordinals. Restricted to fine ultrafilters, it is a partial order. Lemma 3.3.15 shows that the Ketonen order is *graded* in the sense that if $\alpha < \beta$, then the fine ultrafilters on α all lie below those on β. In particular, the Ketonen order is setlike. We finally prove the wellfoundedness of the Ketonen order (Theorem 3.3.8). The general proof of the wellfoundedness of the Ketonen order is due to the author. (Ketonen's proof applies only to ultrafilters that extend the closed unbounded filter.)

Section 3.4. We explore the relationship between the Ketonen order and two well-known orders on ultrafilters. Section 3.4.1 concerns the Mitchell order, which is shown to coincide with the Ketonen order on normal ultrafilters. Section 3.4.2 turns to perhaps the best-known order on ultrafilters: the Rudin-Keisler order. We take this opportunity to set down some basic facts about this order, sometimes with proofs. The Ketonen order is not Rudin-Keisler invariant, so it cannot extend the Rudin-Keisler order. To explain these orders' relationship better, we define an auxiliary order called the *revised Rudin-Keisler order* which is contained in the intersection of the Rudin-Keisler order and the Ketonen order. Moreover we introduce the concept of an *incompressible ultrafilter*, an ultrafilter U such that id_U is as small as possible among all ultrafilters Rudin-Keisler equivalent to U (see Lemma 3.4.18). An argument due to Solovay shows that the strict Rudin-Keisler order and the revised Rudin-Keisler order coincide on incompressible ultrafilters. Thus the Ketonen order extends the strict Rudin-Keisler order on countably complete incompressible ultrafilters.
The next two subsections are devoted to combinatorial generalizations of the Ketonen order. One does not need to read them to understand the rest of this monograph. In Section 3.4.3, we introduce a generalized version of the Lipschitz order, and show that this order extends the Ketonen order on countably complete ultrafilters. Therefore under UA, the two orders coincide, which gives a strange analog of the linearity of the Lipschitz order in determinacy theory. Section 3.4.4 introduces a combinatorial generalization of the Ketonen order to filters, which demonstrates a relationship between the Ketonen order and the

canonical order on stationary sets due to Jech [15].

Section 3.5. This section contains Theorem 3.5.1, the most substantive result of the chapter: the linearity of the Ketonen order is equivalent to the Ultrapower Axiom. The fact that UA implies the linearity of the Ketonen order is immediate. (The proof appears in Section 3.3.1.) The converse, however, is subtle. Since we will mostly work under the assumption of UA, this equivalence is itself not that important (although it does show that all of our results can be proved from an a priori weaker premise). More important is the proof, which identifies a canonical way to compare a pair of ultrafilters assuming the linearity of the Ketonen order.

3.2 Preliminary definitions

3.2.1 Fine ultrafilters on ordinals

Suppose F is a filter on X and C is a class. We say F *concentrates on* C if $C \cap X \in F$. A class C is said to be *F-positive* if F does not concentrate on the complement of C. The set of F-positive subsets of X is denoted by F^+.

The following construction allows us to change the underlying set of a filter:

Definition 3.2.1. If C is an F-positive set, the *projection of F to C* is the filter $F \mid C$ consisting of all sets $A \subseteq C$ such that for F-almost all x, if $x \in C$ then $x \in A$.

Clearly if $C \in F$, then $F \mid C = F \cap P(C)$, but sometimes we will want to consider $F \mid C$ when $C \in F^+ \setminus F$ or when C is not even contained in the underlying set of F (in which case the meaning of the word "projection" is perhaps slightly strained).

Every ultrafilter U is attached to an underlying set X, but the choice of this set is frequently irrelevant. To help ignore this extraneous detail, we introduce the *change of space relation*:

Definition 3.2.2. The *change-of-space relation* is defined on filters F and G by setting $F =_{\Bbbk} G$ if there is a set $X \in F \cap G$ such that $F \mid X = G \mid X$.

The change-of-space relation is obviously an equivalence relation.

Lemma 3.2.3. *Suppose U and W are ultrafilters. Then the following are equivalent:*

(1) $U =_{\Bbbk} W$.

(2) Let Y be the underlying set of W. Then $U \mid Y = W$.

(3) For all sets A, $\text{id}_U \in j_U(A)$ if and only if $\text{id}_W \in j_W(A)$.

(4) There is a comparison (k, h) of (j_U, j_W) such that $k(\text{id}_U) = h(\text{id}_W)$. □

While we will define the Ketonen order on arbitrary countably complete ultrafilters on ordinals, the (nonstrict) Ketonen order is only a preorder on this class. (In fact, Lemma 3.3.16 states that if U and W are ultrafilters on ordinals, then $U \leq_{\Bbbk} W$ and $W \leq_{\Bbbk} U$ if and only if $U =_{\Bbbk} W$, which should explain the notation we have chosen.) It is therefore sometimes notationally convenient to restrict further to fine ultrafilters:

Definition 3.2.4. A filter F on an ordinal δ is *fine* if it contains $\delta \setminus \alpha$ for every $\alpha < \delta$.

For example, the principal ultrafilter on $\alpha + 1$ concentrated at α is fine.

For any ordinal δ, the *tail filter* on δ is the filter generated by sets of the form $\delta \setminus \alpha$ for $\alpha < \delta$. A filter on an ordinal is fine if and only if it extends the tail filter. Equivalently, F is fine if every F-positive set is cofinal in α. The concept of a fine filter on an ordinal is a special case of the more general concept of a fine filter introduced in Definition 4.4.7.

Definition 3.2.5. If F is a filter that concentrates on ordinals, then δ_F denotes the least ordinal δ on which F concentrates.

Lemma 3.2.6. *If F is an ultrafilter on an ordinal, then F is fine if and only if δ_F is the underlying set of F.* ☐

The key property of fine ultrafilters, which is quite obvious, is that they yield canonical representatives of $=_{\Bbbk}$ equivalence classes of ultrafilters concentrating on ordinals.

Lemma 3.2.7. *For any filter F on an ordinal, $F \mid \delta_F$ is the unique fine filter G such that $F =_{\Bbbk} G$. In particular, if F and G are fine ultrafilters on ordinals such that $F =_{\Bbbk} G$, then $F = G$.* ☐

There is an obvious but useful characterization of δ_U in terms of elementary embeddings:

Lemma 3.2.8. *If U is an ultrafilter on an ordinal, then δ_U is the least ordinal δ such that M_U satisfies $\mathrm{id}_U < j_U(\delta)$.* ☐

Definition 3.2.9. For any ordinal δ, let $\mathbf{Fine}(\leq\delta)$, $\mathbf{Fine}(<\delta)$, and $\mathbf{Fine}(\delta)$ denote the sets of countably complete fine ultrafilters U such that $\delta_U \leq \delta$, $\delta_U < \delta$, and $\delta_U = \delta$ respectively. Let $\mathbf{Fine} = \bigcup_{\delta \in \mathrm{Ord}} \mathbf{Fine}(\delta)$.

Fineness and uniformity (Definition 2.2.26) are not the same concept, and moreover neither is a strengthening of the other. The simplest way to separate these concepts is by considering the Fréchet and tail filters themselves. For any set X, let F_X denote the Fréchet filter on X. For any ordinal α, let T_α denote the tail filter on α.

Lemma 3.2.10. *Suppose λ is an ordinal.*

- *$T_\lambda \subseteq F_\lambda$ if and only if λ is a cardinal.*

- *$F_\lambda \subseteq T_\lambda$ if and only if $|\lambda| = \mathrm{cf}(\lambda)$.*

Thus $T_\lambda = F_\lambda$ if and only if λ is a regular cardinal. If λ is a singular cardinal, T_λ is fine but not uniform. If λ is not a cardinal, then F_λ is uniform but not fine. □

One can easily obtain ultrafilters that are counterexamples to the equivalence of fineness and true uniformity by extending T_λ and F_λ to ultrafilters.

3.2.2 Limits of ultrafilters

The following definition comes from classical ultrafilter theory:

Definition 3.2.11. Suppose W is an ultrafilter, I is a set in W, and $\langle U_i : i \in I\rangle$ is a sequence of ultrafilters on a set X. The *W-limit of $\langle U_i : i \in I\rangle$* is the ultrafilter

$$W\text{-}\lim_{i\in I} U_i = \{A \subseteq X : \{i \in I : A \in U_i\} \in W\}$$

It is often easier to think about limits in terms of elementary embeddings:

Lemma 3.2.12. *Suppose W is an ultrafilter, I is a set in W, and $\langle U_i : i \in I\rangle$ is a sequence of ultrafilters on a fixed set X. Then*

$$W\text{-}\lim_{i\in I} U_i = j_W^{-1}[Z]$$

where $Z = [\langle U_i : i \in I\rangle]_W$.

Proof. Suppose $A \subseteq X$. Then

$$\begin{aligned} A \in W\text{-}\lim_{i\in I} U_i &\iff A \in U_i \text{ for } W\text{-almost all } i \in I \\ &\iff j_W(A) \in [\langle U_i : i \in I\rangle]_W \\ &\iff A \in j_W^{-1}[Z] \end{aligned}$$

where the middle equivalence follows from Łoś's Theorem. □

Limits generalize the usual derived ultrafilter and pushforward constructions:

Definition 3.2.13. Suppose X is a set and $a \in X$. The *principal ultrafilter on X concentrated at a* is the ultrafilter $\mathrm{p}_a^X = \{A \subseteq X : a \in A\}$.

Definition 3.2.14. Suppose W is an ultrafilter, I is a set in W, and $f : I \to X$ is a function. Then the *pushforward of W by f* is the ultrafilter $f_*(W) = \{A \subseteq X : f^{-1}[A] \in W\}$.

The following lemmas relate the derived ultrafilter construction to inverse images, limits, and pushforwards.

Lemma 3.2.15. *Suppose N and P are transitive models of* ZFC*, X is a set in N, $i : N \to P$ is an elementary embedding, and $a \in i(X)$. Then the N-ultrafilter on X derived from i using a is $i^{-1}[\mathrm{p}_a^{i(X)}]$.* □

Lemma 3.2.16. *Suppose W is an ultrafilter, I is a set in W, and $f : I \to X$ is a function. Then*

$$f_*(W) = W\text{-}\lim_{i\in I} \mathrm{p}^X_{f(i)} = j_W^{-1}[\mathrm{p}^{j_W(X)}_{[f]_W}]$$

In other words, $f_*(W)$ is the ultrafilter on X derived from j_W using $[f]_W$. □

These lemmas are trivial, but it turns out that many calculations are significantly simpler when one treats limits and derived ultrafilters uniformly as inverse images.

A pedantic reader might point out that for example in Lemma 3.2.16, $\mathrm{p}^{j_W(X)}_{[f]_W}$ is not an M_W-ultrafilter but a V-ultrafilter. Moreover if M_W is not wellfounded, the statement $[f]_W \in j_W(X)$ is meaningless, so technically $\mathrm{p}^{j_W(X)}_{[f]_W}$ is not well-defined. Of course, $\mathrm{p}^{j_W(X)}_{[f]_W}$ really denotes $(\mathrm{p}^{j_W(X)}_{[f]_W})^{M_W}$. For the benefit of all involved, we will try to omit all these superscripts in our notation for principal ultrafilters when they can be guessed from context. For example, in Lemma 3.2.16, we would usually write:

$$f_*(W) = W\text{-}\lim_{i\in I} \mathrm{p}_{f(i)} = j_W^{-1}[\mathrm{p}_{[f]_W}]$$

The key to understanding derived ultrafilters is to consider the natural factor embeddings associated to them. There is a generalization of the factor embedding construction to the case of limits. In fact, this works somewhat more generally for arbitrary inverse images of ultrafilters:

Lemma 3.2.17. *Suppose N and P are transitive models of* ZFC*, X is a set in N, $i : N \to P$ is an elementary embedding, and U_* is a P-ultrafilter on $i(X)$. Let $U = i^{-1}[U_*]$. There is a unique elementary embedding $i_* : M^N_U \to M^P_{U_*}$ such that $i_*(\mathrm{id}_U) = \mathrm{id}_{U_*}$ and $i_* \circ j^N_U = j^P_{U_*} \circ i$.*

Proof. For any function $f \in N$ defined on a set in U, set

$$i_*([f]^N_U) = [i(f)]^P_{U_*}$$

See Fig. 3.1 It is immediate from this definition that $i_*(\mathrm{id}_U) = \mathrm{id}_{U_*}$ and $i_* \circ j^N_U = j^P_{U_*} \circ i$. We must show that i_* is well-defined and elementary. This follows from the usual calculation:

$$\begin{aligned} M^N_U \vDash \varphi([f]^N_U) &\iff N \vDash \varphi(f(x)) \text{ for } U\text{-almost all } x \\ &\iff P \vDash \varphi(i(f)(x)) \text{ for } U_*\text{-almost all } x \\ &\iff M^P_{U_*} \vDash \varphi([i(f)]^P_{U_*}) \end{aligned}$$

□

3.3 The Ketonen order

3.3.1 Characterizations of the Ketonen order

Let us begin our investigation of the Ketonen order with a purely combinatorial definition.

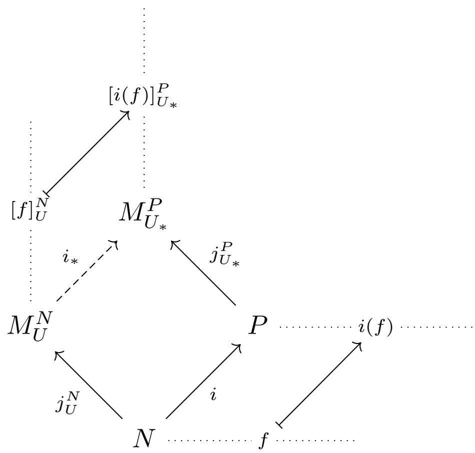

Figure 3.1: The factor embedding associated to a limit

Definition 3.3.1. Suppose X is a set and A is a class. Then $\mathbf{UF}(X)$ denotes the set of countably complete ultrafilters on X, and $\mathbf{UF}(X, A)$ denotes the set of countably complete ultrafilters on X that concentrate on A. Finally $\mathbf{UF}$ is the class of all countably complete ultrafilters.

Definition 3.3.2. Suppose δ is an ordinal. The *Ketonen order* is defined on $\mathbf{UF}(\delta)$ as follows. For $U, W \in \mathbf{UF}(\delta)$, $U <_{\Bbbk} W$ (resp. $U \leq_{\Bbbk} W$) if for some $I \in W$ and $\langle U_\alpha : \alpha \in I\rangle \in \prod_{\alpha\in I} \mathbf{UF}(\delta, \alpha)$ (resp. $\prod_{\alpha\in I} \mathbf{UF}(\delta, \alpha{+}1)$), $U = W\text{-}\lim_{i\in I} U_\alpha$.

We refer to $<_{\Bbbk}$ and $\leq_{\Bbbk}$ as the *strict* and *non-strict* Ketonen orders.

In the context of Definition 3.3.2, one can arrange by padding that $I = \delta\setminus\{0\}$ (resp. $I = \delta$), but the combinatorics are typically clearer if one does not make this demand.

There is perhaps a potential ambiguity in our notation, since the order $<_{\Bbbk}$ depends on the ordinal δ, which we suppress in our notation. This dependence is always immaterial, however, since there are canonical embeddings between the various Ketonen orders. These embeddings will later allow us to spin all these orders together into one (Definition 3.3.13).

Let us first explain the straightforward relationship between the strict and nonstrict Ketonen orders.

Proposition 3.3.3. *Suppose δ is an ordinal and $U, W \in \mathbf{UF}(\delta)$. Then $U \leq_{\Bbbk} W$ if and only if $U <_{\Bbbk} W$ or $U = W$.*

Proof. Let $\langle U_\alpha : \alpha \in I\rangle \in \prod_{\alpha\in I} \mathbf{UF}(\delta, \alpha + 1)$ witness $U \leq_{\Bbbk} W$. Let

$$J = \{i \in I : U_\alpha \in \mathbf{UF}(\delta, \alpha)\}$$

If $J \in W$, then $\langle U_\alpha : \alpha \in J\rangle \in \prod_{\alpha\in J} \mathbf{UF}(\delta, \alpha)$ witnesses $U <_{\Bbbk} W$.

Assume therefore that $J \notin W$. For all $\alpha \in I \setminus J$, $U_\alpha \in \mathbf{UF}(\delta, \alpha+1) \setminus \mathbf{UF}(\delta, \alpha)$. Note that $\mathbf{UF}(\delta, \alpha+1) \setminus \mathbf{UF}(\delta, \alpha)$ contains only the principal ultrafilter p^δ_α, and hence $U_\alpha = \mathrm{p}^\delta_\alpha$ for $\alpha \in I \setminus J$. Thus

$$U = W\text{-}\lim_{\alpha \in I} U_\alpha = W\text{-}\lim_{\alpha \in I \setminus J} \mathrm{p}_\alpha = W$$

where the final equality follows easily from the definitions (or Lemma 3.2.16). □

We therefore focus our attention on the strict Ketonen order $<_\Bbbk$ for now. Before establishing its basic order-theoretic properties, let us give some fairly obvious alternate characterizations of it. We think the characterization Lemma 3.3.4 (2) is quite elegant in that it demonstrates a basic relationship between the Ketonen order, the covering properties of ultrapowers, and extensions of filter bases to countably complete ultrafilters, foreshadowing the powerful interactions between strong compactness and the Ultrapower Axiom that we will see in Chapter 7 and Chapter 8. Lemma 3.3.4 (3) and (4) are more useful, though, linking the Ketonen order and the Ultrapower Axiom through the concept of a comparison (Definition 2.3.7).

Lemma 3.3.4. *Suppose δ is an ordinal and $U, W \in \mathbf{UF}(\delta)$. The following are equivalent:*

(1) $U <_\Bbbk W$.

(2) $j_W[U]$ *extends to an* M_W*-ultrafilter* $Z \in \mathbf{UF}^{M_W}(j_W(\delta), \mathrm{id}_W)$.

(3) There is a comparison $(k, h) : (M_U, M_W) \to P$ *of* (j_U, j_W) *such that* h *is an internal ultrapower embedding of* M_W *and* $k(\mathrm{id}_U) < h(\mathrm{id}_W)$.

(4) There is a comparison $(k, h) : (M_U, M_W) \to P$ *of* (j_U, j_W) *such that* h *is close to* M_W *and* $k(\mathrm{id}_U) < h(\mathrm{id}_W)$.

Proof. (1) implies (2): Fix $I \in W$ and $\langle U_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathbf{UF}(\delta, \alpha)$ witnessing $U <_\Bbbk W$. Let $Z = [\langle U_\alpha : \alpha \in I \rangle]_W$. By Łoś's Theorem, $Z \in \mathbf{UF}^{M_W}(j_W(\delta), \mathrm{id}_W)$, and by Lemma 3.2.12, $j_W^{-1}[Z] = W\text{-}\lim_{i \in I} U_i = U$. This implies $j_W[U] \subseteq Z$.

(2) implies (1): Similar.

(2) implies (3): Fix $Z \in \mathbf{UF}^{M_W}(j_W(\delta), \mathrm{id}_W)$ such that $j_W[U] \subseteq Z$. Because of the basic structure of ultrafilters, the fact that $j_W[U] \subseteq Z$ implies that $j_W^{-1}[Z] = U$. Let $h : M_W \to N$ be the ultrapower of M_W by Z. Since Z concentrates on id_W, $\mathrm{id}_Z < h(\mathrm{id}_W)$. By Lemma 3.2.17, there is a unique elementary embedding $k : M_U \to N$ such that $k(\mathrm{id}_U) = \mathrm{id}_Z$ and $k \circ j_U = h \circ j_W$. The former equation implies $k(\mathrm{id}_U) < h(\mathrm{id}_W)$, while the latter equation says that (k, h) is a comparison of (j_U, j_W). Therefore (3) holds.

(3) implies (4): Internal ultrapower embeddings are close.

(4) implies (2): Let Z be the M_W-ultrafilter on $j_W(\delta)$ derived from h using $k(\mathrm{id}_U)$. Thus $Z = h^{-1}[\mathrm{p}_{k(\mathrm{id}_U)}]$. (Here $\mathrm{p}_{k(\mathrm{id}_U)}$ denotes the principal ultrafilter

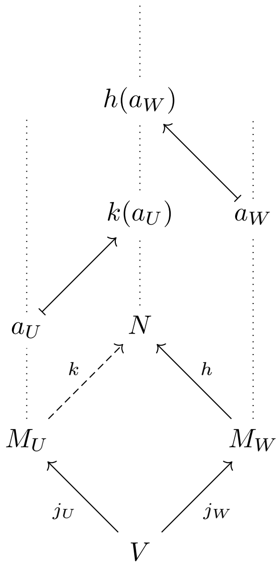

Figure 3.2: A comparison witnessing $U <_{\Bbbk} W$

on $k(j_U(\delta))$ concentrated at $k(\mathrm{id}_U)$; see Definition 3.2.13 and the ensuing discussion.) Since h is close, Z belongs to M_W, and since $k(\mathrm{id}_U) < h(\mathrm{id}_W)$, Z concentrates on id_W. Thus $Z \in \mathbf{UF}^{M_W}(j_W(\delta), \mathrm{id}_W)$. Moreover,

$$j_W^{-1}[Z] = j_W^{-1}[h^{-1}[\mathrm{p}_{k(\mathrm{id}_U)}]] = j_U^{-1}[k^{-1}[\mathrm{p}_{k(\mathrm{id}_U)}]] = j_U^{-1}[\mathrm{p}_{\mathrm{id}_U}] = U$$

In particular, $j_W[U] \subseteq Z$, which shows (2). □

Of course, there are identical characterizations for the nonstrict Ketonen order as well:

Lemma 3.3.5. *Suppose δ is an ordinal and $U, W \in \mathbf{UF}(\delta)$. The following are equivalent:*

(1) $U \leq_{\Bbbk} W$.

(2) $j_W[U]$ extends to an M_W-ultrafilter $Z \in \mathbf{UF}^{M_W}(j_W(\delta), \mathrm{id}_W + 1)$.

(3) There is a comparison $(k, h) : (M_U, M_W) \to P$ of (j_U, j_W) such that h is an internal ultrapower embedding of M_W and $k(\mathrm{id}_U) \leq h(\mathrm{id}_W)$.

(4) There is a comparison $(k, h) : (M_U, M_W) \to P$ of (j_U, j_W) such that h is close to M_W and $k(\mathrm{id}_U) \leq h(\mathrm{id}_W)$. □

Lemma 3.3.4 and Lemma 3.3.5 lead to the central linearity theorem for the Ketonen order under UA:

Theorem 3.3.6 (UA). *Suppose δ is an ordinal and $U, W \in \mathbf{UF}(\delta)$. Either $U <_{\Bbbk} W$ or $W \leq_{\Bbbk} U$.*

Proof. Let $(k, h) : (M_U, M_W) \to N$ be an internal ultrapower comparison of (j_U, j_W). If $k(\mathrm{id}_U) < h(\mathrm{id}_W)$, then Lemma 3.3.4 (3) implies $U <_{\Bbbk} W$. Otherwise, $h(\mathrm{id}_W) \leq k(\mathrm{id}_U)$ and so $W \leq_{\Bbbk} U$ by Lemma 3.3.5. □

This linearity theorem is only interesting, of course, if we know that the Ketonen order is "well defined" in the sense that both $U <_{\Bbbk} W$ and $W <_{\Bbbk} U$ cannot occur. We now show that in fact the Ketonen order is a wellfounded partial order.

3.3.2 Basic properties of the Ketonen order

We state the main theorem of this section, which we will prove in pieces:

Theorem. *For any ordinal δ, $(\mathbf{UF}(\delta), <_{\Bbbk})$ is a strict wellfounded partial order.*

Thus we must show the following facts:

Proposition 3.3.7. *For any ordinal δ, $<_{\Bbbk}$ is a transitive relation on $\mathbf{UF}(\delta)$.*

Theorem 3.3.8. *For any ordinal δ, $<_{\Bbbk}$ is a wellfounded relation on $\mathbf{UF}(\delta)$.*

Let us warm up to this by proving irreflexivity:

Proposition 3.3.9. *For any ordinal δ, $<_{\Bbbk}$ is an irreflexive relation on $\mathbf{UF}(\delta)$.*

Proof. Suppose towards a contradiction that $U \in \mathbf{UF}(\delta)$ satisfies $U <_{\Bbbk} U$. Fix $I \in U$, and $\langle U_\alpha : \alpha \in I\rangle \in \prod_{\alpha \in I} \mathbf{UF}(\delta, \alpha)$ such that

$$U = U\text{-}\lim_{\alpha \in I} U_\alpha$$

Define $A \subseteq \delta$ by induction: put $\alpha \in A$ if and only if $A \cap \alpha \notin U_\alpha$. Then

$$\begin{aligned} A \in U &\iff \{\alpha \in I : A \in U_\alpha\} \in U \\ &\iff \{\alpha \in I : A \cap \alpha \in U_\alpha\} \in U \\ &\iff \{\alpha \in I : \alpha \notin A\} \in U \\ &\iff I \setminus A \in U \end{aligned}$$

Since $I \in U$ and U is an ultrafilter, either A or $I \setminus A$ must belong to U. Thus both belong to U, contradicting that U is closed under intersections. □

Notice that the proof does not use the countable completeness of U. We now give two proofs of the transitivity of the Ketonen order.

Proof of Proposition 3.3.7. Suppose $U <_{\Bbbk} W \leq_{\Bbbk} Z$. We will show that $U <_{\Bbbk} Z$. Fix the following objects:

- A set $I \in W$ and a sequence $\langle U_\alpha : \alpha \in I\rangle \in \prod_{\alpha\in I} \mathbf{UF}(\delta,\alpha)$ such that $U = W\text{-}\lim_{\alpha\in I} U_\alpha$.
- A set $J \in Z$ and a sequence $\langle W_\beta : \beta \in J\rangle \in \prod_{\beta\in J} \mathbf{UF}(\delta,\beta)$ such that $W = Z\text{-}\lim_{\beta\in J} Z_\beta$.

Since $I \in W = Z\text{-}\lim_{\beta\in J} W_\alpha$, the set $J' = \{\beta \in J : I \in W_\beta\}$ belongs to Z. For $\beta \in J'$, we can define $U'_\beta = W_\beta\text{-}\lim_{\alpha\in I} U_\alpha$. Thus:

$$\begin{aligned} U &= W\text{-}\lim_{\alpha\in I} U_\alpha \\ &= (Z\text{-}\lim_{\beta\in J} W_\alpha)\text{-}\lim_{\alpha\in I} U_\alpha \\ &= Z\text{-}\lim_{\beta\in J'} (W_\alpha\text{-}\lim_{\alpha\in I} U_\alpha) \\ &= Z\text{-}\lim_{\beta\in J'} U'_\alpha \end{aligned}$$

Finally, if $\beta \in J'$, then $\{\alpha \in I : U_\alpha \in \mathbf{UF}(\delta,\beta)\} \supseteq I \cap (\beta+1) \in W_\beta$, so

$$\langle U'_\beta : \beta \in J'\rangle \in \prod_{\beta\in J'} \mathbf{UF}(\delta,\beta)$$

Therefore $\langle U'_\beta : \beta \in J'\rangle$ witnesses $U <_{\Bbbk} Z$. $\square$

Our second proof of the transitivity of the Ketonen order is more diagrammatic:

Alternate Proof of Proposition 3.3.7. Using Lemma 3.3.4, fix the following objects:

- A comparison $(k_0, h_0) : (M_U, M_W) \to N_0$ of (j_U, j_W) such that h_0 is an internal ultrapower embedding of M_W and $k_0(\text{id}_U) < h_0(\text{id}_W)$.
- A comparison $(k_1, h_1) : (M_W, M_Z) \to N_1$ of (j_W, j_Z) such that h_1 is an internal ultrapower embedding of M_Z and $k_1(\text{id}_W) \leq h_1(\text{id}_Z)$.

The rest of the proof is contained in Fig. 3.3. Consider the embeddings $h_0 : M_W \to N_0$ and $k_1 : M_W \to N_1$. There is a very general construction that yields a comparison of (h_0, k_1). Since h_0 is amenable to M_W, one can define $k_1(h_0) : N_1 \to k_1(N_0)$ by shifting the fragments of h_0 using k_1. The identity $(k_1 \restriction N_0) \circ h_0 = k_1(h_0) \circ k_1$ implies that $(k_1 \restriction N_0, k_1(h_0)) : (N_0, N_1) \to k_1(N_0)$ is a comparison of (h_0, k_1).

It follows easily that $((k_1 \restriction N_0) \circ k_0, k_1(h_0) \circ h_1)$ is a comparison of (j_U, j_Z). Easily $k_1(h_0) \circ h_0$ is an internal ultrapower embedding of M_Z. Finally

$$(k_1 \restriction N_0) \circ k_0(\text{id}_U) < (k_1 \restriction N_0) \circ h_0(\text{id}_W) = k_1(h_0) \circ k_1(\text{id}_W) \leq k_1(h_0) \circ h_1(\text{id}_Z)$$

Thus $U <_{\Bbbk} Z$ by Lemma 3.3.4. $\square$

$$\begin{tikzcd}
 & & P & & \\
 & N_0 \arrow[ur, "k_1 \restriction N_0"] & & N_1 \arrow[ul, "k_1(h_0)"'] & \\
M_U \arrow[ur, "k_0"] & & M_W \arrow[ul, "h_0"'] \arrow[ur, "k_1"] & & M_Z \arrow[ul, "h_1"'] \\
 & & V \arrow[ull, "j_U"] \arrow[u, "j_W"] \arrow[urr, "j_Z"'] & &
\end{tikzcd}$$

Figure 3.3: The transitivity of the Ketonen order

We finally turn to wellfoundedness. The proof proceeds by iterating the following *strong transitivity lemma* for the Ketonen order, abstracted from the proof of Proposition 3.3.7:

Lemma 3.3.10. *Suppose δ is an ordinal, $U, W \in \mathbf{UF}(\delta)$, and $U <_{\Bbbk} W$. Suppose Z is an ultrafilter, J is a set in Z, $\langle W_x : x \in J\rangle$ is a sequence of countably complete ultrafilters on δ, and*

$$W = Z\text{-}\lim_{x \in J} W_x$$

Then there is a set $J' \subseteq J$ in Z and a sequence $\langle U_x : x \in J'\rangle$ of countably complete ultrafilters on δ with $U_x <_{\Bbbk} W_x$ for all $x \in J'$ such that

$$U = Z\text{-}\lim_{x \in J'} U_x$$

Sketch. Fix $I \in W$ and $\langle U'_\alpha : \alpha \in I\rangle \in \prod_{\alpha \in I} \mathbf{UF}(\delta, \alpha)$ such that $U = W\text{-}\lim_{\alpha \in I} U'_\alpha$. Let $J' = \{x \in J : I \in W_x\}$. For $x \in J'$, let $U_x = W_x\text{-}\lim_{\alpha \in I} U'_\alpha$. Then $\langle U'_\alpha : \alpha \in I\rangle$ witnesses that $U_x <_{\Bbbk} W_x$. Moreover the calculation in Proposition 3.3.7 shows that $U = Z\text{-}\lim_{x \in J'} U_x$. □

This is more elegantly stated using elementary embeddings:

Lemma 3.3.11. *Suppose δ is an ordinal, $U, W \in \mathbf{UF}(\delta)$, and $U <_{\Bbbk} W$. Suppose $j : V \to M$ is an elementary embedding and $W_* \in j(\mathbf{UF}(\delta))$ extends $j[W]$. Then there is some $U_* \in j(\mathbf{UF}(\delta))$ extending $j[U]$ such that $M \vDash U_* <_{\Bbbk} W_*$.* □

Recall now the notation $U \mid C$ from Definition 3.2.1, denoting the projection of an ultrafilter U to a set C on which it concentrates. We will need the following trivial lemma, which is also implicit in the proof of Proposition 3.3.7:

Lemma 3.3.12. *Suppose ϵ and δ are ordinals, $U \in \mathbf{UF}(\delta)$, and $W \in \mathbf{UF}(\delta, \epsilon)$. If $U \leq_{\Bbbk} W$, then $U \in \mathbf{UF}(\delta, \epsilon)$ and $U \mid \epsilon \leq_{\Bbbk} W \mid \epsilon$ in the Ketonen order on $\mathbf{UF}(\epsilon)$.*

Proof. Fix $I \in W$ and $\langle U_\alpha : \alpha \in I\rangle \in \prod \mathbf{UF}(\delta, \alpha)$ such that $U = W\text{-}\lim_{\alpha\in I} U_\alpha$. Then $U = W\text{-}\lim_{\alpha\in I} U_\alpha = W\text{-}\lim_{\alpha\in I\cap\epsilon} U_\alpha$ is a limit of ultrafilters concentrating on ϵ, so U itself concentrates on ϵ. Moreover $\langle U_\alpha \mid \epsilon : \alpha \in I \cap \epsilon\rangle$ witnesses that $U \mid \epsilon <_{\Bbbk} W \mid \epsilon$ in the Ketonen order on $\mathbf{UF}(\epsilon)$. □

As we prove Theorem 3.3.8, the reader may profit from the observation that the proof consists of the combinatorial core of the proof of the wellfoundedness of the Mitchell order on normal ultrafilters, stripped of all applications of normality and Łoś's Theorem.

Proof of Theorem 3.3.8. Assume towards a contradiction that there is an ordinal δ such that $<_{\Bbbk}$ is illfounded on $\mathbf{UF}(\delta)$. Fix the least such δ. Choose countably complete ultrafilters $\langle U_n : n < \omega\rangle$ on δ such that

$$U_0 >_{\Bbbk} U_1 >_{\Bbbk} U_2 >_{\Bbbk} \cdots$$

For each positive number n, we will define by recursion a set $J_n \in U_0$ and a sequence of ultrafilters $\langle U^n_\alpha : \alpha \in J_n\rangle \in \prod_{\alpha\in J_n} \mathbf{UF}(\delta, \alpha)$ such that for all $n < \omega$, the following hold:

- $U_n = U\text{-}\lim_{\alpha\in J_n} U^n_\alpha$.
- If $n > 1$, then $J_n \subseteq J_{n-1}$ and for all $\alpha \in J_n$, $U^n_\alpha <_{\Bbbk} U^{n-1}_\alpha$.

To start, fix $J_1 \in U_0$ and $\langle U^1_\alpha : \alpha \in J_1\rangle \in \prod_{\alpha\in J_1} \mathbf{UF}(\delta, \alpha)$ witnessing that $U_1 <_{\Bbbk} U_0$; that is,

$$U_1 = U_0\text{-}\lim_{\alpha\in J_1} U^1_\alpha$$

Suppose $n > 1$ and $J_{n-1} \in U_0$ and $\langle U^{n-1}_\alpha : \alpha \in J_{n-1}\rangle \in \prod_{\alpha\in J_{n-1}} \mathbf{UF}(\delta, \alpha)$ have been defined so that $U_{n-1} = U\text{-}\lim_{\alpha\in J_{n-1}} U^{n-1}_\alpha$. Lemma 3.3.10 (with $U = U_n$, $W = U_{n-1}$, and $Z = U_0$) yields a set $J_n \subseteq J_{n-1}$ and a sequence $\langle U^n_\alpha : \alpha \in J_n\rangle$ of countably complete ultrafilters on δ such that the two bullet points above are satisfied. We must verify that $\langle U^n_\alpha : \alpha \in J_n\rangle \in \prod_{\alpha\in J_n} \mathbf{UF}(\delta, \alpha)$. But for any $\alpha \in J^n$, $U^n_\alpha <_{\Bbbk} U^{n-1}_\alpha \in \mathbf{UF}(\delta, \alpha)$, and therefore $U^n_\alpha \in \mathbf{UF}(\delta, \alpha)$ by Lemma 3.3.12, as desired. This completes the recursive definition.

Now let $J = \bigcap_{n<\omega} J_n$. For any $\alpha \in J$, we have

$$U^1_\alpha >_{\Bbbk} U^2_\alpha >_{\Bbbk} U^3_\alpha >_{\Bbbk} \cdots$$

by the second bullet point above. Since $U^n_\alpha \in \mathbf{UF}(\delta, \alpha)$ for all $n < \omega$, Lemma 3.3.12 implies

$$U^1_\alpha \mid \alpha >_{\Bbbk} U^2_\alpha \mid \alpha >_{\Bbbk} U^3_\alpha \mid \alpha >_{\Bbbk} \cdots$$

Thus the restriction of $<_{\Bbbk}$ to $\mathbf{UF}(\alpha)$ is illfounded. This contradicts the minimality of δ. $\square$

Observe that the proof of Theorem 3.3.8 can be carried out in the theory ZF + DC. The structure of countably complete ultrafilters on ordinals is of great interest in the context of the Axiom of Determinacy, and so the existence of a combinatorial analog of the Mitchell order in that context raises a number of very interesting structural questions that we will not pursue seriously in this monograph.

3.3.3 The global Ketonen order

Lemma 3.3.12 above suggests extending the Ketonen order to an order on ultrafilters that is agnostic about the underlying sets of the ultrafilters involved:

Definition 3.3.13. Suppose U and W are countably complete ultrafilters on ordinals. The (global) *Ketonen order* is defined as follows:

- $U <_{\Bbbk} W$ if $U \mid \delta <_{\Bbbk} W \mid \delta$.
- $U \leq_{\Bbbk} W$ if $U \mid \delta \leq_{\Bbbk} W \mid \delta$.

where δ is any ordinal such that U and W both concentrate on δ.

By Lemma 3.3.12, this definition does not conflict with our original definition of the Ketonen order on $\mathbf{UF}(\delta)$. In fact, various characterizations of the Ketonen order from Lemma 3.3.4 translate smoothly to this context:

Lemma 3.3.14. *Suppose ϵ and δ are ordinals, $U \in \mathbf{UF}(\epsilon)$, and $W \in \mathbf{UF}(\delta)$. Then the following are equivalent:*

(1) $U <_{\Bbbk} W$.

(2) There exist $I \in W$ and $\langle U_\alpha : \alpha \in I\rangle \in \prod_{\alpha\in I} \mathbf{UF}(\epsilon, \alpha)$ such that $U = W\text{-}\lim_{\alpha\in I} U_\alpha$.

(3) $j_W[U]$ extends to an element of $\mathbf{UF}^{M_W}(j_W(\epsilon), \mathrm{id}_W)$.

(4) There is a comparison $(k, h) : (M_U, M_W) \to P$ of (j_U, j_W) such that h is an internal ultrapower embedding of M_W and $k(\mathrm{id}_U) < h(\mathrm{id}_W)$.

(5) There is a comparison $(k, h) : (M_U, M_W) \to P$ of (j_U, j_W) such that h is close to M_W and $k(\mathrm{id}_U) < h(\mathrm{id}_W)$. $\square$

We have the following simple relationship between the space of an ultrafilter and its position in the Ketonen order:

Lemma 3.3.15. *Suppose U and W are countably complete ultrafilters on ordinals.*

- *If $\delta_U < \delta_W$, then $U <_{\Bbbk} W$.*
- *If $U \leq_{\Bbbk} W$, then $\delta_U \leq \delta_W$.*

Proof. To see the first bullet point, note that for any $\alpha \in [\delta_U, \delta_W)$, $\alpha \geq \delta_U$ and hence U concentrates on α. Thus the constant sequence $\langle U : \alpha \in [\delta_U, \delta_W)\rangle$ belongs to $\prod_{\alpha \in [\delta_U, \delta_W)} \mathbf{UF}(\epsilon, \alpha)$, and clearly $U = W\text{-}\lim_{\alpha\in[\delta_U,\delta_W)} U$. By Lemma 3.3.14, $U <_{\Bbbk} W$.

The second bullet point is immediate from Lemma 3.3.12. ☐

The one issue with the global Ketonen order, which presents only notational difficulties, is that in this generalized context, $<_{\Bbbk}$ is no longer the irreflexive part of $\leq_{\Bbbk}$. Instead we have the following fact, where $=_{\Bbbk}$ is the change-of-space relation defined in Definition 3.2.2:

Lemma 3.3.16. *Suppose U and W are countably complete ultrafilters on ordinals. Then $U \leq_{\Bbbk} W$ if and only if $U <_{\Bbbk} W$ or $U =_{\Bbbk} W$.* ☐

It is often notationally convenient to restrict the global Ketonen order to the class of countably complete fine ultrafilters **Fine**, since one then has the exact analog of of Proposition 3.3.3:

Lemma 3.3.17. *If $U, W \in$ **Fine**, $U \leq_{\Bbbk} W$ if and only if $U <_{\Bbbk} W$ or $U = W$.* ☐

Lemma 3.3.18. *For any ultrafilter U on an ordinal, let*

$$\phi(U) = U \mid \delta_U$$

Then for all ordinals δ, ϕ restricts to an isomorphism from $(\mathbf{UF}(\delta), <_{\Bbbk})$ to $(\mathbf{Fine}(\leq\delta), <_{\Bbbk})$. Thus the Ketonen order is a set-like wellfounded partial order on **Fine**. ☐

The coherence of the various Ketonen orders on $\mathbf{UF}(\delta)$ is a special case of the following lemma, which states that order embeddings between ordinals induce order embeddings on their associated Ketonen orders:

Lemma 3.3.19. *Suppose $\epsilon \leq \delta$ are ordinals and $f : \epsilon \to \delta$ is an order embedding. Then the pushforward map $f_* : \mathbf{UF}(\epsilon) \to \mathbf{UF}(\delta)$ is an order embedding.*

Sketch. For ultrafilters U and W in $\mathbf{UF}(\epsilon)$, we must show that $U <_{\Bbbk} W$ if and only if $f_*(U) <_{\Bbbk} f_*(W)$. We show the forwards direction since the converse is similar.

Fix $I \in W$ and $\langle U_\alpha : \alpha \in I\rangle \in \prod_{\alpha\in I} \mathbf{UF}(\epsilon, \alpha)$ such that $U = W\text{-}\lim_{\alpha\in I} U_\alpha$. Let $J = f[I]$, and for $\alpha \in I$, let $Z_{f(\alpha)} = f_*(W_\alpha)$. (This makes sense because f is injective.) Thus $J \in f_*(W)$. Moreover, for all $\alpha \in I$,

$$f(\alpha) \supseteq f[\alpha] \in Z_{f(\alpha)}$$

since f is an order embedding, so $Z_{f(\alpha)} \in \mathbf{UF}(\delta, f(\alpha))$. Thus $\langle Z_\beta : \beta \in J\rangle \in \prod_{\beta\in J} \mathbf{UF}(\delta, \beta)$. Finally

$$f_*(U) = W\text{-}\lim_{\alpha\in I} f_*(W_\alpha) = f_*(W)\text{-}\lim_{\beta\in f[I]} Z_\alpha$$

It follows that $f_*(U) <_{\Bbbk} f_*(W)$. □

3.4 Orders on ultrafilters

In this section, we compare the Ketonen order with a number of better-known orders: the Mitchell order, the Rudin-Keisler order, the Lipschitz order, and the canonical order on stationary sets.

3.4.1 The Mitchell order

The Ketonen order can be seen as a combinatorial generalization of the Mitchell order on normal ultrafilters. We will discuss the relationship between the Ketonen order and the generalization of the Mitchell order to arbitrary countably complete ultrafilters at length in Chapter 4, but for now, we satisfy ourselves by proving that the Ketonen and Mitchell orders coincide on normal ultrafilters.

Theorem 3.4.1. *Suppose U and W are normal ultrafilters. Then $U \lhd W$ if and only if $U <_{\Bbbk} W$.*

Proof. Suppose first that U and W are normal ultrafilters on distinct cardinals κ and λ. Clearly $U \lhd W$ if and only if $\kappa < \lambda$. Moreover by Lemma 3.3.15, $U <_{\Bbbk} W$ if and only if $\kappa < \lambda$. Thus $U \lhd W$ if and only if $U <_{\Bbbk} W$.

Assume instead that U and W lie on the same cardinal κ. By Lemma 2.2.37, U and W are κ-complete and $\kappa = \mathrm{id}_U = \mathrm{id}_W$. The key fact we use is that since $\mathrm{crit}(j_W) = \kappa$, $j_W(A) \cap \kappa = A$ for all $A \subseteq \kappa$.

First, suppose $U \lhd W$. Then $U \in M_W$. Working in M_W, consider the projection $Z = U \mid j_W(\kappa) \in \mathbf{UF}^{M_W}(j_W(\kappa), \kappa)$. For any $A \subseteq \kappa$, $j_W(A) \cap \kappa = A \in U$, or in other words, $j_W(A) \in Z$. In other words, $j_W[U] \subseteq Z$, so by Lemma 3.3.4, $U <_{\Bbbk} W$.

Conversely, suppose $U <_{\Bbbk} W$. Fix $Z \in \mathbf{UF}^{M_W}(j_W(\kappa), \kappa)$ such that $U = j_W^{-1}[Z]$. Suppose $A \subseteq \kappa$. Then $A \in U$ if and only if $j_W(A) \cap \kappa \in Z$ if and only if $A \in Z$. Therefore $U = Z \mid \kappa$, so $U \in M_W$. This implies $U \lhd W$. □

Thus the wellfoundedness of the Mitchell order follows from the wellfoundedness of the Ketonen order. Notice that this theorem combined with the linearity of the Ketonen order under UA (Theorem 3.3.6) gives another proof of the linearity of the Mitchell order on normal ultrafilters under UA. Finally, the proof has the following consequence:

Corollary 3.4.2. *Suppose κ is a cardinal, $U \in \mathbf{UF}(\kappa)$, and W is a normal ultrafilter on κ. Then $U <_{\Bbbk} W$ if and only if $U \lhd W$.* □

It is remarkable, given the combinatorial nature of the structures involved, that for any normal ultrafilter U, the set of Ketonen predecessors of U is equal to $\mathbf{UF}(\kappa) \cap M$ where M is an inner model of ZFC containing $P(\kappa)$.

We will see various nontrivial generalizations of this fact to more general types of ultrafilters than normal ones.

3.4.2 The Rudin-Keisler order

In this section, we briefly recall the theory of the Rudin-Keisler order and explain its relationship with the Ketonen order. We also introduce the notion of an *incompressible ultrafilter*, which will be a useful technical tool.

The Rudin-Keisler order is defined in terms of pushforward ultrafilters (Definition 3.2.14).

Definition 3.4.3. Suppose U and W are ultrafilters. The *Rudin-Keisler order* is defined by setting $U \leq_{\text{RK}} W$ if there is a function $f : I \to X$ such that $f_*(W) = U$ where $I \in W$ and X is the underlying set of U.

We could of course take I to be the underlying set of W above. The Rudin-Keisler order is a (nonstrict) preorder on the class of ultrafilters. For us, the most important characterization of the Rudin-Keisler order uses elementary embeddings:

Lemma 3.4.4. *Suppose U and W are ultrafilters. Then $U \leq_{\text{RK}} W$ if and only if there is an elementary embedding $k : M_U \to M_W$ such that $k \circ j_U = j_W$.*

Proof. Let X be the underlying set of U.

First assume $U \leq_{\text{RK}} W$. Fix $I \in W$ and $f : I \to X$ such that $f_*(W) = U$. Let $a = [f]_W$, so by Lemma 3.2.16, U is the ultrafilter on X derived from j_W using a. Let $k : M_U \to M_W$ be the factor embedding, so $k(\text{id}_U) = a$ and $k \circ j_U = j_W$. Then k witnesses the conclusion of the lemma.

Conversely, assume there is an elementary embedding $k : M_U \to M_W$ such that $k \circ j_U = j_W$. Let $b = k(\text{id}_U)$. Then $b \in j_W(X)$. On the one hand, U is equal to the ultrafilter on X derived from j_W using b. (Explicitly: $U = j_U^{-1}[\text{p}_{\text{id}_U}] = j_U^{-1}[k^{-1}[k(\text{p}_{\text{id}_U})]] = j_W^{-1}[\text{p}_b]$.) Fix $I \in W$ and $f : I \to X$ such that $[f]_W = b$. Then by Lemma 3.2.16, $f_*(W)$ is the ultrafilter on X derived from j_W using b, or in other words $f_*(W) = U$. Thus $U \leq_{\text{RK}} W$ as desired. □

A second combinatorial formulation of the Rudin-Keisler order is in terms of partitions which will become relevant when we study indecomposability (especially in Theorem 7.5.26):

Lemma 3.4.5. *Suppose U and W are ultrafilters. Let X be the underlying set of U. Then $U \leq_{\text{RK}} W$ if and only if there is a sequence of pairwise disjoint sets $\langle Y_x : x \in X\rangle$ such that $U = \left\{A \subseteq X : \bigcup_{x\in A} Y_x \in W\right\}$.* □

The fundamental theorem of the Rudin-Keisler order explains its relationship between with the notion of Rudin-Keisler equivalence introduced in Definition 2.2.28:

Theorem 3.4.6. *Suppose U and W are ultrafilters. Then $U \equiv_{\text{RK}} W$ if and only if $U \leq_{\text{RK}} W$ and $W \leq_{\text{RK}} U$.*

We sketch the proof even though we do not need it in what follows. The key is a very interesting rigidity theorem for pushforwards. (The proof we give is a reformulation of the proof from [16].)

Lemma 3.4.7. *Suppose U is an ultrafilter on X and $f : X \to X$ is a function. If $f_*(U) = U$ then $f(x) = x$ for U-almost all $x \in X$.*

Proof. Assume $f : X \to X$ is such that $f(x) \neq x$ for all $x \in X$. We will show that $f_*(U) \neq U$.

Claim. *There is a partition of X into three pieces $(A_n)_{n<3}$ such that $f[A_n] \subseteq X \setminus A_n$ for all $n < 3$.*

Sketch. Consider the graph G with vertex set X formed by drawing an edge between x and $f(x)$ for each $x \in X$. Our claim above amounts to the fact that G is 3-colorable. It suffices to show that each connected subgraph $H \subseteq G$ is 3-colorable. Therefore suppose H is a connected subgraph of G. The key point is that H contains at most one cycle: by the construction of G, no subgraph of G can have more edges than vertices, and therefore no two cycles in G can be connected.

If H does contain a cycle, one can remove an edge from H to obtain an acyclic graph H'; otherwise let $H' = H$. Since H' is acyclic, H' is 2-colorable.[1] By changing the color of at most one vertex in the coloring of H', one obtains a 3-coloring of H. □

Since $A_0 \cup A_1 \cup A_2 = X \in U$, either A_0, A_1, or A_2 belongs to U. Assume without loss of generality that $A_0 \in U$. Then the set $f[A_0]$, which belongs to $f_*(U)$, is included in $X \setminus A_0$ by the key property of the partition $\{A_0, A_1, A_2\}$, so $X \setminus A_0 \in f_*(U)$.

Since $A_0 \in U$ and $X \setminus A_0 \in f_*(U)$, $f_*(U) \neq U$. □

Let us reformulate this in terms of ultrapowers:

Theorem 3.4.8. *Suppose U is an ultrafilter and $k : M_U \to M_U$ is an elementary embedding such that $k \circ j_U = j_U$. Then k is the identity.*

[1]Let $\{x_n\}_{n<k}$ enumerate the vertices of H', and set $x_n \prec x_m$ if $n < m$ and x_n and x_m are adjacent in H'. Since $\prec$ is an acyclic relation on a finite set, it is wellfounded. One obtains a 2-coloring c of H' by setting $c(x_n)$ equal to the parity of the E-rank of x_n.

Proof. Let X be the underlying set of U. Fix a function $f : X \to X$ such that $[f]_U = k(\mathrm{id}_U)$. Then by Lemma 3.2.16, $f_*(U)$ is the ultrafilter on X derived from j_U using $k(\mathrm{id}_U)$, which is easily seen to equal U. (Yet another inverse image calculation: $j_U^{-1}[\mathrm{p}_{k(\mathrm{id}_U)}] = (k \circ j_U)^{-1}[\mathrm{p}_{k(\mathrm{id}_U)}] = j_U^{-1}[k^{-1}[\mathrm{p}_{k(\mathrm{id}_U)}]] = j_U^{-1}[\mathrm{p}_{\mathrm{id}_U}] = U$.) Therefore by Lemma 3.4.7, $f \restriction I = \mathrm{id}$ for some $I \in U$. Thus $k(\mathrm{id}_U) = [f]_U = \mathrm{id}_U$. It follows that $k \restriction j_U[V] \cup \{\mathrm{id}_U\}$ is the identity, so $k \restriction M_U$ is the identity since $M_U = H^{M_U}(j_U[V] \cup \{\mathrm{id}_U\})$. $\square$

Lemma 3.4.7 immediately implies Theorem 3.4.6:

Proof of Theorem 3.4.6. Let X be the underlying set of U and Y be the underlying set of W. The proof that $U \equiv_{\mathrm{RK}} W$ implies $U \leq_{\mathrm{RK}} W$ and $W \leq_{\mathrm{RK}} U$ is quite easy. Fix $I \in U$, $J \in W$, and a bijection $f : I \to J$ such that for all $A \subseteq I$, $A \in U$ if and only if $f[A] \in W$. Viewing f as a function $p : I \to Y$, we have $W = p_*(U)$. Viewing f^{-1} as a function $p : J \to X$, we have $U = p_*(W)$. This implies implies $W \leq_{\mathrm{RK}} U$ and $U \leq_{\mathrm{RK}} W$.

Conversely assume $U \leq_{\mathrm{RK}} W$ and $W \leq_{\mathrm{RK}} U$. Fix $I \in U$ and $f : I \to Y$ such that $f_*(U) = W$. Fix $J \in W$ and $g : J \to X$ such that $g_*(W) = U$. We claim there is a set $I' \subseteq I$ such that $I' \in U$ and $g \circ f \restriction I'$ is the identity. To see this, note that $(g \circ f)_*(U) = g_*(f_*(U)) = g_*(W) = U$. Therefore by Lemma 3.4.7, there is a set $I' \subseteq I$ such that $I' \in U$ and $g \circ f$ is the identity. $\square$

Theorem 3.4.6 motivates the following definition:

Definition 3.4.9. The *strict Rudin-Keisler order* is defined on ultrafilters U and W by setting $U <_{\mathrm{RK}} W$ if $U \leq_{\mathrm{RK}} W$ and $U \not\equiv_{\mathrm{RK}} W$.

We now discuss the structure of the Rudin-Keisler order on countably complete ultrafilters and its relationship to the Ketonen order. To facilitate this discussion, we introduce a revised version of the Rudin-Keisler order. Recall that a function f defined on a set of ordinals I is *regressive* if $f(\alpha) < \alpha$ for all $\alpha \in I$.

Definition 3.4.10. Suppose U and W are ultrafilters on ordinals. Let X be the underlying set of U. The *revised Rudin-Keisler order* is defined by setting $U <_{\mathrm{rk}} W$ if there is a set $I \in W$ and a regressive function $f : I \to X$ such that $f_*(W) = U$.

Lemma 3.4.11. *If U and W are ultrafilters on ordinals, then $U <_{\mathrm{rk}} W$ if and only if there is an elementary embedding $k : M_U \to M_W$ such that $k \circ j_U = j_W$ and $k(\mathrm{id}_U) < \mathrm{id}_W$.* $\square$

Corollary 3.4.12. *The Ketonen order and the Rudin-Keisler order extend the revised Rudin-Keisler order.* $\square$

Lemma 3.4.13. *For any ultrafilter U, the intersection of Rudin-Keisler equivalence class of U with* **Fine** *is linearly ordered by the revised Rudin-Keisler order.*

Proof. For $W_0, W_1 \in [U]_{\text{RK}} \cap \textbf{Fine}$, $W_0 <_{\text{rk}} W_1$ if and only if $M_U \vDash \text{id}_{W_0} < \text{id}_{W_1}$. ☐

We now introduce a concept that is very useful in the study of countably complete ultrafilters. (The same concept was considered by Ketonen [17], who used the term "normalized ultrafilters.")

Definition 3.4.14. A fine ultrafilter U on an ordinal λ is *incompressible* if for any set $I \in U$, no regressive function on I is one-to-one.

One can reformulate incompressibility in terms of an ideal. A set of ordinals is *compressible* if it carries a one-to-one regressive function. The *compressible ideal on* λ is the set of all subsets of λ that can be covered by a finite union of compressible sets. (This ideal can be incompatible with the bounded ideal on λ; for example, this is the case when $\text{cf}(\lambda) = \omega$ or when λ is not a cardinal. In this case, λ carries no incompressible ultrafilters.) A fine ultrafilter on λ is incompressible if and only if it is disjoint from the compressible ideal.

Lemma 3.4.15. *Suppose U is a fine ultrafilter on an ordinal. The following are equivalent:*

(1) U is incompressible.

(2) If $W <_{\text{rk}} U$, then $W <_{\text{RK}} U$. ☐

Lemma 3.4.16. *A fine ultrafilter U on an ordinal is incompressible if and only if it is the $<_{\text{rk}}$-minimum element of $C = \{U' \in \textbf{Fine} : U' \equiv_{\text{RK}} U\}$.*

Proof. By Lemma 3.4.15, U is an $<_{\text{rk}}$-minimal element of C. Since $<_{\text{rk}}$ linearly orders C by Lemma 3.4.13, U is the $<_{\text{rk}}$-minimum element of C. ☐

Corollary 3.4.17. *An ultrafilter is Rudin-Keisler equivalent to at most one incompressible ultrafilter.* ☐

Lemma 3.4.18. *Suppose U is a fine ultrafilter on δ. Then the following are equivalent:*

(1) U is incompressible.

(2) id_U is the least ordinal a of M_U such that $M_U = H^{M_U}(j_U[V] \cup \{a\})$.

(3) id_U is the largest ordinal a of M_U such that $a \neq j_U(f)(b)$ for any function $f : \delta \to \delta$ and $b < a$. ☐

If U is countably complete, then the ultrafilters in **Fine** that are Rudin-Keisler equivalent to U are *wellordered* by $<_{\text{rk}}$, and therefore there is a least such ultrafilter. The following is the key existence theorem for incompressible ultrafilters:

Lemma 3.4.19. *Any countably complete ultrafilter U is Rudin-Keisler equivalent to a unique incompressible ultrafilter W which can be obtained in any of the following ways:*

- *W is the $<_{\text{rk}}$-least element of the Rudin-Keisler equivalence class of U.*
- *$W = f_*(U)$ where $f : \delta_U \to \delta_U$ is the least one-to-one function modulo U.*
- *W is the fine ultrafilter derived from j_U using α where α is the ordinal defined in either of the following ways:*
  - *α is least such that $M_U = H^{M_U}(j_U[V] \cup \{\alpha\})$.*
  - *α is largest such that $\alpha \neq j_U(f)(\beta)$ for any $\beta < \alpha$.* □

What makes incompressible ultrafilters useful is the following dual version of Lemma 3.4.15:

Proposition 3.4.20. *Suppose U is incompressible and W is an ultrafilter on an ordinal. If $U <_{\text{RK}} W$, then $U <_{\text{rk}} W$.*

Proof. Assume $U <_{\text{RK}} W$. Fix $k : M_U \to M_W$ such that $k \circ j_U = j_W$. Since $U \not\equiv_{\text{RK}} W$, k is not an isomorphism. It follows that $\text{id}_W \notin k[M_U]$: otherwise $j_W[V] \cup \{\text{id}_W\} \subseteq k[M_U]$ and so $M_W = H^{M_W}(j_W[V] \cup \{\text{id}_W\}) \subseteq k[M_U]$, and therefore k is surjective and hence an isomorphism.

To show that $U <_{\text{rk}} W$, it suffices by Lemma 3.4.11 to show that $k(\text{id}_U) < \text{id}_W$. Suppose not. Then $\text{id}_W \leq k(\text{id}_U)$, and since $\text{id}_W \notin k[M_U]$, in fact $\text{id}_W < k(\text{id}_U)$. Since $M_W = H^{M_W}(j_W[V] \cup \{\text{id}_W\})$ we can fix a function $f : \delta_W \to \delta_W$ such that $j_W(f)(\text{id}_W) = k(\text{id}_U)$. Since $\text{id}_W < k(\text{id}_U)$,

$$M_W \vDash \exists \xi < k(\text{id}_U)\ j_W(f)(\xi) = k(\text{id}_U)$$

Since $j_W(f) = k(j_U(f))$, the elementarity of $k : M_U \to M_W$ implies

$$M_U \vDash \exists \xi < \text{id}_U\ j_U(f)(\xi) = \text{id}_U$$

This contradicts Lemma 3.4.18 (3), which in particular states that $\text{id}_U \neq j_U(f)(\xi)$ for any $\xi < \text{id}_U$. □

Corollary 3.4.21. *The strict Rudin-Keisler order and the revised Rudin-Keisler order coincide on incompressible ultrafilters.* □

Corollary 3.4.22. *The Ketonen order extends the strict Rudin-Keisler order on countably complete incompressible ultrafilters.* □

Given Corollary 3.4.22, one might guess that (on **UF**), $<_{\text{rk}} = \leq_{\text{RK}} \cap <_{\Bbbk}$, but it is not hard to construct a counterexample under weak large cardinal assumptions.

Corollary 3.4.23 (Solovay)**.** *The strict Rudin-Keisler order is wellfounded on countably complete ultrafilters.*

Proof. Suppose towards a contradiction that

$$U_0 >_{\mathrm{RK}} U_1 >_{\mathrm{RK}} U_2 >_{\mathrm{RK}} \cdots$$

is a descending sequence of countably complete ultrafilters in the strict Rudin-Keisler order. For each n, let W_n be the unique incompressible ultrafilter Rudin-Keisler equivalent to U_n. Then

$$W_0 >_{\mathrm{RK}} W_1 >_{\mathrm{RK}} W_2 >_{\mathrm{RK}} \cdots$$

since the strict Rudin-Keisler order is invariant under Rudin-Keisler equivalence. But by Corollary 3.4.22, the Ketonen order extends the strict Rudin-Keisler order on countably complete incompressible ultrafilters, and therefore

$$W_0 >_{\Bbbk} W_1 >_{\Bbbk} W_2 >_{\Bbbk} \cdots$$

This contradicts the wellfoundedness of the Ketonen order (Lemma 3.3.18). □

Note that this yields another proof of Theorem 3.4.6 in the case that U and W are countably complete.

3.4.3 The Lipschitz order

In this short subsection, we describe a generalization of the Ketonen order that connects the Ultrapower Axiom to the determinacy of long games. Throughout the section, we fix an infinite ordinal δ.

Definition 3.4.24. A function $f : P(\delta) \to P(\delta)$ is:

- *Lipschitz* if for $A \subseteq \delta$ and $\alpha < \delta$, $f(A) \cap \alpha$ depends only on $A \cap \alpha$.
- *super-Lipschitz* if for $A \subseteq \delta$ and $\alpha < \delta$, $f(A) \cap (\alpha + 1)$ depends only on $A \cap \alpha$.

If $X, Y \subseteq P(\delta)$, then f is a *reduction from X to Y* if $f^{-1}[Y] = X$.

We say X is (super-)Lipschitz reducible to Y if there is a (super-)Lipschitz reduction from X to Y.

These concepts are best thought of in terms of long games:

Definition 3.4.25. In the *Lipschitz game of length δ* associated to sets $X, Y \subseteq P(\delta)$, denoted $G_\delta(X, Y)$, two players I and II alternate playing 0s or 1s with I playing at limit stages:

$$\begin{array}{lcccccccc} \mathrm{I} & x(0) & & x(1) & & \ldots & x(\alpha) & & \ldots \\ \mathrm{II} & & y(0) & & y(1) & & & y(\alpha) & \end{array}$$

The play lasts for $\delta \cdot 2$ moves, so that I and II produce sequences x and y, respectively, with $x, y \in 2^\delta$. Identifying x and y with subsets of δ, II wins if and only if

$$x \in X \iff y \in Y$$

Player II has a winning strategy in $G_\delta(X, Y)$ if and only if X is Lipschitz reducible to Y, and Player I has a winning strategy if and only if Y is super-Lipschitz reducible to $P(\delta) \setminus X$.

Definition 3.4.26. The *strict Lipschitz order* is defined on $X, Y \subseteq P(\delta)$ by setting $X <_L Y$ if X and $P(\delta) \setminus X$ are both super-Lipschitz reducible to Y. The *Lipschitz order* is defined on $X, Y \subseteq P(\delta)$ by setting $X \leq_L Y$ if X is Lipschitz reducible to Y.

This notation is perhaps misleading since it might suggest that $X <_L Y$ if and only if $X \leq_L Y$ and $Y \not\leq_L X$. Under the Axiom of Determinacy, this is true when X and Y are contained in $P(\omega)$.

The Lipschitz order is transitive in the following strong sense:

Lemma 3.4.27. *The composition of a super-Lipschitz function and a Lipschitz function is a super-Lipschitz function. Therefore if X super-Lipschitz reduces to Y and Y Lipschitz reduces to Z, then X super-Lipschitz reduces to Z. In particular, if $X <_L Y \leq_L Z$ then $X <_L Z$.* □

A generalization of the proof of Proposition 3.3.9 shows that the Lipschitz order is irreflexive:

Lemma 3.4.28. *Suppose $X \subseteq P(\delta)$. Then X does not super-Lipschitz reduce to $P(\delta) \setminus X$.*

Proof. It suffices to show that every super-Lipschitz function $f : P(\delta) \to P(\delta)$ has a fixed point A: then $A \in X$ if and only if $f(A) \in X$ so f is not a super-Lipschitz reduction from X to $P(\delta) \setminus X$.

We define A by recursion. Suppose $\alpha < \delta$ and we have defined $A \cap \alpha$. We then put $\alpha \in A$ if and only if $\alpha \in f(A \cap \alpha)$. Then for any $\alpha < \delta$,

$$\begin{aligned} \alpha \in A &\iff \alpha \in f(A \cap \alpha) \\ &\iff \alpha \in f(A) \end{aligned}$$

The final equivalence follows from the fact that f is super-Lipschitz. Thus $f(A) = A$, as desired. □

Another (ultimately equivalent) way to prove Lemma 3.4.28 is to note that since Player II has a winning strategy in $G_\delta(X, X)$, Player I does not. The nonexistence of a winning strategy for Player I is equivalent to the statement that X does not super-Lipschitz reduce to $P(\delta) \setminus X$.

Corollary 3.4.29. *The strict Lipschitz order is a strict partial order.* □

By the proof of the Martin-Monk theorem (see [18]) descending sequences in the Lipschitz order give rise to pathological subsets of Cantor space:

Theorem 3.4.30 (ZF + DC)**.** *The following are equivalent:*

(1) There is a flip set.[2]

(2) The strict Lipschitz order on $P(\omega)$ is illfounded.

(3) The strict Lipschitz order on $P(\delta)$ is illfounded.

Proof. To see (1) implies (2), suppose $F \subseteq 2^\omega$ is a flip set. Define $(E_n)_{n<\omega}$ by recursion, setting $E_0 = F$ and $E_{n+1} = \{s \in 2^\omega : 0s \in E_n\}$. It is easy to see that E_{n+1} and $2^\omega \setminus E_{n+1}$ both super-Lipschitz reduce to E_n, via the super-Lipschitz reductions $s \mapsto 0s$ and $s \mapsto 1s$ respectively.

(2) trivially implies (3).

We finally show (3) implies (1). Fix $X_0 >_L X_1 >_L X_2 >_L \cdots$ a descending sequence of subsets of $P(\delta)$. For $n < \omega$, fix super-Lipschitz reductions f^0_n from X_{n+1} to X_n and f^1_n from X_{n+1} to $P(\delta) \setminus X_n$. For each $s \in 2^\omega$, we define sets $A^s_n \subseteq \delta$ such that

$$A^s_n = f^{s(n)}_n(A^s_{n+1})$$

Suppose $A^s_n \cap \alpha$ has been defined for all $n < \omega$. Then

$$A^s_n \cap (\alpha + 1) = f^{s(n)}_n(A^s_{n+1} \cap \alpha) \cap (\alpha + 1)$$

Since f^i_n is super-Lipschitz for all $n < \omega$ and $i \in \{0, 1\}$, A^s_n is well-defined and $A^s_n = f^{s(n)}_n(A^s_{n+1})$.

Define $F_n \subseteq 2^\omega$ by putting $s \in F_n$ if and only if $A^s_n \in X_n$. Whether $s \in F_n$ depends only on $s \restriction (\omega \setminus n)$. Moreover, if $s \in F_{n+1}$ then $s \in F_n$ if and only if $s(n) = 0$. It is easy to show by induction that if s and s' agree on $\omega \setminus n$ and $\sum_{k<n} s(k) = \sum_{k<n} s'(k) \mod 2$, then $s \in F_0$ if and only if $s' \in F_0$. Similarly, if s and s' agree on $\omega \setminus n$ and $\sum_{k<n} s(k) \neq \sum_{k<n} s'(k) \mod 2$, then $s \in F_0$ if and only if $s' \notin F_0$. It follows that F_0 is a flip set. $\square$

Of course, (1), (2), and (3) are all provable in ZFC. In the choiceless context of ZF + DC, however, there may be no flip sets (for example, if every subset of Cantor space has the Baire property or is Lebesgue measurable). In this case, Theorem 3.4.30 shows that the Lipschitz order is wellfounded not only on subsets of Cantor space but also on subsets of $P(\delta)$.[3] The proof also shows that the wellfounded part of the Lipschitz order is equal to the collection of sets that do not lie above a flip set.

We turn now to the relationship between the Lipschitz order and the Ketonen order.

Definition 3.4.31. A set $Z \subseteq P(\delta)$ *concentrates* on a set S if for all $A, B \subseteq \delta$ with $A \cap S = B \cap S$, $A \in Z$ if and only if $B \in Z$.

In the case that Z is an ultrafilter, Definition 3.4.31 reduces to Definition 3.2.1.

[2]A set $X \subseteq P(\omega)$ is a *flip set* if for any $x, y \in X$ such that $|x \triangle y|$ is odd, $x \in X$ if and only if $y \notin X$.

[3]Under the same hypotheses, one can show that the Lipschitz order on S^δ is wellfounded for any set S after generalizing the definition of the Lipschitz order in the natural way.

Lemma 3.4.32. *Suppose $X \subseteq P(\delta)$ and W is an ultrafilter on δ. Then the following are equivalent:*

(1) For some $Z \in M_W$ that concentrates on id_W, $X = j_W^{-1}[Z]$.

(2) X is super-Lipschitz reducible to W.

(3) $X <_L W$.

Proof. (1) implies (2): Fix $Z \in M_W$ concentrating on id_W such that $X = j_W^{-1}[Z]$. Let $\langle X_\alpha : \alpha \in I\rangle$ be such that $Z = [\langle X_\alpha : \alpha \in I\rangle]_W$ and X_α concentrates on α for all $\alpha \in I$. Define $f : P(\delta) \to P(\delta)$ by setting $f(A) = \{\alpha \in I : A \in X_\alpha\}$. Then f is a super-Lipschitz function since X_α concentrates on α for all $\alpha \in I$. Moreover,

$$\begin{aligned} A \in X &\iff j_W(A) \in Z \\ &\iff \{\alpha < \delta : A \in X_\alpha\} \in W \\ &\iff f(X) \in W \end{aligned}$$

so f Lipschitz reduces X to W.

(2) implies (3): Assume X is super-Lipschitz reducible to W. Since W is an ultrafilter, W Lipschitz reduces to $P(\delta) \setminus W$. Since X is super-Lipschitz to W and W Lipschitz reduces to $P(\delta) \setminus W$, X is super-Lipschitz to $P(\delta) \setminus W$ by Lemma 3.4.27. Therefore $X <_L W$.

(3) implies (1): Let $f : P(\delta) \to P(\delta)$ be a super-Lipschitz function from X to W. For each α, let $X_\alpha = \{A \subseteq \delta : \alpha \in f(A)\}$. Since f is super-Lipschitz, X_α concentrates on α. Let $Z = [\langle X_\alpha : \alpha < \delta\rangle]_W$. By Łoś's Theorem, Z concentrates on id_W. Then

$$\begin{aligned} A \in X &\iff f(A) \in W \\ &\iff \{\alpha < \delta : A \in X_\alpha\} \in W \\ &\iff j_W(A) \in Z \end{aligned}$$

Thus $j_W^{-1}[Z] = X$. ☐

Using Lemma 3.3.4, this has the following corollary:

Corollary 3.4.33. *The Lipschitz order extends the Ketonen order on* $\mathbf{UF}(\delta)$.

Proof. This is immediate from Lemma 3.4.32. ☐

Under UA, it follows that the two orders coincide:

Corollary 3.4.34 (UA)**.** *The Lipschitz order and the Ketonen order coincide on $\mathbf{UF}(\delta)$. In particular, the Lipschitz order wellorders $\mathbf{UF}(\delta)$.*

Proof. Since $<_L$ is a strict partial order extending the total relation $<_{\Bbbk}$ (Theorem 3.3.6), the two relations must be equal. ☐

The linearity of the Lipschitz order is a determinacy consequence of UA:

Corollary 3.4.35 (UA)**.** *For all ordinals δ, for any $U, W \in \mathbf{UF}(\delta)$, the game $G_\delta(U, W)$ is determined.*

Question 3.4.36. *Assume that for any ordinal δ, for any $U, W \in \mathbf{UF}(\delta)$, the game $G_\delta(U, W)$ is determined. Does the Ultrapower Axiom hold?*

If this were true then the Ultrapower Axiom would be a long determinacy principle. In Section 3.5, we give partial positive answer.

Finally, we note that the Ketonen order is an algebraic version of the Lipschitz order.

Theorem 3.4.37. *If U and W are countably complete ultrafilters on δ, then $U \leq_{\Bbbk} W$ if and only if there is a countably complete Lipschitz homomorphism $h : P(\delta) \to P(\delta)$ such that $h^{-1}[W] = U$.*

Proof. □

3.4.4 The Ketonen order on filters

We briefly discuss a generalization of the Ketonen order to a wellfounded partial order on arbitrary countably complete filters that is suggested by the proof of Theorem 3.3.8. This order will not appear elsewhere in this monograph, but it seems potentially quite interesting since it identifies a connection between the Ketonen order and stationary reflection.

Definition 3.4.38. Suppose F is a filter, $I \in F$, and $\langle G_i : i \in I\rangle$ is a sequence of filters on a fixed set Y. The *F-limit of* $\langle G_i : i \in I\rangle$ is the filter

$$F\text{-}\lim_{i\in I} G_i = \{A \subseteq Y : \{i \in I : A \in G_i\} \in F\}$$

Definition 3.4.39. If F is a filter on a set X and C is a class, then F *concentrates* on C if $C \cap X \in F$.

Definition 3.4.40. Suppose X is a set and C is a class. Let $\mathbf{F}(X)$ denote the set of countably complete filters on X and let $\mathbf{F}(X, C)$ denote the set of filters on X that concentrate on C.

Definition 3.4.41. Suppose ϵ and δ are ordinals, $F \in \mathbf{F}(\epsilon)$, and $G \in \mathbf{F}(\delta)$. The *Ketonen order on filters* is defined on by setting $F <_{\Bbbk} G$ if there is a set $I \in G$ and a sequence $\langle F_\alpha : \alpha \in I\rangle \in \prod_{\alpha\in I} \mathbf{F}(\epsilon, \alpha)$ such that $F \subseteq G\text{-}\lim_{\alpha\in I} F_\alpha$.

Under the Ultrapower Axiom, the restriction to ultrafilters of the Ketonen order on filters coincides with the Ketonen order as it is defined in Section 3.3.1. We do not know whether this is provable in ZFC.

Note that the proof of Proposition 3.3.9 breaks down when we consider filters instead of ultrafilters. In fact, in a sense this simple proof cannot be remedied, since irreflexivity fails if we allow filters that are countably incomplete, and it is not clear how countable completeness could come in to the argument of Proposition 3.3.9. It is somewhat surprising that one can in fact prove the irreflexivity of the Ketonen order by instead using countable completeness and the argument of Theorem 3.3.8:

Theorem 3.4.42. *The Ketonen order on filters is wellfounded.*

We include the proof, which is closely analogous to that of Theorem 3.3.8.

Lemma 3.4.43. *Suppose H is a filter and $F <_{\Bbbk} G$ are countably complete filters on ordinals ϵ and δ. Suppose $J \in H$ and $\langle G_x : x \in J\rangle$ is a sequence of countably complete filters such that $G \subseteq H\text{-}\lim_{x\in J} G_x$. Then there is a set $J' \subseteq J$ in H and a sequence $\langle F_x : x \in J'\rangle$ of countably complete filters such that $F_x <_{\Bbbk} G_x$ for all $x \in K$ and $F \subseteq H\text{-}\lim_{x\in K} F_x$.*

Proof. Since $F <_{\Bbbk} G$, we can fix $I \in G$ and countably complete filters $\langle D_\alpha : \alpha \in I\rangle \in \prod_{\alpha\in I} \mathbf{F}(\epsilon, \alpha)$ such that $F \subseteq G\text{-}\lim_{\alpha\in I} D_\alpha$.

Let $J' = \{b \in J : I \in G_x\}$. Since $I \in G \subseteq H\text{-}\lim_{x\in J} G_x$, we have $J' \in H$ by the definition of a limit. For each $x \in J'$, let

$$F_x = G_x\text{-}\lim_{\alpha\in I} D_\alpha$$

Then $F_x \in \mathbf{UF}(\epsilon)$, and the sequence $\langle D_\alpha : \alpha \in I\rangle$ witnesses $F_b <_{\Bbbk} G_b$.

Finally,

$$\begin{aligned} F &\subseteq G\text{-}\lim_{\alpha\in I} D_\alpha \\ &\subseteq (H\text{-}\lim_{x\in J} G_x)\text{-}\lim_{\alpha\in I} D_\alpha \\ &= H\text{-}\lim_{x\in J'} (G_x\text{-}\lim_{\alpha\in I} D_\alpha) \\ &= H\text{-}\lim_{x\in J'} F_x \end{aligned}$$

Thus $F \subseteq H\text{-}\lim_{x\in K} F_x$, as desired. □

Proof of Theorem 3.4.42. Suppose towards a contradiction that δ is the least ordinal such that the Ketonen order is illfounded below a countably complete filter that concentrates on δ. Fix a descending sequence $F_0 >_{\Bbbk} F_1 >_{\Bbbk} F_2 >_{\Bbbk} \cdots$ such that F_0 concentrates on δ.

We will define sets of ordinals $I_1 \supseteq I_2 \supseteq \cdots$ in F and sequences $\langle F^m_\alpha : \alpha \in I_m\rangle$ of countably complete filters such that

$$F_m \subseteq F\text{-}\lim_{\alpha\in I_m} F^m_\alpha$$

for all $1 \leq m < \omega$. We will have:

- For all $\alpha \in I_1$, F^1_α concentrates on α.
- For all $1 \leq m < \omega$, for all $\alpha \in I_{m+1}$, $F^{m+1}_\alpha <_{\Bbbk} F^m_\alpha$.

Since $F_1 <_{\Bbbk} F$, there is a set of ordinals $I_1 \in F$ and a sequence $\langle F^1_\alpha : \alpha \in I_1\rangle$ of countably complete ultrafilters such that $F_1 \subseteq F\text{-}\lim_{\alpha\in I_1} F^1_\alpha$ and F^1_α concentrates on α for all $\alpha \in I_1$.

Suppose $1 \leq m < \omega$ and $\langle F^m_\alpha : \alpha \in I_m\rangle$ has been defined. We now apply Lemma 3.4.43 with $H = F$, $F = F_{m+1}$, and $G = F_m$. This yields a set $I_{m+1} \subseteq I_m$ in F and a sequence $\langle F^{m+1}_\alpha : \alpha \in I_m\rangle$ of countably complete filters on δ such that $F^{m+1}_\alpha <_{\Bbbk} F^m_\alpha$ for all $\alpha \in I_{m+1}$ and

$$F_{m+1} \subseteq F\text{-}\lim_{\alpha\in I_{m+1}} F^{m+1}_\alpha$$

This completes the definition of the sets $I_1 \supseteq I_2 \supseteq \cdots$ and sequences $\langle F^m_\alpha : \alpha \in I_m\rangle$ for $1 \leq m < \omega$.

Now let $I = \bigcap_{1\leq m<\omega} I_m$. Since F_0 is countably complete, I is nonempty, so we can fix an ordinal $\alpha \in I$. Then since $\alpha \in I_m$ for all $1 \leq m < \omega$,

$$F^1_\alpha >_{\Bbbk} F^2_\alpha >_{\Bbbk} F^3_\alpha >_{\Bbbk} \cdots$$

Since F^1_α concentrates on $\alpha < \delta$, this contradicts the minimality of δ. $\square$

Recall the following definition, due to Jech [15]:

Definition 3.4.44. Assume δ is a regular cardinal. The *canonical order on stationary sets* is defined on stationary sets $S, T \subseteq \delta$ by setting $S < T$ if there is a closed unbounded set $C \subseteq \delta$ such that $S \cap \alpha$ is stationary in α for all $\alpha \in C \cap T$.

Definition 3.4.45. For any ordinal α, let $\mathscr{C}_\alpha$ denote the filter of closed cofinal subsets of α.

The following proposition connects the canonical order on stationary sets and the Ketonen order on filters:

Proposition 3.4.46. *Suppose δ is a regular cardinal and S and T are stationary subsets of δ. Then $S < T$ implies $\mathscr{C}_\delta \mid S <_{\Bbbk} \mathscr{C}_\delta \mid T$.*

Proof. Fix a closed unbounded set $C \subseteq \delta$ such that $S \cap \alpha$ is stationary in α for all $\alpha \in C \cap T$. Note that $C \cap T \in \mathscr{C}_\delta \mid T$, and for all $\alpha \in C \cap T$, $\mathscr{C}_\alpha \mid S$ is a countably complete filter concentrating on ordinals less than α.

Claim 1. $C_\delta \mid S \subseteq (\mathscr{C}_\delta \mid T)\text{-}\lim_{\alpha\in C\cap T} \mathscr{C}_\alpha \mid S$.

Proof. Suppose $A \in \mathscr{C}_\delta \mid S$. We will show that $A \in (\mathscr{C}_\delta \mid T)\text{-}\lim_{\alpha\in C\cap T} \mathscr{C}_\alpha \mid S$. Fix $E \in C_\delta$ such that $S \cap E \subseteq A$. Let E' be the set of accumulation points of E. Then for any $\alpha \in E'$, $S \cap (E \cap \alpha) \subseteq A$ and $E \cap \alpha \in \mathscr{C}_\alpha$, so $A \in \mathscr{C}_\alpha \mid S$. Thus

$$E' \cap C \cap T \subseteq \{\alpha \in C \cap T : A \in \mathscr{C}_\alpha \mid S\}$$

Since $E' \cap C \in \mathscr{C}_\delta$, $E' \cap C \cap T \in \mathscr{C}_\delta \mid T$, and therefore $\{\alpha \in C \cap T : A \in \mathscr{C}_\alpha \mid S\} \in \mathscr{C}_\delta \mid T$. It follows that $A \in (\mathscr{C}_\delta \mid T)\text{-}\lim_{\alpha\in C\cap T} \mathscr{C}_\alpha \mid S$, as desired. $\square$

The claim implies $C_\delta \mid S <_{\Bbbk} \mathscr{C}_\delta \mid T$, as desired. $\square$

As a corollary of Theorem 3.4.42 and Proposition 3.4.46, we have the following theorem of Jech:

Corollary 3.4.47. *The canonical order on stationary sets is wellfounded.* □

3.5 The linearity of the Ketonen order

In this final section, we prove a converse to Theorem 3.3.6, which can also be seen as a partial positive answer to Question 3.4.36. We say that *the Ketonen order is linear* if for all ordinals δ, the Ketonen order on $\mathbf{UF}(\delta)$ is a linear order. The Ketonen order is linear if and only if its restriction to **Fine** is a linear order.

Theorem 3.5.1. *The Ketonen order is linear if and only if the Ultrapower Axiom holds.*

Definition 3.5.2. Suppose M_0, M_1, and N are transitive models of ZFC and

$$(k_0, k_1) : (M_0, M_1) \to N$$

are elementary embeddings.

- (k_0, k_1) is *left-internal* if k_0 is definable over M_0.
- (k_0, k_1) is *right-internal* if k_1 is definable over M_1.
- (k_0, k_1) is *internal* if it is both left-internal and right-internal.

Given Lemma 3.3.4, the linearity of the Ketonen order would appear to be a weaker assumption than UA: given a pair of ultrapower embeddings, the linearity of the Ketonen order only guarantees a right-internal comparison, while UA asserts the existence of a fully internal one. How can one transform partially internal comparisons into the fully internal comparisons required by UA? In fact, it is simply impossible to do this in general, since partially internal comparisons can be proved to exist in ZFC alone:

Proposition 3.5.3. *Any pair of ultrapower embeddings of V has a left-internal ultrapower comparison and a right-internal ultrapower comparison.* □

Thus the true power of the linearity of the Ketonen order lies not in the mere existence of right-internal comparisons (k, h) but rather in the existence of (k, h) witnessing $U <_{\Bbbk} W$ (or $W \leq_{\Bbbk} U$); that is, with the additional property $k(\mathrm{id}_U) < h(\mathrm{id}_W)$.

Theorem 3.5.1 is an immediate consequence of our next theorem, which shows how to define an internal ultrapower comparison of a pair of ultrafilters explicitly:

Theorem 3.5.4. *Assume the Ketonen order is linear. Suppose ϵ and δ are ordinals. Suppose $U \in \mathbf{UF}(\epsilon)$ and $W \in \mathbf{UF}(\delta)$.*

- *Let W_* be the least element of $j_U(\mathbf{UF}(\delta), <_{\Bbbk})$ extending $j_U[W]$.*

- *Let U_* be the least element of $j_W(\mathbf{UF}(\epsilon), <_{\Bbbk})$ extending $j_W[U]$.*

Then $(j^{M_U}_{W_*}, j^{M_W}_{U_*})$ is a comparison of (j_U, j_W).

The definitions of W_* and U_* rely on the fact that $j_U(\mathbf{UF}(\delta), <_{\Bbbk})$ and $j_W(\mathbf{UF}(\epsilon), <_{\Bbbk})$ are wellorders, not only in M_U and M_W but also, by absoluteness, in the true universe V. This, however, is not the main use of the linearity of the Ketonen order in the proof. Indeed, it is consistent that there is a pair of countably complete ultrafilters U and W such that the minimum extensions W_* and U_* are well-defined yet (j_U, j_W) admits no internal comparison.[4] Instead we will use the linearity of the Ketonen order to compare $(j^{M_U}_{W_*} \circ j_U, j^{M_W}_{U_*} \circ j_W)$.

Lemma 3.5.5. *Suppose ϵ and δ are ordinals. Suppose $U \in \mathbf{UF}(\epsilon)$ and $W \in \mathbf{UF}(\delta)$.*

- *Let W_* be an element of $j_U(\mathbf{UF}(\delta))$ extending $j_U[W]$.*
- *Let U_* be a minimal element of $j_W(\mathbf{UF}(\epsilon), <_{\Bbbk})$ extending $j_W[U]$.*

For any right-internal ultrapower comparison

$$(k, h) : (M^{M_U}_{W_*}, M^{M_W}_{U_*}) \to P$$

of $(j^{M_U}_{W_*} \circ j_U, j^{M_W}_{U_*} \circ j_W)$, the following hold:

$$h(j^{M_W}_{U_*}(\mathrm{id}_W)) \leq k(\mathrm{id}_{W_*}) \tag{3.1}$$

$$h(\mathrm{id}_{U_*}) \leq k(j^{M_U}_{W_*}(\mathrm{id}_U)) \tag{3.2}$$

Proof. Let us direct the reader's attention to the key diagram, Fig. 3.4.

We first prove (3.1). By Lemma 3.2.17, there is an elementary embedding $e : M_W \to M^{M_U}_{W_*}$ such that $e \circ j_W = j^{M_U}_{W_*} \circ j_U$ and $e(\mathrm{id}_W) = \mathrm{id}_{W_*}$. Note that

$$(k \circ e, h \circ j^{M_W}_{U_*}) : (M_W, M_W) \to P$$

is a right internal comparison of (j_W, j_W). Thus by the irreflexivity of the Ketonen order, $h(j^{M_W}_{U_*}(\mathrm{id}_W)) \leq k(e(\mathrm{id}_W)) = k(\mathrm{id}_{W_*})$, proving (3.1).

We now prove (3.2). To reduce subscripts, we define:

$$\alpha = j^{M_U}_{W_*}(\mathrm{id}_U)$$

Let Z be the M_W-ultrafilter on $j_W(\epsilon)$ derived from $h \circ j^{M_W}_{U_*}$ using $k(\alpha)$, so

$$Z = (h \circ j^{M_W}_{U_*})^{-1}\left[\mathrm{p}_{k(\alpha)}\right]$$

[4]Take U and W to be Mitchell incomparable normal ultrafilters. Apply Theorem 3.4.1 and Lemma 8.2.11 to see that $j_U(W)$ and $j_W(U)$ are the *only* extensions of $j_U[W]$ and $j_W[U]$ in M_U and M_W respectively.

$$\begin{tikzcd}
 & N & & \\
M_{W_*} \arrow[ur, dashed, "k"] & & M_{U_*} \arrow[ul, "h"'] & M_Z \arrow[ull, dashed, bend right, "i"'] \\
M_U \arrow[u, "W_*"] \arrow[urr, dashed] & & M_W \arrow[ull, dashed, "e"] \arrow[u, "U_*"'] \arrow[ur, "Z"'] & \\
 & V \arrow[ul, "U"] \arrow[ur, "W"'] & &
\end{tikzcd}$$

Figure 3.4: The proof of Lemma 3.5.5

Since $h \circ j_{U_*}^{M_W}$ is an internal ultrapower embedding of M_W, Z is a countably complete ultrafilter of M_W; in other words, $Z \in j_W(\mathbf{UF}(\epsilon))$. Moreover, it is not hard to compute that Z extends $j_W[U]$, or equivalently $j_W^{-1}[Z] = U$:

$$\begin{aligned}
j_W^{-1}[Z] &= j_W^{-1}[(h \circ j_{U_*}^{M_W})^{-1}[\mathrm{p}_{k(\alpha)}]] \\
&= (h \circ j_{U_*}^{M_W} \circ j_W)^{-1}[\mathrm{p}_{k(\alpha)}] \\
&= (k \circ j_{W_*}^{M_U} \circ j_U)^{-1}[\mathrm{p}_{k(\alpha)}] \\
&= (j_{W_*}^{M_U} \circ j_U)^{-1}[k^{-1}[\mathrm{p}_{k(\alpha)}]] \\
&= (j_{W_*}^{M_U} \circ j_U)^{-1}[\mathrm{p}_\alpha] \\
&= j_U^{-1}[(j_{W_*}^{M_U})^{-1}[\mathrm{p}_{j_{W_*}^{M_U}(\mathrm{id}_U)}]] \\
&= j_U^{-1}[\mathrm{p}_{\mathrm{id}_U}] = U
\end{aligned}$$

Since U_* is a minimal element of $j_W(\mathbf{UF}(\epsilon), <_{\Bbbk})$ extending $j_W[U]$, M_W satisfies $Z \not<_{\Bbbk} U_*$.

Since Z is derived from $h \circ j_{U_*}^{M_W}$ using $k(\alpha)$, there is a factor embedding $i : (M_Z)^{M_W} \to P$ specified by the following properties:

$$i \circ j_Z^{M_W} = h \circ j_{U_*}^{M_W} \tag{3.3}$$

$$i(\mathrm{id}_Z) = k(\alpha) \tag{3.4}$$

Note that these properties define i over M_W. Therefore by (3.3), (i, h) is a right-internal ultrapower comparison of $(j_Z^{M_W}, j_{U_*}^{M_W})$ in M_W. The fact that $Z \not<_{\Bbbk} U_*$ in M_W implies

$$h(\mathrm{id}_{U_*}) \leq i(\mathrm{id}_Z) = k(\alpha) = k(j_{W_*}^{M_U}(\mathrm{id}_U))$$

proving (3.2). $\square$

Lemma 3.5.5 can be read as asserting that the natural ultrafilter representing the embedding $j^{M_W}_{U_*} \circ j_W$ is not strictly above the one representing $j^{M_U}_{W_*} \circ j_U$ in the Ketonen order. To make this precise, we need to define what these natural ultrafilters. This is related to the well-known notion of an ultrafilter sum:

Definition 3.5.6. Suppose U is an ultrafilter on X, I is a set in U, and $\langle W_i : i \in I\rangle$ is a sequence of ultrafilters on Y. The *U-sum of* $\langle W_i : i \in I\rangle$ is the ultrafilter defined by

$$U\text{-}\sum_{i\in I} W_i = \{A \subseteq X \times Y : \{i \in I : A_i \in W_i\} \in U\}$$

In the definition above, if $A \subseteq X \times Y$ and $i \in X$, $A_i = \{j \in Y : (i, j) \in A\}$.

There is an obvious connection between sums and limits: the projection of a sum of ultrafilters onto its second coordinate is precisely equal to the limit of those ultrafilters.

Lemma 3.5.7. *Suppose U is an ultrafilter, I is a set in U, and $\langle W_i : i \in I\rangle$ is a sequence of ultrafilters on Y. Let $Z = [\langle W_i : i \in I\rangle]_U$ and let $D = U\text{-}\sum_{i\in I} W_i$. Then $M_D = M^{M_U}_Z$, $j_D = j^{M_U}_Z \circ j_U$, and $\mathrm{id}_D = (j^{M_U}_Z(\mathrm{id}_U), \mathrm{id}_Z)$.* $\square$

Motivated this lemma, we introduce the following nonstandard notation.

Definition 3.5.8. Suppose U is an ultrafilter on X, and W_* is an M_U-ultrafilter on $j_U(Y)$. Then $[U, W_*]$ denotes the ultrafilter $\{A \subseteq X \times Y : [x \mapsto A_x]_U \in W_*\}$.

Here $A_x = \{y \in Y : (x, y) \in A\}$.

In this section, we will only require sums of ultrafilters where $W_* \in M_U$, but it is just more convenient not to choose a representative for W_*.

Lemma 3.5.9. *Suppose U is an ultrafilter and W_* is an M_U-ultrafilter on $j_U(Y)$. Then $j_{[U,W_*]} = j^{M_U}_{W_*} \circ j_U$, and $\mathrm{id}_{[U,W_*]} = (j^{M_U}_{W_*}(\mathrm{id}_U), \mathrm{id}_{W_*})$.* $\square$

In the context of Theorem 3.5.4, we would like to use Lemma 3.5.5 to conclude that the ultrafilters $[U, W_*]$ and $[W, U_*]$ are either equal or incomparable in the Ketonen order, and thus conclude by the linearity of the Ketonen order that $[U, W_*] = [W, U_*]$. The only remaining problem is that $[U, W_*]$ and $[W, U_*]$ are not ultrafilters on ordinals. But obviously we can associate Ketonen orders to an arbitrary wellorder:

Definition 3.5.10. Suppose $(X, \prec)$ is a wellorder. The *Ketonen order associated to* $(X, \prec)$ is the order $(\mathbf{UF}(X), \prec^{\Bbbk})$ defined on $U, W \in \mathbf{UF}(X)$ by setting $U \prec^{\Bbbk} W$ if there exist $I \in W$ and $\langle U_x : x \in I\rangle \in \prod_{x\in I} \mathbf{UF}(X, X_{\prec x})$ such that $U = W\text{-}\lim_{x\in I} U_x$.

If $(X, \prec)$ and $(X', \prec')$ are isomorphic wellorders, then the associated Ketonen orders are also isomorphic, so in particular all the characterizations of the Ketonen order generalize to arbitrary wellorders:

Lemma 3.5.11. *Suppose $(X, \prec)$ is a wellorder and $U, W \in \mathbf{UF}(X)$. Then the following are equivalent:*

(1) $U \prec^{\Bbbk} W$.

(2) There is a right-internal ultrapower comparison $(k, h) : (M_U, M_W) \to N$ of (j_U, j_W) such that $k(\mathrm{id}_U) \prec^* h(\mathrm{id}_W)$ where $\prec^* = k(j_U(\prec)) = h(j_W(\prec))$. ☐

It is convenient to introduce some notation for the statement of Lemma 3.5.13:

Definition 3.5.12. Let flip : $\mathrm{Ord} \times \mathrm{Ord} \to \mathrm{Ord} \times \mathrm{Ord}$ be defined by $\mathrm{flip}(\alpha, \beta) = (\beta, \alpha)$. Let $\prec$ denote the Gödel order on $\mathrm{Ord} \times \mathrm{Ord}$.

The only property of the Gödel order that we need is that $(\alpha_0, \beta_0) \prec (\alpha_1, \beta_1)$ implies that either $\alpha_0 < \alpha_1$ or $\beta_0 < \beta_1$.

Lemma 3.5.13. *Suppose ϵ and δ are ordinals. Suppose $U \in \mathbf{UF}(\epsilon)$ and $W \in \mathbf{UF}(\delta)$. Assume the Ketonen order $(\mathbf{UF}(\epsilon \times \delta), \prec^{\Bbbk})$ is linear.*

- *Let W_* be the least element of $j_U(\mathbf{UF}(\delta), <_{\Bbbk})$ extending $j_U[W]$.*
- *Let U_* be the least element of $j_W(\mathbf{UF}(\epsilon), <_{\Bbbk})$ extending $j_W[U]$.*

Then $[U, W_*] = \mathrm{flip}_*([W, U_*])$.

Proof. Assume towards a contradiction that $[U, W_*] \prec^{\Bbbk} \mathrm{flip}_*([W, U_*])$. The following identities are easily verified using Lemma 3.5.9:

$$j_{[U,W_*]} = j^{M_U}_{W_*} \circ j_U \qquad j_{\mathrm{flip}_*([W,U_*])} = j^{M_W}_{U_*} \circ j_W$$
$$\mathrm{id}_{[U,W_*]} = (j^{M_U}_{W_*}(\mathrm{id}_U), \mathrm{id}_{W_*}) \qquad \mathrm{id}_{\mathrm{flip}_*([W,U_*])} = (\mathrm{id}_{U_*}, j^{M_W}_{U_*}(\mathrm{id}_W))$$

By Lemma 3.5.11, the assumption that $[U, W_*] \prec^{\Bbbk} \mathrm{flip}_*([W, U_*])$ is equivalent to the existence of a right-internal comparison

$$(k, h) : (M^{M_U}_{W_*}, M^{M_W}_{U_*}) \to N$$

of $(j^{M_U}_{W_*} \circ j_U, j^{M_W}_{U_*} \circ j_W)$ such that

$$k(j^{M_U}_{W_*}(\mathrm{id}_U), \mathrm{id}_{W_*}) \prec h(\mathrm{id}_{U_*}, j^{M_W}_{U_*}(\mathrm{id}_W))$$

Therefore either $k(j^{M_U}_{W_*}(\mathrm{id}_U)) < h(\mathrm{id}_{U_*})$ or $k(\mathrm{id}_{W_*}) < h(j^{M_W}_{U_*}(\mathrm{id}_W))$, contradicting Lemma 3.5.5.

A symmetric argument shows that we cannot have $\mathrm{flip}_*([W, U_*]) \prec^{\Bbbk} [U, W_*]$ either. Thus by the linearity of $(\mathbf{UF}(\epsilon \times \delta), \prec^{\Bbbk})$, we must have $[U, W_*] = \mathrm{flip}_*([W, U_*])$, which proves the theorem. ☐

As an immediate consequence, we can prove Theorem 3.5.4:

Proof of Theorem 3.5.4. Let α be the ordertype of the Gödel order on $\epsilon \times \delta$. Since the Ketonen order is linear on $\mathbf{UF}(\alpha)$, the isomorphic order $(\mathbf{UF}(\epsilon\times\delta), \prec^{\Bbbk})$ is also linear. Thus we can apply Lemma 3.5.13 to conclude that $[U, W_*] = \text{flip}_*([W, U_*])$. In particular, $[U, W_*] \equiv [W, U_*]$, so applying Lemma 3.5.9,

$$j_{W_*}^{M_U} \circ j_U = j_{[U,W_*]} = j_{[W,U_*]} = j_{U_*}^{M_W} \circ j_W$$

Thus $(j_{W_*}^{M_U}, j_{U_*}^{M_W})$ is a comparison of (j_U, j_W), as desired. □

Let us make some comments on this theorem. It is not immediately obvious from the definition that the linearity of the Ketonen order on $\mathbf{UF}(\lambda)$ implies the linearity of the Ketonen order on $\mathbf{UF}(\delta)$ for all ordinals $\delta < \lambda^+$.[5]

Definition 3.5.14. Suppose λ is a cardinal.

- $\text{UA}_{<\lambda}$ is the assertion that any pair of ultrapower embeddings of width less than λ have an internal ultrapower comparison.
- $\text{UA}_{\leq\lambda}$ is another way of writing $\text{UA}_{<\lambda^+}$.

Corollary 3.5.15. *Suppose λ is an infinite cardinal and the Ketonen order is linear on $\mathbf{UF}(\lambda)$. Then $\text{UA}_{\leq\lambda}$ holds. In particular, the Ketonen order is linear on $\mathbf{UF}(\delta)$ for all $\delta < \lambda^+$.*

Proof. Suppose U and W are ultrafilters on λ. To see $\text{UA}_{\leq\lambda}$, it suffices to show that (j_U, j_W) has a comparison. Since the Ketonen order $(\mathbf{UF}(\lambda), <_{\Bbbk})$ is linear, so is $(\mathbf{UF}(X), \prec^{\Bbbk})$ whenever $(X, \prec)$ is a wellorder of ordertype λ. Since λ is an infinite cardinal, the Gödel order on $\lambda\times\lambda$ has ordertype λ. Thus $(\mathbf{UF}(\lambda\times\lambda), \prec^{\Bbbk})$ is linear, and so we can apply Lemma 3.5.13 and the proof of Theorem 3.5.4 to conclude that (j_U, j_W) has a comparison. □

Surely with some extra work one can prove the following conjecture:

Conjecture 3.5.16. *If the Ketonen order is linear on countably complete incompressible ultrafilters, then the Ultrapower Axiom holds.*

The proof of Theorem 3.5.1 that we have given here uses Łoś's Theorem, which makes significant use of the Axiom of Choice. With care, however, the combinatorial content of Theorem 3.5.1, namely Lemma 3.5.13, can actually be established in ZF + DC alone. This makes the following question seem interesting:

Question 3.5.17. *Assume $\text{AD} + V = L(\mathbb{R})$. Is the Ketonen order linear?*

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[5]Note that if κ is regular, then for any $n < \omega$, the collection of subsets of κ^n of ordertype less than κ^n forms a κ-complete ideal; this is closely related to the Milner-Rado Paradox. Therefore for example if κ is 2^κ-strongly compact, there is a κ-complete ultrafilter on κ^n that does not concentrate on a set of ordertype less than κ^n. (It suffices that κ is measurable.) This suggests it may be nontrivial to reduce the linearity of $(\mathbf{UF}(\kappa^2), <_{\Bbbk})$ to that of $(\mathbf{UF}(\kappa), <_{\Bbbk})$ by a direct combinatorial argument.

Chapter 4

The Generalized Mitchell Order

4.1 Introduction

4.1.1 The linearity of the generalized Mitchell order

The topic of this section is the generalized Mitchell order, which is defined simply by extending the definition of the Mitchell order on normal ultrafilters (Definition 2.2.38) to all countably complete ultrafilters:

Definition 4.1.1. The *generalized Mitchell order* is defined on countably complete ultrafilters U and W by setting $U \lhd W$ if $U \in M_W$.

The main question we investigate here is to what extent this generalized order is linear assuming the Ultrapower Axiom. Recall that UA implies the linearity of the Mitchell order on normal ultrafilters (Theorem 2.3.11). On the other hand, the generalized Mitchell order is obviously not a linear order on arbitrary countably complete ultrafilters. (The nonlinearity of the Mitchell order is discussed in Section 4.2.5.) The main theorem of this chapter is the generalization of Theorem 2.3.11 to the ultrafilters associated to supercompact and huge cardinals:

Definition 4.1.2. For any ordinal λ, the *bounded powerset of* λ is the set $P_{\text{bd}}(\lambda) = \bigcup_{\alpha<\lambda} P(\alpha)$.

Theorem 4.4.2 (UA)**.** *Suppose λ is a cardinal such that $2^{<\lambda} = \lambda$. Then the Mitchell order is linear on normal fine ultrafilters on $P_{\text{bd}}(\lambda)$.*

This amounts to the most general form of the linearity of the Mitchell order on normal fine ultrafilters that one could hope for (Proposition 4.4.12), except for the cardinal arithmetic assumption on λ (which we dispense with much later in Theorem 7.5.42).

4.1.2 Outline of Chapter 4

We now outline the rest of this chapter.

Section 4.2. This contains various folklore facts about large cardinals and the generalized Mitchell order. None of these results is due to the author. We give a brief exposition of the theory of strong embeddings (Section 4.2.1) and supercompact embeddings (Section 4.2.2) centered around the relationship between these concepts and the generalized Mitchell order. We also exposit the Kunen Inconsistency Theorem, which is closely related to the wellfoundedness properties of the Mitchell order. Finally we establish the basic order theoretic properties of the generalized Mitchell order, especially its transitivity, wellfoundedness (Theorem 4.2.45), and nonlinearity (Section 4.2.5).

Section 4.3. This section introduces the notion of Dodd soundness. This concept first arose in inner model theory, and our exposition is the first to put it into a general context. We begin by giving a very simple definition of Dodd soundness that will hopefully help the reader view it as a natural refinement of supercompactness. We then prove the equivalence of this notion with the definition of Dodd soundness from fine structure theory (Theorem 4.3.22). A theorem of Schlutzenberg [8] (stated as Theorem 4.3.1 below) shows that the Mitchell order is linear on Dodd sound ultrafilters in the canonical inner models. We prove this theorem (Theorem 4.3.29) here under the much weaker assumption of UA and by a completely different and much simpler argument directly generalizing the proof of the linearity of the Mitchell order on normal ultrafilters.

Section 4.4. We finally turn to the Mitchell order on normal fine ultrafilters, the natural generalization of normal ultrafilters to the realm of supercompact cardinals. Our analysis proceeds by showing that normal fine ultrafilters are Rudin-Keisler equivalent to Dodd sound ultrafilters, and then citing the linearity of the Mitchell order on Dodd sound ultrafilters. To do this, we introduce the notion of an *isonormal ultrafilter* and prove that every normal fine ultrafilter is Rudin-Keisler equivalent to an isonormal ultrafilter (Theorem 4.4.37). The main difficulty is the "singular case" (Section 4.4.4) which amounts to generalizing Solovay's Lemma [19] (proved as Theorem 4.4.27) to singular cardinals. Theorem 4.4.25 states that if $2^{<\lambda} = \lambda$, then isonormal ultrafilters on λ are Dodd sound. Putting these theorems together, we obtain that under the Generalized Continuum Hypothesis, normal fine ultrafilters are Rudin-Keisler equivalent to Dodd sound ultrafilters, yielding the main theorem of the chapter (Theorem 4.4.2), the linearity of the Mitchell order on normal fine ultrafilters.

4.2 Folklore of the generalized Mitchell order

4.2.1 Strength and the Mitchell order

The generalized Mitchell order is often viewed as a more finely calibrated generalization of the concept of the strength of an elementary embedding. In this subsection, we set down the basic theory of strength and discuss its relationship with the Mitchell order.

Definition 4.2.1. If X is a set, M is an inner model and $j : V \to M$ is an elementary embedding, then j is *X-hypermeasurable* if $X \in M$.

Notice that the property of being X-hypermeasurable depends only on M, and most of what we will prove about hypermeasurable embeddings really applies to inner models in general.

It would be strange to define hypermeasurable embeddings without defining hypermeasurable cardinals, so let us include the definition even though we will have little to say about the concept:

Definition 4.2.2. Suppose $\kappa \leq \lambda$ are cardinals. Then κ is *X-hypermeasurable* if there is an X-hypermeasurable embedding from the universe of sets into an inner model with critical point κ and *strong* if it is Y-hypermeasurable for all sets Y.

An elementary embedding $j : V \to M$ is said to be *α-strong* if $V_{\text{crit}(j)+\alpha} \subseteq M$, which is why we are forced to use the term "hypermeasurable."

The notion of X-hypermeasurability is not very natural for arbitrary sets X, and we will be most interested in it when $X = H(\lambda)$ for some cardinal λ:

Definition 4.2.3. If x is a set, $\text{tc}(x)$ denotes the smallest transitive set y with $x \subseteq y$. The *hereditary cardinality* of x is the cardinality of $\text{tc}(x)$. For any cardinal λ, $H(\lambda)$ denotes the collection of sets of hereditary cardinality less than λ.

Lemma 4.2.4. *For any infinite cardinal λ,*

- *$H(\lambda^+)$ is a transitive set.*
- *$H(\lambda^+)$ is bi-interpretable with $P(\lambda)$.*
- *$H(\lambda)$ is bi-interpretable with $P_{\text{bd}}(\lambda)$.* □

The bi-interpretability of $H(\lambda^+)$ and $P(\lambda)$ yields the following lemma:

Lemma 4.2.5. *An embedding $j : V \to M$ is $H(\lambda)$-hypermeasurable (resp. $H(\lambda^+)$-hypermeasurable) if and only if it is $P_{\text{bd}}(\lambda)$-hypermeasurable (resp. $P(\lambda)$-hypermeasurable).* □

Definition 4.2.6. The *strength* of an elementary embedding $j : V \to M$, denoted $\textsc{str}(j)$, is the largest cardinal λ such that j is $H(\lambda)$-hypermeasurable.

The following fact specifies exactly which powersets are contained in the target model of an elementary embedding in terms of its strength:

Lemma 4.2.7. *Suppose $j : V \to M$ is an elementary embedding and λ is a cardinal. Then the following are equivalent:*

(1) $\text{STR}(j) = \lambda$.

(2) For all $X \in M$, $P(X) \subseteq M$ if and only if $|X|^M < \lambda$. □

The main limitation on the strength of an elementary embedding is known as the *Kunen Inconsistency Theorem* [20]:

Theorem 4.2.8 (Kunen)**.** *Suppose $j : V \to M$ is a nontrivial elementary embedding and λ is the first fixed point of j above* $\text{crit}(j)$*. Then* $\text{STR}(j) \leq \lambda$. □

We prove this and other related facts in Section 4.2.3.

The basic relationship between strength and the Mitchell order is given by the following two lemmas:

Lemma 4.2.9. *Suppose U and W are countably complete ultrafilters and $U \lhd W$. Then M_W is $P(\lambda)$-hypermeasurable where λ is the cardinality of the underlying set X of U. In fact, $P(X) \subseteq M_W$.*

Proof. Clearly $X \in M_W$ since $X \in U \in M_W$. It suffices to show that $P(X) \subseteq M_W$. Fix $A \subseteq X$, and we will show $A \in M_W$. Since U is an ultrafilter, either $A \in U$ or $X \setminus A \in U$. If $A \in U$, then $A \in U \in M_W$, so $A \in M_W$. If $X \setminus A \in U$, then similarly $X \setminus A \in M_W$, and since $X \in M_W$, it follows that $A = X \setminus (X \setminus A) \in M_W$. Therefore in either case, $A \in M_W$. □

Lemma 4.2.10. *Suppose W is a countably complete ultrafilter and j_W is $P(2^\lambda)$-hypermeasurable. Then for any countably complete ultrafilter U on λ, $U \lhd W$.*

Proof. Since $U \subseteq P(\lambda)$, $U \in H((2^\lambda)^+) \subseteq M_W$. □

This strength requirement implicit in the definition of the generalized Mitchell order may seem somewhat unnatural. What if one modified the Mitchell order, considering for example the *amenability relation* defined on countably complete ultrafilters by setting $U \lessdot W$ if and only if U concentrates on M_W and $U \cap M_W \in M_W$? Such modified Mitchell orders are the subject of Section 5.5.

For the time being, we must point out some irritating properties of the generalized Mitchell order that suggest that in some sense it may be a little bit *too* general. The issue is that the definition of $U \lhd W$ above has a strong dependence on the choice of the underlying set of U. For example, if W is nonprincipal, then the following hold:

- There is a principal ultrafilter D on an ordinal such that $D \not\lhd W$.
- There is a set x such that the principal ultrafilter $\{\{x\}\} \not\lhd W$.

For the first bullet point, let λ be the strength of j_W, and let D be any principal ultrafilter on λ. For the second bullet point, let x be any set that does not belong to M_W.

These silly counterexamples suggest that the generalized Mitchell order is only a well-behaved relation on a restricted class of ultrafilters. Recall that for any ultrafilter U on a set X, λ_U is defined to be the least cardinality of a set in U, and U is said to be uniform if $|X| = \lambda_U$. Hereditary uniformity is a strengthening of uniformity:

Definition 4.2.11. An ultrafilter is *hereditarily uniform* if its size is equal to the hereditary cardinality of its underlying set. We let $\mathbf{HU}(\lambda)$ denote the set of hereditarily uniform ultrafilters U such that $\lambda_U < \lambda$.

That is, an ultrafilter U on X is hereditarily uniform if $\lambda_U = |\text{tc}(X)|$.

Any ultrafilter U is Rudin-Keisler equivalent to a hereditarily uniform ultrafilter since in fact U is Rudin-Keisler equivalent to an ultrafilter on λ_U (Lemma 2.2.32). The following lemma argues that the generalized Mitchell order is a reasonable relation on the class of hereditarily uniform ultrafilters:

Lemma 4.2.12. *Suppose $U' \leq_{\text{RK}} U \lhd W$ are countably complete ultrafilters. Let X and X' be the underlying sets of U and U', and assume $X' \in M_W$ and M_W satisfies $|X'| \leq |X|$. Then $U' \lhd W$ and M_W satisfies $U' \leq_{\text{RK}} U$. If $U' \equiv_{\text{RK}} U$, then M_W satisfies $U' \equiv_{\text{RK}} U$.*

Lemma 4.2.13. *Suppose $U' \leq_{\text{RK}} U \lhd W$ are countably complete ultrafilters and U' is hereditarily uniform. Then $U' \lhd W$ and M_W satisfies $U' \leq_{\text{RK}} U$. If $U' \equiv_{\text{RK}} U$, then M_W satisfies $U' \equiv_{\text{RK}} U$. In particular, the restriction of the generalized Mitchell order to hereditarily uniform ultrafilters is invariant under Rudin-Keisler equivalence.*

Lemma 4.2.12 and Lemma 4.2.13 follow from a fact that is both more general and easier to prove:

Lemma 4.2.14. *Suppose M is an inner model of* ZFC*, λ is a cardinal, and $X \in M$ is a set of cardinality λ such that $P(X) \subseteq M$.*

- *For any set $Y \in M$ such that $M \vDash |Y| \leq |X|$, $P(Y) \subseteq M$.*
- *For any set $Y \in M$ such that $M \vDash |Y| \leq |X|$, $P(X \times Y) \subseteq M$.*
- *For any set $Y \in M$ such that $M \vDash |Y| \leq |X|$, every function from X to $P(Y)$ belongs to M.*
- *$P(\lambda) \subseteq M$.*
- *Every set of hereditary cardinality at most λ belongs to M and has hereditary cardinality at most λ in M.* $\square$

The bullet points are arranged in such a way that the reader should have no trouble proving each one in turn.[1]

Proof of Lemma 4.2.12. Fix $f : X \to X'$ such that $f_*(U) = U'$. By Lemma 4.2.14, $f \in M_W$, and hence $U' = f_*(U) \in M_W$. Moreover f witnesses $U' \leq_{\mathrm{RK}} U$ in M_W. Finally if $U' \equiv_{\mathrm{RK}} U$, then this is also witnessed by some $g \in M_W$. □

Proof of Lemma 4.2.13. By Lemma 4.2.14, the underlying set of U' belongs to M_W and has hereditary cardinality at most $\lambda_{U'} \leq \lambda_U \leq |X|$ in M_W, so the lemma follows from Lemma 4.2.12. □

4.2.2 Supercompactness and the Mitchell order

We now turn to a concept that is more pertinent to this monograph than strength: supercompactness.

Definition 4.2.15. Suppose M is a transitive class and X is a set. An elementary embedding $j : V \to M$ is *X-supercompact* if $j[X] \in M$.

The following lemma allows us to focus solely on the case of λ-supercompact embeddings for λ a cardinal:

Lemma 4.2.16. *Suppose X and Y are sets such that $|X| = |Y|$. Then an elementary embedding $j : V \to M$ is X-supercompact if and only if it is Y-supercompact. In particular, j is X-supercompact if and only if j is $|X|$-supercompact.*

Proof. Suppose j is X-supercompact and $f : X \to Y$ is a surjection. Then

$$j(f)[j[X]] = j[Y]$$

so j is Y-supercompact. □

Definition 4.2.17. Suppose $\kappa \leq \lambda$ are cardinals. Then κ is *λ-supercompact* if there is a λ-supercompact embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$; κ is supercompact if κ is λ-supercompact for all cardinals $\lambda \geq \kappa$.

The results of this monograph (Section 8.4.3) single out a class of ultrapower embeddings that are just shy of λ-supercompact, so the following is an important definition:

Definition 4.2.18. Suppose λ is a cardinal. An elementary embedding $j : V \to M$ is *$<\lambda$-supercompact* if j is δ-supercompact for all cardinals $\delta < \lambda$.

The definition of supercompactness is motivated by its relationship with the closure of M under λ-sequences:

[1] It is likely, however, that the second bullet-point cannot be established if M is not assumed to satisfy the Axiom of Choice.

Lemma 4.2.19. *Suppose $j : V \to M$ is an elementary embedding and λ is a cardinal.*

(1) j is λ-supercompact if and only if $j \restriction \lambda \in M$.

(2) If j is λ-supercompact, then j is $P(\lambda)$-hypermeasurable.

(3) If j is λ-supercompact, then $j[X] \in M$ for all X of cardinality λ.

(4) If j is λ-supercompact and $M = H^M(j[V] \cup S)$ for some $S \subseteq M$ such that $S^\lambda \subseteq M$, then $M^\lambda \subseteq M$.

Proof. For (1), note that $j \restriction \lambda$ is the inverse of the transitive collapse of $j[\lambda]$.

For (2), suppose $A \subseteq \lambda$. Then $A = (j \restriction \lambda)^{-1}[j(A)]$, so since $j \restriction \lambda$ and $j(A)$ both belong to M, so does A.

(3) is immediate from Lemma 4.2.16.

For (4), fix $\langle x_\alpha : \alpha < \lambda\rangle \in M^\lambda$. Fix $\langle f_\alpha : \alpha < \lambda\rangle$ and $\langle a_\alpha : \alpha < \lambda\rangle \in S^\lambda$ such that $x_\alpha = j(f_\alpha)(a_\alpha)$ for all $\alpha < \lambda$. The function $G : j[\lambda] \to M$ defined by $G(j(\alpha)) = j(f_\alpha)$ belongs to M by (3), since

$$G = j[\{(\alpha, f_\alpha) : \alpha < \lambda\}]$$

Therefore the sequence $\langle j(f_\alpha) : \alpha < \lambda\rangle$ can be computed from G and $j \restriction \lambda$:

$$j(f_\alpha) = G \circ (j \restriction \lambda)(\alpha)$$

Since both G and $j \restriction \lambda$ belong to M by (1), $\langle j(f_\alpha) : \alpha < \lambda\rangle \in M$. Finally,

$$\langle x_\alpha : \alpha < \lambda\rangle = \langle j(f_\alpha)(a_\alpha) : \alpha < \lambda\rangle$$

can be computed from $\langle j(f_\alpha) : \alpha < \lambda\rangle$ and $\langle a_\alpha : \alpha < \lambda\rangle$. Both these sequences belong to M, since $\langle a_\alpha : \alpha < \lambda\rangle \in S^\lambda \subseteq M$, so $\langle x_\alpha : \alpha < \lambda\rangle \in M$, as desired. $\square$

For the purposes of this monograph, the relevant corollary of Lemma 4.2.19 is its application to ultrapower embeddings:

Corollary 4.2.20. *An ultrapower embedding $j : V \to M$ is λ-supercompact if and only if $M^\lambda \subseteq M$.*

Proof. Fix $a \in M$ such that $M = H^M(j[V] \cup \{a\})$. The corollary follows from applying Lemma 4.2.19 (4) in the case $S = \{a\}$. $\square$

We can make good use of Corollary 4.2.20 since it is always possible to derive a λ-supercompact ultrapower embeddings from a λ-supercompact embedding:

Lemma 4.2.21. *Suppose $j : V \to M$ is an X-supercompact embedding, $V \xrightarrow{i} N \xrightarrow{k} M$ are elementary embeddings, $k \circ i = j$, and $j[X] \in k[N]$. Then i is X-supercompact and $k(i[X]) = j[X]$. In particular, letting $\lambda = |X|$, $k \restriction \lambda + 1$ is the identity.*

Proof. Fix $S \in M$ such that $k(S) = j[X]$. Then

$$S = k^{-1}[k(S)] = k^{-1}[j[X]] = k^{-1} \circ j[X] = i[X]$$

Thus $i[X] = S \in M$, so i is X-supercompact, and moreover, $k(i[X]) = k(S) = j[X]$.

Since $k(i[X]) = j[X]$, the argument of Lemma 4.2.16 shows $k(i[\lambda]) = j[\lambda]$. But then if $\alpha \leq \lambda$, $k(\alpha) = k(\mathrm{ot}(i[\lambda] \cap i(\alpha))) = \mathrm{ot}(k(i[\lambda]) \cap k(i(\alpha))) = \mathrm{ot}(j[\lambda] \cap j(\alpha)) = \alpha$. □

Definition 4.2.22. The *supercompactness* of an elementary embedding is the least cardinal λ such that it is not λ-supercompact.

Which cardinals are the supercompactness of an elementary embedding? Which are the supercompactness of an ultrapower embedding? This turns out to be a major distinction:

Proposition 4.2.23. *Suppose λ is a singular cardinal and $j : V \to M$ is an elementary embedding such that $M^{<\lambda} \subseteq M$. Then $M^\lambda \subseteq M$.* □

Thus the supercompactness of an ultrapower embedding is always regular. On the other hand if there is a $\kappa^{+\omega}$-supercompact cardinal κ, one can easily produce an elementary embedding whose supercompactness is $\kappa^{+\omega}$.

An important point is that if the cofinality of λ is small, λ-supercompactness is equivalent to λ^+-supercompactness:

Lemma 4.2.24. *Suppose λ is a cardinal, $j : V \to M$ is elementary embedding, and $\kappa = \mathrm{crit}(j)$. If j is λ-supercompact, then j is $\lambda^{<\kappa}$-supercompact.*

Proof. Assume $j[\lambda] \in M$, and we will show that $j[P_\kappa(\lambda)] \in M$. Note that for $\sigma \in P_\kappa(\lambda)$, $j(\sigma) = j[\sigma]$. Thus

$$j[P_\kappa(\lambda)] = \{j[\sigma] : \sigma \in P_\kappa(\lambda)\} = P_\kappa(j[\lambda])$$

One consequence of this is that $P_\kappa(j[\lambda]) \subseteq M$, since $j[P_\kappa(\lambda)] \subseteq M$, and therefore $P_\kappa(j[\lambda]) = (P_\kappa(j[\lambda]))^M \in M$. It follows that $j[P_\kappa(\lambda)] \in M$, as desired. □

It follows for example that the supercompactness of an elementary embedding is never the successor of a singular cardinal γ of countable cofinality, since $\gamma^\omega \geq \gamma^+$. This is an important component in the proof of Kunen's Inconsistency Theorem (Theorem 4.2.35).

We now begin to examine the relationship between supercompactness and the Mitchell order, which turns out to be central to the rest of this monograph. The key point is that if $U \lhd W$, then the supercompactness of M_W determines the extent to which the ultrapower of M_W by U is correctly computed by M_W.

Lemma 4.2.25. *Suppose $U \lhd W$ are countably complete ultrafilters. Then there is a unique elementary embedding $k : (M_U)^{M_W} \to j_U(M_W)$ such that $k \circ (j_U)^{M_W} = j_U \restriction M_W$ and $k(\mathrm{id}_U^{M_W}) = \mathrm{id}_U$. Let X be the underlying set of U. Then $k(\alpha) = \alpha$ for all $\alpha \leq j_U((2^\lambda)^{M_W})$ where $\lambda = |X|$.*

Proof. Since $P(X) \subseteq M_W$, U is the ultrafilter derived from $j_U \restriction M_W$ using id_U. Thus there is a unique factor embedding $k : (M_U)^{M_W} \to j_U(M_W)$ such that $k \circ (j_U)^{M_W} = j_U \restriction M_W$ and $k(\mathrm{id}_U^{M_W}) = \mathrm{id}_U$. This establishes the first part of the lemma.

As for the second part, since $U \lhd W$, we have $P(X) \subseteq M_W$ and hence by Lemma 4.2.14, $P(\lambda) \subseteq M_W$ and every function from X to $P(\lambda)$ belongs to M_W. It follows that $j_U(P(\lambda)) \subseteq \mathrm{ran}(k)$: if $A \in j_U(P(\lambda))$, then $A = j_U(f)(\mathrm{id}_U)$ for some $f : X \to P(\lambda)$, and therefore

$$A = k(j_U^{M_W}(f)(\mathrm{id}_U^{M_W})) \in \mathrm{ran}(k)$$

Since there is a surjection $g : P(\lambda) \to (2^\lambda)^{M_W}$ in M_W,

$$j_U(g)[j_U(P(\lambda))] = j_U((2^\lambda)^{M_W}) \subseteq \mathrm{ran}(k)$$

Moreover $j_U((2^\lambda)^{M_W}) \in j_U[M_W] \subseteq \mathrm{ran}(k)$. Thus $j_U((2^\lambda)^{M_W}) + 1 \subseteq \mathrm{ran}(k)$, or in other words, $k(\alpha) = \alpha$ for all $\alpha \leq j_U((2^\lambda)^{M_W})$. □

We will refer to the embedding of Lemma 4.2.25 as a *factor embedding.*

Lemma 4.2.26. *Suppose U and W are countably complete ultrafilters with $U \lhd W$. Let X be the underlying set of U, let $\lambda = |X|$ and let $\delta = ((2^\lambda)^+)^{M_W}$. Then*

$$j_U^{M_W} \restriction H^{M_W}(\delta) = j_U \restriction H^{M_W}(\delta)$$

Proof. Let $k : (M_U)^{M_W} \to j_U(M_W)$ be the factor embedding with $k \circ (j_U)^{M_W} = j_U \restriction M_W$ and $k(\mathrm{id}_U^{M_W}) = \mathrm{id}_U$. Then Lemma 4.2.25 implies $k \restriction j_U^{M_W}(\delta)$ is the identity, and therefore $k \restriction j_U^{M_W}(H^{M_W}(\delta))$ is the identity. Now

$$j_U^{M_W} \restriction H^{M_W}(\delta) = (k \restriction j_U^{M_W}(H^{M_W}(\delta))) \circ (j_U^{M_W} \restriction H^{M_W}(\delta)) = j_U \restriction H^{M_W}(\delta) \quad \square$$

Our next proposition, Proposition 4.2.27, suggests that the Mitchell order on ultrafilters be seen as a generalization of supercompactness that asks for one ultrapower M_W how much it can see of another embedding j_U. (On this view supercompactness is the special case in which we ask how much of j_U is seen by M_U itself.)

Proposition 4.2.27. *Suppose U and W are countably complete ultrafilters. Let X be the underlying set of U, let $\lambda = |X|$ and let $\delta = ((2^\lambda)^+)^{M_W}$. Then the following are equivalent:*

(1) $U \lhd W$.

(2) $j_U \restriction H^{M_W}(\delta) \in M_W$.

(3) $j_U \restriction P(\lambda) \in M_W$.

(4) $j_U \restriction P(X) \in M_W$.

Proof. (1) implies (2). Immediate from Lemma 4.2.26.

(2) implies (3). Immediate since $P(\lambda) \subseteq H^{M_W}(\delta)$.

(3) implies (4). This is probably clear enough (and in any case, (1) implies (1) is easy), but let us just make sure. By Lemma 4.2.14, $|X|^M = \lambda$. Let $\rho : \lambda \to X$ be a surjection in M_W. For $A \in P(X)$,

$$j_U(A) = j_U(\rho)[j_U(\rho^{-1}[A])]$$

(4) implies (1). If $j_U \restriction P(X)$ belongs to M_W, then $U = \{A \subseteq X : \mathrm{id}_U \in j_U(A)\}$ belongs to M_W as well. □

Given Lemma 4.2.26, it is reasonable to wonder whether the entire embedding $j_U \restriction M_W$ might be correctly computed by M_W as well; that is, perhaps the factor embedding k is always trivial. We provide a counterexample in Proposition 5.5.6.[2] This is equivalent to the supercompactness of j_W, a phenomenon we exploit later:

Proposition 4.2.28. *Suppose $U \lhd W$ are countably complete ultrafilters. Then the following are equivalent:*

(1) $(j_U)^{M_W} = j_U \restriction M_W$.

(2) j_W *is* λ_U*-supercompact.*

Proof. (1) implies (2): Let $k : (M_U)^{M_W} \to j_U(M_W)$ be the factor embedding of Lemma 4.2.25, with $k \circ j_U^{M_W} = j_U \restriction M_W$ and $k(\mathrm{id}_U^{M_W}) = \mathrm{id}_U$. Since $(j_U)^{M_W} = j_U \restriction M_W$, we have that $k : j_U(M_W) \to j_U(M_W)$ and $k \circ j_U \circ j_W = j_U \circ j_W$. Hence by the basic theory of the Rudin-Keisler order (Theorem 3.4.8), k is the identity.

It follows in particular that $j_U(j_W)(\mathrm{id}_U) \in \mathrm{ran}(k)$. Fix $f : X \to M_W$ in M_W such that

$$k(j_U^{M_W}(f)(\mathrm{id}_U^{M_W})) = j_U(j_W)(\mathrm{id}_U)$$

Thus $j_U(f)(\mathrm{id}_U) = j_U(j_W)(\mathrm{id}_U)$, so by Łoś's Theorem, there is a set $A \in U$ such that $f \restriction A = j_W \restriction A$. Since $P(X) \subseteq M_W$, $A \in M_W$, and hence $j_W \restriction A = f \restriction A \in M_W$. In particular, $j_W[A] \in M_W$, so j_W is A-supercompact. By Lemma 4.2.16, j_W is $|A|$-supercompact, and since $\lambda_U \leq |A|$, it follows that j_W is λ_U-supercompact.

(2) implies (1): Obvious. □

After building up enough machinery, we will show that under UA, whenever $U \lhd W$, in fact j_W is λ_U-supercompact (Theorem 8.3.29), and thus $j_W^{M_U} = j_W \restriction M_U$. For now, let us mention a generalization of Proposition 4.2.28, which actually follows from the proof given above:

[2] This counterexample also shows that in the context of Lemma 4.2.25, the lower bound given there on $\mathrm{crit}(k)$ can be tight in the sense that (consistently) one can have

$$\mathrm{crit}(k) = j_U\left((2^\lambda)^{M_W}\right)^{+(M_U)^{M_W}}$$

Proposition 4.2.29. *Suppose U and W are countably complete ultrafilters such that U concentrates on a set in M_W. The following are equivalent:*

(1) $j^{M_W}_{U\cap M_W} = j_U \restriction M_W$

(2) There is a function $f \in M_W$ such that $f \restriction A = j_W \restriction A$ for some $A \in U$.

(3) For all $f : I \to M_W$ where $I \in U$, there is some $g \in M_W$ such that $g \restriction A = f \restriction A$ for some $A \in U$. □

We finish this section with a restriction on the supercompactness of an ultrafilter:

Proposition 4.2.30. *Suppose U is an ultrafilter and j_U is λ_U^+-supercompact. Then U is principal.*

We use the following lemma:

Lemma 4.2.31. *Suppose $j : V \to M$ is an elementary embedding that is discontinuous at the infinite cardinal λ. Let $\lambda_* = \sup j[\lambda]$. Then*

$$\lambda^+ \leq \lambda_*^{+M} < j(\lambda)^{+M} = j(\lambda^+)$$

If j is continuous at λ^+, then $j(\lambda^+)$ is a singular ordinal of cofinality λ^+, so $j(\lambda^+) < j(\lambda)^+$.

Proof. We first show that $\lambda^+ \leq \lambda_*^{+M}$. Suppose $\alpha < \lambda^+$. Let $\prec$ be a wellorder of λ such that $\text{ot}(\prec) = \alpha$. Then $\prec_* = j(\prec) \restriction \lambda_*$ is a wellorder of λ_* and j restricts to an order-preserving embedding from $(\lambda, \prec)$ into $(\lambda_*, \prec_*)$. Therefore

$$\alpha \leq \text{ot}(\lambda_*, \prec_*) < \lambda_*^{+M}$$

The final inequality follows from the fact that $(\lambda_*, \prec_*)$ belongs to M. Since $\alpha < \lambda^+$ was arbitrary, it follows that $\lambda^+ \leq \lambda_*^{+M}$.

To prove $\lambda_*^{+M} < j(\lambda)^{+M}$, it is of course enough to show $\lambda_*^{+M} \leq j(\lambda)$. But $j(\lambda)$ is a cardinal of M that is greater than λ_*, and hence $\lambda_*^{+M} \leq j(\lambda)$.

Finally, assume that j is continuous at λ^+. Obviously $j(\lambda^+)$ has cofinality λ^+, but the point is that this implies $j(\lambda^+)$ is *singular*, since the inequalities above show $\lambda^+ < j(\lambda^+)$. We can therefore conclude $j(\lambda)^{+M} < j(\lambda)^+$: obviously $j(\lambda)^{+M} \leq j(\lambda)^+$, and equality cannot hold since $j(\lambda^+)$ is singular and $j(\lambda)^+$ is regular. □

Proof of Proposition 4.2.30. Let $\lambda = \lambda_U$. Without loss of generality, we may assume that U is a uniform ultrafilter on λ and λ is infinite. Thus j_U is discontinuous at λ. Assume towards a contradiction that $j_U[\lambda^+] \in M_U$. By Lemma 4.2.31, $j_U(\lambda^+) > \lambda^+$. But by Lemma 2.2.34, j_U is continuous at λ^+, and therefore $j_U[\lambda^+] \in M_U$ is a cofinal subset of $j_U(\lambda^+)$ of ordertype λ^+. Hence $\text{cf}^{M_U}(j_U(\lambda)^{+M_U}) = \lambda^+ < j_U(\lambda^+)$, and this contradicts that $j_U(\lambda^+)$ is regular in M_U. □

4.2.3 The Kunen Inconsistency

The story of the Kunen Inconsistency Theorem is often cast as a cautionary tale with the moral that a large cardinal hypothesis may turn out to be false for nontrivial combinatorial reasons:

Theorem 4.2.32 (Kunen)**.** *There is no nontrivial elementary embedding from the universe to itself.* □

A more pragmatic perspective is to view the Kunen Inconsistency as a proof technique, providing at least some constraint on the elementary embeddings a large cardinal theorist is bound to analyze. Examples pervade this work, but for example, the Kunen Inconsistency will form a key component of the proof of the wellfoundedness of the Mitchell order in Section 4.2.4. Since our applications of Kunen's theorem will require the basic concepts from the proof (especially the notion of a critical sequence), we devote this subsection for a brief exposition of this topic.

We first give a proof of a version of Kunen's inconsistency Theorem that is due to Harada. (Another writeup of this proof appears in Kanamori's textbook [21].) The methods are purely ultrafilter-theoretic and very much in the spirit of this monograph.

Definition 4.2.33. Suppose N and P are transitive models of ZFC and $j : N \to P$ is a nontrivial elementary embedding. The *critical sequence* of j is defined by recursion: set $\kappa_0(j) = \text{crit}(j)$, and for $n < \omega$, if $\kappa_n(j) \in N$, set $\kappa_{n+1}(j) = j(\kappa_n(j))$; otherwise $\kappa_{n+1}(j)$ is undefined. If $\kappa_n(j)$ is defined for all $n < \omega$, then $\kappa_\omega(j) = \sup_{n<\omega} \kappa_n(j)$.

In the context of Definition 4.2.33, if $\kappa_\omega(j)$ is defined, it is the least ordinal greater than $\text{crit}(j)$ such that $j[\lambda] \subseteq \lambda$. If $\lambda \in N$ and $\text{cf}^N(\lambda) = \omega$, then j is continuous at λ, so $j(\lambda) = \lambda$. In particular, if $N = V$, which is the case of interest in this section, then λ is the first fixed point of j above $\text{crit}(j)$.

In the case $n > 1$, the conclusion of the following lemma is a considerable understatement:

Lemma 4.2.34. *Suppose $j : V \to M$ is a nontrivial elementary embedding and $\langle \kappa_n : n < \omega \rangle$ is its critical sequence. For any $n < \omega$, if j is $P(\kappa_n)$-hypermeasurable then κ_n is measurable.*

Proof. The proof is by induction on n. Certainly $\kappa_0 = \text{crit}(j)$ is measurable. Assume the lemma is true for $n = m$, and we will show it is true for $n = m + 1$. Therefore assume j is $P(\kappa_{m+1})$-hypermeasurable. In particular, j is $P(\kappa_m)$-hypermeasurable, so by our induction hypothesis, κ_m is measurable. By elementarity, $\kappa_{m+1} = j(\kappa_m)$ is measurable in M. Since j is $P(\kappa_{m+1})$-hypermeasurable, $P(\kappa_{m+1}) \subseteq M$. Thus the measurability of κ_{m+1} in M is upwards absolute to V, so κ_{m+1} is measurable. □

Theorem 4.2.35 (Kunen)**.** *Suppose λ is an ordinal, $j : V \to M$ is λ-supercompact, and $j[\lambda] \subseteq \lambda$. Then $j \restriction \lambda$ is the identity.*

Proof. Let λ be the least ordinal greater than $\text{crit}(j)$ such that $j[\lambda] \subseteq \lambda$. It suffices to show that j is not λ-supercompact, so assume towards a contradiction that it is.

Let U be the ultrafilter on $P(\lambda)$ derived from j using $j[\lambda]$. Then $j_U : V \to M_U$ is λ-supercompact and $j_U \restriction \lambda = j \restriction \lambda$ by Lemma 4.2.21. Since $\text{crit}(j) < \lambda$, it follows that $\text{crit}(j_U)$ exists, so in particular, U is nonprincipal.

Note that $\lambda = \kappa_\omega(j)$ and j is $P(\lambda)$-hypermeasurable by Lemma 4.2.19. Therefore by Lemma 4.2.34, $\kappa_n(j)$ is measurable. In particular, λ is a limit of inaccessible cardinals, and so in particular, λ is a strong limit cardinal. It follows from Lemma 4.2.24 that j_U is λ^ω-supercompact. Since λ is a strong limit cardinal of countable cofinality, $\lambda^\omega = 2^\lambda$. Therefore j_U is 2^λ-supercompact.

Consider the cardinal $\delta = \lambda_U$. Since the underlying set of U is $P(\lambda)$, $\delta \leq 2^\lambda$. On the other hand, $j_U(2^\lambda) = (2^\lambda)^{M_U} = 2^\lambda$ since M_U is closed under 2^λ-sequences. Since U is Rudin-Keisler equivalent to a fine ultrafilter on δ, Lemma 3.2.8 implies that $j_U(\delta) \neq \delta$. Hence $\delta \neq 2^\lambda$, so $\delta < 2^\lambda$. Since j_U is 2^λ-supercompact, in particular j_U is δ^+-supercompact. By Proposition 4.2.30, this implies that U is principal, which is a contradiction. □

The following lemma is a useful consequence of Kunen's Inconsistency Theorem:

Lemma 4.2.36. *Suppose γ is a cardinal, $j : V \to M$ is a nontrivial elementary embedding,* $\text{crit}(j) \leq \gamma$, *and $P(\gamma) \subseteq M$. Then there is a measurable cardinal $\kappa \leq \gamma$ such that $j(\kappa) > \gamma$.*

Proof. Let $\langle \kappa_n : n < \omega \rangle$ be the critical sequence of j and $\lambda = \sup_{n<\omega} \kappa_n$. Thus λ is the least ordinal with $j[\lambda] \subseteq \lambda$. By Theorem 4.2.35, $P(\lambda) \not\subseteq M$, so since $P(\gamma) \subseteq M$, we have $\gamma < \lambda$. Let $n < \omega$ be least such that $\kappa_n \leq \gamma < \kappa_{n+1}$. Lemma 4.2.34 implies κ_n is measurable, and $j(\kappa_n) = \kappa_{n+1} > \gamma$. Thus taking $\kappa = \kappa_n$ proves the lemma. □

In one instance (Theorem 4.4.36), we will need a strengthening of Lemma 4.2.36 which has essentially the same proof:

Lemma 4.2.37. *Suppose $\gamma \leq \lambda$ are cardinals and $j : V \to M$ is a nontrivial λ-supercompact elementary embedding with* $\text{crit}(j) \leq \gamma$. *Then there is a λ-supercompact cardinal $\kappa \leq \gamma$ such that $j(\kappa) > \gamma$.* □

4.2.4 The wellfoundedness of the generalized Mitchell order

The main theorem of this subsection states that the generalized Mitchell order is a wellfounded partial order when restricted to a reasonable class of countably complete ultrafilters. In fact, the wellfoundedness of the generalized Mitchell order on countably complete ultrafilters is a special case of Steel's wellfoundedness theorem for the Mitchell order on extenders [22], since countably complete ultrafilters are amenable extenders in the sense of [22], but we will give a much simpler proof here.

We start with the fundamental fact that the Mitchell order is irreflexive:

Lemma 4.2.38. *Suppose U is a countably complete nonprincipal ultrafilter. Then $U \not\lhd U$.*

Proof. Suppose towards a contradiction that $U \lhd U$. By Lemma 4.2.13, if $U' \equiv_{\text{RK}} U$ is a uniform ultrafilter on a cardinal (as given by Lemma 2.2.32) then $U' \lhd U'$ as well. We can therefore assume without loss of generality that U is a uniform ultrafilter on a cardinal λ. By Proposition 4.2.27, $j_U \restriction P(\lambda) \in M_U$. In particular, $j_U \restriction \lambda \in M_U$, so $M_U^\lambda \subseteq M_U$ by Lemma 4.2.19. Therefore $j_U^{M_U} = j_U \restriction M_U$, for example as a consequence of Proposition 4.2.28. Thus j_U is δ-supercompact for all cardinals δ. This contradicts Proposition 4.2.30. $\square$

We now turn to the transitivity and wellfoundedness of the generalized Mitchell order. The following lemma (which in the language of [22] states that countably complete ultrafilters are *amenable*), is the key to the proof.

Lemma 4.2.39. *Suppose U is a nonprincipal countably complete ultrafilter on a set X. Suppose λ is a cardinal such that $P(\lambda) \subseteq M_U$. Then $M_U \vDash 2^\lambda < j_U(|X|)$.*

Proof. The proof proceeds by finding a measurable cardinal $\kappa \leq |X|$ such that $2^\lambda < j_U(\kappa)$.

If $\lambda < \text{crit}(j_U)$, then $\kappa = \text{crit}(j_U)$ works. Therefore assume $\text{crit}(j_U) \leq \lambda$. By Lemma 4.2.36, there is an measurable cardinal $\kappa \leq \lambda$ such that $j_U(\kappa) > \lambda$. We claim that $\kappa \leq |X|$, which completes the proof. Assume not. Then $|X|$ is smaller than the inaccessible cardinal κ, and hence $j_U(\kappa) = \kappa \leq \lambda$, a contradiction. $\square$

We really only use the following consequence of Lemma 4.2.39:

Corollary 4.2.40. *Suppose $U_0 \lhd U_1$ are countably complete nonprincipal hereditarily uniform ultrafilters. Then $M_{U_1} \vDash 2^{\lambda_{U_0}} < j_{U_1}(\lambda_{U_1})$.*

Proof. This is immediate from Lemma 4.2.39, using the fact (Lemma 4.2.9) that if $U_0 \lhd U_1$ then $P(\lambda_{U_0}) \subseteq M_{U_1}$. $\square$

Corollary 4.2.41. *Suppose $U_0 \lhd U_1$ are countably complete nonprincipal hereditarily uniform ultrafilters. Let $\lambda = \lambda_{U_1}$. Then $U_0 \in j_{U_1}(H(\lambda))$.*

Proof. Since U_0 is hereditarily uniform, $M_{U_1} \vDash |\text{tc}(U_0)| = 2^{\lambda_{U_0}}$ By Corollary 4.2.40, $M_{U_1} \vDash 2^{\lambda_{U_0}} < j_{U_1}(\lambda)$. Therefore $U_0 \in H^{M_{U_1}}(j_{U_1}(\lambda)) = j_{U_1}(H(\lambda))$. $\square$

Proposition 4.2.42. *Suppose $U_0 \lhd U_1 \lhd U_2$ are countably complete nonprincipal hereditarily uniform ultrafilters. Then $U_0 \lhd U_2$ and $M_{U_2} \vDash U_0 \lhd U_1$.*

Proof. Let $\lambda = \lambda_{U_1}$. Then $U_0 \in j_{U_1}(H(\lambda))$. By Lemma 4.2.26, M_{U_2} contains $j_{U_1}(H(\lambda))$, so $U_0 \in M_{U_2}$, which yields $U_0 \lhd U_2$. In fact, by Lemma 4.2.26, $j_{U_1}(H(\lambda)) = j_{U_1}^{M_{U_2}}(H(\lambda))$, and so $U_0 \in j_{U_1}^{M_{U_2}}(H(\lambda)) \subseteq M_{U_1}^{M_{U_2}}$. Thus $U_0 \in M_{U_1}^{M_{U_2}}$, or other words, $M_{U_2} \vDash U_0 \lhd U_1$. $\square$

Corollary 4.2.43. *The generalized Mitchell order is transitive on countably complete nonprincipal hereditarily uniform ultrafilters.* □

The generalized Mitchell order on extenders is *not* transitive if there is a cardinal that is $P(\kappa)$-hypermeasurable where κ is a measurable cardinal. The counterexample is described in [22]. (The generalized Mitchell order is not transitive on arbitrary countably complete ultrafilters either as a consequence of the silly counterexamples in Section 4.2.1.) The failure of transitivity is what makes it so much more difficult to prove the wellfoundedness of the Mitchell order on extenders.

Proposition 4.2.44. *The generalized Mitchell order is wellfounded on countably complete nonprincipal hereditarily uniform ultrafilters.*

Proof. Suppose not, and let λ be the least cardinal such that there is a descending sequence

$$U_0 \rhd U_1 \rhd U_2 \rhd \cdots$$

of countably complete hereditarily uniform ultrafilters with $\lambda_{U_0} = \lambda$.

By Proposition 4.2.42 and the closure of M_{U_0} under countable sequences, the sequence $\langle U_n : 1 \leq n < \omega \rangle$ belongs to M_{U_0} and

$$M_{U_0} \vDash U_1 \rhd U_2 \rhd \cdots$$

Note that in M_{U_0}, U_1 is a countably complete nonprincipal hereditarily uniform ultrafilter, and by Corollary 4.2.40, $\lambda_{U_1} < j_{U_0}(\lambda)$.

On the other hand, by the elementarity of j_{U_0}, from the perspective of M_{U_0}, $j_{U_0}(\lambda)$ is the least cardinal λ' such that there is a descending sequence

$$W_0 \rhd W_1 \rhd W_2 \rhd \cdots$$

of countably complete hereditarily uniform ultrafilters such that $\lambda_{W_0} = \lambda'$. This is a contradiction. □

One can prove a slightly more general result than Proposition 4.2.44 although this generality is never useful.

Theorem 4.2.45. *The generalized Mitchell order is wellfounded on nonprincipal countably complete ultrafilters.*

Proof. Assume towards a contradiction that $U_0 \rhd U_1 \rhd \cdots$ are nonprincipal countably complete ultrafilters. For each $n < \omega$, let U'_n be a hereditarily uniform ultrafilter such that $U'_n \equiv_{\text{RK}} U_n$. Then by Lemma 4.2.13, $U'_0 \rhd U'_1 \rhd \cdots$. This contradicts Proposition 4.2.44. □

4.2.5 The nonlinearity of the generalized Mitchell order

Before we discuss the extent to which the generalized Mitchell order is linear under UA, it is worth pointing out the obvious counterexamples to linearity and the maximal amount of linearity one could reasonably hope for.

The fact is that if there is a measurable cardinal, then the generalized Mitchell order is not linear, even restricting to countably complete incompressible ultrafilters. The known counterexamples to the linearity of the generalized Mitchell order are closely related to the *Rudin-Frolík order* (the subject of Chapter 5):

Definition 4.2.46. The *Rudin-Frolík order* is defined on countably complete ultrafilters U and W by setting $U \leq_{\text{RF}} W$ if there is an internal ultrapower embedding $i : M_D \to M_W$ such that $i \circ j_D = j_W$.

By Lemma 3.4.4, the Rudin-Keisler order can be defined in exactly the same way except omitting the requirement that i be internal.

Proposition 4.2.47. *If $U \leq_{\text{RF}} W$ are nonprincipal countably complete ultrafilters, then U and W are incomparable in the generalized Mitchell order.*

Proof. We first show $U \not\lhd W$. Since $U \leq_{\text{RF}} W$, $M_W \subseteq M_U$. Therefore the fact that $U \notin M_U$ implies that $U \notin M_W$, and hence $U \not\lhd W$.

We now show $W \not\lhd U$. Assume towards a contradiction that $W \lhd U$. Assume without loss of generality that U is a uniform ultrafilter on a cardinal λ. (Since the Mitchell order is Rudin-Keisler invariant in its second argument, this does not change our situation.) Since $U \leq_{\text{RF}} W$, we have $U \leq_{\text{RK}} W$ by Lemma 3.4.4. Since U is hereditarily uniform and $U \leq_{\text{RK}} W \lhd U$, our lemma on the invariance of the Mitchell order (Lemma 4.2.13) yields that $U \lhd U$. This contradicts Lemma 4.2.38. □

A similar argument shows the following:

Proposition 4.2.48. *Suppose U and W are countably complete ultrafilters and there is a nonprincipal $D \leq_{\text{RF}} U, W$. Then U and W are incomparable in the generalized Mitchell order.* □

Even this does not exhaust the known counterexamples to the linearity of the generalized Mitchell order:

Proposition 4.2.49. *Suppose $U_0 \lhd U_1 \lhd U_2$. Suppose $U_0, U_2 \leq_{\text{RF}} W$. Then U_1 and W are incomparable in the Mitchell order.* □

We omit the proof. The hypotheses of the proposition are satisfied if U_0, U_1, U_2 are normal ultrafilters on measurable cardinals $\kappa_0 < \kappa_1 < \kappa_2$ respectively and $W = U_0 \times U_2$.

All known examples of nonlinearity in the generalized Mitchell order are accompanied by nontrivial relations in the Rudin-Frolík order. A driving question in this work is whether assuming UA, these are the only counterexamples.

Definition 4.2.50. A nonprincipal countably complete ultrafilter W is *irreducible* if for all $U \leq_{\text{RF}} W$, either U is principal or U is Rudin-Keisler equivalent to W.

The Irreducible Ultrafilter Hypothesis (IUH) essentially states that the sort of counterexamples to the linearity of the Mitchell order that we have described are the only ones.

Irreducible Ultrafilter Hypothesis. Suppose U and W are hereditarily uniform irreducible ultrafilters. Either $U \equiv_{\text{RK}} W$, $U \lhd W$, or $W \lhd U$.

We can now make precise the question of the extent of the linearity of the Mitchell order under UA:

Question 4.2.51. *Does UA imply IUH?*

With this in mind, let us turn to the positive results on linearity.

4.3 Dodd soundness

4.3.1 Introduction

Dodd soundness is a fine-structural generalization of supercompactness, introduced by Steel [3] in the context of inner model theory as a strengthening of the initial segment condition. The following remarkable theorem is due to Schlutzenberg [8]:

Theorem 4.3.1 (Schlutzenberg). *Suppose $L[\mathbb{E}]$ is an iterable Mitchell-Steel model and U is a countably complete ultrafilter of $L[\mathbb{E}]$. Then the following are equivalent:*

(1) U is irreducible.

(2) U is Rudin-Keisler equivalent to a Dodd sound ultrafilter.

(3) U lies on the sequence $\mathbb{E}$.[3] □

Since the total extenders on $\mathbb{E}$ are linearly ordered by the Mitchell order, this has the following consequence:

Theorem 4.3.2 (Schlutzenberg). *Suppose $L[\mathbb{E}]$ is an iterable Mitchell-Steel model. Then $L[\mathbb{E}]$ satisfies the Irreducible Ultrafilter Hypothesis.* □

It is open whether this theorem can be extended to the Woodin models at the finite levels of supercompactness. The main result of this section (Theorem 4.3.29) states that UA alone suffices to prove the linearity of the generalized Mitchell order on Dodd sound ultrafilters.

[3] That is, the amenable code of the trivial completion of the extender of U lies on $\mathbb{E}$.

4.3.2 Dodd sound embeddings, extenders, and ultrafilters

In this subsection, we present a definition of Dodd soundness due to the author that is simpler than the one given in [3, 8] and easier to use in certain contexts. (The other definition is also useful.) We then show that the two definitions are equivalent.

Definition 4.3.3. Suppose M is a transitive class, $j : V \to M$ is an elementary embedding, and α is an ordinal. Let δ be the least ordinal such that $j(\delta) \geq \alpha$. Then

$$j^\alpha : P(\delta) \to M$$

is the function defined by $j^\alpha(X) = j(X) \cap \alpha$. The embedding j is said to be *α-sound* if j^α belongs to M.

Recall that the bounded powerset of an ordinal δ is defined by $P_{\text{bd}}(\delta) = \bigcup_{\xi<\delta} P(\xi)$. In the context of Definition 4.3.3, if $\alpha = \sup j[\delta]$, it would have been natural to define $j^\alpha = j \restriction P_{\text{bd}}(\delta)$. With this alternate definition, $j^\alpha \in M$ is an a priori weaker requirement. The next lemma shows that this does not actually make a difference:

Lemma 4.3.4. *Suppose M is a transitive class, $j : V \to M$ is an elementary embedding, and δ is an ordinal. Let $\delta_* = \sup j[\delta]$. Then the following are equivalent:*

(1) j is δ_*-sound.

(2) $j[P_{\text{bd}}(\delta)] \in M$ or equivalently j is $2^{<\delta}$-supercompact.

(3) $j \restriction P_{\text{bd}}(\delta) \in M$.

(4) $j \restriction P^M_{\text{bd}}(\delta) \in M$.

Proof. (1) implies (2): Trivial. (The equivalence of $j[P_{\text{bd}}(\delta)] \in M$ with $2^{<\delta}$-supercompactness is immediate from Lemma 4.2.16.)

(2) implies (3): $j \restriction P_{\text{bd}}(\delta)$ is the inverse of the transitive collapse of $j[P_{\text{bd}}(\delta)]$.

(3) implies (4): Trivial.

(4) implies (1): Assume $j \restriction P^M_{\text{bd}}(\delta) \in M$. Since $\delta \subseteq P^M_{\text{bd}}(\delta)$,

$$j \restriction \delta = (j \restriction P^M_{\text{bd}}(\delta)) \restriction \delta \in M$$

Therefore j is δ-supercompact. Since supercompactness implies hypermeasurability (Lemma 4.2.19), $P(\delta) \subseteq M$. In particular $j \restriction P^M_{\text{bd}}(\delta) = j \restriction P_{\text{bd}}(\delta)$. Finally for $X \subseteq \delta$, $j^{\delta_*}(X) = \bigcup_{\xi<\delta} j(X \cap \xi)$, so j^{δ_*} is definable from $j \restriction P_{\text{bd}}(\delta)$ and hence $j^{\delta_*} \in M$, which shows (1). $\square$

Lemma 4.3.5. *Suppose M is a transitive class, $j : V \to M$ is an elementary embedding, and α is an ordinal. Then j is α-sound if and only if $\{j(X) \cap \alpha : X \in V\} \in M$.*

Proof. The forward direction is immediate since $\{j(X)\cap\alpha : X \in V\} = \operatorname{ran}(j^\alpha)$. The reverse direction follows from the fact that j^α is the inverse of the transitive collapse of $\{j(X)\cap\alpha : X\in V\}$. $\square$

Our next lemma states that the fragments j^α "pull back" under elementary embeddings.

Lemma 4.3.6. *Suppose $V \xrightarrow{i} N \xrightarrow{k} M$ are elementary embeddings and $j = k\circ i$. Suppose $j^\alpha \in \operatorname{ran}(k)$. Then $k^{-1}(j^\alpha) = i^{k^{-1}(\alpha)}$.*

Proof. Let δ be the least ordinal such that $j(\delta)\geq\alpha$. Note that $j[\delta] = j^\alpha[\delta]\in \operatorname{ran}(k)$, so by our analysis of derived embeddings (Lemma 4.2.21), $k\restriction\delta+1$ is the identity and i is δ-supercompact. In particular, $P(\delta)\subseteq M$ and $k(P(\delta)) = P(\delta)$.

Let $h = k^{-1}(j^\alpha)$. Then $\operatorname{dom}(h) = k^{-1}(P(\delta)) = P(\delta)$. Thus for $X\in\operatorname{dom}(h)$, $k(X) = X$, and hence

$$k(h(X)) = k(h)(k(X)) = k(h)(X) = j^\alpha(X) = j(X)\cap\alpha = k(i(X))\cap\alpha$$

By the elementarity of k, this implies that $h(X) = i(X)\cap k^{-1}(\alpha)$, or in other words $k^{-1}(j^\alpha) = h = i^{k^{-1}(\alpha)}$, as desired. $\square$

We now turn to Dodd soundness.

Definition 4.3.7. If $j : V\to M$ is an extender embedding, the *Dodd length of* j, denoted $\alpha(j)$, is the least ordinal α such that every element of M is of the form $j(f)(\xi)$ for some $\xi<\alpha$.

On first glance, one might believe that the Dodd length of an elementary embedding j is the same as its *natural length*, denoted $\nu(j)$, the least ν such that $M = H^M(j[V]\cup\nu)$. In fact, equality may fail: the (admittedly minor) issue is that $\nu(j)$ is the least ordinal such that every element of M is of the form $j(f)(p)$ for a finite set $p\subseteq\nu$, whereas in the definition of $\alpha(j)$, one must write every element of M in the form $j(f)(\xi)$ where ξ is not a finite set but a single ordinal below $\alpha(j)$. Thus $\nu(j)\leq\alpha(j)$.

Our main focus, of course, is on ultrafilters, and in this case the Dodd length has an obvious characterization:[4]

Lemma 4.3.8. *If $j : V\to M$ is an ultrapower embedding, then $\alpha(j) = \xi+1$ where ξ is the least ordinal such that $M = H^M(j[V]\cup\{\xi\})$. Therefore U is incompressible if and only if U is a fine ultrafilter on an ordinal and $\alpha(j_U) = \mathrm{id}_U + 1$.* $\square$

Our next lemma establishes a limit on the soundness of an extender embedding. (It is equivalent to the statement that no extender belongs to its own ultrapower.)

[4] This gives us a counterexample to the equality of Dodd length and natural length. Suppose U is a normal ultrafilter on κ. Let $W = U^2$. Then $\nu(j_W) = j_U(\kappa)+1$ but $\alpha(j_W) = j_U(\kappa)+\kappa+1$.

Lemma 4.3.9. *Suppose $j : V \to M$ is an extender embedding and $\alpha = \alpha(j)$. Then j is not α-sound.*

Proof. Let us first show that if U is a countably complete fine ultrafilter on an ordinal δ, then j_U is not $\mathrm{id}_U + 1$-sound. Note that

$$U = \{A \subseteq \delta : \mathrm{id}_U \in j(A)\} = \{A \subseteq \delta : \mathrm{id}_U \in j_U^{\mathrm{id}_U+1}(A)\}$$

so since $U \notin M_U$, $j_U^{\mathrm{id}_U+1} \notin M_U$. Thus j_U is not $\mathrm{id}_U + 1$-sound, as claimed.

We now handle the case where j is an arbitrary extender embedding. By the definition of Dodd length, there is some $\xi < \alpha$ and some function $f \in V$ such that $j^\alpha = j(f)(\xi)$. Let U be the fine ultrafilter derived from j using ξ, and let $k : M_U \to M$ be the factor embedding. Then $\xi \in \mathrm{ran}(k)$ and so $j^\alpha \in \mathrm{ran}(k)$. Applying our lemma on pullbacks of the fragments j^α (Lemma 4.3.6), $k^{-1}(j^\alpha) = j_U^{k^{-1}(\alpha)}$. Therefore j_U is $k^{-1}(\alpha)$-sound. But note that $\mathrm{id}_U = k^{-1}(\xi) < k^{-1}(\alpha)$. Hence j_U is $\mathrm{id}_U + 1$-sound, and this contradicts the first paragraph. □

An embedding is *Dodd sound* if it is as sound as it can possibly be:

Definition 4.3.10. Suppose M is a transitive class and $j : V \to M$ is an elementary embedding. Then j is said to be *Dodd sound* if j is β-sound for all $\beta < \alpha(j)$.

We now prove the equivalence between the Dodd soundness of an extender E as it is defined in [3] and the Dodd soundness of its associated embedding j_E as it is defined in Definition 4.3.3.

Definition 4.3.11.

- A *parameter* is a finite set of ordinals.
- The *parameter order* is defined on parameters p and q by

$$p < q \iff \max(p \mathbin{\triangle} q) \in q$$

- If p is a parameter, then $\langle p_i : i < |p| \rangle$ denotes the *descending* enumeration of p.
- For any $k \leq |p|$, $p \restriction k$ denotes the parameter $\{p_i : i < k\}$.

The point of enumerating parameters in descending order is that the parameter order is then transformed into the lexicographic order:

Lemma 4.3.12. *Suppose p and q are parameters of length n and m respectively. Then $p < q$ if and only if $\langle p_0, \ldots, p_{n-1} \rangle <_{\mathrm{lex}} \langle q_0, \ldots, q_{m-1} \rangle$.* □

Lemma 4.3.13. *The parameter order is a set-like wellorder.* □

Definition 4.3.14. If $j : V \to M$ is an elementary embedding and p is a parameter, then $\mu_j(p)$ is the least ordinal μ such that $p \subseteq j(\mu)$.

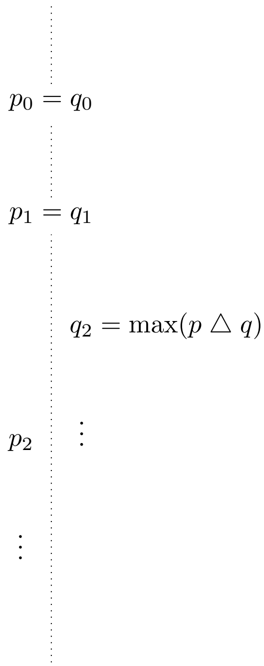

Figure 4.1: The parameter order

Definition 4.3.15. Suppose $j : V \to M$ is an elementary embedding, p is a parameter, and $\nu < \min(p)$ is an ordinal. Let $\delta = \mu_j(p)$. Then the *extender of j below* (p, ν) is the set

$$E^j \restriction p \cup \nu = \{(q, A) : q \in [\nu]^{<\omega},\ A \subseteq [\delta]^{<\omega}, \text{ and } p \cup q \in j(A)\}$$

The restriction $E^j \restriction p \cup \nu$ can be thought of as an extender *relativized* to the parameter p. It is possible to axiomatize relativized extenders as directed systems of ultrafilters and associate to them ultrapower embeddings, namely the direct limit of these systems. Instead we make the following definition:

Definition 4.3.16. A *relativized extender* is a set of the form $E^j \restriction p \cup \nu$ for some elementary embedding j. The *extender embedding* associated to a relativized extender E, denoted

$$j_E : V \to M_E$$

is the unique $j : V \to M$ such that $E = E^j \restriction p \cup \nu$ for some p, ν and $M = H^M(j[V] \cup p \cup \nu)$.

If E is a relativized extender, ν is an ordinal, and p is a parameter, then

$$E \restriction p \cup \nu = E^j \restriction p \cup \nu$$

where $j = j_E$.

The *Dodd parameter* of an extender is the key to the fine-structural proofs of Dodd soundness, which are motivated by the fundamental solidity proofs from fine structure theory.

Definition 4.3.17. Suppose $j : V \to M$ is an extender embedding. Then the *Dodd projectum of* j, denoted $\eta(j)$, is the least ordinal η such that for some parameter p,

$$M = H^M(j[V] \cup p \cup \eta)$$

The *Dodd parameter of* j, denoted $p(j)$, is the least parameter p such that

$$M = H^M(j[V] \cup p \cup \eta(j))$$

If j is an ultrapower embedding, as it always will be in our applications, then $\eta(j) = 0$. More generally, $\eta(j)$ is obviously always a limit ordinal.

The Dodd parameter can also be defined recursively using the concept of an x-generator of an elementary embedding:

Definition 4.3.18. Suppose M and N are transitive models of ZFC, $j : M \to N$ is an elementary embedding, and $x \in N$. Then an ordinal $\xi \in N$ is an *x-generator of* j if $\xi \notin H^N(j[M] \cup \xi \cup \{x\})$.

Lemma 4.3.19. *Suppose $j : V \to M$ is an extender embedding. Let q be the $\subseteq$-maximum parameter with the property that q_k is the largest $q \restriction k$-generator of j for all $k < |q|$. Then $p(j) = q$ and $\eta(j)$ is the strict supremum of the q-generators of j.*

Proof. Let $p = p(j)$, $n = |p|$, and $\eta = \eta(j)$. Fix $k < n$. We will show p_k is the largest $p \restriction k$-generator.

Since $M = H^M(j[V] \cup p \cup \eta) \subseteq H^M(j[V] \cup p \restriction k \cup (p_k + 1))$, there are no $p \restriction k$-generators strictly above p_k. It therefore suffices to show that p_k is a $p \restriction k$-generator. Assume not. Then $p_k \in H^M(j[V] \cup p \restriction k \cup p_k)$. Fix $u \subseteq p_k$ such that $p_k = j(f)(p \cup r)$ for some function $f \in V$. Let $r = p \setminus \{p_k\} \cup u$. Then $r < p$ in the parameter order, but $p \subseteq H^M(j[V] \cup r)$, and hence $M = H^M(j[V] \cup r \cup \eta)$, contrary to the minimality of the Dodd parameter p.

By the maximality of q, this shows that $p = q \restriction n$. We now show that η is the strict supremum of the p-generators of j. Since $M = H^M(j[V] \cup p \cup \eta)$, there are no p-generators greater than or equal to η. It therefore suffices to show that for any $\alpha < \eta$, there is a p-generator of j above α. Suppose $\alpha < \eta$. By the minimality of η, $M \neq H^M(j[V] \cup p \cup \alpha)$, and so there is a p-generator of j above α, as desired.

Since η is a limit ordinal, there is no largest p-generator, and hence $p = q$. □

Corollary 4.3.20. *Suppose $j : V \to M$ is an extender embedding and $p = p(j)$. Then for all $i < |p|$, p_i is a $\{p_0, \ldots, p_{i-1}\}$-generator.* □

The following is Steel's definition of the Dodd soundness of an extender:

Definition 4.3.21. Suppose E is an extender, $p = p(j_E)$, and $\eta = \eta(j_E)$.

- E is *Dodd solid* if
$$E \restriction \{p_0, \dots, p_{i-1}\} \cup p_i \in M_E$$
for all $i < |p|$.

- E is *Dodd sound* if E is Dodd solid and
$$E \restriction p \cup \nu \in M_E$$
for all $\nu < \eta$.

If E is an extender such that j_E is an ultrapower embedding, then E is Dodd solid if and only if E is Dodd sound, simply because $\eta(j_E) = 0$ (so the extra requirement for Dodd soundness holds vacuously).

The following fact is essentially a matter of rearranging definitions:

Theorem 4.3.22. *Suppose E is an extender. Then E is Dodd sound in the sense of Definition 4.3.21 if and only if j_E is Dodd sound in the sense of Definition 4.3.10.*

Proof. Before we prove the equivalence, we prove three preliminary claims.

Let $j = j_E$ and $M = M_E$. Let $\eta = \eta(j)$ and let $p = p(j)$ be the Dodd parameter of j.

Claim 1. *$p \cup \{\eta\}$ is the least parameter s such that every element of M is of the form $j(f)(q)$ for some $q < s$.*

Proof. Suppose not. Then fix $s < p \cup \{\eta\}$ such that every element of M is of the form $j(f)(q)$ for some $q < s$. Fix $q < s$ such that $p = j(f)(q)$ for some f. Then $M = H^M(j[V] \cup q \cup \eta)$. Since p is the least parameter with this property (by the definition of the Dodd parameter), it follows that $p \leq q$. In particular $p < s$. Since $p < s < p \cup \{\eta\}$, $s = p \cup r$ for some $r \in [\eta]^{<\omega}$. Now let $\xi < \eta$ be a p-generator such that $r \subseteq \xi$. Then $p \cup \{\xi\} = j(f)(u)$ for some $u < s$. Since u generates p, we must have $p \leq u$. Since $p \leq u \leq p \cup r$, $u = p \cup t$ for some $t < r$. In particular, since $r \subseteq \xi$, $t \subseteq \xi$. Now $\xi = j(f)(p \cup r)$ where $r \in [\xi]^{<\omega}$, contradicting that ξ is not a p-generator. □

Let φ be the function that sends a parameter to its rank in the parameter order.

Claim 2. *Suppose $x \in M$ and q is a parameter. Then $x = j(f)(q)$ for some function $f \in V$ if and only if $x = j(g)(\varphi(q))$ for some function $g \in V$.*

Proof. For the forwards direction, let $g = f \circ \varphi^{-1}$, and for the reverse direction, let $f = g \circ \varphi$. □

From Claim 1 and Claim 2, we obtain the following key identity:

$$\varphi(p \cup \{\eta\}) = \alpha(j) \tag{4.1}$$

(Recall that $\alpha(j)$ denotes the Dodd length of j, the least ordinal α such that every element of M is of the form $j(f)(\xi)$ for some $\xi < \alpha$.)

Claim 3. *Suppose q is a parameter and $m = |q|$. For $i < m$, let*

$$F_i = E \upharpoonright \{q_0, \dots, q_{i-1}\} \cup q_i$$

Then for any transitive model N of ZFC*, the following are equivalent:*

(1) $F_0, \dots, F_{m-1} \in N$.

(2) $j^{\varphi(q)} \in N$.

Sketch. (1) implies (2): Let $\mu = \mu_j(q) = \mu_j(\{q_0\})$. If $F_0, \dots, F_{m-1} \in N$, then so is the function $e : P([\mu]^{<\omega}) \to M$ defined by $e(X) = \{r < q : r \in j(X)\}$. ($e$ is the parameter version of j^q.) This is because $r \in e(X)$ if and only if $(r, X) \in F_i$ where i is such that $\max(q \triangle r) = q_i$.

Let δ be least such that $j(\delta) \geq \varphi(q)$. Then $\varphi[\delta] \subseteq \mu$ and for $A \subseteq \delta$, $j(A) \cap \varphi(q) = \varphi^{-1}[e(\varphi[A])]$. This shows $j^{\varphi(q)} \in N$.

(2) implies (1): Similar. □

Having proved the three claims, we finally turn to the equivalence of the two notions of Dodd soundness. (We will leave some of the parameter order combinatorics to the reader.)

Assume first that E is Dodd sound in the sense of Definition 4.3.21. Suppose $\beta < \alpha(j)$, and we will show that j is β-sound. It suffices to show that j is β'-sound for some $\beta' \geq \beta$, which allows us to increase β throughout the argument if necessary. By (4.1), by increasing β, we may assume $\varphi(p) \leq \beta$. Thus $p \leq \varphi^{-1}(\beta) < \varphi^{-1}(\alpha(j)) = p \cup \{\eta\}$, as a consequence of (4.1). Let $q = \varphi^{-1}(\beta)$. Then $p \leq q < p \cup \{\eta\}$, so $q = p \cup r$ for some $r \subseteq \eta$. Since η is a limit ordinal, by increasing β if necessary, we may assume $|r| \leq 1$. By the Dodd soundness of E, for all $i < |q|$,

$$E \upharpoonright \{q_0, \dots, q_{i-1}\} \cup q_i \in M$$

This is because either $\{q_0, \dots, q_{i-1}\} \cup q_i = \{p_0, \dots, p_{i-1}\} \cup p_i$ or $\{q_0, \dots, q_{i-1}\} \cup q_i = p \cup \xi$ for some $\xi < \eta$. Therefore by Claim 3, $j^\beta \in M$ so j is β-sound.

Conversely, assume that j is Dodd sound as an elementary embedding. Let $\beta = \varphi(p)$. Since $p < p \cup \{\eta\}$, by (4.1), $\beta < \alpha$. Therefore $j^\beta \in M$ by the Dodd soundness of j. By Claim 3, it follows that $E \upharpoonright \{p_0, \dots, p_{i-1}\} \cup p_i$ for all $i < |p|$, so E is Dodd solid. If $\eta = 0$, it follows that E is Dodd sound. Assume instead that $\eta > 0$. Fix $\xi < \eta$, and we will show $E \upharpoonright p \cup \xi \in M$. Let $q = p \cup \{\xi\}$. Then $q < p \cup \{\eta\}$, so $\varphi(q) < \alpha$. Therefore by the Dodd soundness of j, $j^{\varphi(q)} \in M$. Applying Claim 3, it follows that $E \upharpoonright p \cup \xi \in M$. □

It is worth remarking that the proof shows that an extender E is Dodd solid if and only if j_E is β-solid where β is the rank of $p(j_E)$ in the parameter order.

We now define Dodd sound ultrafilters. One could define an ultrafilter to be Dodd sound if its ultrapower embedding is Dodd sound, but then there would be many Rudin-Keisler equivalent Dodd sound ultrafilters all with the same associated embedding, which complicates the statements of our theorems and adds no real generality. Instead, we ensure that a Dodd sound ultrafilter is the canonical element of its Rudin-Keisler equivalence class:

Definition 4.3.23. A countably complete ultrafilter is *Dodd sound* if it is incompressible and its ultrapower embedding is Dodd sound.

The following alternate characterization of Dodd soundness for ultrafilters is immediate from Lemma 4.3.8 and Lemma 4.3.9:

Lemma 4.3.24. *A fine ultrafilter U on an ordinal δ is Dodd sound if and only if j_U is id_U-sound. That is, U is Dodd sound if and only if the function $h : P(\delta) \to M_U$ defined by $h(X) = j_U(X) \cap \mathrm{id}_U$ belongs to M_U.* □

We finally provide a combinatorial characterization of Dodd soundness for ultrafilters:

Definition 4.3.25. Suppose U is an ultrafilter on an ordinal δ.

- A sequence of sets $S_\alpha \subseteq \alpha$, for $\alpha < \delta$, is *U-threadable* if there is a set $S \subseteq \delta$ such that $S \cap \alpha = S_\alpha$ for U-almost all $\alpha < \delta$.
- A *soundness sequence* is a sequence $\langle \mathcal{A}_\alpha : \alpha < \delta\rangle$ such that for any sequence $\langle S_\alpha : \alpha < \delta\rangle$, the following are equivalent:
  (1) $\langle S_\alpha : \alpha < \delta\rangle$ is U-threadable.
  (2) $S_\alpha \in \mathcal{A}_\alpha$ for U-almost all α.

Theorem 4.3.26. *A fine ultrafilter U on an ordinal δ is Dodd sound if and only if it has a soundness sequence.*

Proof. Note that a sequence $\langle S_\alpha : \alpha < \delta\rangle$ is U-threadable if and only if

$$[\langle S_\alpha : \alpha < \delta\rangle]_U = j_U(S) \cap a_U$$

some $S \subseteq \delta$. Thus $\langle \mathcal{A}_\alpha : \alpha < \delta\rangle$ is a soundness sequence for U if and only if

$$[\langle \mathcal{A}_\alpha : \alpha < \delta\rangle]_U = \{j_U(S) \cap \mathrm{id}_U : S \subseteq \delta\}$$

By Lemma 4.3.5, it follows that U has a soundness sequence if and only if j_U is id_U-sound, or in other words (applying Lemma 4.3.24) U is Dodd sound. □

4.3.3 The generalized Mitchell order on Dodd sound ultrafilters

In this short section, we prove the linearity of the Mitchell order on Dodd sound ultrafilters from UA. We first prove a stronger statement that characterizes $P(P(\lambda)) \cap M_W$ when W is Dodd solid in terms of the Lipschitz order on subsets of $P(\lambda)$.

Proposition 4.3.27. *Suppose W is a Dodd sound ultrafilter on a cardinal λ. Then*

$$P(P(\lambda)) \cap M_W = \{X \subseteq P(\lambda) : X <_L W\}$$

Proof. Suppose $X \subseteq P(\lambda)$.

Assume first that $X <_L W$. By our characterization of the Lipschitz order where the second argument is an ultrafilter (Lemma 3.4.32), this means that there is a set $Z \in M_W$ such that for all $A \subseteq \delta$, $A \in X$ if and only if $j_W(A) \cap \text{id}_W \in Z$. But then $X = (j^{\text{id}_W})^{-1}[Z]$, so $X \in M_W$.

Conversely, suppose $X \in M_W$. Let $Z = j^{\text{id}_W}[X]$. Then $Z \in M_W$ and for all $A \subseteq \delta$, $A \in X$ if and only if $j_W(A) \cap \text{id}_W = j^{\text{id}_W}(A) \in Z$. It follows that $X <_L W$. ☐

Corollary 4.3.28. *Suppose U and W are countably complete ultrafilters on λ and W is Dodd sound. Then $U <_L W$ if and only if $U \lhd W$. In particular, if $U <_{\Bbbk} W$ then $U \lhd W$.* ☐.

In particular, the Lipschitz order is wellfounded on Dodd sound ultrafilters.

Theorem 4.3.29 (UA)**.** *The generalized Mitchell order is linear on Dodd sound ultrafilters.*

Proof. Suppose U and W are Dodd sound ultrafilters. By the linearity of the Lipschitz order on **Fine** (Corollary 3.4.34), either $U = W$ $U <_L W$, or $W <_L U$. Therefore by Proposition 4.3.27, either $U = W$, $U \lhd W$, or $W \lhd U$, as desired. ☐

Notice that the linearity of the Mitchell order on Dodd sound ultrafilters actually follows from the linearity of the Lipschitz order, which perhaps is weaker than UA.

As a consequence of Corollary 4.3.28, if W is Dodd sound and $U <_{\Bbbk} W$, then $U \lhd W$. We now prove a strong converse, which is closely related to Proposition 4.2.28:

Proposition 4.3.30. *Suppose U is a countably complete ultrafilter on a cardinal λ and W is a nonprincipal uniform ultrafilter on a cardinal δ such that j_W is λ-supercompact. If $U \lhd W$, then $U <_S W$.*

Proof. Note that $(j_U(j_W), j_U \restriction M_W)$ is a left-internal comparison of (j_U, j_W) by the standard identity:

$$j_U(j_W) \circ j_U = j_U \circ j_W$$

Since j_W is λ-supercompact, $j_U \restriction M_W = j_U^{M_W}$, which is definable over M_W since $U \lhd W$.

Since j_W is λ-supercompact, $\lambda \leq \delta$ by Proposition 4.2.30. Therefore for all $\alpha < \lambda$, $j_W(\alpha) < \mathrm{id}_W$. Applying Łoś's Theorem,

$$j_U(j_W)(\mathrm{id}_U) = [j_W \restriction \lambda]_U < j_U(\mathrm{id}_W)$$

Thus $(j_U(j_W), j_U \restriction M_W)$ witnesses that $U <_S W$. $\square$

This raises the question of whether the Ketonen order extends the generalized Mitchell order. One should restrict attention here to countably complete uniform ultrafilters on cardinals, or else there are silly counterexamples. Even for such ultrafilters, this is consistently false:

Proposition 5.5.6. *Suppose κ is 2^κ-supercompact and $2^\kappa = 2^{\kappa^+}$. Then there are κ-complete uniform ultrafilters U and W on κ and κ^+ respectively such that $W \lhd U$.*

Thus $W \lhd U$ but $U <_{\Bbbk} W$ simply because $\delta_U < \delta_W$. (This is a consequence of Lemma 3.3.15.) Given Proposition 4.3.30, it is not surprising that the counterexample has this form: if U and W are uniform ultrafilters on the *same* cardinal λ and both j_U and j_W are λ-supercompact, then $U \lhd W$ implies $U <_{\Bbbk} W$.

Lemma 4.3.31. *Suppose λ is a cardinal, W is a countably complete ultrafilter on λ, and Z is a countably complete ultrafilter such that $W \lhd Z$. Assume that for all $\alpha < \lambda$, $\mathbf{UF}(\lambda, \alpha) \subseteq M_Z$ and $M_Z \vDash \mathbf{UF}(\lambda, \alpha) \leq 2^\lambda$. Then for any $U <_{\Bbbk} W$, $U \lhd Z$.*

Proof. Since $W \lhd Z$, $P(\lambda) \subseteq M_Z$ and in fact $P(\lambda)^\lambda \subseteq M_Z$. Moreover

$$M_Z \vDash \left| \bigcup_{\alpha<\lambda} \mathbf{UF}(\lambda, \alpha) \right| \leq 2^\lambda = |P(\lambda)|$$

Hence $(\bigcup_{\alpha<\lambda} \mathbf{UF}(\lambda, \alpha))^\lambda \subseteq M_Z$, so $\prod_{\alpha \in I} \mathbf{UF}(\lambda, \alpha) \in M_Z$ for any set $I \subseteq \lambda$.

Now suppose $U <_{\Bbbk} W$. Fix $I \in W$ and $\langle U_\alpha : \alpha \in I\rangle \in \prod_{\alpha\in I} \mathbf{UF}(\lambda, \alpha)$ such that $U = W\text{-}\lim_{\alpha \in I} U_\alpha$. Then the sequence $\langle U_\alpha : \alpha \in I\rangle \in M_Z$, so $U \in M_Z$, so $U \lhd Z$, as desired. $\square$

In fact, this lemma yields the somewhat stronger result that for any $I \in W$ and sequence $\langle U_\alpha : \alpha \in I\rangle$ of ultrafilters with $\delta_{U_\alpha} < \lambda$, $W\text{-}\lim_{\alpha\in I} U_\alpha \lhd Z$.

Corollary 4.3.32 (UA)**.** *Assume λ is a cardinal such that $2^{<\lambda} = \lambda$. If W and Z are countably complete ultrafilters on λ such that $W \lhd Z$, then $W <_{\Bbbk} Z$.*

Proof. Given the assumption that $2^{<\lambda} = \lambda$ and the fact that $P(\lambda) \subseteq M_Z$, it is not hard to show that $\mathbf{Fine}(\alpha) \in M_Z$ and $M_Z \vDash |\mathbf{Fine}(\alpha)| \leq 2^\lambda$ for all $\alpha < \lambda$. Therefore we are in a position to apply Lemma 4.3.31 to any ultrafilter $U <_\Bbbk W$. Assume towards a contradiction that $W \not<_\Bbbk Z$. By the linearity of the Ketonen order, $Z <_\Bbbk W$. Now $Z <_\Bbbk W \lhd Z$, so by Lemma 4.3.31, $Z \lhd Z$. This contradicts the strictness of the Mitchell order (Lemma 4.2.38). □

Corollary 4.3.33 (UA + GCH)**.** *The Ketonen order extends the generalized Mitchell order on countably complete uniform ultrafilters on infinite cardinals.* □

Corollary 8.3.30 shows that the same conclusion can be deduced from UA alone. This will be achieved by proving from UA that if $W \lhd Z$, then Z is λ_W-supercompact. The result then follows from Proposition 4.3.30.

4.4 Generalizations of normality

In this section, we develop the theory of normal fine ultrafilters, the natural combinatorial generalization of normal ultrafilters, and a central component of the classical theory of supercompact cardinals. The main result of the section (Theorem 4.4.2) states roughly that UA + GCH implies that all these ultrafilters are linearly ordered by the Mitchell order.

Definition 4.4.1. For any infinite cardinal λ, let $\mathscr{N}_\lambda$ be set of normal fine ultrafilters on $P_{\text{bd}}(\lambda)$. Let $\mathscr{N} = \bigcup_\lambda \mathscr{N}_\lambda$.

We provide the definitions of normality and fineness in Section 4.4.1.

Theorem 4.4.2 (UA)**.** *Suppose λ is a cardinal such that $2^{<\lambda} = \lambda$. Then $\mathscr{N}_\lambda$ is wellordered by the Mitchell order. Therefore assuming the Generalized Continuum Hypothesis, $\mathscr{N}$ is linearly ordered by the Mitchell order.*

We asserted that UA + GCH would roughly imply that the Mitchell order is linear on the class of all normal fine ultrafilters, but our theorem only mentions the subclass $\mathscr{N}$. In fact, the class of all normal fine ultrafilters is not literally linearly ordered by the Mitchell order for a number of reasons: one reason is that distinct normal fine ultrafilters can be Rudin-Keisler equivalent and hence Mitchell incomparable. Proposition 4.4.12 below, however, shows that every normal fine ultrafilter is Rudin-Keisler equivalent to an element of $\mathscr{N}$, so Theorem 4.4.2 essentially covers all the bases.

A key concept in the proof of Theorem 4.4.2 is that of an isonormal ultrafilter.

Definition 4.4.3. Suppose λ is a cardinal. An ultrafilter U on λ is *isonormal* if U is weakly normal and j_U is λ-supercompact.

We define weak normality in Section 4.4.2. The concept dates back to Solovay and Ketonen [17]. The other main theorem of this section explains how isonormal ultrafilters get their name:

Theorem 4.4.37. *Suppose U is a nonprincipal ultrafilter. Then U is isonormal if and only if U is the incompressible ultrafilter Rudin-Keisler equivalent to a normal fine ultrafilter. In particular, every normal fine ultrafilter is Rudin-Keisler equivalent to a unique isonormal ultrafilter.*

The proof appears in Section 4.4.4. The forwards direction is quite easy, but the reverse implication requires quite a bit of work amounting to a generalization of the theorem of [19] known as *Solovay's Lemma* to singular cardinals. This generalization constitutes a fundamental new fact about supercompactness whose proof requires some basic notions from PCF theory.

The investigation of isonormal ultrafilters is related back to the linearity of the Mitchell order by the following proposition:

Theorem 4.4.25. *Suppose $2^{<\lambda} = \lambda$. Then every isonormal ultrafilter U on λ is Dodd sound.*

We can actually prove our main theorem (Theorem 4.4.2) right now granting Theorem 4.4.25 and Theorem 4.4.37. We also need a lemma that shows $\mathscr{N}$ is well-behaved under the Mitchell order assuming GCH:

Lemma 4.4.4. *If $2^{<\lambda} = \lambda$, then any $\mathcal{U} \in \mathscr{N}_\lambda$ is hereditarily uniform and satisfies $\lambda_\mathcal{U} = \lambda$.*

Proof. Since $P_{\text{bd}}(\lambda)$ is transitive, $|\text{tc}(P_{\text{bd}}(\lambda))| = |P_{\text{bd}}(\lambda)| = 2^{<\lambda} = \lambda$. On the other hand, since $j_\mathcal{U}$ is λ-supercompact, Proposition 4.2.30 implies $\lambda_\mathcal{U} \geq \lambda$. Thus $|\text{tc}(P_{\text{bd}}(\lambda))| = \lambda_\mathcal{U}$, so $\mathcal{U}$ is hereditarily uniform. $\square$

We finally prove Theorem 4.4.2 assuming Theorem 4.4.25 and Theorem 4.4.37.

Proof of Theorem 4.4.2. Suppose $\mathcal{U}$ and $\mathcal{W}$ are elements of $\mathscr{N}_\lambda$. We show that either $\mathcal{U} \lhd \mathcal{W}$, $\mathcal{U} = \mathcal{W}$, or $\mathcal{U} \rhd \mathcal{W}$. Applying Theorem 4.4.37, let U be the isonormal ultrafilter Rudin-Keisler equivalent to $\mathcal{U}$ and let W be the isonormal ultrafilter Rudin-Keisler equivalent to $\mathcal{W}$. Note that U and W are uniform ultrafilters on the cardinal $\lambda_\mathcal{U} = \lambda_\mathcal{W} = \lambda$ (Lemma 4.4.4). We have $2^{<\lambda} = \lambda$ by assumption, so Theorem 4.4.25 yields that U and W are Dodd sound. By the linearity of the Mitchell order on Dodd sound ultrafilters (Theorem 4.3.29), we are in one of the following cases:

Case 1. $U = W$.

Proof in Case 1. Since $\mathcal{U} \equiv_{\text{RK}} U = W \equiv_{\text{RK}} \mathcal{W}$, Lemma 4.4.11 below implies $\mathcal{U} = \mathcal{W}$. $\square$

Case 2. $U \lhd W$.

Proof in Case 2. Since $W \equiv_{\text{RK}} \mathcal{W}$, we have $U \lhd \mathcal{W}$. Since $\mathcal{U}$ is hereditarily uniform (Lemma 4.4.4) and Rudin-Keisler equivalent to U, the invariance of the generalized Mitchell order on hereditarily uniform ultrafilters under Rudin-Keisler equivalence (Lemma 4.2.13) implies $\mathcal{U} \lhd \mathcal{W}$. $\square$

Case 3. *$U \rhd W$.*

Proof in Case 3. Proceeding as in Case 2, we obtain $\mathscr{U} \rhd \mathscr{W}$. $\square$

This shows that either $\mathscr{U} \lhd \mathscr{W}$, $\mathscr{U} = \mathscr{W}$, or $\mathscr{U} \rhd \mathscr{W}$, as desired.

We finally sketch the proof that $\mathscr{N}$ is linearly ordered by the Mitchell order assuming UA + GCH. It suffices to show the following: suppose $\mathscr{U} \in \mathscr{N}_\gamma$, $\mathscr{W} \in \mathscr{N}_\lambda$, and $2^\gamma \leq \lambda$. Then $\mathscr{U} \lhd \mathscr{W}$. Let U be the isonormal ultrafilter of $\mathscr{U}$, so by the proof of Lemma 4.4.4, U is an ultrafilter on γ. Since $2^\gamma \leq \lambda$, $U \in H_{(2^\gamma)^+} \subseteq H_{\lambda^+} \subseteq M_{\mathscr{W}}$. Since $\mathscr{U}$ is hereditarily uniform and $\mathscr{U} \equiv_{\text{RK}} U \lhd \mathscr{W}$, Lemma 4.2.13 implies $\mathscr{U} \lhd \mathscr{W}$. $\square$

4.4.1 Normal fine ultrafilters

In this section, we give the general definition of a normal fine ultrafilter, which is the natural combinatorial generalization of the notion of a normal ultrafilter on a cardinal. This begins with the generalized diagonal intersection operation:

Definition 4.4.5. Suppose X is a set and $\langle A_x : x \in X\rangle$ is a sequence with $A_x \subseteq P(X)$ for all $x \in X$. The *diagonal intersection* of $\langle A_x : x \in X\rangle$ is the set

$$\triangle_{x\in X} A_x = \left\{\sigma \in P(X) : \sigma \in \textstyle\bigcap_{x\in\sigma} A_x\right\}$$

Definition 4.4.6. If X is a set, a *family over* X is a family Y of subsets of X such that every element of X belongs to some element of Y.

Thus any set Y is a family on a unique set (namely $X = \bigcup Y$).

Definition 4.4.7. Suppose Y is a family over X. A filter $\mathscr{F}$ on Y is:

- *fine* if for any $x \in X$, $\mathscr{F}$ concentrates on $\{\sigma : x \in \sigma\}$.
- *normal* if for any $\{A_x : x \in X\} \subseteq \mathscr{F}$, $\triangle_{x\in X} A_x \in \mathscr{F}$.

Remark 4.4.8. Let us make some remarks regarding this definition.

(1) It makes sense to discuss normal fine filters on Y without mention of X, since $X = \bigcup Y$ is determined from Y.

(2) The structure of the underlying set Y is usually not that important since a normal fine ultrafilter $\mathscr{U}$ on Y can always be lifted to a normal fine ultrafilter on $P(X)$ where $X = \bigcup_{\sigma\in Y}\sigma$. Therefore it is tempting to restrict consideration to normal fine ultrafilters on $P(X)$ for some X. It is often important for technical reasons, however, that the underlying set Y be small; usually we want $|Y| = |\bigcup Y|$.

(3) The structure of the set X is also usually irrelevant, but sometimes it is useful that X be transitive or that X be a cardinal. Suppose X and X' are sets and $f : X \to X'$ is a surjection. If Y is a family over X, then $Y' = \{f[\sigma] : \sigma \in Y\}$ is a family over X' and $g(\sigma) = f[\sigma]$ defines

a surjection from Y to Y'. If $\mathcal{U}$ is an ultrafilter on Y, then $g_*(\mathcal{U})$ is a Rudin-Keisler equivalent ultrafilter on Y' and moreover $\mathcal{U}'$ is normal (fine) if and only if $\mathcal{U}$ is normal (fine). (This is the ultrafilter theoretic analog of Lemma 4.2.16.)

(4) An ultrafilter on an ordinal is fine in the sense of Definition 4.4.7 if and only if it is fine in the sense of Definition 3.2.4. Thus a normal fine ultrafilter on κ is the same thing as a normal ultrafilter on κ.

The connection between normality and supercompactness is clear from the following lemma:

Lemma 4.4.9. *Suppose Y is a family over X and $\mathcal{U}$ is an ultrafilter on Y.*

(1) $\mathcal{U}$ is fine if and only if $j_\mathcal{U}[X] \subseteq \mathrm{id}_\mathcal{U}$.

(2) $\mathcal{U}$ is normal if and only if $\mathrm{id}_\mathcal{U} \subseteq j_\mathcal{U}[X]$. □

Thus $\mathcal{U}$ is a normal fine ultrafilter on Y over X if and only if $\mathrm{id}_\mathcal{U} = j_\mathcal{U}[X]$, or in other words, $\mathrm{id}_\mathcal{U}$ witnesses that $j_\mathcal{U}$ is X-supercompact.

Lemma 4.4.9 yields the main source of normal fine ultrafilters.

Lemma 4.4.10. *Suppose $j : V \to M$ is an X-supercompact elementary embedding and $Y \subseteq P(X)$ is such that $j[X] \in j(Y)$.*

- *Y is a family over X.*
- *The ultrafilter $\mathcal{U}$ on Y derived from j using $j[X]$ is a normal fine ultrafilter on Y.*
- *Let $k : M_\mathcal{U} \to M$ be the factor embedding. Then $k(\alpha) = \alpha$ for all $\alpha \leq |X|$.*

Proof. Immediate from Lemma 4.2.21 and Lemma 4.4.9. □

Another consequence of Lemma 4.4.9 is the following fact, which does not seem to have a simple combinatorial proof:

Lemma 4.4.11. *Suppose $\mathcal{U}$ and $\mathcal{W}$ are normal fine ultrafilters on Y. If $\mathcal{U} \equiv_{\text{RK}} \mathcal{W}$ then $\mathcal{U} = \mathcal{W}$.*

Proof. Let $X = \bigcup Y$. Since $\mathcal{U} \equiv_{\text{RK}} \mathcal{W}$, $j_\mathcal{U} = j_\mathcal{W}$. By Lemma 4.4.9, $\mathrm{id}_\mathcal{U} = j_\mathcal{U}[X] = j_\mathcal{W}[X] = \mathrm{id}_\mathcal{W}$. Thus $\mathcal{U} = \{A \subseteq Y : \mathrm{id}_\mathcal{U} \in j_\mathcal{U}(A)\} = \{A \subseteq Y : \mathrm{id}_\mathcal{W} \in j_\mathcal{W}(A)\} = \mathcal{W}$. □

It also follows that any normal fine ultrafilter is countably complete. This is because the proof that an ω-supercompact ultrapower embedding $j : V \to M$ has the property that $M^\omega \subseteq M$ does not really require that M is wellfounded. (The reader will lose nothing by simply appending countable completeness to the definition of normality, rather than proving it from the definition we have given.)

Recall the class $\mathscr{N}$ defined in the previous section. We finish this section by proving that every normal fine ultrafilter is Rudin-Keisler equivalent to a unique element of $\mathscr{N}$.

Proposition 4.4.12. *Any nonprincipal normal fine ultrafilter $\mathcal{D}$ is Rudin-Keisler equivalent to a unique ultrafilter $\mathcal{U} \in \mathscr{N}$.*

For this we will use a basic lemma about supercompactness:

Lemma 4.4.13. *Suppose $j : V \to M$ is λ-supercompact and $\sup j[\lambda] = j(\lambda)$. Then j is λ^ι-supercompact where $\iota = \text{cf}(\lambda)$. In particular, j is λ^+-supercompact.*

Proof. Let $\kappa = \text{crit}(j)$. Lemma 4.2.24 states that j is $\lambda^{<\kappa}$-supercompact. It suffices to show that $\iota < \kappa$: then since j is $\lambda^{<\kappa}$-supercompact, j is λ^ι-supercompact, and so since $\lambda^\iota > \lambda$, j is λ^+-supercompact.

We now show $\iota < \kappa$. Since $\sup j[\lambda] = j(\lambda)$ and $j[\lambda] \in M$, $\text{cf}^M(j(\lambda)) = \text{cf}(\lambda) = \iota$. On the other hand, by elementarity $\text{cf}^M(j(\lambda)) = j(\text{cf}(\lambda)) = j(\iota)$. It follows that $j(\iota) = \iota$. Since j is ι-supercompact, the Kunen Inconsistency Theorem (Theorem 4.2.35) implies $\iota < \kappa$ where $\kappa = \text{crit}(j)$. $\square$

Actually, we always have $\lambda^{<\kappa} = \lambda^+$ in the context of Lemma 4.4.13.

Proof of Proposition 4.4.12. Obviously, any normal fine ultrafilter is Rudin-Keisler equivalent to a normal fine ultrafilter on $P(\lambda)$ for some cardinal λ. Therefore assume $\mathcal{D}$ is a normal ultrafilter on $P(\lambda)$, and we will show that $\mathcal{D}$ is Rudin-Keisler equivalent to a normal fine ultrafilter on $P_{\text{bd}}(\lambda')$ for some cardinal λ'.

If $\mathcal{D}$ concentrates on $P_{\text{bd}}(\lambda)$, we are done, since $\mathcal{D}$ is then Rudin-Keisler equivalent to $\mathcal{D} \mid P_{\text{bd}}(\lambda)$. So assume $\mathcal{D}$ does not concentrate on $P_{\text{bd}}(\lambda)$. By Łoś's Theorem, $\text{id}_{\mathcal{D}} = j_{\mathcal{D}}[\lambda]$ is unbounded in $j_{\mathcal{D}}(\lambda)$. In other words, $j_{\mathcal{D}}$ is continuous at λ. Therefore by Lemma 4.4.13, $j_{\mathcal{D}}$ is λ^+-supercompact. Note that $j_{\mathcal{D}}[\lambda^+]$ is not cofinal in $j_{\mathcal{D}}(\lambda^+)$: otherwise $j_{\mathcal{D}}(\lambda^+) = \text{cf}^{M_{\mathcal{D}}}(j_{\mathcal{D}}(\lambda^+)) = \lambda^+$, so $\text{crit}(j_{\mathcal{D}}) > \lambda^+$ by Theorem 4.4.32, which implies that $\mathcal{D}$ is principal. Therefore let $\mathcal{U}$ be the normal fine ultrafilter on $P_{\text{bd}}(\lambda^+)$ derived from $j_{\mathcal{D}}$ using $j_{\mathcal{D}}[\lambda^+]$. Then $\mathcal{U}$ is Rudin-Keisler equivalent to $\mathcal{D}$: by construction $\mathcal{U} \leq_{\text{RK}} \mathcal{D}$, and on the other hand, the map $f : P_{\text{bd}}(\lambda^+) \to Y$ defined by $f(\sigma) = \sigma \cap \lambda$ pushes $\mathcal{U}$ forward to $\mathcal{D}$ so $\mathcal{D} \leq_{\text{RK}} \mathcal{U}$. $\square$

4.4.2 Weakly normal ultrafilters

Another combinatorial generalization of the notion of a normal ultrafilter, due to Solovay and Ketonen [17], is that of a weakly normal ultrafilter.

Definition 4.4.14. A uniform ultrafilter U on a cardinal λ is *weakly normal* if for any set $A \in U$, if $f : A \to \lambda$ is regressive, then there is some $B \subseteq A$ such that $B \in U$ and $f[B]$ has cardinality less than λ.

Solovay's definition of a weakly normal ultrafilter applied only to regular cardinals λ, asserting that every regressive function on λ is *bounded* on a set of full measure. The generalization of the concept of weak normality to singular cardinals is due to Ketonen.

Lemma 4.4.15. *Suppose U is a uniform ultrafilter on a cardinal λ. Then the following are equivalent:*

(1) U is weakly normal.

(2) Suppose $\langle A_\alpha : \alpha < \lambda\rangle$ is a sequence of subsets of λ such that $\bigcap_{\alpha\in\sigma} A_\alpha \in U$ for all nonempty $\sigma \in P_\lambda(\lambda)$. Then $\triangle_{\alpha<\lambda} A_\alpha \in U$. ☐

Corollary 4.4.16. *A uniform ultrafilter on a regular cardinal is weakly normal if and only if it is closed under decreasing diagonal intersections.* ☐

Weakly normal ultrafilters on regular cardinals have a simple characterization in terms of their ultrapowers:

Lemma 4.4.17. *Suppose λ is a regular cardinal. An ultrafilter U on λ is weakly normal if and only if* $\text{id}_U = \sup j_U[\lambda]$.

Proof. Suppose U is weakly normal. Since U is a tail uniform ultrafilter on λ, $\text{id}_U > j_U(\alpha)$ for all $\alpha < \lambda$. We will show that $j_U[\lambda]$ is cofinal in id_U, which proves $\text{id}_U = \sup j_U[\lambda]$. Suppose $\xi < \text{id}_U$. Then $\xi = [f]_U$ for some $f : \lambda \to \lambda$ that is regressive on a set in U. Since U is weakly normal, there is a set $A \in U$ such that $|f[A]| < \lambda$. Since λ is regular, $f[A]$ is bounded below λ. Fix $\alpha < \lambda$ such that $f(\xi) < \alpha$ for all $\xi \in A$. Then $[f]_U < j_U(\alpha)$.

Conversely suppose $\text{id}_U = \sup j_U[\lambda]$. Since $\text{id}_U > j_U(\alpha)$ for all $\alpha < \lambda$, $\delta_U \geq \lambda$, and hence U is tail uniform. Since λ is regular, it follows that λ is uniform. Next, suppose $A \in U$ and $f : A \to \lambda$ is regressive. Then $[f]_U < \text{id}_U$. Since $j_U[\lambda]$ is cofinal in id_U, fix $\alpha < \lambda$ with $[f]_U < j_U(\alpha)$. Then for a set $B \in U$ with $B \subseteq A$, $f(\beta) < \alpha$ for all $\beta \in B$. In particular, f takes fewer than λ values on B. ☐

Lemma 4.4.17 yields the main source of weakly normal ultrafilters on regular cardinals:

Corollary 4.4.18. *Suppose $j : V \to M$ is an elementary embedding and λ is a regular cardinal such that $\sup j[\lambda] < j(\lambda)$. Then the ultrafilter on λ derived from j using $\sup j[\lambda]$ weakly normal.* ☐

To help motivate the concept of weak normality on singular cardinals, let us explain its relationship to a Rudin-Keisler invariant notion:

Definition 4.4.19. Suppose λ is an infinite cardinal. An ultrafilter U is λ-*minimal* if $\lambda_U = \lambda$ and for any $W <_{\text{RK}} U$, $\lambda_W < \lambda$.

If $2^\lambda = \lambda^+$, there is a λ-minimal (countably incomplete) ultrafilter on λ, according to a result of Comfort-Negrepontis [16, Theorem 9.13]. On the other hand, the existence of a weakly normal ultrafilter (with no completeness assumptions) implies the existence of an inner model with a measurable cardinal [23]. Weakly normal ultrafilters, however, are the revised Rudin-Keisler analog (Definition 3.4.10) of λ-minimal ones:

Lemma 4.4.20. *An ultrafilter U on a cardinal λ is weakly normal if and only if $\lambda_U = \lambda$ and for all $W <_{\text{rk}} U$, $\lambda_W < \lambda$.* $\square$

Lemma 4.4.20 yields a generalization of Scott's theorem that every countably complete ultrafilter has a derived normal ultrafilter:

Corollary 4.4.21. *If Z is a countably complete uniform ultrafilter on λ, there is a weakly normal ultrafilter U on λ such that $U \leq_{\text{RK}} Z$.*

Proof. Since $<_{\text{rk}}$ is wellfounded on countably complete ultrafilters, there is a countably complete ultrafilter U that is $<_{\text{rk}}$-minimal with the property that $\lambda_U = \lambda$ and $U \leq_{\text{RK}} Z$. Then U satisfies the conditions of Lemma 4.4.20: if $W <_{\text{rk}} U$, then $W \leq_{\text{RK}} Z$, so by the $<_{\text{rk}}$-minimality of U, it must be the case that $\lambda_W < \lambda$ $\square$

The following theorem shows that every countably complete λ-minimal ultrafilter is Rudin-Keisler equivalent to a weakly normal ultrafilter.

Proposition 4.4.22. *A countably complete uniform ultrafilter U on a cardinal λ is weakly normal if and only if it is λ-minimal and incompressible.*

Proof. Suppose U is weakly normal. To see U is incompressible, note that any function that is regressive on a set in U takes less than λ-many values on a set in U, and hence is not one-to-one. To see U is λ-minimal, suppose $W <_{\text{RK}} U$ and we will show that $\lambda_W < \lambda$. Since $W \leq_{\text{RK}} U$, W is countably complete, and hence W is Rudin-Keisler equivalent to an incompressible ultrafilter. We can therefore assume without loss of generality that W is incompressible. Then by the key lemma about the strict Rudin-Keisler order on incompressible ultrafilters (Proposition 3.4.20) the fact that $W <_{\text{RK}} U$ implies $W <_{\text{rk}} U$. Now by Lemma 4.4.20, $\lambda_W < \lambda$.

Conversely suppose U is λ-minimal and incompressible. Suppose $W <_{\text{rk}} U$, and we will show $\lambda_W < \lambda$. We can then conclude that U is weakly normal using Lemma 4.4.20. Since U is incompressible, $W <_{\text{rk}} U$ implies $W <_{\text{RK}} U$ (Lemma 3.4.15, essentially the definition of incompressibility). Therefore by the definition of λ-minimality, $\lambda_W < \lambda$, as desired. $\square$

It is not clear whether Proposition 4.4.22 can be proved without the assumption of countable completeness, though of course countable completeness is not required if λ is regular.

The following characterization of weak normality is the one that is most relevant to our investigations of supercompactness.

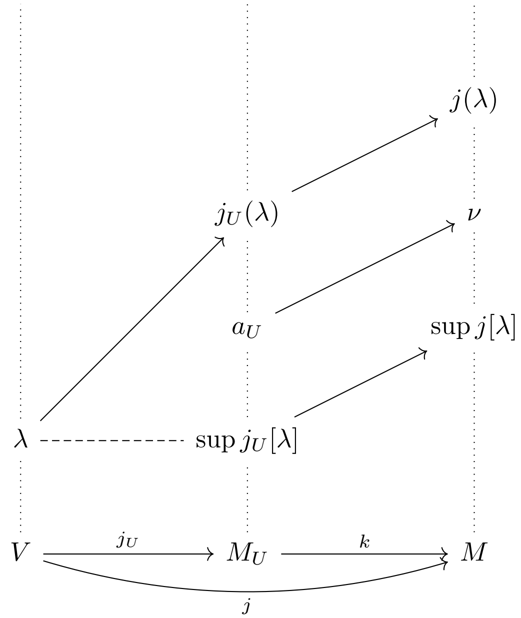

Figure 4.2: Deriving a weakly normal ultrafilter on a singular cardinal

Proposition 4.4.23. *Suppose λ is an infinite cardinal. A countably complete ultrafilter U on λ is weakly normal if and only if* id_U *is the unique generator of j_U that lies above $j(\delta)$ for all $\delta < \lambda$.*

For the proof, we will need an obvious lemma:

Lemma 4.4.24. *Suppose λ is an infinite cardinal. An ultrafilter U on λ is uniform if and only if* $\text{id}_U \notin H^{M_U}(j_U[V] \cup j_U(\delta))$ *for any $\delta < \lambda$.* ☐

Proof of Proposition 4.4.23. We begin with the forwards direction. Suppose U is weakly normal.

We first show that for any ordinal ξ such that $\xi < \text{id}_U$, $\xi \in H^{M_U}(j_U[V] \cup j_U(\delta))$ for some $\delta < \lambda$. Assume not, towards a contradiction. Let W be the tail uniform ultrafilter derived from j_U using ξ. Then $W <_{\text{rk}} U$, as witnessed by the factor embedding $k : M_W \to M_U$. By Lemma 4.4.20, it follows that W is not a uniform ultrafilter on λ, and so by Lemma 4.4.24, there is some $\delta < \lambda$ such that $\xi \in H^{M_W}(j_W[V] \cup j_W(\delta))$. It follows that $\xi \in H^{M_U}(j_U[V] \cup j_U(\delta))$.

Next we show that id_U is a generator of j_U. Since U is uniform, Lemma 4.4.24 implies $\text{id}_U \notin H^{M_U}(j_U[V] \cup j_U(\delta))$ for any $\delta < \lambda$. But by the previous paragraph, for all $\xi < \text{id}_U$, $\xi \in H^{M_U}(j_U[V] \cup j_U(\delta))$ for some $\delta < \lambda$. Thus $\text{id}_U \notin H^{M_U}(j_U[V] \cup \vec{\xi})$ for any $\vec{\xi} \in [\text{id}_U]^{<\omega}$. In other words, id_U is a generator of j_U. By the previous paragraph, id_U is clearly the unique generator above $j_U(\delta)$ for all $\delta < \lambda$.

We now turn to the converse. Assume id_U is the unique generator of j_U that lies above $j(\delta)$ for all $\delta < \lambda$. We will show U is weakly normal by verifying the conditions of Proposition 4.4.22. Since id_U is a generator, U is incompressible. Since M_U is wellfounded, there is a least ordinal that does not belong to $H^{M_U}(j_U[V] \cup j_U(\delta))$ for any δ, and clearly this ordinal is a generator of j_U that lies above $j_U(\delta)$ for all $\delta < \lambda$. Thus it must equal id_U. In other words, for any $\xi < \mathrm{id}_U$, $\xi \in H^{M_U}(j_U[V] \cup j_U(\delta))$ for some $\delta < \lambda$.

Fix an ultrafilter W on λ such that $W <_{\mathrm{rk}} U$. We will show $\lambda_W < \lambda$, verifying the second condition of Proposition 4.4.22. Let $k : M_W \to M_U$ be an elementary embedding with $k \circ j_W = j_U$ and $k(\mathrm{id}_W) < \mathrm{id}_U$. Then by the previous paragraph, $k(\mathrm{id}_W) \in H^{M_U}(j_U[V] \cup j_U(\delta))$ for some $\delta < \lambda$. It follows that $\mathrm{id}_W \in H^{M_W}(j_W[V] \cup j_W(\delta))$. (To see this, fix a function f on δ such that $k(\mathrm{id}_W) = j_U(f)(\xi)$ for some $\xi < j_U(\delta)$, and note that by the elementarity of k, $\mathrm{id}_W = j_W(f)(\bar{\xi})$ for some $\bar{\xi} < j_W(\delta)$.) By Lemma 4.4.24, this implies W is not uniform on λ, or in other words, $\lambda_W < \lambda$. $\square$

Using Proposition 4.4.23, we can prove the Dodd soundness of isonormal ultrafilters on $\lambda = 2^{<\lambda}$.

Theorem 4.4.25. *Suppose $2^{<\lambda} = \lambda$. Then every isonormal ultrafilter U on λ is Dodd sound.*

Proof. Let $j : V \to M$ be the ultrapower of the universe by U. Since j is λ-supercompact, j is $2^{<\lambda}$-supercompact. By Lemma 4.3.4, j is λ_*-sound where $\lambda_* = \sup j[\lambda]$.

We now show that j is ξ-sound where ξ is the least generator of j such that $\xi \geq \lambda_*$. Since λ_* is closed under pairing, the λ_*-soundness of j implies that the extender

$$E = E_j \restriction \lambda_* = \{(p, X) : p \in [\lambda_*]^{<\omega},\ X \subseteq [\lambda]^{<\omega}, \text{ and } p \in j(X)\}$$

belongs to M_U. Let $j_E : V \to M_E$ be the associated extender embedding and let $k : M_E \to M$ be the factor embedding. Then

$$\mathrm{crit}(k) = \min\{\alpha : \alpha \notin H^M(j[V] \cup \lambda_*)\} = \xi$$

by the definition of a generator. Therefore $j_E^\xi = j^\xi$. Moreover since M is closed under λ-sequences by Corollary 4.2.20, $j_E^M = j_E \restriction M$. Therefore $j^\xi = j_E^\xi = (j_E^M)^\xi \in M$, so j is ξ-sound.

By Proposition 4.4.23, $\xi = \mathrm{id}_U$. Therefore j is id_U-sound, which implies that U is Dodd sound. $\square$

We should point out that the assumption that $2^{<\lambda} = \lambda$ is *necessary*:

Lemma 4.4.26. *Suppose λ is a cardinal that carries a Dodd sound ultrafilter U. Then $2^{<\lambda} = \lambda$.*

Proof. Since U is Dodd sound, j_U is id_U-sound. In particular, j_U is $\sup j_U[\lambda]$-sound since $\sup j_U[\lambda] \leq \text{id}_U$. Therefore by Lemma 4.3.4, j_U is $2^{<\lambda}$-supercompact. By Proposition 4.2.30, j_U is not λ^+-supercompact. It follows that $2^{<\lambda} < \lambda^+$, or in other words $2^{<\lambda} = \lambda$. $\square$

4.4.3 Solovay's Lemma

A special case of our main theorem, Theorem 4.4.37, was known long before our work.

Theorem 4.4.27 (Solovay's Lemma)**.** *Suppose λ is a regular cardinal. Then there is a set $B \subseteq P(\lambda)$ such that the following hold:*

- *For any family Y over λ, any normal fine ultrafilter $\mathcal{U}$ on Y concentrates on B.*
- *If σ and τ are elements of B with the same supremum, then $\sigma = \tau$.*

Before proving Solovay's Lemma, let us explain its relevance to isonormal ultrafilters. Essentially, Solovay's Lemma yields the "regular case" of the key isomorphism theorem for isonormal ultrafilters (Theorem 4.4.37):

Corollary 4.4.28. *Suppose λ is a regular cardinal, Y is a family over λ, and $\mathcal{U}$ is a nonprincipal normal fine ultrafilter on Y. Then $\mathcal{U}$ is Rudin-Keisler equivalent to the ultrafilter*

$$U = \{A \subseteq \lambda : \{\sigma \in Y : \sup \sigma \in A\} \in \mathcal{U}\}$$

Moreover, U is an isonormal ultrafilter.

Proof. To see $\mathcal{U} \equiv_{\text{RK}} U$, let $f : P(\lambda) \to \lambda + 1$ be the function $f(\sigma) = \sup \sigma$. Then $f_*(\mathcal{U}) = U$ and by Theorem 4.4.27, f is one-to-one on a set in $\mathcal{U}$.

To see U is isonormal, we must verify that U is weakly normal and j_U is λ-supercompact. The latter is trivial: $j_{\mathcal{U}}$ is λ-supercompact by Lemma 4.4.9, and $j_{\mathcal{U}} = j_U$ since $\mathcal{U}$ and U are Rudin-Keisler equivalent. As for weak normality, by Lemma 3.2.16, $U = f_*(\mathcal{U})$ is the ultrafilter on λ derived from $j_{\mathcal{U}}$ using $[f]_{\mathcal{U}}$ so U is weakly normal by Corollary 4.4.18. $\square$

The proof of Solovay's lemma uses the observation that if $j : V \to M$ is an elementary embedding, $j[\lambda]$ is definable from the action of j on a stationary partition:[5]

Lemma 4.4.29. *Suppose λ is a cardinal, $j : V \to M$ is an elementary embedding, and $\mathcal{P} \subseteq P(\lambda)$ is a partition of $S^\lambda_\omega = \{\alpha < \lambda : \text{cf}(\alpha) = \omega\}$ into stationary sets. Then*

$$j[\mathcal{P}] = \{T \in j(\mathcal{P}) : T \textit{ is stationary in } \sup j[\lambda]\}$$

[5] Solovay's published proof [19] uses the combinatorics of ω-Jonsson algebras instead of stationary sets. Woodin rediscovered the proof using stationary sets, which was already known to Solovay.

It is worth noting that Lemma 4.4.29 is perfectly general; we really do allow j to be an arbitrary elementary embedding of V.

Proof. Let $\lambda_* = \sup j[\lambda]$.

Claim 1. $j[\mathcal{P}] \subseteq \{T \in j(\mathcal{P}) : T$ *is stationary in* $\lambda_*\}$.

Proof. Fix $S \in \mathcal{P}$. We will show that $j(S)$ intersects every closed cofinal subset of λ_*. Suppose $C \subseteq \lambda_*$ is closed cofinal in λ_*. Then $j^{-1}[C]$ is ω-closed cofinal in λ. Since S is a stationary subset of S^λ_ω, $S \cap j^{-1}[C] \neq \emptyset$. But $j(S) \cap C = j(S) \cap C \supseteq j[S \cap j^{-1}[C]] \neq \emptyset$. So $j(S) \cap C \neq \emptyset$, as desired. □

Claim 2. $\{T \in j(\mathcal{P}) : T$ *is stationary in* $\lambda_*\} \subseteq j[\mathcal{P}]$.

Proof. Fix $T \in j(\mathcal{P})$ such that T is stationary in λ_*. We will show that there is some $S \in \mathcal{P}$ such that $j(S) = T$. Since $j[\lambda]$ is ω-closed cofinal in λ_*, $T \cap j[\lambda] \neq \emptyset$. Take $\xi < \lambda$ such that $j(\xi) \in T$. Since $j(\xi) \in T \subseteq j(S^\lambda_\omega)$, $\xi \in S^\lambda_\omega$. Therefore $\xi \in S$ for some $S \in \mathcal{P}$, since $\bigcup \mathcal{P} = S^\lambda_\omega$. Now $j(\xi) \in j(S) \cap T$. Therefore $j(S)$ and T are not disjoint, so since $j(\mathcal{P})$ is a partition, $j(S) = T$, as desired. □

Combining the two claims yields the lemma. □

Lemma 4.4.29 leads to a characterization of supercompactness that looks surprisingly weak:

Corollary 4.4.30. *Suppose $j : V \to M$ is an elementary embedding and λ is a regular cardinal. The following are equivalent:*

(1) j is λ-supercompact.

(2) M is correct about stationary subsets of $\lambda_* = \sup j[\lambda]$.

Proof. (1) implies (2): Assume j is λ-supercompact. Suppose M satisfies that S is stationary in λ_*, and we will show that S is truly stationary in λ_*. Fix a closed cofinal set $C \subseteq \lambda_*$. We will show $S \cap C \neq \emptyset$. Note that $C \cap j[\lambda] \in M$ by Lemma 4.2.19 (3). Let E be the closure of $C \cap j[\lambda]$ in λ_*. Then $E \in M$, $E \subseteq C$, and E is closed cofinal in λ_*. Since $E \in M$ and S is stationary from the perspective of M, $S \cap E \neq \emptyset$. In particular, $S \cap C \neq \emptyset$.

(2) implies (1): Since λ is regular, there is a partition $\mathcal{P}$ of S^λ_ω into stationary sets such that $|\mathcal{P}| = \lambda$. By Lemma 4.4.29, $j[\mathcal{P}] = \{T \in j(\mathcal{P}) : T \text{ is stationary in } \lambda_*\}$, which is definable over M since M is correct about stationary subsets of λ_*. Thus j is $\mathcal{P}$-supercompact, so by Lemma 4.2.16, j is λ-supercompact, as desired. □

Of course the implication from (1) to (2) is not very surprising, but it allows us to restate Lemma 4.4.29 in a useful way:

Corollary 4.4.31. *Suppose λ is a regular cardinal, $j : V \to M$ is a λ-supercompact elementary embedding, and $\langle S_\alpha : \alpha < \lambda\rangle$ is a partition of S^λ_ω into stationary sets. Let $\langle T_\beta : \beta < j(\lambda)\rangle = j(\langle S_\alpha : \alpha < \lambda\rangle)$. Then $j[\lambda] = \{\beta < j(\lambda) : M \vDash T_\beta$ is stationary in $\lambda_*\}$.* □

We now prove Solovay's Lemma.

Proof of Theorem 4.4.27. Let $\langle S_\alpha : \alpha < \lambda\rangle$ be a partition of $S^\lambda_\omega = \{\alpha < \lambda : \text{cf}(\alpha) = \omega\}$ into stationary sets. Let

$$B = \{\sigma \subseteq \lambda : \sigma = \{\beta < \lambda : S_\beta \text{ is stationary in } \sup(\sigma)\}\}$$

By construction, any two elements of B with the same supremum are equal.

To finish, suppose Y is a family over λ and $\mathcal{U}$ is a normal fine on Y. We must show that $\mathcal{U}$ concentrates on B, or equivalently, that $\text{id}_\mathcal{U} \in j_\mathcal{U}(B)$. Since $\text{id}_\mathcal{U} = j_\mathcal{U}[\lambda]$ (Lemma 4.4.9), this amounts to showing

$$j_\mathcal{U}[\lambda] = \{\beta < j_\mathcal{U}(\lambda) : M_\mathcal{U} \vDash j_\mathcal{U}(S)_\beta \text{ is stationary in } \sup j_\mathcal{U}[\lambda]\}$$

which is of course a consequence of Corollary 4.4.31. □

Another corollary of Solovay's Lemma is Woodin's proof of the Kunen Inconsistency Theorem:

Theorem 4.4.32. *Suppose $j : V \to M$ is an elementary embedding, ι is a regular cardinal, j is ι-supercompact, and $j(\iota) = \sup j[\iota]$. Then $j \restriction \iota + 1$ is the identity.*

Proof. Let $\langle S_\alpha : \alpha < \iota\rangle$ be a partition of S^ι_ω into stationary sets. By Corollary 4.4.31, and using the fact that $j(\iota) = \sup j[\iota]$,

$$j[\iota] = \{\beta < j(\iota) : M \vDash j(S)_\beta \text{ is stationary in } j(\iota)\} = j(\iota)$$

But this means $j \restriction \iota + 1$ is the identity, as desired. □

Applying Theorem 4.4.32 at $\iota = \lambda^+$ where λ is the first fixed point of j above $\text{crit}(j)$ yields a proof of the Kunen Inconsistency (Theorem 4.2.35).

4.4.4 Supercompactness and singular cardinals

In this section, we finish the proof of Theorem 4.4.37. We do this by proving an analog of Solovay's Lemma at singular cardinals. One basic issue, however, is that Theorem 4.4.27 itself cannot generalize: in fact, if λ is a singular cardinal, Y is a family over λ, and $\mathcal{U}$ is a normal fine ultrafilter on Y, then the supremum function is *not* one-to-one on any set in $\mathcal{U}$.

Proposition 4.4.33. *Suppose λ is a cardinal of cofinality ι, Y is a family over λ, and $\mathcal{U}$ is a normal fine ultrafilter on Y. Define $f : Y \to \lambda + 1$ by*

$$f(\sigma) = \sup \sigma$$

Define $g : Y \to \iota + 1$ by

$$g(\sigma) = \sup(\sigma \cap \iota)$$

Then $f_*(\mathcal{U}) \equiv_{\text{RK}} g_*(\mathcal{U})$.

It is a bit easier to prove the following equivalent statement first (which in any case turns out to be more useful):

Proposition 4.4.34. *Suppose $j : V \to M$ is an elementary embedding and λ is a cardinal of cofinality ι. Then $\sup j[\lambda]$ and $\sup j[\iota]$ are interdefinable in M from parameters in $j[V]$.*

Proof. Let $h : \iota \to \lambda$ be an increasing cofinal function. Then

$$\sup j[\lambda] = \sup j[h[\iota]] = \sup j(h) \circ j[\iota] = \sup j(h)[\sup j[\iota]]$$

Therefore $\sup j[\lambda]$ is definable in M from $j(h)$ and $\sup j[\iota]$. Moreover,

$$\sup j[\iota] = \sup j(h)^{-1}[\sup j[\lambda]]$$

so $\sup j[\iota]$ is definable in M from $j(h)$ and $\sup j[\lambda]$. $\square$

Proof of Proposition 4.4.33. Let $j : V \to M$ be the ultrapower of the universe by $\mathcal{U}$. Then (by Lemma 3.2.16) $f_*(\mathcal{U})$ is the ultrafilter on $\lambda + 1$ derived from j using $[f]_{\mathcal{U}} = \sup j[\lambda]$ and $g_*(\mathcal{U})$ is the ultrafilter on $\iota + 1$ derived from j using $[g]_{\mathcal{U}} = \sup j[\iota]$. By Proposition 4.4.34,

$$H^M(j[V] \cup \{\sup j[\lambda]\}) = H^M(j[V] \cup \{\sup j[\iota]\})$$

But

$$(M_{f_*(\mathcal{U})}, j_{f_*(\mathcal{U})}) \cong (H^M(j[V] \cup \{\sup j[\lambda]\}), j) \cong (M_{g_*(\mathcal{U})}, j_{g_*(\mathcal{U})})$$

It follows that $f_*(\mathcal{U}) \equiv_{\text{RK}} g_*(\mathcal{U})$. $\square$

Corollary 4.4.35. *Suppose λ is a cardinal of cofinality ι, Y is a family over λ, and $\mathcal{U}$ is a normal fine ultrafilter on Y. Then there is a set $B \in \mathcal{U}$ on which the supremum function takes at most ι-many values.*

Proof. Let $f : Y \to \lambda$ be the supremum function. Since $f_*(\mathcal{U})$ is Rudin-Keisler equivalent to an ultrafilter on $\iota + 1$, f takes at most ι-many values on a set in $\mathcal{U}$. $\square$

What we show instead is that an analog of Lemma 4.4.29 holds:

Theorem 4.4.36. *Suppose λ is a cardinal and $j : V \to M$ is a λ-supercompact elementary embedding. Let θ be the least generator of j with $\theta \geq \sup j[\lambda]$. Then*

$$j[\lambda] \in H^M(j[V] \cup \{\theta\})$$

Moreover if $\sup j[\lambda] < j(\lambda)$, *then* $\theta < j(\lambda)$.

As a corollary, we prove the second of the main theorems of this section:

Theorem 4.4.37. *Suppose U is a nonprincipal ultrafilter. Then U is isonormal if and only if U is the incompressible ultrafilter Rudin-Keisler equivalent to a normal fine ultrafilter. In particular, every normal fine ultrafilter is Rudin-Keisler equivalent to a unique isonormal ultrafilter.*

Proof. We begin with the forward direction, which turns out to follow from Proposition 4.4.22. Suppose U is an isonormal ultrafilter on a cardinal λ. We will show that U is incompressible and Rudin-Keisler equivalent to a normal fine ultrafilter on $P_{\text{bd}}(\lambda)$. Since U is weakly normal, Proposition 4.4.22 implies U is incompressible.

Since U is uniform on λ, $\sup j_U[\lambda] < j_U(\lambda)$ and thus $j_U \restriction \lambda \in j_U(P_{\text{bd}}(\lambda))$. Let $\mathcal{U}$ be the ultrafilter on $P_{\text{bd}}(\lambda)$ derived from j_U using $j_U \restriction \lambda$. Then $\mathcal{U} \leq_{\text{RK}} U$ and $\mathcal{U}$ is a normal fine ultrafilter on $P_{\text{bd}}(\lambda)$ by Lemma 4.4.9. It follows that $j_{\mathcal{U}}$ is λ-supercompact, and therefore $\lambda_{\mathcal{U}} \geq \lambda$ by Proposition 4.2.30. Since U is weakly normal, Proposition 4.4.22 implies U is λ-minimal and therefore $\mathcal{U} \not<_{\text{RK}} U$. Since $\mathcal{U} \leq_{\text{RK}} U$ and $\mathcal{U} \not<_{\text{RK}} U$, we must have $\mathcal{U} \equiv_{\text{RK}} U$ (by definition).

Conversely, suppose U is incompressible and Rudin-Keisler equivalent to a normal fine ultrafilter, and we will show that U is isonormal. Since every normal fine ultrafilter is Rudin-Keisler equivalent to an element of $\mathscr{N}$ (Proposition 4.4.12), for some cardinal λ, U is Rudin-Keisler equivalent to a normal fine ultrafilter $\mathcal{U}$ on $P_{\text{bd}}(\lambda)$. In particular $j_U = j_{\mathcal{U}}$ is λ-supercompact. To show that U is isonormal, it therefore suffices to show that U is a weakly normal ultrafilter on λ.

Let $j : V \to M$ be the ultrapower of the universe by U. Let θ be the least generator of j with $\theta \geq \sup j[\lambda]$. Since $P_{\text{bd}}(\lambda) \in \mathcal{U}$, $\sup j[\lambda] < j(\lambda)$, and so by Theorem 4.4.36, $\theta < j(\lambda)$. Since θ is a generator of $j = j_U$, $\theta \leq \text{id}_U$. In fact, we claim $\text{id}_U = \theta$. On the other hand, by Theorem 4.4.36,

$$M = H^M(j[V] \cup \{j[\lambda]\}) = H^M(j[V] \cup \{\theta\})$$

The ultrapower theoretic characterization of incompressibility (Lemma 3.4.18) implies that id_U is the least ordinal α such that $M = H^M(j[V] \cup \{\alpha\})$. Thus $\text{id}_U \leq \theta$. Hence $\text{id}_U = \theta$, as desired.

Since U is tail uniform (by the definition of incompressibility) and $\text{id}_U < j_U(\lambda)$, U is an ultrafilter on λ. Since id_U is the least generator of j above $\sup j[\lambda]$, the characterization of weakly normal ultrafilters in terms of generators (Proposition 4.4.23) implies that U is a weakly normal ultrafilter on λ. □

We conclude this chapter by proving Theorem 4.4.36. The proof relies on some basic notions from PCF theory.

Definition 4.4.38. Suppose ι is an ordinal. We denote by J^{ι}_{bd} the ideal of bounded subsets of ι, omitting the superscript ι when it is clear from context. If f and g are functions from ι to Ord,

- $f <_{\text{bd}} g$ if $\{\alpha < \iota : f(\alpha) \geq g(\alpha)\} \in J_{\text{bd}}$.
- $f =_{\text{bd}} g$ if $\{\alpha < \iota : f(\alpha) \neq g(\alpha)\} \in J_{\text{bd}}$.

Definition 4.4.39. Suppose C is a set of functions from ι to Ord. A function $s : \iota \to \text{Ord}$ is an *exact upper bound* of C if the following hold:

- For all $f \in C$, $f <_{\text{bd}} s$.
- For all $g <_{\text{bd}} s$, for some $f \in C$, $g <_{\text{bd}} f$.

The following trivial fact plays a key role in the proof of Theorem 4.4.36:

Lemma 4.4.40. *Suppose C is a set of functions from ι to* Ord *and s and t are exact upper bounds of C. Then $s =_{\text{bd}} t$.*

Proof. Suppose s and t are exact upper bounds of C. Suppose towards a contradiction that $s \neq_{\text{bd}} t$. Without loss of generality, we can assume that there is an unbounded set $A \subseteq \iota$ such that $s(\alpha) < t(\alpha)$ for all $\alpha \in A$. Define $g : \iota \to \text{Ord}$ by setting

$$g(\alpha) = \begin{cases} s(\alpha) & \text{if } \alpha \in A \\ 0 & \text{otherwise} \end{cases}$$

Then $g < t$, so since t is an exact upper bound of C, there is some $f \in C$ such that $g <_{\text{bd}} f$. Since s is an upper bound of C, $f <_{\text{bd}} s$. Therefore $g <_{\text{bd}} s$. This contradicts that $A = \{\alpha < \iota : g(\alpha) = s(\alpha)\}$ is unbounded in ι. ☐

Definition 4.4.41. If $s : \iota \to \text{Ord}$ is a function and δ is an ordinal, a *scale of length δ in $\prod_{\alpha<\iota} s(\alpha)$* is a $<_{\text{bd}}$-increasing cofinal sequence $\langle f_\alpha : \alpha < \delta\rangle \subseteq \prod_{\alpha<\iota} s(\alpha)$.

Shelah's Representation Theorem [24] states that if λ is a singular cardinal of cofinality ι, then there is a cofinal continuous sequence $u : \iota \to \lambda$ such that $\prod_{\alpha<\iota} u(\alpha)^+$ has a scale of length λ^+. This is a deep theorem in the context of ZFC, but since we are assuming large cardinals, we will have enough SCH to get away with using only the following trivial version of Shelah's theorem:

Lemma 4.4.42. *Suppose λ is a singular cardinal of cofinality ι such that $\lambda^\iota = \lambda^+$. Suppose $\langle \delta_\alpha : \alpha < \iota\rangle$ is a sequence of regular cardinals cofinal in λ. Then there is a scale of length λ^+ in $\prod_{\alpha<\iota} \delta_\alpha$.*

Proof. We start by proving the standard fact that $\mathbb{P} = (\prod_{\alpha<\iota} \delta_\alpha, <_{\text{bd}})$ is a $\leq\lambda$-directed partial order. The proof proceeds in two steps.

First, we prove that $\mathbb{P}$ is $<\lambda$-directed. Suppose $\gamma < \lambda$ and $\{g_i : i < \gamma\} \subseteq \mathbb{P}$. We will find a $<_{\text{bd}}$-upper bound g of $\{g_i : i < \gamma\}$. Fix α_0 such that $\gamma < \delta_{\alpha_0}$. For $\alpha < \iota$, define

$$g(\alpha) = \begin{cases} \sup_{i<\gamma} g_i(\alpha) + 1 & \text{if } \alpha_0 \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

If $\alpha_0 \leq \alpha$, then δ_α is a regular cardinal greater than γ, so $\sup_{i<\gamma} g_i(\alpha) < \delta_\alpha$. Hence $g \in \prod_{\alpha<\iota} \delta_\alpha$ and g is a $<_{\text{bd}}$-upper bound of $\{g_i : i < \gamma\}$.

Second, we prove that $\mathbb{P}$ is λ-directed. Fix $\{g_i : i < \lambda\} \subseteq \mathbb{P}$. For $\alpha < \iota$, let $h_\alpha \in \mathbb{P}$ be a $<_{\text{bd}}$-upper bound of $\{g_i : i < \delta_\alpha\}$. Finally let $g \in \mathbb{P}$ be a $<_{\text{bd}}$-upper bound of $\{g_i : i < \iota\}$. Then g is a $<_{\text{bd}}$-upper bound of $\{g_i : i < \lambda\}$, as desired.

Enumerate $\prod_{\alpha<\iota} \delta_\alpha$ as $\{g_\xi : \xi < \lambda^+\}$. We define $\langle f_\xi : \xi < \lambda^+\rangle$ recursively. If $\langle f_\xi : \xi < \theta\rangle$ has been defined, choose a $<_{\text{bd}}$-upper bound $f_\theta \in \mathbb{P}$ of $\{f_\xi : \xi < \theta\} \cup \{g_\xi\}$. (Such a function exists by the λ-directedness of $\mathbb{P}$.) By construction $\langle f_\xi : \xi < \lambda^+\rangle$ is a scale in $\prod_{\alpha<\iota} \delta_\alpha$. $\square$

This concludes our summary of the basic notions from PCF theory used in the proof of Theorem 4.4.36, which we now commence.

Proof of Theorem 4.4.36. For the purposes of the proof, let us say that x is *weakly definable from y (in M)* if x is definable in M from parameters in $j[V] \cup \{y\}$, or in other words, $x \in H^M(j[V] \cup \{y\})$. Note that weak definability is a transitive relation.

By Lemma 4.4.29, we may assume λ is a singular cardinal. Let ι be the cofinality of λ. Let $\lambda_* = \sup j[\lambda]$.

Claim 1. *Suppose $\langle\delta_\alpha : \alpha < \iota\rangle$ is an increasing sequence of regular cardinals cofinal in λ. Let e be the equivalence class of $\langle\sup j[\delta_\alpha] : \alpha < \iota\rangle$ modulo J_{bd}. Then $j[\lambda]$ is weakly definable from e and $j \restriction \iota$ in M.*

Proof of Claim 1. Fix a sequence $\langle S^\alpha : \alpha < \iota\rangle$ such that $S^\alpha = \{S^\alpha_\beta : \beta < \delta_\alpha\}$ is a partition of $S^{\delta_\alpha}_\omega$ into stationary sets. Note that $\langle j(S^\alpha) : \alpha < \iota\rangle = j(\langle S^\alpha : \alpha < \iota\rangle) \circ j \restriction \iota$ is weakly definable from $j \restriction \iota$.

Solovay's Lemma (Corollary 4.4.31) implies that for all $\alpha < \iota$, $j[\delta_\alpha]$ is equal to the set $\{\beta < j(\delta_\alpha) : M \vDash j(S^\alpha)_\beta \text{ is stationary in } \sup j[\delta_\alpha]\}$. It follows that

$$\begin{aligned} \beta \in j[\lambda] &\iff \{\alpha < \iota : M \vDash j(S^\alpha)_\beta \text{ is stationary in } \sup j[\delta_\alpha]\} \notin J_{\text{bd}} \\ &\iff \exists s \in e\ \{\alpha < \iota : M \vDash j(S^\alpha)_\beta \text{ is stationary in } s(\alpha)\} \notin J_{\text{bd}} \end{aligned}$$

Thus $j[\lambda]$ is weakly definable from e and $\langle j(S^\alpha) : \alpha < \iota\rangle$. Since $\langle j(S^\alpha) : \alpha < \iota\rangle$ is weakly definable from $j \restriction \iota$, this proves the claim. $\square$

It is not hard to see that $j \restriction \iota$ is itself weakly definable from e, but we will not need this. The following observation, however, will be crucial:

Observation 1. *$j \restriction \iota$ and $j[\iota]$ are weakly definable from* $\sup j[\iota]$.

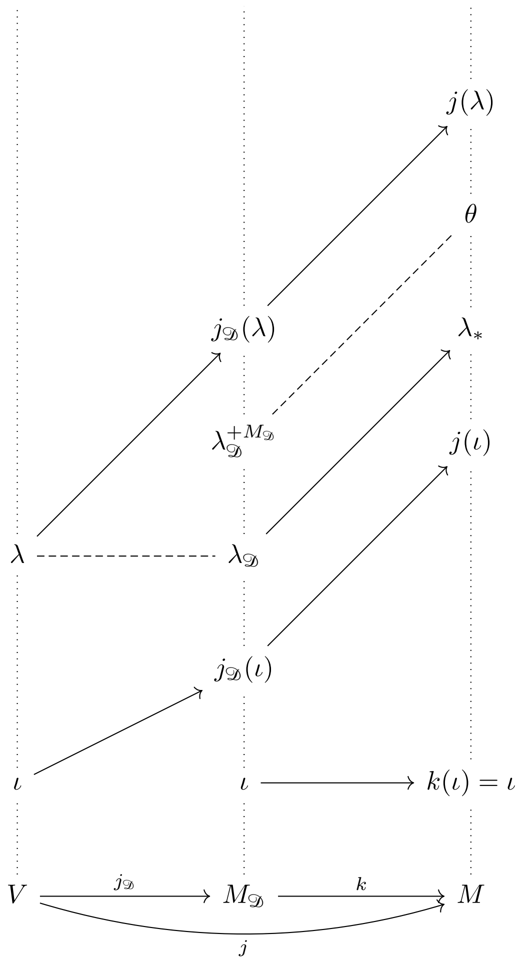

Figure 4.3: A generalization of Solovay's Lemma

This is an immediate consequence of Solovay's Lemma (Corollary 4.4.31).

Let $\mathscr{D}$ be the normal fine ultrafilter on $P(\iota)$ derived from j using $j[\iota]$ and let $k : M_{\mathscr{D}} \to M$ be the factor embedding. Let $\lambda_{\mathscr{D}} = \sup j_{\mathscr{D}}[\lambda]$.[6] By Lemma 4.4.10, $\text{crit}(k) > \iota$ and hence $k(\lambda_{\mathscr{D}}) = \sup k[\lambda_{\mathscr{D}}] = \lambda_*$.

Observation 2. *$k[M_{\mathscr{D}}]$ consists of all $x \in M$ that are weakly definable from $\sup j[\iota]$.*

Observation 2 follows from the fact that $k[M_{\mathscr{D}}] = H^M(j[V] \cup \{j[\iota]\})$ combined with the fact (Observation 1) that $j[\iota]$ and $\sup j[\iota]$ are weakly definable from each other.

Let

$$\theta = \sup k[\lambda_{\mathscr{D}}^{+M_{\mathscr{D}}}]$$

The ordinal θ will turn out to be the least generator of j above λ_*. For now, let us just show that there is no smaller generator:

Claim 2. *$\theta \subseteq H^M(j[V] \cup \lambda_*)$.*

Proof. Suppose $\alpha < \theta$. The claim amounts to showing that α is weakly definable from a finite set of ordinals below λ_*. By the definition of θ, $\alpha < k(\xi)$ for some $\xi < \lambda_{\mathscr{D}}^{+M_{\mathscr{D}}}$. Fix a surjection $p : \lambda_{\mathscr{D}} \to \xi$ with $p \in M_{\mathscr{D}}$. Observation 2 implies $k(p)$ is weakly definable from $\sup j[\iota]$. Since $k(p)$ is a surjection from λ_* onto $k(\xi)$, for some $\nu < \lambda_*$, $\alpha = k(p)(\nu)$. Thus α is weakly definable from $\sup j[\iota]$ and ν, which both lie below λ_*, proving the claim. $\square$

Fix a sequence $\langle \delta_\alpha : \alpha < \iota \rangle$ of regular cardinals greater than ι that is increasing and cofinal in λ.

Claim 3. *In $M_{\mathscr{D}}$, there is a scale $\vec{f} = \langle f_\alpha : \alpha < \lambda_{\mathscr{D}}^{+M_{\mathscr{D}}} \rangle$ in $\prod_{\alpha<\iota} j_{\mathscr{D}}(\delta_\alpha)$.*

Proof. Applying Lemma 4.4.42 in $M_{\mathscr{D}}$, it suffices to show that $M_{\mathscr{D}}$ satisfies $\lambda_{\mathscr{D}}^\iota = \lambda_{\mathscr{D}}^{+\mathscr{D}}$. By the critical sequence analysis given by the Kunen Inconsistency Theorem (Lemma 4.2.37), there is a λ-supercompact cardinal $\kappa \leq \iota$ such that $j_{\mathscr{D}}(\kappa) > \iota$. Thus $\iota < j_{\mathscr{D}}(\kappa) < \lambda_*$ and $j_{\mathscr{D}}(\kappa)$ is $\lambda_{\mathscr{D}}^{+\mathscr{D}}$-supercompact in $M_{\mathscr{D}}$. By the local version of Solovay's theorem [19] (which appears as Theorem 7.2.19) applied in $M_{\mathscr{D}}$, it follows that in $M_{\mathscr{D}}$, $\lambda_{\mathscr{D}}^\iota \leq (\lambda_{\mathscr{D}}^+)^{<j_{\mathscr{D}}(\kappa)} = \lambda_{\mathscr{D}}^{+M_{\mathscr{D}}}$, as desired. $\square$

Claim 4. *$\langle \sup j[\delta_\alpha] : \alpha < \iota \rangle$ is an exact upper bound of $k(\vec{f}) \restriction \theta$.*

Before proving Claim 4, let us show how it implies the theorem.

Let e be the equivalence class of $\langle \sup j[\delta_\alpha] : \alpha < \iota \rangle$ modulo the bounded ideal on ι. Then Claim 4 and Lemma 4.4.40 imply that e is definable in M from the parameters θ and $k(\vec{f})$. Thus by Claim 1, $j[\lambda]$ is weakly definable from θ, $k(\vec{f})$, and $j \restriction \iota$.

Note that λ_* is definable in M from θ: λ_* is the largest M-cardinal below θ. By Proposition 4.4.34, $\sup j[\iota]$ is weakly definable from λ_* and hence from θ. Thus by Observation 1 $j \restriction \iota$ is weakly definable from θ, and by by Observation 2,

[6] If $\eta^\iota < \lambda$ for all $\eta < \lambda$, then $\lambda_{\mathscr{D}} = \lambda$, but we do not assume this.

$k(\vec{f})$ is weakly definable from θ. Combining this with the previous paragraph, $j[\lambda]$ is weakly definable from θ alone. This yields:

$$j[\lambda] \in H^M(j[V] \cup \{\theta\})$$

We now show θ is the least generator of j above λ_*. It suffices by Claim 2 to show that θ is a generator of j. Assume towards a contradiction that this fails. Then $\theta \in H^M(j[V]\cup\theta) = H^M(j[V]\cup\lambda_*)$ by Claim 2. Thus $j[\lambda] \in H^M(j[V]\cup\lambda_*)$. Fix $\xi < \lambda_*$ such that $j[\lambda] \in H^M(j[V] \cup \{\xi\})$. Let W be the ultrafilter derived from j using ξ. Then by Lemma 4.2.21, j_W is λ-supercompact, yet $\lambda_W < \lambda$, and this contradicts Proposition 4.2.30. Thus our assumption was false, and in fact θ is a generator of j.

Thus $j[\lambda] \in H^M(j[V] \cup \{\theta\})$ where θ is the least generator of j greater than or equal to λ_*. To finish, we must show that if $\lambda_* < j(\lambda)$ then $\theta < j(\lambda)$. But $\theta \leq \lambda_*^{+M}$ while $j(\lambda)$ is a limit cardinal of M above λ_*. Hence $\lambda_*^{+M} < j(\lambda)$, as desired.

We now turn to the proof of Claim 4. It will be important here that for any $s : \iota \to \text{Ord}$, $k(s) = k \circ s$ since $\text{crit}(k) > \iota$.

Proof of Claim 4. We first show that for all $\nu < \theta$,

$$k(\vec{f})_\nu <_{\text{bd}} \langle \sup j[\delta_\alpha] : \alpha < \iota\rangle$$

For any $\nu < \theta$, there is some $\xi < \lambda_{\mathscr{D}}^{+M_{\mathscr{D}}}$ such that $\nu < k(\xi)$. Therefore

$$k(\vec{f})_\nu <_{\text{bd}} k(\vec{f})_{k(\xi)} = k(f_\xi)$$

Hence it suffices to show that for any $\xi < \lambda_{\mathscr{D}}^{+M_{\mathscr{D}}}$, $k(f_\xi) < \langle \sup j[\delta_\alpha] : \alpha < \iota\rangle$. For all $\alpha < \iota$, we have that δ_α is a regular cardinal above ι. By Corollary 4.4.28, $\lambda_{\mathscr{D}} = \iota$, so since ultrapower embeddings are continuous at regular cardinals above their size (Lemma 2.2.34),

$$j_{\mathscr{D}}(\delta_\alpha) = \sup j_{\mathscr{D}}[\delta_\alpha]$$

Since $f_\xi \in \prod_{\alpha<\iota} j_{\mathscr{D}}(\delta_\alpha)$, we therefore have $f_\xi(\alpha) < \sup j_{\mathscr{D}}[\delta_\alpha]$ and hence $k(f_\xi)(\alpha) = k(f_\xi(\alpha)) < \sup j[\delta_\alpha]$ for all $\alpha < \iota$, as desired.

We finish by showing that for any $g : \iota \to \text{Ord}$ such that $g <_{\text{bd}} \langle \sup j[\delta_\alpha] : \alpha < \iota\rangle$, there is some $\xi < \lambda_{\mathscr{D}}^{+M_{\mathscr{D}}}$ such that $g <_{\text{bd}} k(f_\xi)$. For $\alpha < \iota$, let $h(\alpha) < \delta_\alpha$ be least such that $g(\alpha) \leq j(h(\alpha))$. Then $j_{\mathscr{D}} \circ h \in M_{\mathscr{D}}$ (since $M_{\mathscr{D}}$ is closed under ι-sequences by Corollary 4.2.20). Since $\langle f_\xi : \xi < \lambda^+\rangle$ is cofinal, in $\prod_{\alpha<\iota} j_{\mathscr{D}}(\delta_\alpha)$, there is some $\xi < \lambda_{\mathscr{D}}^{+M_{\mathscr{D}}}$ such that $j_{\mathscr{D}} \circ h <_{\text{bd}} f_\xi$. It follows that

$$g \leq j \circ h = k \circ j_{\mathscr{D}} \circ h = k(j_{\mathscr{D}} \circ h) <_{\text{bd}} k(f_\xi)$$

as desired. □

This completes the proof of Theorem 4.4.36. □

Chapter 5

The Rudin-Frolík Order

5.1 Introduction

5.1.1 Ultrafilters on the least measurable cardinal

This chapter is motivated by a single simple question. Chapter 2 established the linearity of the Mitchell order on normal ultrafilters assuming UA. As a consequence, the least measurable cardinal κ carries a unique normal ultrafilter. But what are the other countably complete ultrafilters on κ? The following theorem of Kunen [20] answers this question under a hypothesis that is much more restrictive than UA:

Theorem 5.1.1 (Kunen)**.** *Suppose U is a normal ultrafilter on κ and $V = L[U]$. Then every countably complete ultrafilter is Rudin-Keisler equivalent to U^n for some $n < \omega$.*

Here U^n is the ultrafilter on $[\kappa]^n$ generated by sets of the form $[A]^n$ where $A \in U$. An even stronger theorem of Kunen characterizes every elementary embedding of the universe when $V = L[U]$:

Theorem 5.1.2 (Kunen)**.** *Suppose $V = L[U]$ for some normal ultrafilter U. Then any elementary embedding $j : V \to M$ is an iterated ultrapower of U.*

Kunen's proofs of these theorems rely heavily on the structure of $L[U]$, so much so that it might seem unlikely UA alone could imply analogous results. The results of this chapter, however, show that UA does just as well:

Theorem 5.3.18 (UA)**.** *Let κ be the least measurable cardinal. Then there is a unique normal ultrafilter U on κ, and every countably complete ultrafilter is Rudin-Keisler equivalent to U^n for some $n < \omega$.*

Theorem 5.3.20 (UA)**.** *Let κ be the least measurable cardinal and let U be the unique normal ultrafilter on κ. Then any elementary embedding $j : V \to M$ such that $M = H^M(j[V] \cup j(\kappa))$ is an iterated ultrapower of U.*

The requirement that $M = H^M(j[V] \cup j(\kappa))$ ensures that j is the embedding associated to a short extender. This assumption is necessary because for example there could be two measurable cardinal, but one could actually make do with the requirement that $M = H^M(j[V] \cup j(\alpha))$ for some ordinal α such that there are no measurable cardinals in the interval $(\kappa, \alpha]$.

Thus there is an abstract generalization of Kunen's analysis of $L[U]$ to arbitrary models of UA. Far more interesting, however, is that this generalization leads to the discovery of new structure high above the least measurable cardinal.

Definition 5.1.3. A nonprincipal countably complete ultrafilter U is *irreducible* if its ultrapower embedding cannot be written nontrivially as a linear iterated ultrapower.

Irreducible ultrafilters arise in the generalization of Kunen's theorem, which really factors into the following two theorems:

Theorem 5.3.11 (UA)**.** *Every irreducible ultrafilter on the least measurable cardinal κ is Rudin-Keisler equivalent to the unique normal ultrafilter on κ.*

Theorem 5.3.13 (UA)**.** *Every ultrapower embedding can be written as a finite linear iterated ultrapower of irreducible ultrafilters.*

The first of these theorems is highly specific to the least measurable cardinal, but the second is a perfectly general fact: under UA, the structure of countably complete ultrafilters in general can be reduced to the structure of irreducible ultrafilters. The nature of irreducible ultrafilters in general is arguably the most interesting problem raised by this monograph, intimately related to the theory of supercompactness and strong compactness under UA.

5.1.2 Outline of Chapter 5

We now outline the rest of this chapter.

Section 5.2. We introduce the fundamental Rudin-Frolík order, which measures how an ultrapower embedding can be factored as a finite iterated ultrapower. We explain how the topological definition of the Rudin-Frolík order is related to the concept of an internal ultrapower embedding (Corollary 5.2.7). We show that the Ultrapower Axiom is equivalent to the directedness of the Rudin-Frolík order on countably complete ultrafilters, and we show that the Rudin-Frolík order is not directed on ultrafilters on ω.

Section 5.3. We characterize the ultrafilters on the least measurable cardinal up to Rudin-Keisler equivalence. It turns out that such a characterization is possible for all ultrafilters below the least μ-measurable cardinal. (In fact, the analysis extends quite a bit further, but we have omitted this work from this monograph.) Towards this, in Section 5.3.2, we introduce irreducible ultrafilters and analyze the irreducible ultrafilters up to Rudin-Keisler equivalence. We then prove that every ultrafilter can be factored into finitely many irreducible ultrafilters in Section 5.3.3.

SECTION 5.4. In this section, we investigate the deeper structural properties of the Rudin-Frolík order assuming UA. We show that the Rudin-Frolík order satisfies the local ascending chain condition (Theorem 5.3.14), which was actually required as a step in the irreducible factorization theorem. We show that the Rudin-Frolík order induces a lattice on the class of countably complete ultrafilters modulo Rudin-Keisler equivalence. This involves showing that every pair of ultrapower embeddings has a minimum comparison, which we call a *pushout*. In Section 5.4.3, we use pushouts to prove the local finiteness of the Rudin-Frolík order: a countably complete ultrafilter has at most finitely many Rudin-Frolík predecessors assuming UA. Finally, in Section 5.4.4, we study the structure of pushouts and their relationship to the minimal covers of Section 3.5. This involves the key notion of a *translation* of ultrafilters.

SECTION 5.5. In this section, we use the theory of comparisons developed in Section 5.4 to investigate a variant of the generalized Mitchell order called the internal relation.

5.2 The Rudin-Frolík order

Irreducible ultrafilters are most naturally studied in the setting of the *Rudin-Frolík order*, an order on ultrafilters introduced by Rudin and Frolík [25] in the late 1960s. The structure of the Rudin-Frolík order on countably complete ultrafilters turns out to encapsulate many of the phenomena we have been studying so far. For example, the Ultrapower Axiom is equivalent to the statement that the Rudin-Frolík order is directed, while irreducible ultrafilters are simply the minimal elements of the Rudin-Frolík order. The deeper properties of this order (especially the existence of least upper bounds) will provide some of the key tools in the analysis of supercompactness under UA.

In this section, we discuss the theory of the Rudin-Frolík order without yet restricting to countably complete ultrafilters. For this reason, this subsection is a bit out of step with the rest of this monograph, and the only fact that will be truly essential going forward is the characterization of the Rudin-Frolík order on countably complete ultrafilters given by Corollary 5.2.8, which the reader who is not interested in ultrafilter combinatorics can take as the definition of the Rudin-Frolík order on countably complete ultrafilters.

Definition 5.2.1. A sequence of ultrafilters $\langle W_i : i \in I\rangle$ is *(strongly) discrete* if there is a sequence of pairwise disjoint sets $\langle Y_i : i \in I\rangle$ such that $Y_i \in W_i$ for all $i \in I$.

Typically (for example, in Definition 5.2.2), we will consider discrete sequences of ultrafilters that all lie on the same set X. Then discreteness says these ultrafilters can be simultaneously separated from each other.

Definition 5.2.2. Suppose U is an ultrafilter on X and W is an ultrafilter on Y. The *Rudin-Frolík order* is defined by setting $U \leq_{\mathrm{RF}} W$ if there is a set $I \in U$ and a discrete sequence of ultrafilters $\langle W_i : i \in I \rangle$ on Y such that $W = U\text{-}\lim_{i\in I} W_i$.

Recall that if U is an ultrafilter on X, I is a set in U, and $\langle W_i : i \in I \rangle$ is a sequence of ultrafilters on Y, then the U-sum of $\langle W_i : i \in I \rangle$ is defined by

$$U\text{-}\sum_{i\in I} W_i = \{A \subseteq X \times Y : \{i \in I : A_i \in W_i\} \in U\}$$

where $A_i = \{y \in Y : (i, y) \in A\}$. The projection $\pi^0 : X \times Y \to X$ defined by $\pi^0(i, j) = i$ satisfies $\pi^0_* \left(U\text{-}\sum_{i\in I} W_i\right) = U$, and the projection $\pi^1 : X \times Y \to Y$ defined by $\pi^1(i, j) = j$ satisfies

$$\pi^1_* \left(U\text{-}\sum_{i\in I} W_i\right) = U\text{-}\lim_{i\in I} W_i$$

We will give a useful model-theoretic characterization of the Rudin-Frolík order, which requires the following lemma:

Lemma 5.2.3. *Suppose U is an ultrafilter, $I \in U$, and $\langle W_i : i \in I \rangle$ is a sequence of ultrafilters on Y. Then the following are equivalent:*

(1) There is a U-large set $J \subseteq I$ such that $\langle W_i : i \in J \rangle$ is discrete.

(2) π^1 is one-to-one on a set in $U\text{-}\sum_{i\in I} W_i$.

(3) $U\text{-}\sum_{i\in I} W_i \equiv_{\mathrm{RK}} U\text{-}\lim_{i\in I} W_i$.

Proof. (1) implies (2): Fix $J \in U$ contained in I and pairwise disjoint sets $\langle Y_i : i \in J \rangle$ with $Y_i \in W_i$ for all $i \in J$. We will show π^1 is one-to-one on a set in $U\text{-}\sum_{i\in I} W_i$. Let

$$A = \{(i, j) : i \in J \text{ and } j \in Y_i\}$$

Then $A \in U\text{-}\sum_{i\in I} W_i$ and π^1 is one-to-one on A.

(2) implies (1): Fix $A \in U\text{-}\sum_{i\in I} W_i$ on which π^1 is one-to-one. For each $i \in I$, let $Y_i = \{j \in Y : (i, j) \in A\}$. Since π^1 is one-to-one on A, the sets Y_i are disjoint. Since $A \in U\text{-}\sum_{i\in I} W_i$, the set $J = \{i \in I : Y_i \in W_i\}$ belongs to U. Thus $J \in U$, $J \subseteq I$, and $\langle W_i : i \in J \rangle$ is witnessed to be discrete by $\langle Y_i : i \in J \rangle$, as desired.

(2) implies (3): Trivial.

(3) implies (2): By Theorem 3.4.8, if $Z \equiv_{\mathrm{RK}} Z'$ and f is such that $f_*(Z) = Z'$, then f is one-to-one on a set in Z. Therefore since $\pi^1_*(U\text{-}\sum_{i\in I} W_i) = U\text{-}\lim_{i\in I} W_i$, π^1 is one-to-one on a set in $U\text{-}\sum_{i\in I} W_i$. □

Corollary 5.2.4. *If U and W are ultrafilters, the following are equivalent:*

(1) $U \leq_{\mathrm{RF}} W$.

(2) There exist $I \in U$ and ultrafilters $\langle W_i : i \in I\rangle$ on a set Y such that $W \equiv_{\text{RK}} U\text{-}\sum_{i\in I} W_i$.

Proof. (1) implies (2): Obvious from Lemma 5.2.3.

(2) implies (1): The proof uses the fact that the Rudin-Frolík order is invariant under Rudin-Keisler equivalence, which should be easy enough to see from the definition.

Let $Y' = I \times Y$. Let $f^i : Y \to Y'$ be the embedding defined by $f^i(y) = (i, y)$, and let $W_i' = f^i_*(W_i)$. Then $W_i' \equiv_{\text{RK}} W_i$ and $\langle W_i' : i \in I\rangle$ is discrete. We have

$$W \equiv_{\text{RK}} U\text{-}\sum_{i\in I} W_i \equiv_{\text{RK}} U\text{-}\sum_{i\in I} W_i' \equiv_{\text{RK}} U\text{-}\lim_{i\in I} W_i'$$

where the last Rudin-Keisler equivalence follows from Lemma 5.2.3. By the definition of the Rudin-Frolík order $U \leq_{\text{RF}} U\text{-}\lim_{i\in I} W_i'$, so by the Rudin-Keisler equivalence invariance of the Rudin-Frolík order, $U \leq_{\text{RF}} W$. □

The following generalization of closeness to possibly illfounded models in our view simplifies the theory of the Rudin-Frolík order on countably incomplete ultrafilters:

Definition 5.2.5. Suppose N and M are models of ZFC. A cofinal elementary embedding $h : N \to M$ is *close* to N if for all $X \in N$ and all $a \in M$ such that $M \vDash a \in h(X)$, the N-ultrafilter on X derived from h using a belongs to N.

It is not quite accurate to say that this derived N-ultrafilter D belongs to N, so what we really mean is that $D = \{A \in N : N \vDash A \in D'\}$ for some $D' \in N$.

Lemma 5.2.6. *If $h : N \to M$ is close and $M = H^M(h[N] \cup \{a\})$ for some $a \in M$, then there is an ultrafilter Z of N and an isomorphism $k : M_Z^N \to M$ such that $k \circ j_Z^N = h$.* □

Corollary 5.2.7. *If U and W are ultrafilters, the following are equivalent:*

(1) $U \leq_{\text{RF}} W$.

(2) There is a close embedding $e : M_U \to M_W$ such that $e \circ j_U = j_W$.

(3) There is a (possibly illfounded) comparison $(k, h) : (M_U, M_W) \to N$ of (j_U, j_W) such that k is close to M_U and $k(\text{id}_U) \in h[M_W]$.

Sketch. (1) implies (2): By Corollary 5.2.4, fix $I \in U$ and a sequence of ultrafilters $\langle W_i : i \in I\rangle$ such that $W \equiv_{\text{RK}} U\text{-}\sum_{i\in I} W_i$. Let $D = U\text{-}\sum_{i\in I} W_i$ and let $Z = [\langle W_i : i \in I\rangle]_U$. We have $(M_Z^{M_U}, j_Z^{M_U} \circ j_U) \cong (M_D, j_D) \cong (M_W, j_W)$, so fix an isomorphism $k : M_Z^{M_U} \to M_W$ such that $k \circ j_Z^{M_U} \circ j_U = j_W$. It is easy to see that $k \circ j_Z^{M_U}$ is close to M_U.

(2) implies (3): Trivial.

(3) implies (1): Recall that $M_U = H^{M_U}(j_U[V] \cup \{\text{id}_U\})$, so since $k(\text{id}_U) \in h[M_W]$ and $k[j_U[V]] = h[j_W[V]] \subseteq h[M_W]$, in fact $k[M_U] \subseteq h[M_W]$. Therefore

letting $e = h^{-1} \circ k$, $e : M_U \to M_W$ is an elementary embedding. Moreover, $h \circ e = k$, so e is close to M_U since k is (by the argument of Lemma 2.2.24).

Let W_* be the ultrafilter derived from e using id_W and let $\ell : (M_{W_*})^{M_U} \to M_W$ be the canonical factor embedding (Lemma 2.2.8). It is easy to see that ℓ is surjective, and hence ℓ is an isomorphism. By Lemma 2.2.30, ℓ witnesses that the iteration $[U, W_*]$ is Rudin-Keisler equivalent to W. By Lemma 5.2.3, it follows that $U \leq_{\mathrm{RF}} W$. □

Note that the close embedding given by Corollary 5.2.7 is "isomorphic to" a (possibly illfounded) internal ultrapower embedding of M_U. But the language of close embeddings makes it possible to work with the Rudin-Frolík order in fairly simple model theoretic terms while keeping our language precise.

In the countably complete case, Corollary 5.2.7 really does imply that there is an internal ultrapower embedding from M_U to M_W:

Corollary 5.2.8. *If U and W are countably complete ultrafilters, then the following are equivalent:*

- *$U \leq_{\mathrm{RF}} W$.*
- *There is an internal ultrapower embedding $e : M_U \to M_W$ such that $e \circ j_U = j_W$.*
- *There is a comparison $(k, h) : (M_U, M_W) \to N$ of (j_U, j_W) such that k is close to M_U and $k(\mathrm{id}_U) \in h[M_W]$.* □

Corollary 5.2.9. *The Ultrapower Axiom holds if and only if the Rudin-Frolík order is directed on countably complete ultrafilters.*

Proof. Assume the Ultrapower Axiom. Suppose U and W are countably complete ultrafilters. Let $j : V \to M$ and $i : V \to N$ be their respective ultrapower embeddings. Using UA, fix an internal ultrapower comparison $(k, h) : (M, N) \to P$. Then the composition $k \circ j = h \circ i$ is an ultrapower embedding of V, associated say to the countably complete ultrafilter D. Then $U \leq_{\mathrm{RF}} D$ since $k : M_U \to M_D$ is an internal ultrapower embedding such that $k \circ j_U = k \circ j = j_D$. Similarly, $W \leq_{\mathrm{RF}} D$. Thus the Rudin-Frolík order is directed on countably complete ultrafilters. The converse is similar. □

Corollary 5.2.7 makes the relationship between the Rudin-Frolík order and the Rudin-Keisler order clear:

Corollary 5.2.10. *The Rudin-Keisler order extends the Rudin-Frolík order.*

Proof. Suppose $U \leq_{\mathrm{RF}} W$. Then by Corollary 5.2.7, there is an elementary embedding $h : M_U \to M_W$ such that $h \circ j_U = j_W$. By Lemma 3.4.4, $U \leq_{\mathrm{RK}} W$. □

Thus by Theorem 3.4.6, if $U \leq_{\mathrm{RF}} W$ and $W \leq_{\mathrm{RF}} U$, then $U \equiv_{\mathrm{RK}} W$. This motivates the following definition:

Definition 5.2.11. The *strict Rudin-Frolík order* is defined on ultrafilters U and W by setting $U <_{\mathrm{RF}} W$ if $U \leq_{\mathrm{RF}} W$ but $U \not\equiv_{\mathrm{RK}} W$.

Lemma 5.2.12. *The strict Rudin-Frolík order is wellfounded on countably complete ultrafilters.*

Proof. This follows from the fact that the strict Rudin-Keisler order extends the strict Rudin-Frolík order (Corollary 5.2.10) and is wellfounded on countably complete ultrafilters (Corollary 3.4.23). □

The Rudin-Frolík order is *not* directed on arbitrary ultrafilters. In fact, the Rudin-Frolík order restricted to ultrafilters on ω already fails to be directed. We sketch a proof of this fact that bears a striking resemblance to many of the comparison arguments used throughout this monograph, especially Theorem 5.3.9 below. We hope it demonstrates that the close embedding approach to the Rudin-Frolík order really yields some simplifications.

Theorem 5.2.13 (Rudin)**.** *If U and W are ultrafilters on ω that have an upper bound in the Rudin-Frolík order, then either $U \leq_{\mathrm{RF}} W$ or $W \leq_{\mathrm{RF}} U$.*

Sketch. By Corollary 5.2.7 (3), the existence of an $\leq_{\mathrm{RF}}$-upper bound of U and W implies the existence of close embeddings $(k, h) : (M_U, M_W) \to N$ such that $k \circ j_U = h \circ j_W$. (See Fig. 5.1.)

Assume without loss of generality that $k(\mathrm{id}_U) < h(\mathrm{id}_W)$. Let Z be the M_W-ultrafilter on $j_W(\omega)$ derived from h using $k(\mathrm{id}_U)$. Then Z concentrates on $\mathrm{id}_W < j_W(\omega)$. Since Z belongs to M_W and concentrates on an M_W-finite set, Z is principal. Since Z is derived from h using $k(\mathrm{id}_U)$, we must in fact have $h(\mathrm{id}_Z) = k(\mathrm{id}_U)$. In particular $k(\mathrm{id}_U) \in h[M_W]$, so $U \leq_{\mathrm{RF}} W$ by Corollary 5.2.7. □

This theorem is often summarized by the statement that "the Rudin-Frolík order forms a tree," but this is only true of the Rudin-Frolík order on ω. The reader should note that this proof is very similar to the proof of the linearity of the Mitchell order from UA. The argument shows that a natural generalization of the seed order to $\beta(\omega)$ is equal to the Rudin-Frolík order, while the natural generalization of the Ketonen order is equal to the revised Rudin-Keisler order.

Corollary 5.2.14. *The Rudin-Frolík order on $\beta(\omega)$ is not directed.*

Proof. Assume towards a contradiction that the Rudin-Frolík order on $\beta(\omega)$ is directed. Then by Theorem 5.2.13, it is linear. This contradicts the well-known theorem of Kunen [26] that the Rudin-Keisler order is not linear on ultrafilters on ω. □

Thus, unsurprisingly, the analog of the Ultrapower Axiom for countably incomplete ultrafilters is false.

We conclude this section with a basic rigidity lemma for the Rudin-Frolík order that apparently had not been noticed:

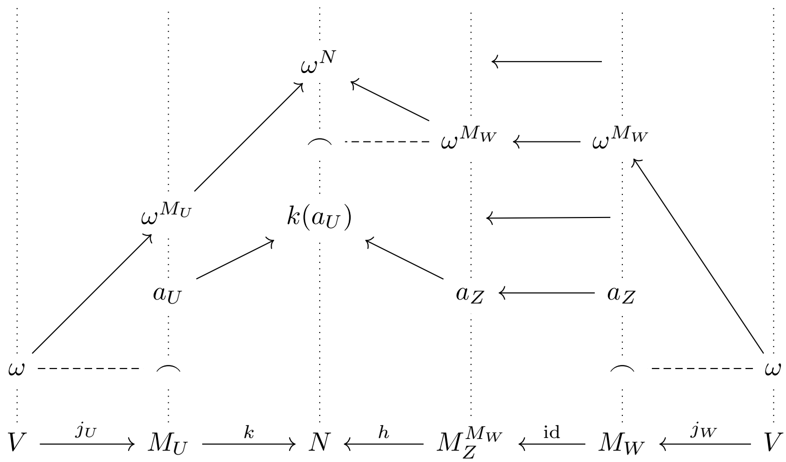

Figure 5.1: The proof of Rudin's Theorem 5.2.13

Theorem 5.2.15. *Suppose U is an ultrafilter, $I, I' \in U$, and $\langle W_i : i \in I\rangle$ and $\langle W'_i : i \in I'\rangle$ are discrete sequences of ultrafilters such that*

$$U\text{-}\lim_{i\in I} W_i = U\text{-}\lim_{i\in I'} W'_i$$

Then for U-almost all $i \in I \cap I'$, $W_i = W'_i$.

In other words, there is at most one way to realize one countably complete ultrafilter as a discrete limit with respect to another.

Lemma 5.2.16. *Suppose U and W are ultrafilters. Then there is at most one close embedding $h : M_U \to M_W$ such that $h \circ j_U = j_W$.*

Sketch. By replacing U with a Rudin-Keisler equivalent ultrafilter, we may assume without loss of generality that U is an ultrafilter on an ordinal δ. Suppose $h_0, h_1 : M_U \to M_W$ are close embeddings such that $h_0 \circ j_U = h_1 \circ j_U$. We will show $h_0 = h_1$. Since $M_U = H^{M_U}(j_U[V] \cup \{\mathrm{id}_U\})$ and $h_0 \restriction j_U[V] = h_1 \restriction j_U[V]$ by assumption, it suffices to show $h_0(\mathrm{id}_U) = h_1(\mathrm{id}_U)$.

Assume towards a contradiction that $h_0(\mathrm{id}_U) \neq h_1(\mathrm{id}_U)$, and without loss of generality assume that $h_0(\mathrm{id}_U) < h_1(\mathrm{id}_U)$. The idea now is that the comparison $(h_0, h_1) : (M_U, M_U) \to M_W$ witnesses $U <_{\Bbbk} U$ in the Ketonen order, contradicting the irreflexivity of the Ketonen order. The problem, however, is that much of the theory of the Ketonen order only goes through for countably complete ultrafilters. Yet there is one proof of the irreflexivity of the Ketonen order (Proposition 3.3.9) that does not actually require countable completeness. By imitating this proof, one reaches a contradiction.

Specifically, let U_* be the M_U-ultrafilter on $j_U(\delta)$ derived from h_1 using $h_0(\mathrm{id}_U)$. Since h_1 is close to M_U, there is a sequence $\langle U_\alpha : \alpha < \delta\rangle$ of ultrafilters on δ such that $U_* = [\langle U_\alpha : \alpha < \delta\rangle]_U$. Since $\text{id}_U \in U_*$, U_α concentrates on α for U-almost all α by Łoś's theorem. By Lemma 3.2.12, $U = U\text{-}\lim_{\alpha<\delta} U_\alpha$. By Lemma 3.4.32, this implies $U <_L U$. This contradicts the irreflexivity of the Lipschitz order (Corollary 3.4.29). □

Translating from the language of close embeddings to the language of ultrafilter sums, Lemma 5.2.16 implies Theorem 5.2.15:

Proof of Theorem 5.2.15. Let $Z = [\langle W_i : i \in I\rangle]_U$ and let $Z' = [\langle W'_i : i \in I'\rangle]_U$. By Lemma 5.2.3, $U\text{-}\sum_{i\in I} Z_i \equiv_{\text{RK}} U\text{-}\sum_{i\in I'} Z'_i$ and their projections to the second coordinate are equal. Using the ultrapower theoretic characterization of sums (Lemma 3.5.9), this means:

$$j_Z^{M_U} \circ j_U = j_{Z'}^{M_U} \circ j_U$$
$$\text{id}_Z = \text{id}_{Z'}$$

Lemma 5.2.16 now implies $j_Z^{M_U} = j_{Z'}^{M_U}$. But Z and Z' are derived from $j_Z^{M_U} = j_{Z'}^{M_U}$ using $\text{id}_Z = \text{id}_{Z'}$. Thus $Z = Z'$. Finally, by Łoś's Theorem we have that $W_i = W'_i$ for U-almost all $i \in I \cap I'$. □

5.3 Below the first μ-measurable cardinal

5.3.1 Introduction

Essentially the first large cardinal that cannot be formulated in terms of the existence of normal ultrafilters is the μ-measurable cardinal:

Definition 5.3.1. A cardinal κ is said to be *μ-measurable* if there is an elementary embedding $j : V \to M$ with critical point κ such that the normal ultrafilter on κ derived from j using κ belongs to M.

The existence of a μ-measurable cardinal is a large cardinal axiom that is stronger than the existence of a measurable cardinal κ such that $o(\kappa) = 2^{2^\kappa}$ but weaker than the existence of a cardinal κ that is $P(2^\kappa)$-hypermeasurable.

As an example of the strength of μ-measurable cardinals, let us show the following fact:

Proposition 5.3.2. *Suppose κ is a μ-measurable cardinal. Then there is a normal ultrafilter on κ that concentrates on cardinals δ such that for any $A \subseteq P(\delta)$, there is a normal ultrafilter D on δ such that $A \in M_D$.*

Proof. Let $j : V \to M$ witness that κ is μ-measurable and let U be the normal ultrafilter on κ derived from j using κ. Thus $U \in M$.

Claim 1. *M_U satisfies the statement that for all $A \subseteq P(\kappa)$, there is a normal ultrafilter D on κ such that $A \in M_D$.*

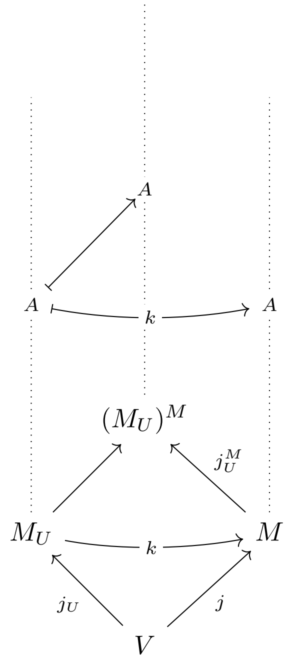

Figure 5.2: Diagram of Proposition 5.3.2

Proof. Suppose not, and fix $A \subseteq P(\kappa)$ such that M_U satisfies that there is no normal ultrafilter D on κ with $A \in (M_D)^{M_U}$. Let $k : M_U \to M$ be the factor embedding. By Lemma 4.4.10, $\text{crit}(k) > \kappa$ and $P(\kappa) \cap M_U = P(\kappa) = P(\kappa) \cap M$, so $k(A) = A$. Therefore since k is elementary, M satisfies that there is no normal ultrafilter D on κ with $A \in (M_D)^M$. But $A \in j_U(V_\kappa) \subseteq (M_U)^M$, and $U \in M$ is a normal ultrafilter. This is a contradiction. $\square$

By Łoś's Theorem, U concentrates on cardinals δ such that for all $A \subseteq P(\delta)$, there is a normal ultrafilter D on δ such that $A \in M_D$. $\square$

Thus a μ-measurable cardinal κ is a limit of cardinals δ such that $o(\delta) = 2^{2^\delta}$, although κ itself need not have order 2^{2^κ}.

5.3.2 Irreducible ultrafilters and μ-measurability

The goal of the next few subsections is to analyze the countably complete ultrafilters in V_κ where κ is the least μ-measurable cardinal. We first analyze simpler ultrafilters called irreducible ultrafilters, and then we reduce the general case to the irreducible case.

Definition 5.3.3. An a nonprincipal countably complete ultrafilter U is *irreducible* if every ultrafilter $D <_{\text{RF}} U$ is principal.

Let us give some examples of irreducible ultrafilters.

Proposition 5.3.4. *If U is a normal ultrafilter on a cardinal κ, then U is irreducible.*

Proof. Suppose $D <_{\text{RF}} U$. By Corollary 5.2.10, $D <_{\text{RK}} U$, and therefore by Proposition 4.4.22, $\lambda_D < \kappa$. But since $D \leq_{\text{RK}} U$, D is κ-complete. Since D is κ-complete and $\lambda_D < \kappa$, D is principal. □

A direct generalization of this yields:

Proposition 5.3.5. *Normal fine ultrafilters are irreducible.*

Proof. Suppose $\mathcal{U}$ is a normal fine ultrafilter. By Theorem 4.4.37, $\mathcal{U}$ is Rudin-Keisler equivalent to an isonormal ultrafilter U on a cardinal λ. It suffices to show that U is irreducible. Suppose $D <_{\text{RF}} U$, and we will show D is principal. By Corollary 5.2.10, $D <_{\text{RK}} U$, and therefore by Proposition 4.4.22, $\lambda_D < \lambda$. Since $D \leq_{\text{RF}} U$, M_U is contained in M_D, and so because j_U is λ-supercompact, using Corollary 4.2.20, $\text{Ord}^\lambda \subseteq M_U \subseteq M_D$. In particular, $j_D \restriction \lambda \in M_D$, so j_D is λ-supercompact. Since $\lambda_D < \lambda$ and j_D is λ-supercompact, D is principal by Proposition 4.2.30. □

Dodd sound ultrafilters are also irreducible:

Proposition 5.3.6. *If U is a Dodd sound ultrafilter, then U is irreducible.*

Proof. Suppose $D <_{\text{RF}} U$, and we will show D is principal. We may assume without loss of generality that D is incompressible. Then since $D <_{\text{RK}} U$, in fact $D <_{\Bbbk} U$ by Corollary 3.4.22. Since the Lipschitz order extends the Ketonen order, $D <_L U$, so by Corollary 4.3.28, $D \lhd U$. But then $D \in M_U \subseteq M_D$, so $D \lhd D$, which implies D is principal by Lemma 4.2.38. □

Finally returning to μ-measurable cardinals, we have the following fact:

Proposition 5.3.7. *Suppose $j : V \to M$ is such that $\text{crit}(j) = \kappa$ and $U_0 \in M$ where U_0 is the normal ultrafilter on κ derived from j. Let U_1 be the ultrafilter on V_κ derived from j using U_0. Then U_1 is irreducible and U_1 is not Rudin-Keisler equivalent to a normal ultrafilter.*

Proof. Let $j_1 : V \to M_1$ be the ultrapower of V by U_1. The key point, which is easily verified, is that $\text{id}_{U_1} = U_0$. Also note that

$$M_1 = H^{M_1}(j_1[V] \cup (2^{2^\kappa})^{M_1})$$

since $\text{id}_{U_1} = U_0 \in H^{M_1}(j_1[V] \cup (2^{2^\kappa})^{M_1})$, U_0 being a subset of $P(\kappa)$.

We now show that U_1 is irreducible. Suppose $D \leq_{\text{RF}} U_1$ and D is nonprincipal. We must show $D \equiv_{\text{RK}} U_1$. Since $\lambda_D = \kappa$, we have $\text{crit}(j_D) = \kappa$. Let

$k : M_D \to M_1$ be the unique internal ultrapower embedding with $k \circ j_D = j_1$. We claim $k(\kappa) = \kappa$. Supposing the contrary, we have that $k(\kappa) > \kappa$ is an inaccessible cardinal that is a generator of j_1, contradicting that $M_1 = H^{M_1}(j_1[V] \cup (2^{2^\kappa})^{M_1})$. Thus $k(\kappa) = \kappa$. Since $M_1 \subseteq M_D$, $U_0 \in M_D$, and since $k(\kappa) = \kappa$, $k(U_0) = U_0$. Since $U_0 = \mathrm{id}_{U_1}$, it follows that k is surjective. Thus k is an isomorphism, and it follows that $D \equiv_{\mathrm{RK}} U_1$.

Finally we show that U_1 is not Rudin-Keisler equivalent to a normal ultrafilter. Suppose towards a contradiction that it is. Then in fact, U_1 is Rudin-Keisler equivalent to the ultrafilter on κ derived from j_{U_1} using κ, namely U_0. In particular, $M_{U_0} = M_{U_1}$, so since $U_0 \in M_{U_1}$, in fact $U_0 \in M_{U_0}$. This contradicts the fact that the Mitchell order is irreflexive (Lemma 4.2.38). $\square$

Under UA, Proposition 5.3.7 has a converse that makes precise the sense in which μ-measurability is the first large cardinal axiom beyond normal ultrafilters:

Theorem 5.3.8 (UA). *Suppose κ is a measurable cardinal. Exactly one of the following holds:*

(1) κ is μ-measurable.

(2) Every irreducible ultrafilter U of completeness κ is Rudin-Keisler equivalent to a normal ultrafilter.

Towards this, we prove the theorem, which can be viewed as yet another generalization of the proof that the Mitchell order is linear under UA.

Theorem 5.3.9 (UA). *Suppose U is a countably complete ultrafilter. Let D be the normal ultrafilter on $\kappa = \mathrm{crit}(j_U)$ derived from j_U using κ. Then either $D \leq_{\mathrm{RF}} U$ or $D \lhd U$.*

Proof. Let $i : M_D \to M_U$ be the factor embedding. Let $(k, h) : (M_D, M_U) \to N$ be an internal ultrapower comparison of (j_D, j_U). We claim that $k(\kappa) \leq h(i(\kappa))$. Suppose not, towards a contradiction. Notice that $(h \circ i, k) : (M_D, M_D) \to N$ is a right-internal comparison of (j_D, j_D). Since $h(i(\kappa)) < k(\kappa)$, this comparison witnesses $D <_{\Bbbk} D$ (Lemma 3.3.4). This contradicts the irreflexivity of the Ketonen order (Proposition 3.3.9). Thus $k(\kappa) \leq h(i(\kappa))$, as claimed. As a consequence, $k(\kappa) \leq h(\kappa)$. (In fact, $i(\kappa) = \kappa$.)

The proof now breaks into two cases:

Case 1. $k(\kappa) = h(\kappa)$

Proof in Case 1. In this case, $k(\mathrm{id}_D) \in h[M_U]$, so Corollary 5.2.8 implies $D \leq_{\mathrm{RF}} U$. $\square$

Case 2. $k(\kappa) < h(\kappa)$

Proof in Case 2. We will show that $D \in M_U$. The key point is that for any $A \subseteq \kappa$,

$$h(j_U(A)) \cap h(\kappa) = h(A) \cap h(\kappa)$$

and therefore

$$\begin{aligned} A \in D &\iff \mathrm{id}_D \in j_D(A) \\ &\iff k(\mathrm{id}_D) \in k(j_D(A)) \\ &\iff k(\mathrm{id}_D) \in h(j_U(A)) \\ &\iff k(\mathrm{id}_D) \in h(A) \end{aligned}$$

Since h is definable over M_U and $P(\kappa) \subseteq M_U$, it follows that D is a definable over M_U, and hence $D \in M_U$. □

Thus in Case 1, $D \leq_{\mathrm{RF}} U$, and in Case 2, $D \lhd U$. This proves the theorem. □

Theorem 5.3.9 leads to the proof of Theorem 5.3.8.

Proof of Theorem 5.3.8. Assume (2) fails, and we will show (1) holds. Let U be an irreducible ultrafilter such that $\mathrm{crit}(j_U) = \kappa$ but U is not Rudin-Keisler equivalent to a normal ultrafilter. Let D be the normal ultrafilter on κ derived from j_U using κ. By Theorem 5.3.9, either $D \leq_{\mathrm{RF}} U$ or $D \lhd U$. If $D \leq_{\mathrm{RF}} U$ then since D is nonprincipal and U is irreducible, $D \equiv_{\mathrm{RK}} U$, contrary to our hypothesis that U is not Rudin-Keisler equivalent to a normal ultrafilter. Therefore $D \lhd U$. Then $j_U : V \to M_U$ has critical point κ and the normal ultrafilter on κ derived from j_U using κ belongs to M_U, so κ is a μ-measurable cardinal. Therefore (1) holds.

If (2) holds, then (1) fails as a consequence of Proposition 5.3.7. □

Corollary 5.3.10 (UA). *Suppose κ is the least μ-measurable cardinal. Then every irreducible ultrafilter in V_κ is Rudin-Keisler equivalent to a normal ultrafilter.*

Proof. This follows from Theorem 5.3.8 applied in V_κ, which is a model of ZFC + UA that also satisfies the statement that there are no μ-measurable cardinals. □

Corollary 5.3.11 (UA). *Let κ be the least measurable cardinal. Then κ carries a unique irreducible ultrafilter up to Rudin-Keisler equivalence.* □

5.3.3 Factorization into irreducibles

The results of the previous section motivate understanding how arbitrary countably complete ultrafilters relate to irreducible ultrafilters. The main theorem of this subsection answers the question in complete generality: every ultrapower embedding can be written as a finite iterated ultrapower of irreducible ultrafilters. To be perfectly precise, let us introduce our conventions for iterated

$$\begin{tikzcd} M_0 \arrow[r, "U_0"] & M_1 \arrow[r, "U_1"] & M_2 \arrow[r, "U_2"] & \cdots & M_{\bar{\nu}} \arrow[r, "U_{\bar{\nu}}"] & M^{\mathcal{J}}_\infty & \nu = \bar{\nu} + 1 \end{tikzcd}$$

$$\begin{tikzcd} M_0 \arrow[r, "U_0"] \arrow[rrrr, bend right, "j^{\mathcal{J}}_{0\infty}"'] & M_1 \arrow[r, "U_1"] \arrow[rrr, bend right, "j^{\mathcal{J}}_{1\infty}"'] & M_2 \arrow[r] \arrow[rr, bend left, "j^{\mathcal{J}}_{2\infty}"] & \cdots & M^{\mathcal{J}}_\infty & \nu \in \mathrm{Lim} \end{tikzcd}$$

Figure 5.3: Iterated ultrapowers of length ν

ultrapowers, which we define by a natural but perhaps somewhat confusing simultaneous recursion.

Definition 5.3.12. Suppose ν is an ordinal and M is a model of set theory. An *iterated ultrapower of M of length ν* is a sequence

$$\mathcal{J} = \langle M_\alpha, U_\alpha, j_{\alpha,\beta} : \alpha \leq \beta < \nu \rangle$$

such that the following hold:

- For all $\bar{\nu} < \nu$, $\mathcal{J} \restriction \bar{\nu} = \langle M_\alpha, U_\alpha, j_{\alpha,\beta} : \alpha \leq \beta < \bar{\nu} \rangle$ is an iterated ultrapower.
- If $\nu = \bar{\nu} + 1$ for some ordinal $\bar{\nu}$, then $M_{\bar{\nu}} = M^{\mathcal{J}\restriction\bar{\nu}}_\infty$, $j_{\alpha,\bar{\nu}} = j^{\mathcal{J}\restriction\bar{\nu}}_{\alpha,\infty}$ for all $\alpha < \bar{\nu}$, $j_{\bar{\nu},\bar{\nu}} = \mathrm{id}$, and $U_{\bar{\nu}}$ is an ultrafilter of $M_{\bar{\nu}}$.

For any iterated ultrapower $\mathcal{J}$ of length ν:

- If $\nu = 0$, $M^{\mathcal{J}}_\infty = M$.
- If $\nu = \bar{\nu} + 1$, $M^{\mathcal{J}}_\infty = \mathrm{Ult}(M_{\bar{\nu}}, U_{\bar{\nu}})$ and for $\alpha < \nu$, $j^{\mathcal{J}}_{\alpha,\infty} = j_{U_{\bar{\nu}}} \circ j_{\alpha,\bar{\nu}}$.
- If ν is a nonzero limit ordinal, then $M^{\mathcal{J}}_\infty$ is the direct limit of the system $\langle M_\alpha, j_{\alpha,\beta} : \alpha \leq \beta < \nu \rangle$, and for $\alpha < \nu$, $j^{\mathcal{J}}_{\alpha,\infty} : M_\alpha \to M^{\mathcal{J}}_\infty$ is the canonical embedding.

This completes the definition of an iterated ultrapower. The one slightly nonstandard aspect of this definition is that the final model $M^{\mathcal{J}}_\infty$ of an iterated ultrapower $\mathcal{J}$ (that is, either the direct limit or the last ultrapower) is not indexed on the sequence $\mathcal{J}$. In Schindler's terminology [13], $\mathcal{J}$ is a *putative* iterated ultrapower.

Let us establish some further notation. If $\mathcal{J} = \langle M_\alpha, U_\alpha, j_{\alpha,\beta} : \alpha \leq \beta < \nu \rangle$ is an iterated ultrapower, then for $\alpha \leq \beta < \nu$, $M^{\mathcal{J}}_\alpha = M_\alpha$, $U^{\mathcal{J}}_\alpha = U_\alpha$, and $j^{\mathcal{J}}_{\alpha,\beta} = j_{\alpha,\beta}$. We also let $M^{\mathcal{J}}_\nu = M^{\mathcal{J}}_\infty$ and $j^{\mathcal{J}}_{0,\nu} = j^{\mathcal{J}}_{0,\infty}$.

Note that an iterated ultrapower $\mathcal{J}$ is uniquely determined by the model $M = M^{\mathcal{J}}_0$ and the sequence $\langle U^{\mathcal{J}}_\alpha : \alpha < \nu \rangle$; we say that $\mathcal{J}$ is the *iterated ultrapower of M given by $\langle U^{\mathcal{J}}_\alpha : \alpha < \nu \rangle$*. Finally, if C is a class of ultrafilters, then we say $\mathcal{J}$ is a *C-iteration* or an *iteration of C-ultrafilters* if for all $\alpha < \mathrm{length}(\mathcal{J})$, $U_\alpha \in j^{\mathcal{J}}_{0,\alpha}(C)$.

Theorem 5.3.13 (UA)**.** *Suppose W is a countably complete ultrafilter. Then there is a finite irreducible ultrafilter iteration $\mathcal{I}$ of V such that $M^{\mathcal{I}}_\infty = M_W$ and $j_W = j^{\mathcal{I}}_{0,\infty}$.*

The proof of this theorem relies on a stronger structural property of the Rudin-Frolík order:

Theorem 5.3.14 (UA)**.** *Suppose W is a countably complete ultrafilter. Then there is no ascending chain $D_0 <_{\text{RF}} D_1 <_{\text{RF}} D_2 <_{\text{RF}} \cdots$ such that $D_n \leq_{\text{RF}} W$ for all $n < \omega$.*

More succinctly, the Rudin-Frolík order satisfies the *local ascending chain condition*. Later we will give a deeper explanation of why this is true (Theorem 5.4.25): a countably complete ultrafilter has only finitely many Rudin-Frolík predecessors up to Rudin-Keisler equivalence.

We defer the proof of Theorem 5.3.14 to the next section. In this section we will derive Theorem 5.3.13 from Theorem 5.3.14, and show how this can be used to analyze ultrafilters on the least measurable cardinal.

Before we can proceed, we need a simple lemma about the pervasiveness of irreducible ultrafilters:

Lemma 5.3.15. *Suppose $D <_{\text{RF}} W$ are countably complete ultrafilters. Then there is a countably complete ultrafilter F with $D <_{\text{RF}} F \leq_{\text{RF}} W$ and an irreducible ultrafilter U of M_D such that $j_F = j^{M_D}_U \circ j_D$.*

Proof. By the wellfoundedness of the Rudin-Frolík order on countably complete ultrafilters (Lemma 5.2.12), let F be $<_{\text{RF}}$-minimal among all ultrafilters Z such that $D <_{\text{RF}} Z \leq_{\text{RF}} W$. By Corollary 5.2.8, fix a countably complete ultrafilter U of M_D such that $j_F = j^{M_D}_U \circ j_D$.

We claim U is an irreducible ultrafilter of M_D. Suppose $\bar{U} <_{\text{RF}} U$ in M_D, and we will show that $\bar{U}$ is principal in M_D. Let $\bar{F}$ be a countably complete ultrafilter such that $j_{\bar{F}} = j^{M_D}_{\bar{U}} \circ j_D$. One easily computes:

$$D \leq_{\text{RF}} \bar{F} <_{\text{RF}} F \leq_{\text{RF}} W$$

Assume towards a contradiction $D <_{\text{RF}} \bar{F}$; then $D <_{\text{RF}} \bar{F} \leq_{\text{RF}} W$ and $\bar{F} <_{\text{RF}} F$, contradicting that F is $<_{\text{RF}}$-minimal among all ultrafilters Z such that $D <_{\text{RF}} Z \leq_{\text{RF}} W$. Therefore $D \not<_{\text{RF}} \bar{F}$, or in other words $D \equiv_{\text{RK}} \bar{F}$. Now

$$M_D = M_{\bar{F}} = M^{M_D}_{\bar{U}}$$

It follows that $\bar{U}$ is principal in M_D. ∎

We now deduce Theorem 5.3.13 from Theorem 5.3.14.

Proof of Theorem 5.3.13 assuming Theorem 5.3.14. By recursion on $k < \omega$, we define a countably complete ultrafilter $D_k \leq_{\text{RF}} W$ and an irreducible ultrafilter iterations $\mathcal{I}_k$ of V such that $\text{length}(\mathcal{I}_k) = k$ and $j^{\mathcal{I}_k}_{0,\infty} = j_{D_k}$. We will maintain that the ultrafilters D_k are strictly increasing in the Rudin-Frolík order.

To begin, let $M_0 = V$ and let D_0 be principal.

Suppose $D_k \leq_{\text{RF}} W$ and $\mathcal{J}_k$ have been defined. If $D_k \equiv_{\text{RK}} W$, we set $\ell = k$ and terminate the construction. Otherwise, $D_k <_{\text{RF}} W$. Using Lemma 5.3.15, fix D_{k+1} with $D_k <_{\text{RF}} D_{k+1} \leq_{\text{RF}} W$ and an irreducible ultrafilter U of $M_{D_k} = M_\infty^{\mathcal{J}_k}$ such that $j_{D_{k+1}} = j_U^{M_k} \circ j_{D_k}$. Let $\mathcal{J}_{k+1}$ be the unique extension $\mathcal{J}$ of $\mathcal{J}_k$ with length$(J) = k + 1$ and $U_k^{\mathcal{J}} = U$.

This recursion must terminate in finitely many steps: otherwise we produce $D_0 <_{\text{RF}} D_1 <_{\text{RF}} \cdots$ with $D_n \leq_{\text{RF}} W$ for all $n < \omega$, contradicting the local ascending chain condition (Theorem 5.3.14). When the process terminates, we have $D_\ell \equiv_{\text{RK}} W$.

In particular, we have produced a finite irreducible ultrafilter iteration $\mathcal{J} = \mathcal{J}_\ell$ of V such that $j_{0,\infty}^{\mathcal{J}} = j_{D_\ell} = j_W$, as desired. □

We now turn our sights back to the countably complete ultrafilters below the least μ-measurable cardinal.

Theorem 5.3.16 (UA). *Assume that there are no μ-measurable cardinals. Suppose W is a countably complete ultrafilter. Then there is a finite normal ultrafilter iteration $\mathcal{J}$ of M such that $M_W = M_\infty^{\mathcal{J}}$ and $j_W = j_{0,\infty}^{\mathcal{J}}$.*

Proof. This is immediate from Theorem 5.3.8 and Theorem 5.3.13. □

Stated more succinctly, if there are no μ-measurable cardinals and the Ultrapower Axiom holds, then every ultrapower embedding is given by a finite iteration of normal ultrafilters. Combining this with the linearity of the Mitchell order on normal ultrafilters, Theorem 5.3.16 comes very close to a complete analysis of all countably complete ultrafilters below the least μ-measurable cardinal on the assumption of the Ultrapower Axiom alone. In any case, it gives as complete an analysis as the Ultrapower Axiom ever will:

Proposition 5.3.17. *The following are equivalent:*

(1) The Mitchell order is linear and every ultrapower embedding is given by a finite iteration of normal ultrafilters.

(2) The Ultrapower Axiom holds and there are no μ-measurable cardinals. □

The proof is as obvious but tedious, and it is omitted. A much more general theorem is proved in Theorem 8.3.36.

We now derive the analog of Kunen's theorem (Theorem 5.1.1 above):

Theorem 5.3.18 (UA). *Suppose κ is the least measurable cardinal. Let U be the unique normal ultrafilter on κ. Then every countably complete ultrafilter on κ is Rudin-Keisler equivalent to U^n for some $n < \omega$.*

Proof. We first prove the theorem assuming κ is the only measurable cardinal. Then U is the only normal ultrafilter. Thus by Theorem 5.3.13, every ultrapower embedding is given by a finite iterated ultrapower of U. In other words, every countably complete ultrafilter is Rudin-Keisler equivalent to U^n for some $n < \omega$.

We now prove the theorem assuming there are two measurable cardinals. Let δ be the second one. Since V_δ is a model of UA and satisfies that κ is the only measurable cardinal, by the previous paragraph V_δ satisfies that every countably complete ultrafilter is Rudin-Keisler equivalent to U^n for some $n < \omega$. Since every countably complete ultrafilter on κ belongs to V_δ, it follows that (in V) every countably complete ultrafilter on κ is Rudin-Keisler equivalent to U^n for some $n < \omega$. □

We sketch how this implies the transfinite version of Kunen's theorem.

Definition 5.3.19. Suppose U is a countably complete ultrafilter. Then $\mathscr{I}(U)$ is the proper class iterated ultrapower $\langle M_\beta, U_\alpha, j_{\alpha,\beta} : \alpha \leq \beta \in \text{Ord}\rangle$ of V defined recursively by setting $U_\alpha = j_{0,\alpha}(U)$ for all $\alpha \in \text{Ord}$.

Theorem 5.3.20 (UA)**.** *Let κ be the least measurable cardinal. Let U be the unique normal ultrafilter on κ. Suppose M is an inner model and $j : V \to M$ is an elementary embedding such that $M = H^M(j[V] \cup j(\kappa))$. Then $j = j^{\mathscr{I}(U)}_{0,\nu}$ for some ordinal ν.*

This uses the following lemma, the notation for which was introduced in Section 4.3.

Lemma 5.3.21. *Suppose M is an inner model, $j : V \to M$ is an elementary embedding, and $\langle \xi_\alpha : \alpha < \nu\rangle$ is the increasing enumeration of the generators of j. For any $p \in [\nu]^{<\omega}$, let U_p be the ultrafilter on $[\mu_j(p)]^{|p|}$ derived from j using $\{\xi_\alpha : \alpha \in p\}$. Then j is uniquely determined by the sequence $\langle U_p : p \in [\nu]^{<\omega}\rangle$.*

Sketch. This follows from the usual extender ultrapower construction. This proof is not intended as an exposition of this construction; we are merely checking, for the sake of the reader already familiar with this construction, that a slightly modified version (i.e., using only generators) works just as well.

For $p \in [\nu]^{<\omega}$, let $j_p : V \to M_p$ be the ultrapower of the universe by U_p and let $k_p : M_p \to M$ be the unique elementary embedding such that $k_p \circ j_p = j$ and $k_p(\text{id}_{U_p}) = \{\xi_\alpha : \alpha \in p\}$.

For $p \subseteq q \in [\nu]^{<\omega}$, define $k_{p,q} : M_p \to M_q$ by setting

$$k_{p,q}([f]_{U_p}) = [f']_{U_q}$$

where, letting $e : |p| \to |q|$ be the unique function such that $q_{e(n)} = p_n$, f' is defined for U_q-almost every s by

$$f'(s) = f(\{s_{e(n)} : n < |p|\})$$

Then

$$\langle M_{U_q}, k_{p,q} : p \subseteq q \in [\nu]^{<\omega}\rangle$$

is a directed system. Let N be its direct limit and let $j_{p,\infty} : M_p \to N$ be the direct limit map.

For any $p \subseteq q \in [\nu]^{<\omega}$, it is easy to check that $k_q \circ k_{p,q} = k_p$. Therefore by the universal property of the direct limit, there is a map $k : N \to M$ such that $k \circ j_{p,\infty}$ is equal to $k_p : M_p \to M$.

We claim k is the identity. Towards a contradiction, suppose not. Let $\xi = \text{crit}(k)$. Then ξ is a generator of j, so $\xi = \xi_\alpha$ for some $\alpha < \nu$. But then letting $a = \text{id}_{U_{\{\alpha\}}}$, we have $\{\xi\} = k_{\{\alpha\}}(a) = k \circ j_{p,\infty}(a) \in \text{ran}(k)$, so $\xi \in \text{ran}(k)$, contradicting that $\xi = \text{crit}(k)$.

Since k is the identity, $j_{0,\infty} = j$. Since the directed system $\langle M_{U_q}, k_{p,q} : p \subseteq q \in [\nu]^{<\omega}\rangle$, and thus the embedding $j_{0,\infty}$, were defined only with reference to the sequence $\langle U_p : p \in [\nu]^{<\omega}\rangle$, the lemma follows. $\square$

Lemma 5.3.22. *Suppose U is a normal ultrafilter, M is an inner model, and $j : V \to M$ is an elementary embedding such that for any $a \in M$, the ultrafilter derived from j using a is Rudin-Keisler equivalent to U^n for some $n < \omega$. Then $M = M_\nu^{\mathscr{I}(U)}$ and $j = j_{0,\nu}^{\mathscr{I}(U)}$ for some ordinal ν.*

Sketch. For all $m < \omega$, let $\kappa_m = j_{U^m}(\kappa)$, so κ_m is the m-th generator of j_{U^n} for any $n > m$. Let W_n be the ultrafilter on $[\kappa]^n$ derived from j_{U^n} using $\{\kappa_{n-1}, \ldots, \kappa_0\}$. Thus W_n is the unique ultrafilter with the following properties:

- $W_n \equiv_{\text{RK}} U^m$ for some $m < \omega$.
- The underlying set of W_n is $[\kappa]^n$.
- Every element of id_{W_m} is a generator of j_{W_m}.

Since every ultrafilter derived from j is Rudin-Keisler equivalent to an ultrafilter on κ, the class of generators of j is contained in $j(\kappa)$, and in particular it forms a set. Let $\langle \xi_\alpha : \alpha < \nu\rangle$ enumerate this set in increasing order. For any finite set $p \subseteq \nu$, the ultrafilter on $[\kappa]^n$ derived from j using $\{\xi_\alpha : \alpha \in p\}$ has the properties enumerated above, and hence is equal to W_n.

Let $\langle \xi'_\alpha : \alpha < \nu\rangle$ denote the sequence of generators of $j_{0,\nu}^{\mathscr{I}(U)}$. Then for any finite set $p \subseteq \nu$, the ultrafilter on $[\kappa]^n$ derived from $j_{0,\nu}^{\mathscr{I}(U)}$ using $\{\xi'_\alpha : \alpha \in p\}$ is equal to W_n.

By Lemma 5.3.21, it follows that $j = j_{0,\nu}^{\mathscr{I}(U)}$. $\square$

Proof of Theorem 5.3.20. The assumption that $M = H^M(j[V] \cup j(\kappa))$ implies that every ultrafilter derived from j is Rudin-Keisler equivalent to an ultrafilter on κ. By Theorem 5.1.1, it follows that every ultrafilter derived from j is Rudin-Keisler equivalent to U^n for some n. By Lemma 5.3.22, $j = j_{0,\nu}^{\mathscr{I}(U)}$ for some ordinal ν. $\square$

This theorem can be generalized considerably, for example, to analyze all elementary embeddings of V assuming UA and no μ-measurable cardinals, but the proofs go beyond the scope of this chapter.

5.4 The structure of the Rudin-Frolík order

5.4.1 The local ascending chain condition

The goal of this subsection is to prove Theorem 5.3.14, the local ascending chain condition for the Rudin-Frolík order. This uses two lemmas, the first of which is often useful in the context of UA. The approach taken here uses the following concept:

Definition 5.4.1. Suppose Y is a set, $W \in \mathbf{UF}(Y)$, and $U \leq_{\text{RF}} W$. Then the *translation of U by W*, denoted $t_U(W)$, is the unique M_U-ultrafilter $Z \in j_U(\mathbf{UF}(Y))$ such that $j_Z^{M_U} \circ j_U = j_W$ and $\text{id}_Z = \text{id}_W$.

The uniqueness of Z follows from the fact (Lemma 5.2.16) that there is at most one internal ultrapower embedding $k : M_U \to M_W$ such that $k \circ j_U = j_W$. Then $t_U(W)$ must be the M_U-ultrafilter on $j_U(Y)$ derived from k using id_W. We view $t_U(W)$ as a version of W inside M_U.

A more elegant, less comprehensible characterization of $t_U(W)$ is immediate from the proof of Theorem 5.2.15:

Lemma 5.4.2. *Suppose U and W are countably complete ultrafilters. Suppose I is a set in U and $\langle W_i : i \in I\rangle$ is a discrete sequence of ultrafilters such that $W = U\text{-}\lim_{i\in I} W_i$. Then $t_U(W) = [\langle W_i : i \in I\rangle]_U$.* □

The following lemma links translations to the minimal covers from the proof of Theorem 3.5.1:

Lemma 5.4.3. *Suppose δ is an ordinal, $W \in \mathbf{UF}(\delta)$, and $U \leq_{\text{RF}} W$ is a countably complete ultrafilter. Then $t_U(W)$ is $<_{\Bbbk}^{M_U}$-minimal among all $Z \in j_U(\mathbf{UF}(\delta))$ such that $j_U[W] \subseteq Z$.*

Proof. Fix $Z \in j_U(\mathbf{UF}(\delta))$ with $j_U[W] \subseteq Z$. For ease of notation, let $N = M_Z^{M_U}$. Then by Lemma 3.2.17, there is a unique embedding $e : M_W \to N$ such that $e \circ j_W = j_Z^{M_U} \circ j_U$ and $e(\text{id}_W) = \text{id}_Z$. Suppose now towards a contradiction that $Z <_{\Bbbk} t_U(W)$ in M_U. Let $(k, h) : (N, M_W) \to P$ be a right-internal comparison of $(j_Z^{M_U}, j_{t_U(W)}^{M_U})$ such that $k(\text{id}_Z) < h(\text{id}_{t_U(W)})$. Thus $(k \circ e, h) : (M_W, M_W) \to P$ is a right-internal comparison of (j_W, j_W) and $k \circ e(\text{id}_W) = k(\text{id}_Z) < h(\text{id}_{t_U(W)}) = h(\text{id}_W)$. The comparison $(k \circ e, h)$ witnesses $W <_{\Bbbk} W$ (Lemma 3.3.4), contradicting the irreflexivity of the Ketonen order (Proposition 3.3.9). □

Lemma 5.4.4. *Suppose $U \leq_{\text{RF}} W$ are countably complete ultrafilters. If U is nonprincipal, then $t_U(W) \neq j_U(W)$.*

Proof. Assume $t_U(W) = j_U(W)$, and we will show that U is principal. By Lemma 5.4.2, fix a set $I \in U$ and a discrete sequence $\langle W_i : i \in I\rangle$ such that $[\langle W_i : i \in I\rangle]_U = t_U(W)$. Since $\langle W_i : i \in I\rangle$ is discrete, in particular the W_i are pairwise distinct. Since $t_U(W) = j_U(W)$, Łoś's Theorem implies that there is a U-large set $J \subseteq I$ such that $W_i = W$ for all $i \in J$. Since the W_i are pairwise distinct, it follows that $|J| = 1$. Thus U contains a set of size 1, so U is principal. □

Proposition 5.4.5 (UA). *Suppose δ is an ordinal, $W \in \mathbf{UF}(\delta)$, and $U \leq_{\text{RF}} W$ is a nonprincipal ultrafilter. Then $t_U(W) <_{\Bbbk} j_U(W)$ in M_U.*

Proof. By Lemma 5.4.3 and the linearity of the Ketonen order, $t_U(W) \leq_{\Bbbk} j_U(W)$. By Lemma 5.4.4, $t_U(W) \neq j_U(W)$. It follows that $t_U(W) <_{\Bbbk} j_U(W)$. □

The following simple lemma on the preservation of the Rudin-Frolík order under translation functions will be used in the proof of Theorem 5.3.14:

Lemma 5.4.6. *Suppose U, W, and Z are countably complete ultrafilters with $U \leq_{\text{RF}} W, Z$.*

- *$W \leq_{\text{RF}} Z$ if and only if $t_U(W) \leq_{\text{RF}} t_U(Z)$ in M_U.*
- *$W <_{\text{RF}} Z$ if and only if $t_U(W) <_{\text{RF}} t_U(Z)$ in M_U.* □

We finally prove the local ascending chain condition.

Proof of Theorem 5.3.14. Assume towards a contradiction that the theorem is false. Let C be the class of countably complete $Z \in \mathbf{Fine}$ such that there is an infinite $<_{\text{RF}}$-ascending sequence $\langle U_n : n < \omega\rangle$ sequence $\leq_{\text{RF}}$-bounded above by Z. Let W be a $<_{\Bbbk}$-minimal element of C, and fix $U_0 <_{\text{RF}} U_1 <_{\text{RF}} \cdots$ such that $U_n \leq_{\text{RF}} W$ for all $n < \omega$. We may assume without loss of generality that U_0 is nonprincipal. By elementarity, $j_{U_0}(W)$ is a $<_{\Bbbk}^{M_{U_0}}$-minimal element of $j_{U_0}(C)$.

Since translation functions preserve the Rudin-Frolík order (Lemma 5.4.6), M_{U_0} satisfies $t_{U_0}(U_0) <_{\text{RF}} t_{U_0}(U_1) <_{\text{RF}} t_{U_0}(U_2) <_{\text{RF}} \cdots$ and $t_{U_0}(U_n) \leq_{\text{RF}} t_{U_0}(W)$ for all $n \leq \omega$. Since M_{U_0} is closed under countable sequences, it follows that $t_{U_0}(W) \in j_{U_0}(C)$. But by Proposition 5.4.5, $t_{U_0}(W) <_{\Bbbk} j_{U_0}(W)$. This contradicts that $j_{U_0}(W)$ is a $<_{\Bbbk}^{M_{U_0}}$-minimal element of $j_{U_0}(C)$. □

5.4.2 Pushouts and the Rudin-Frolík lattice

The main theorem of this section states that under UA, the Rudin-Frolík order induces a lattice structure on the Rudin-Keisler equivalence classes of countably complete ultrafilters:

Theorem (UA). *The Rudin-Frolík order is a lattice preorder:*

- *If U_0 and U_1 are countably complete ultrafilters, there is an $\leq_{\text{RF}}$-minimum countably complete ultrafilter $W \geq_{\text{RF}} U_0, U_1$.*
- *If U_0 and U_1 are countably complete ultrafilters, there is an $\leq_{\text{RF}}$-maximum countably complete ultrafilter $D \leq_{\text{RF}} U_0, U_1$.*

$$\begin{tikzcd}
 & P & \\
 & N \arrow[u, dashed, "h"] & \\
M_0 \arrow[uur, "k_0"] \arrow[ur, "i_0"] & & M_1 \arrow[uul, "k_1"'] \arrow[ul, "i_1"'] \\
 & V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}$$

Figure 5.4: The pushout of (j_0, j_1).

The two parts of this theorem will be proved as Corollary 5.4.16 and Proposition 5.4.18 below.

We begin by establishing the existence of least upper bounds in the Rudin-Frolík order, which is by far the most important part of the theorem. Here it is cleaner to work with elementary embeddings rather than ultrafilters:

Definition 5.4.7. Suppose $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are ultrapower embeddings. An internal ultrapower comparison $(i_0, i_1) : (M_0, M_1) \to N$ is a *pushout* of (j_0, j_1) if for any internal ultrapower comparison $(k_0, k_1) : (M_0, M_1) \to P$ of (j_0, j_1), there is a unique internal ultrapower embedding $h : N \to P$ such that $h \circ i_0 = k_0$ and $h \circ i_1 = k_1$.

Pushout comparisons are simply the model theoretic manifestation of least upper bounds in the Rudin-Frolík order. The uniqueness of pushouts is a standard category theoretic fact: the pushout of a pair of embeddings is what a category theorist, deposited by unnatural forces on page 132 of this monograph, would call the pushout of these arrows in the category of internal ultrapower embeddings. In general, if two arrows in a category have a pushout, it is unique up to isomorphism. Since the only isomorphisms of transitive models are identity functions, this implies the uniqueness of ultrapower pushouts up to equality. We will prove:

Theorem 5.4.8 (UA)**.** *Every pair of ultrapower embeddings has a pushout.*

We now begin the proof of Theorem 5.4.8. The proof involves the following auxiliary concept:

Definition 5.4.9. Suppose M_0 and M_1 are transitive models of ZFC. A pair of elementary embeddings $(i_0, i_1) : (M_0, M_1) \to N$ to a transitive model N is *minimal* if $N = H^N(i_0[M_0] \cup i_1[M_1])$.

In the context of ultrapower embeddings, minimality has the following alternate characterization:

Lemma 5.4.10. *Suppose $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are elementary embeddings and $(i_0, i_1) : (M_0, M_1) \to N$ is a comparison of (j_0, j_1). Suppose $a \in M_1$ is such that $M_1 = H^{M_1}(j_1[V] \cup \{a\})$. Then (i_0, i_1) is minimal if and only if $N = H^N(i_0[M_0] \cup \{i_1(a)\})$.* □

Embedded in any pair $(k_0, k_1) : (M_0, M_1) \to P$, there is a unique minimal pair $(i_0, i_1) : (M_0, M_1) \to N$. This follows from a trivial hull argument:

Lemma 5.4.11. *Suppose $(k_0, k_1) : (M_0, M_1) \to P$ is a pair of elementary embeddings. Then there exists a unique minimal $(i_0, i_1) : (M_0, M_1) \to N$ admitting an elementary embedding $h : N \to P$ such that $h \circ i_0 = k_0$ and $h \circ i_1 = k_1$.*

Proof. Let $H = H^P(k_0[M_0] \cup k_1[M_1])$. Let N be the transitive collapse of H. Let $h : N \to P$ be the inverse of the transitive collapse. Let $i_0 = h^{-1} \circ k_0$ and $i_1 = h^{-1} \circ k_1$. Then $(i_0, i_1) : (M_0, M_1) \to N$ and $h \circ i_0 = k_0$ and $h \circ i_1 = k_1$. Moreover

$$h[H^N(i_0[M_0] \cup i_1[M_1])] = H^P(k_0[M_0] \cup k_1[M_1]) = h[N]$$

which implies $H^N(i_0[M_0] \cup i_1[M_1]) = N$ since h is injective. Thus (i_0, i_1) is minimal.

Uniqueness is obvious; we omit the proof. □

Corollary 5.4.12 (UA)**.** *Every pair of ultrapower embeddings of V has a minimal internal ultrapower comparison.*

Proof. Suppose $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are ultrapower embeddings. Fix an internal ultrapower comparison $(k_0, k_1) : (M_0, M_1) \to P$ of (j_0, j_1). By Lemma 5.4.11, there is a minimal pair $(i_0, i_1) : (M_0, M_1) \to N$ and an elementary $h : N \to P$ with $h \circ i_0 = k_0$ and $h \circ i_1 = k_1$. It follows immediately that (i_0, i_1) is a comparison of (j_0, j_1). By Lemma 5.4.10, i_0 is an ultrapower embedding of M_0. Since k_0 is close to M_0 and $h \circ i_0 = k_0$, i_0 is close to M_0. Thus i_0 is a close ultrapower embedding of M_0, so i_0 is an internal ultrapower embedding of M_0. Similarly i_1 is an internal ultrapower embedding of M_1. Thus (i_0, i_1) is a minimal internal ultrapower comparison of (j_0, j_1). □

Lemma 5.4.13. *Suppose $(k_0, k_1) : (M_0, M_1) \to P$ is a pair of elementary embeddings and $(i_0, i_1) : (M_0, M_1) \to N$ is a minimal pair. Then there is at most one elementary embedding $h : N \to P$ such that $h \circ i_0 = k_0$ and $h \circ i_1 = k_1$.*

Proof. Suppose $h, h' : N \to P$ satisfy $h \circ i_0 = h' \circ i_0 = k_0$ and $h \circ i_1 = h' \circ i_1 = k_1$. Then $h \restriction i_0[M_0] = h' \restriction i_0[M_0]$ and $h \restriction i_1[M_1] = h' \restriction i_1[M_1]$. Since $N = H^N(i_0[M_0] \cup i_1[M_1])$, it follows that $h = h'$. □

$$
\begin{tikzcd}
 & P & \\
N \arrow[ur, "e"] & & N' \arrow[ul, "e'"'] \\
M_0 \arrow[u, "k_0"] \arrow[urr, "k_0'"' near end] & & M_1 \arrow[u, "k_1'"'] \arrow[ull, "k_1" near end] \\
 & V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}
$$

Figure 5.5: Comparing comparisons.

Lemma 5.4.14 (UA)**.** *Suppose $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are ultrapower embeddings and $(k_0, k_1) : (M_0, M_1) \to N$ and $(k_0', k_1') : (M_0, M_1) \to N'$ are internal ultrapower comparisons of (j_0, j_1). Then there is an internal ultrapower comparison $(e, e') : (N, N') \to P$ such that $e \circ k_0 = e' \circ k_0'$ and $e \circ k_1 = e' \circ k_1'$.*

Proof. For $n = 0, 1$, fix ordinals α_n, such that $M_n = H^{M_n}(j_n[V] \cup \{\alpha_n\})$. Applying UA, let $(e, e') : (N, N') \to P$ be an internal ultrapower comparison of $(k_0 \circ j_0, k_0' \circ j_0)$. We must show that for $n = 0, 1$, $e \circ k_n = e' \circ k_n'$.

Clearly $e \circ k_n \restriction j_n[V] = e' \circ k_n' \restriction j_n[V]$. Moreover $e(k_n(\alpha_n)) = e'(k_n'(\alpha_n))$: otherwise we contradict the irreflexivity of the Ketonen order (Proposition 3.3.9), since the comparison $(e \circ k_n, e' \circ k_n') : (M_n, M_n) \to P$ witnesses $U_n <_{\Bbbk} U_n$ where U_n is the ultrafilter derived from j_n using α_n. Since $M_n = H^{M_n}(j_n[V] \cup \{\alpha_n\})$, it follows that $e \circ k_n = e' \circ k_n'$. $\square$

Lemma 5.4.15 (UA)**.** *Suppose $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are ultrapower embeddings and $(k_0, k_1) : (M_0, M_1) \to N$ is a minimal comparison of (j_0, j_1). Then (k_0, k_1) is the pushout of (j_0, j_1).*

Proof. Suppose $(k_0', k_1') : (M_0, M_1) \to N'$ is a comparison of (j_0, j_1). We must show that there is an internal ultrapower embedding $h : N \to N'$ such that $k_0' = h \circ k_0$ and $k_1' = h \circ k_1$. Fix $(e, e') : (N, N') \to P$ as in Lemma 5.4.14.

By Lemma 5.4.11, (k_0, k_1) is the unique minimal pair that embeds into $(e \circ k_0, e \circ k_1)$. But by Lemma 5.4.11, there is a minimal pair that embeds into (k_0', k_1'), and composing with e', this pair embeds into $(e' \circ k_0', e' \circ k_1')$, and hence into $(e \circ k_0, e \circ k_1)$. It follows that (k_0, k_1) is equal to the minimal pair that embeds into (k_0', k_1'). Therefore fix an embedding $h : N \to N'$ such that $k_0' = h \circ k_0$ and $k_1' = h \circ k_0$.

We finish by showing that h is an internal ultrapower embedding. Notice that $e' \circ h : N \to P$ and $e : N \to P$ both embed the minimal pair (k_0, k_1) into the pair $(e \circ k_0, e \circ k_1)$. Thus by the uniqueness of such embeddings (Lemma 5.4.13),

$e' \circ h = e$. Since e is a close embedding, Lemma 2.2.24 implies that h is a close embedding. Since $h \circ k_0 = k_0'$ and k_0' is an ultrapower embedding, Lemma 2.2.19 implies that h is an ultrapower embedding. By Lemma 2.2.25, h is an internal ultrapower embedding. □

Proof of Theorem 5.4.8. The existence of pushouts is an immediate consequence of Corollary 5.4.12 and Lemma 5.4.15. □

Pushouts of course yield least upper bounds in the Rudin-Frolík order:

Corollary 5.4.16. *Suppose U_0 and U_1 are countably complete ultrafilters. Suppose $(i_0, i_1) : (M_{U_0}, M_{U_1}) \to N$ is the pushout of (j_{U_0}, j_{U_1}). Suppose W is a countably complete ultrafilter such that $j_W = i_0 \circ j_0 = i_1 \circ j_1$. Then W is the $\leq_{\text{RF}}$-minimum countably complete ultrafilter $W \geq_{\text{RF}} U_0, U_1$.*

Proof. The internal ultrapower embeddings i_0 and i_1 witness that $U_0 \leq_{\text{RF}} W$ and $U_1 \leq_{\text{RF}} W$. Suppose $U_0 \leq_{\text{RF}} Z$ and $U_1 \leq_{\text{RF}} Z$. We will show $W \leq_{\text{RF}} Z$. Let $k_0 : M_{U_0} \to M_Z$ and $k_1 : M_{U_1} \to M_Z$ witness $U_0 \leq_{\text{RF}} Z$ and $U_1 \leq_{\text{RF}} Z$. Then since (i_0, i_1) is a pushout and $(k_0, k_1) : (M_{U_0}, M_{U_1}) \to M_Z$, there is an internal ultrapower embedding $h : M_W \to M_Z$ such that $h \circ i_0 = k_0$ and $h \circ i_1 = k_1$. In particular $h \circ j_W = h \circ i_0 \circ j_{U_0} = k_0 \circ j_{U_0} = j_Z$, so h witnesses that $W \leq_{\text{RF}} Z$. □

It is worth noting the following bound here:

Proposition 5.4.17. *Suppose U_0 and U_1 are countably complete ultrafilters. If W is a minimal upper bound of U_0 and U_1 in the Rudin-Frolík order, then $\lambda_W = \max\{\lambda_{U_0}, \lambda_{U_1}\}$.*

Proof. Let $\lambda = \max\{\lambda_{U_0}, \lambda_{U_1}\}$. Since $U_0, U_1 \leq_{\text{RF}} W$, $\lambda \leq \lambda_W$. We will prove the reverse inequality. We may assume that λ is the underlying set of U_0 and U_1. Let $j_0 : V \to M_0$ and $j_1 : V \to M_1$ be the ultrapowers by U_0 and U_1 respectively. There is a minimal comparison $(i_0, i_1) : (M_0, M_1) \to N$ of (j_0, j_1) such that $i_0 \circ j_0 = i_1 \circ j_1 = j_W$. By Lemma 5.4.10, $N = H^N(j_W[V] \cup \{i_0(\text{id}_{U_0}), i_1(\text{id}_{U_1})\})$. Thus W is Rudin-Keisler equivalent to the ultrafilter on $\lambda \times \lambda$ derived from j_W using $\langle i_0(\text{id}_{U_0}), i_1(\text{id}_{U_1})\rangle$. It follows that $\lambda_W \leq |\lambda \times \lambda| = \lambda$. □

We now show the existence of greatest lower bounds in the Rudin-Frolík order. In fact we do a bit better:

Proposition 5.4.18. *Suppose A is a nonempty class of countably complete ultrafilters. Then A has a greatest lower bound in the Rudin-Frolík order.*

This follows purely abstractly from what we have proved so far. Recall that a partial order $(P, \leq)$ has the local ascending chain condition if for any $p \in P$, there is no ascending sequence $a_0 < a_1 < \cdots$ in P with $a_n \leq p$ for all $n < \omega$.

Lemma 5.4.19. *Suppose $(P, \leq)$ is a join semi-lattice with a minimum element that satisfies the local ascending chain condition. For any nonempty set $A \subseteq P$, A has a greatest lower bound in P.*

Proof. Consider the set $B \subseteq P$ of lower bounds of A. In other words,

$$B = \{b \in P : \forall a \in A\ b \leq a\}$$

Since P has a minimum element, B is nonempty. Since A is nonempty, fixing $p \in A$, every element of B lies below p. Therefore by the local ascending chain condition, B has a maximal element b_0. (The ascending chain condition says that the relation $>$ is wellfounded on $\{c \in P : c \leq p\}$, so the nonempty set B has a $>$-minimal element, or equivalently a $<$-maximal element.)

We claim B is a directed subset of $(P, \leq)$. Suppose $b, c \in B$. For any $a \in A$, by the definition of B, $b, c \leq a$, and therefore their least upper bound $b \vee c \leq a$. In other words, $b \vee c \leq a$ for all $a \in A$, so $b \vee c \in B$. This shows that B is directed.

Finally since b_0 is a maximal element of the directed set B, in fact b_0 is its maximum element. □

Proof of Proposition 5.4.18. The Rudin-Frolík order induces a partial order on the class of countably complete ultrafilters modulo Rudin-Keisler equivalence. This partial order is a join semi-lattice by Corollary 5.4.16, and it has the local ascending chain condition by Theorem 5.3.14. It has a minimum element, namely the Rudin-Keisler equivalence class of a principal ultrafilter. Therefore the conditions of Lemma 5.4.19 are met (except that we are considering a set-like partial order instead of a set, which makes no difference). This implies the proposition. □

Let us give another application of pushouts to the Rudin-Frolík order. The following characterization of the internal ultrapower embeddings of a pushout is remarkably easy to prove:

Theorem 5.4.20. *Suppose $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are ultrapower embeddings and $(i_0, i_1) : (M_0, M_1) \to N$ is their pushout. Suppose $h : N \to P$ is an ultrapower embedding. Then the following are equivalent:*

(1) h is amenable to both M_0 and M_1.

(2) h is an internal ultrapower embedding of N.

Proof. (1) implies (2): Let $k_0 = h \circ i_0$ and $k_1 = h \circ i_1$. Since h is an ultrapower embedding of N, k_0 is an ultrapower embedding of M_0. Since h is amenable to M_0, k_0 is amenable to M_0, and hence k_0 is close to M_0. Since k_0 is a close ultrapower embedding of M_0, in fact k_0 is an internal ultrapower embedding of M_0. Similarly k_1 is an internal ultrapower embedding of M_1. Thus (k_0, k_1) is an internal ultrapower comparison of (j_0, j_1). Since (i_0, i_1) is a pushout, there is an internal ultrapower embedding $h' : N \to P$ such that $h' \circ i_0 = k_0$ and $h' \circ i_1 = k_1$. By Lemma 5.4.13, however, h is the unique elementary embedding from N to P such that $h \circ i_0 = k_0$ and $h \circ i_1 = k_1$. Thus $h = h'$, so h is an internal ultrapower embedding, as desired.

(2) implies (1): Trivial. □

An elegant way to restate this is in terms of the ultrafilters amenable to a pushout:

Corollary 5.4.21. *Suppose $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are ultrapower embeddings and $(i_0, i_1) : (M_0, M_1) \to N$ is their pushout. Suppose W is a countably complete N-ultrafilter. Then $W \in N$ if and only if $W \in M_0 \cap M_1$.* □

Corollary 5.4.22 (UA)**.** *Suppose U and W are countably complete ultrafilters. Then the following are equivalent:*

(1) $U \leq_{\text{RF}} W$.

(2) $M_W \subseteq M_U$.

Proof. (1) implies (2): Trivial.

(2) implies (1): Let $(h.k) : (M_U, M_W) \to N$ be the pushout of (j_U, j_W). Since $M_W \subseteq M_U$ and k is an internal ultrapower of M_U, k is amenable to M_U. In particular, $k \restriction N$ is amenable to both M_U and M_W. Therefore $k \restriction N$ is an internal ultrapower of N. Thus k is γ-supercompact for all ordinals γ. It follows from Proposition 4.2.30 that k is the identity. Hence $h : M_U \to M_W$ is an internal ultrapower embedding with $h \circ j_U = j_W$, so $U \leq_{\text{RF}} W$. □

Corollary 5.4.23 (UA)**.** *Suppose $j_0, j_1 : V \to M$ are ultrapower embeddings with the same target model. Then $j_0 = j_1$.* □

One can actually show that under UA, if $j_0, j_1 : V \to M$ are arbitrary elementary embeddings with the same target model, then $j_0 = j_1$, but the proof is much more involved.

5.4.3 The finiteness of the Rudin-Frolík order

The goal of this subsection is to prove the central structural fact about the Rudin-Frolík order under UA: any countably complete ultrafilter has at most finitely many predecessors in the Rudin-Frolík order up to Rudin-Keisler equivalence. The following terminology allows us to state this more precisely:

Definition 5.4.24. The *type* of an ultrafilter U is its Rudin-Keisler equivalence class $\{U' : U' \equiv_{\text{RK}} U\}$.

Theorem 5.4.25 (UA)**.** *The set of Rudin-Frolík predecessors of a countably complete ultrafilter is the union of finitely many types.*

The proof heavily uses the concept of a Dodd parameter, introduced in Section 4.3 in a slightly more general context. Let us just remind the reader what this is in the special case of ultrapower embeddings. We identify finite sets of ordinals with their *decreasing* enumerations: if $p \subseteq \text{Ord}$ and $|p| = \ell$, then $\langle p_n : n < \ell \rangle$ denotes the unique decreasing sequence such that $p = \{p_0, \ldots, p_{\ell-1}\}$. The canonical order on finite sets of ordinals is then the lexicographic order on their decreasing enumerations.

Definition 5.4.26. Suppose $j : V \to M$ is an ultrapower embedding. The *Dodd parameter* of j, denoted $p(j)$, is the least finite set of ordinals p such that $H^M(j[V] \cup p) = M$.

Note that since j is an ultrapower embedding, $M = H^M(j[V] \cup \{\alpha\})$ for some ordinal α, so $p(j)$ certainly exists.

Recall the notion of an x-generator of an elementary embedding: if $j : M \to N$ is an elementary embedding between transitive models of ZFC and $x \in N$, then an ordinal $\xi \in N$ is an x-generator of j if $\xi \notin H^N(j[V] \cup \xi \cup \{x\})$. We need a basic but not completely trivial lemma about x-generators:

Lemma 5.4.27. *Suppose $M \xrightarrow{j} N \xrightarrow{i} P$ are elementary embeddings between transitive models and ξ is an x-generator of j. Then $i(\xi)$ is an $i(x)$-generator of $i \circ j$.*

Proof. Suppose not, and fix a function f and a finite set $p \subseteq i(\xi)$ such that

$$i(\xi) = i(j(f))(p, i(x))$$

Then P satisfies the statement that for some finite set $q \subseteq i(\xi)$, $i(\xi) = i(j(f))(q, i(x))$. Since i is elementary, N satisfies that for some finite set $q \subseteq \xi$, $\xi = j(f)(q, x)$, and this contradicts that ξ is an x-generator of j. □

The key lemma regarding the Dodd parameter is that each of its elements is a generator in a strong sense:

Lemma 5.4.28. *Suppose $j : V \to M$ is an ultrapower embedding. Let $p = p(j)$. Let $\ell = |p|$. Then for any $n < \ell$, p_n is the largest $p \restriction n$-generator of j.*

Proof. We first show that p_n is a $p \restriction n$-generator of j. Suppose not, towards a contradiction. Fix a finite set $q \subseteq p_n$ such that $p_n \in H^M(j[V] \cup p \restriction n \cup q)$. Let $r = (p \setminus \{p_n\}) \cup q$. Then $r < p$ but $p \in H^M(j[V] \cup r)$. Therefore

$$M = H^M(j[V] \cup p) \subseteq H^M(j[V] \cup r)$$

so $H^M(j[V] \cup r) = M$, contrary to the minimality of the Dodd parameter p.

Now let ξ be the largest $p \restriction n$-generator of j. Suppose towards a contradiction that $p_n < \xi$. Then $p \subseteq \xi \cup \{p_0, \ldots, p_{n-1}\}$, so since $\xi \notin H^M(j[V] \cup \xi \cup p \restriction n)$, in fact $\xi \notin H^M(j[V] \cup p)$. This contradicts the definition of $p(j)$. □

The proof of the finiteness of the Rudin-Frolík order proceeds by partitioning the Rudin-Frolík predecessors of a countably complete ultrafilter according to their relationship with its Dodd parameter.

Definition 5.4.29. Suppose $U <_{\text{RF}} W$ are countably complete ultrafilters. Let $p = p(j_W)$. Let $i : M_U \to M_W$ be the unique internal ultrapower embedding such that $i \circ j_U = j_W$. Then $n(U, W)$ is the least number n such that $p_n \notin i[M_U]$.

Note that $n(U, W)$ depends only on the types of U and W. Note moreover that $n(U, W)$ exists whenever $U <_{\text{RF}} W$: otherwise $p \subseteq i[M_U]$, so $M_W = H^{M_W}(j_W[V] \cup p) \subseteq i[M_U]$, which implies that i is surjective; thus i is an isomorphism, so $U \equiv_{\text{RK}} W$, contrary to the assumption that $U <_{\text{RF}} W$.

Lemma 5.4.30. *Suppose $U <_{\text{RF}} W$ are countably complete ultrafilters. Let $p = p(j_W)$. Let $i : M_U \to M_W$ be the unique internal ultrapower embedding such that $i \circ j_U = j_W$. Let $n = n(U, W)$. Then*

$$i[M_U] \subseteq H^{M_W}(j_W[V] \cup p \restriction n \cup p_n)$$

Proof. Suppose towards a contradiction that the lemma fails. Let ξ be the least ordinal such that $i(\xi) \notin H^{M_W}(j_W[V] \cup p \restriction n \cup p_n)$. Then $i[\xi] \subseteq H^{M_W}(j_W[V] \cup p \restriction n \cup p_n)$.

By the minimality of n, $p \restriction n \in i[M_U]$. Therefore let $q \in M_U$ be such that $i(q) = p \restriction n$. We claim ξ is a q-generator of j_U. Supposing the contrary, we have $\xi \in H^{M_U}(j_U[V] \cup \xi \cup q)$, so

$$i(\xi) \in i[H^{M_U}(j_U[V] \cup \xi \cup q)] \subseteq H^{M_W}(j_W[V] \cup p \restriction n \cup p_n)$$

which contradicts the definition of ξ.

Since ξ is a q generator of j_U, $i(\xi)$ is an $i(q)$-generator of $i \circ j_U$ by Lemma 5.4.27. In other words, $i(\xi)$ is a $p \restriction n$-generator of j_W. By Lemma 5.4.28, p_n is the largest $p \restriction n$-generator of j_W, so $i(\xi) \leq p_n$. This contradicts that $i(\xi) \notin H^{M_W}(j_W[V] \cup p \restriction n \cup p_n)$. □

Definition 5.4.31. Suppose W is a countably complete ultrafilter and $p = p(j_W)$. For any $n < |p|$, $\mathscr{D}_n(W) = \{U <_{\text{RF}} W : n(U, W) = n\}$.

Lemma 5.4.32. *For any countably complete ultrafilter W,*

$$\{U : U <_{\text{RF}} W\} = \bigcup_{n < |p(j_W)|} \mathscr{D}_n(W)$$

Proof. See the remarks following Definition 5.4.29. □

The following fact is the key to the proof of the finiteness of the Rudin-Frolík order:

Lemma 5.4.33. *Suppose $U_0, U_1 \in \mathscr{D}_n(W)$ and D is the $\leq_{\text{RF}}$-minimum countably complete ultrafilter such that $U_0, U_1 \leq_{\text{RF}} D$. Then $D \in \mathscr{D}_n(W)$.*

Proof. Let $M_0 = M_{U_0}$ and let $M_1 = M_{U_1}$. Let $(i_0, i_1) : (M_0, M_1) \to M_D$ be internal ultrapower embeddings witnessing that $U_0, U_1 \leq_{\text{RF}} D$ and let $(k_0, k_1) : (M_0, M_1) \to M_W$ be internal ultrapower embeddings witnessing that $U_0, U_1 \leq_{\text{RF}} W$.

Since D is the $\leq_{\text{RF}}$-minimum countably complete ultrafilter with $U_0, U_1 \leq_{\text{RF}} D$, in fact $D \leq_{\text{RF}} W$. Let $h : M_D \to M_W$ be the unique internal ultrapower embedding such that $h \circ j_D = j_W$. Notice that

$$h \circ i_0 = k_0$$
$$h \circ i_1 = k_1$$

by Lemma 5.2.16.

Since D is the $\leq_{\text{RF}}$-minimum ultrafilter with $U_0, U_1 \leq_{\text{RF}} D$, $(i_0, i_1) : (M_0, M_1) \to M_D$ must be minimal in the sense of Definition 5.4.9:

$$M_D = H^{M_D}(i_0[M_0] \cup i_1[M_1])$$

(The proof is a trivial diagram chase. Let $(\bar{i}_0, \bar{i}_1) : (M_0, M_1) \to N$ be the unique minimal pair admitting $e : N \to M_D$ such that $e \circ \bar{i}_0 = i_0$ and $e \circ \bar{i}_1 = i_1$. By the proof of Corollary 5.4.12, N is an internal ultrapower of M_0 and M_1, so since D is a least upper bound of U_0, U_1, there is an internal ultrapower embedding $d : M_D \to N$ such that $d \circ i_0 = \bar{i}_0$ and $d \circ i_1 = \bar{i}_1$. Then $d \circ e : N \to N$ satisfies $d \circ e \circ \bar{i}_0 = \bar{i}_0$ and $d \circ e \circ \bar{i}_1 = \bar{i}_1$, and hence by Lemma 5.4.13, $d \circ e$ must be the identity map. Hence d and e are inverses, so by transitivity $N = M_D$ and e is the identity. Now $\bar{i}_0 = e \circ \bar{i}_0 = i_0$ and $\bar{i}_1 = e \circ \bar{i}_1 = i_1$ so $(\bar{i}_0, \bar{i}_1) = (i_0, i_1)$. Since $(\bar{i}_0, \bar{i}_1)$ is minimal, so is (i_0, i_1).) Therefore

$$h[M_D] = h[H^{M_D}(i_0[M_0] \cup i_1[M_1])] = H^{M_W}(k_0[M_0] \cup k_1[M_1])$$

Let $p = p(j_W)$. Since $U_0 \in \mathscr{D}_n(W)$, $k_0[M_0] \subseteq H^{M_W}(j_W[V] \cup p \restriction n \cup p_n)$ by Lemma 5.4.30. Similarly, $k_1[M_1] \subseteq H^{M_W}(j_W[V] \cup p \restriction n \cup p_n)$. Thus

$$k_0[M_0] \cup k_1[M_1] \subseteq H^{M_W}(j_W[V] \cup p \restriction n \cup p_n)$$

It follows that $h[M_D] = H^{M_W}(k_0[M_0] \cup k_1[M_1]) \subseteq H^{M_W}(j_W[V] \cup p \restriction n \cup p_n)$. In particular, since p_n is a $p \restriction n$-generator of j_W by Lemma 5.4.28, $p_n \notin h[M_D]$. Clearly

$$p \restriction n \in k_0[M_0] \subseteq h[M_D]$$

so n is the least number such that $p_n \notin h[M_D]$. It follows that $n(D, W) = n$. In other words, $D \in \mathscr{D}_n(W)$. □

The point now is that by Theorem 5.3.14 and Corollary 5.4.16, we can find a maximum element of $\mathscr{D}_n$:

Proposition 5.4.34 (UA)**.** *Suppose W is a countably complete ultrafilter and $n < |p(j_W)|$. If $\mathscr{D}_n(W)$ is nonempty, then $\mathscr{D}_n(W)$ has an $\leq_{\text{RF}}$-maximum element.*

Proof. By Corollary 5.4.16, every pair of countably complete ultrafilters has a least upper bound in the Rudin-Frolík order. Combining this with Lemma 5.4.33, the class $\mathscr{D}_n(W)$ is directed under $\leq_{\text{RF}}$. Moreover it is bounded below W in $\leq_{\text{RF}}$. Therefore by Theorem 5.3.14, it has a maximal element U. By the $\leq_{\text{RF}}$-directedness of $\mathscr{D}_n(W)$, this maximal element is a maximum element. □

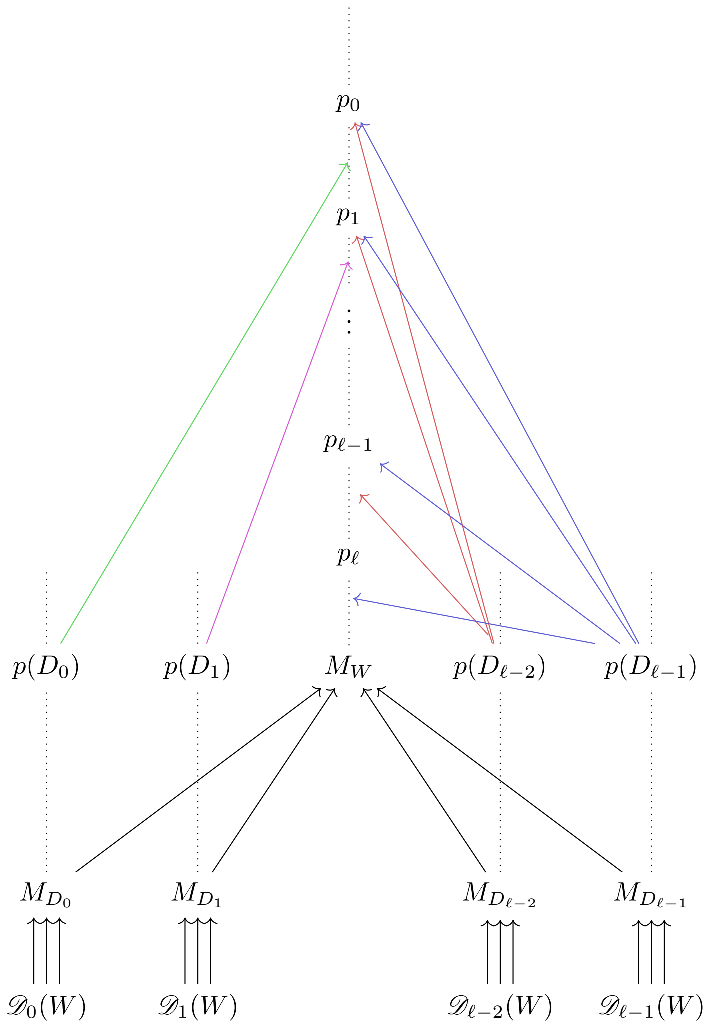

Figure 5.6: The maximal $\leq_{\mathrm{RF}}$-predecessors of W, $D_n = \max(\mathscr{D}_n(W))$.

We finally prove Theorem 5.4.25.

Proof of Theorem 5.4.25. The proof is by induction on the wellfounded relation $<_{\mathrm{RF}}$. (See Lemma 5.2.12.) Assume W is a countably complete ultrafilter. Our induction hypothesis is that for all $U <_{\mathrm{RF}} W$, $\{D : D \leq_{\mathrm{RF}} U\}$ is the union of finitely many types. We aim to show that $\{U : U \leq_{\mathrm{RF}} W\}$ is the union of finitely many types.

Let $p = p(j_W)$ and let $\ell = |p(j_W)|$. By Lemma 5.4.32,

$$\{U : U <_{\mathrm{RF}} W\} = \bigcup_{n<\ell} \mathscr{D}_n(W)$$

We claim that for any $n < \ell$, $\mathscr{D}_n(W)$ is the union of finitely many types. If $\mathscr{D}_n(W)$ is empty, this is certainly true. If $\mathscr{D}_n(W)$ is nonempty, then by Proposition 5.4.34, there is an $\leq_{\mathrm{RF}}$-maximum element U of $\mathscr{D}_n(W)$. Since $U \in \mathscr{D}_n(W)$, $U <_{\mathrm{RF}} W$, and so by our induction hypothesis $\{D : D \leq_{\mathrm{RF}} U\}$ is the union of finitely many types. But since U is an $\leq_{\mathrm{RF}}$-maximum element of $\mathscr{D}_n(W)$, $\mathscr{D}_n(W) \subseteq \{D : D \leq_{\mathrm{RF}} U\}$. Thus $\mathscr{D}_n(W)$ is the union of finitely many types.

Since $\{U : U <_{\mathrm{RF}} W\} = \bigcup_{n<\ell} \mathscr{D}_n(W)$ is a finite union of classes $\mathscr{D}_n(W)$ each of which is itself the union of finitely many types, $\{U : U <_{\mathrm{RF}} W\}$ is the union of finitely many types. The collection $\{U : U \leq_{\mathrm{RF}} W\}$ contains just one more type than $\{U : U <_{\mathrm{RF}} W\}$, namely that of W. So $\{U : U \leq_{\mathrm{RF}} W\}$ is the union of finitely many types, completing the induction. □

5.4.4 Translations and limits

In this section we explain the relationship between pushouts, ultrafilter translations, and the minimal covers defined for the proof of UA from the linearity of the Ketonen order in Section 3.5.

Recall Definition 5.4.1, which defined for any countably complete ultrafilters $U \leq_{\mathrm{RF}} W$ the translation of W by U, the canonical countably complete ultrafilter of M_U that leads from M_U into M_W. It turns out that there is a natural generalization of $t_U(W)$ for any ultrafilters that admit a pushout:

Definition 5.4.35. Suppose U and $W \in \mathbf{UF}(Y)$ are countably complete ultrafilters. Suppose $(k, h) : (M_U, M_W) \to N$ is the pushout of (j_U, j_W). Then $t_U(W)$ denotes the M_U-ultrafilter on $j_U(Y)$ derived from k using $h(\mathrm{id}_W)$.

The point of this definition is that $t_U(W)$ is the canonical ultrafilter of M_U giving rise to the M_U-side of the pushout of (j_U, j_W):

Lemma 5.4.36. *Suppose U and $W \in \mathbf{UF}(Y)$ are countably complete ultrafilters. Suppose $(k, h) : (M_U, M_W) \to N$ is the pushout of (j_U, j_W). Then $t_U(W)$ is the unique ultrafilter $Z \in j_U(\mathbf{UF}(Y))$ such that $j_Z^{M_U} = k$ and $\mathrm{id}_Z = h(\mathrm{id}_W)$.* □

We will try to omit superscripts when we can:

Corollary 5.4.37. *If U and W are countably complete ultrafilters, then if (j_U, j_W) has a pushout, it is equal to $(j_{t_U(W)}, j_{t_W(U)})$.*

The notation $t_U(W)$ generalizes the notation $t_U(W)$ introduced in Definition 5.4.1 when $U \leq_{\mathrm{RF}} W$. To see this, assume $U \leq_{\mathrm{RF}} W$ and let $k : M_U \to M_W$ be the unique internal ultrapower embedding of M_U such that $k \circ j_U = j_W$. Then $(k, \mathrm{id}) : (M_U, M_W) \to M_W$ is the pushout of (j_U, j_W), and hence $t_U(W)$ as we

$$\begin{tikzcd}
M_U \arrow[r, "t_U(W)"] \arrow[rr, bend left, "t_U(Z)"] & N \arrow[r, dashed, "h"] & P \\
V \arrow[u, "U"] \arrow[r, "W"'] & M_W \arrow[u, "t_W(U)"'] \arrow[r, "t_W(Z)"'] & M_Z \arrow[u, "t_Z(U)"']
\end{tikzcd}$$

Figure 5.7: The proof of Lemma 5.4.40.

have defined it here is just the M_U-ultrafilter derived from k using id_W, which is precisely $t_U(W)$ as defined in Definition 5.4.1.

It turns out that in the definition of a translation, one does not need to use the pushout (as long as the pushout exists):

Lemma 5.4.38. *Suppose U and $W \in \mathbf{UF}(Y)$ are countably complete ultrafilters such that the pair (j_U, j_W) has a pushout. Let $(k, h) : (M_U, M_W) \to P$ be a close comparison of (j_U, j_W). Then $t_U(W)$ is the M_U-ultrafilter on $j_U(Y)$ derived from k using $h(\mathrm{id}_W)$.* $\square$

It is not hard to see that translations are invariant under Rudin-Keisler equivalence:

Lemma 5.4.39. *Suppose $U \equiv_{\mathrm{RK}} U'$ and $W \equiv_{\mathrm{RK}} W'$. Then $t_U(W) \equiv_{\mathrm{RK}} t_{U'}(W')$ in M_U.*

In fact, we can do quite a bit better than this: translation functions preserve the Rudin-Frolík order.

Lemma 5.4.40. *Suppose U, W, and Z are countably complete ultrafilters. If $W \leq_{\mathrm{RF}} Z$, then $t_U(W) \leq_{\mathrm{RF}} t_U(Z)$ in M_U.*

Proof. Let $N = M^{M_U}_{t_U(W)}$ and let $P = M^{M_U}_{t_U(Z)}$. The proof is contained in Fig. 5.7. By Corollary 5.4.37:

- $(j_{t_U(W)}, j_{t_W(U)}) : (M_U, M_W) \to N$ is the pushout of (j_U, j_W).
- $(j_{t_U(Z)}, j_{t_Z(U)} \circ j_{t_W(Z)}) : (M_U, M_W) \to P$ is an internal ultrapower comparison of (j_U, j_W).

Since $(j_{t_U(W)}, j_{t_W(U)})$ is a pushout, there is an internal ultrapower embedding $h : N \to P$ such that $h \circ j_{t_U(W)} = j_{t_U(Z)}$ and $h \circ j_{t_W(U)} = j_{t_Z(U)} \circ j_{t_W(Z)}$. In particular, the first of these equations says that h witnesses $t_U(W) \leq_{\mathrm{RF}} t_U(Z)$ in M_U. $\square$

We occasionally use the following fact, which is an immediate consequence of Lemma 5.4.36:

Lemma 5.4.41. *Suppose U and W are countably complete ultrafilters on X and Y. Then the following are equivalent:*

(1) $U \leq_{\text{RF}} W$.

(2) For some $I \in U$ and some discrete sequence $\langle W_i : i \in I\rangle$ of countably complete ultrafilters on Y, $t_U(W) = [\langle W_i : i \in I\rangle]_U$.

(3) $j_{t_U(W)} \circ j_U = j_W$.

(4) $t_W(U)$ is a principal ultrafilter of M_W.

(5) $t_W(U) = \mathrm{p}^{j_W(X)}_{h(\mathrm{id}_U)}$ where $h : M_U \to M_W$ is the unique internal ultrapower embedding such that $h \circ j_U = j_W$. □

The following fundamental fact connects translations back to the minimal covers of Section 3.5:

Theorem 5.4.42 (UA)**.** *Suppose δ is an ordinal, U is a countably complete ultrafilter, and $W \in \mathbf{UF}(\delta)$. Then $t_U(W)$ is the least element of $j_U(\mathbf{UF}(\delta), <_{\Bbbk})$ that extends $j_U[W]$.*

Proof. By replacing U with an Rudin-Keisler equivalent ultrafilter, we may assume that for some ordinal ϵ, $U \in \mathbf{UF}(\epsilon)$, putting us in a position to apply the results of Section 3.5.

Let W_* be the least element of $j_U(\mathbf{UF}(\delta), <_{\Bbbk})$ that extends $j_U[W]$ and let U_* be the least element of $j_W(\mathbf{UF}(\epsilon), <_{\Bbbk})$ that extends $j_W[U]$. By Theorem 3.5.4,

$$(j^{M_U}_{W_*}, j^{M_W}_{U_*}) : (M_U, M_W) \to N$$

is a comparison of (j_U, j_W). Moreover, as a consequence of Lemma 3.5.13, $\mathrm{id}_{W_*} = j^{M_W}_{U_*}(\mathrm{id}_W)$. In particular,

$$N = H^N(j^{M_U}_{W_*}[M_U] \cup \{\mathrm{id}_{W_*}\}) = H^N(j^{M_U}_{W_*}[M_U] \cup \{j^{M_W}_{U_*}(\mathrm{id}_W)\})$$

It follows from Lemma 5.4.10 that $(j^{M_U}_{W_*}, j^{M_W}_{U_*})$ is minimal. Therefore by Lemma 5.4.15, $(j^{M_U}_{W_*}, j^{M_W}_{U_*})$ is the pushout of (j_U, j_W). Since W_* is the M_U-ultrafilter on $j_U(\delta)$ derived from $j^{M_U}_{W_*}$ using $j^{M_W}_{U_*}(\mathrm{id}_W)$, by definition $W_* = t_U(W)$. □

This yields the following bound on $t_U(W)$ that is not a priori obvious:

Corollary 5.4.43 (UA)**.** *Suppose U is a countably complete ultrafilter and W is a countably complete ultrafilter on an ordinal. Then $t_U(W) \leq_{\Bbbk} j_U(W)$ in M_U.*

Proof. Let δ be the underlying ordinal of W. Then $j_U(W) \in j_U(\mathbf{UF}(\delta))$ and $j_U[W] \subseteq j_U(W)$. Thus $t_U(W) \leq_{\Bbbk} j_U(W)$ in M_U by Theorem 5.4.42. □

We finally show that translation functions preserve the Ketonen order:

Theorem 5.4.44 (UA)**.** *Translation functions preserve the Ketonen order. More precisely, suppose Z is a countably complete ultrafilter and U and W are countably complete ultrafilters on ordinals. Then $U <_{\Bbbk} W$ if and only if $M_Z \vDash t_Z(U) <_{\Bbbk} t_Z(W)$.*

For this we need the strong transitivity of the Ketonen order (Lemma 3.3.10). We actually use the following immediate corollary of Lemma 3.3.10 and the characterization of limits in terms of inverse images (Lemma 3.2.12):

Lemma 5.4.45. *Suppose Z is an ultrafilter, δ is an ordinal, and $U, W \in \mathbf{UF}(\delta)$ satisfy $U <_{\Bbbk} W$. For any $W_* \in j_Z(\mathbf{UF}(\delta))$ with $j_Z[W] \subseteq W_*$, there is some $U_* \in j_Z(\mathbf{UF}(\delta))$ with $U_* <_{\Bbbk}^{M_Z} W_*$ and $j_Z[U] \subseteq U_*$.* □

With Theorem 5.4.42 and Lemma 5.4.45 in hand, we can prove Theorem 5.4.44.

Proof of Theorem 5.4.44. Assume that $U <_{\Bbbk} W$ are countably complete ultrafilters on ordinals. We will show $t_Z(U) <_{\Bbbk}^{M_Z} t_Z(W)$. For ease of notation, we will assume (without real loss of generality) that $U, W \in \mathbf{UF}(\delta)$ for a fixed ordinal δ.

Let $W_* = t_Z(W)$. Theorem 5.4.42 implies that $j_Z[W] \subseteq W_*$. (This is actually much easier to prove that Theorem 5.4.42.) By Lemma 5.4.45, it follows that there is some $U_* \in j_Z(\mathbf{UF}(\delta))$ with

$$U_* <_{\Bbbk}^{M_Z} W_*$$

and $j_Z[U] \subseteq U_*$. Since $t_Z(U)$ is the minimal extension of $j_Z[U]$ by Theorem 5.4.42, we have

$$t_Z(U) \leq_{\Bbbk}^{M_Z} U_*$$

By the transitivity of the Ketonen order, $t_Z(U) \leq_{\Bbbk}^{M_Z} t_Z(W)$, as desired. □

5.5 The internal relation

5.5.1 A generalized Mitchell order

In this section, we introduce a variant of the generalized Mitchell order that will serve as a powerful tool in the theory of countably complete ultrafilters. The trouble with using the Mitchell order itself to prove general theorems about countably complete ultrafilters is that the Mitchell order is only meaningful for ultrafilters that have a certain amount of strength: a precondition for $U \lhd W$ is that $P(\lambda_U) \subseteq M_W$. In order to analyze a more general class of ultrafilters, we need a way to talk about the Mitchell order on ultrafilters that are not assumed to be strong.

There are a number of possible approaches, but the one that has proved most successful is called the *internal relation:*

Definition 5.5.1. The *internal relation* is defined on countably complete ultrafilters U and W by setting $U \sqsubset W$ if $j_U \restriction M_W$ is an internal ultrapower embedding of M_W.

The topic of this section is the theory of the internal relation under UA. The reason that we have saved it for this chapter is that it is closely related to the theory of pushouts from Section 5.4.2.

Before proceeding to the basic theory below, let us mention that the supercompactness analysis of Chapter 7 and Chapter 8 yields a more familiar description of the internal relation on a very large class of ultrafilters assuming UA. In fact, the internal relation and the Mitchell order are essentially one and the same:

Theorem 8.3.33 (UA)**.** *Suppose U and W are hereditarily uniform irreducible ultrafilters. Then the following are equivalent:*

(1) $U \sqsubset W$.

(2) Either $U \lhd W$ or $W \in V_\kappa$ where $\kappa = \text{crit}(j_U)$.

The second part of condition (2) should be compared with Kunen's commuting ultrapowers lemma (Theorem 5.5.19).

5.5.2 The Mitchell order versus the internal relation

To understand the nature of the internal relation, it helps to consider its relationship with the Mitchell order.

Proposition 5.5.2. *Suppose U is a countably complete ultrafilter on a set X and W is a countably complete ultrafilter such that $X \in M_W$ and $U \sqsubset W$. Then the M_W-ultrafilter $U \cap M_W$ belongs to M_W. In particular, if $P(X) \subseteq M_W$, then $U \lhd W$.* □

In general, however, $U \sqsubset W$ does not imply $U \lhd W$. This is a consequence of Kunen's commuting ultrapowers lemma (Theorem 5.5.19):

Proposition 5.5.3. *Suppose κ is a measurable cardinal, $U \in V_\kappa$ is a countably complete ultrafilter and W is a κ-complete ultrafilter. Then $W \sqsubset U$.* □

Note that in the situation above, if W is nonprincipal, then $\lambda_W \geq \kappa$, and in particular $W \not\lhd U$ since $P(\kappa) \nsubseteq M_U$.

Whether $U \lhd W$ always implies $U \sqsubset W$ is a considerably subtler question.

Proposition 5.5.4. *Suppose U is an ultrafilter on X and W is an ultrafilter such that M_W is closed under X-sequences. If $U \lhd W$, then $U \sqsubset W$.*

Proof. If $U \lhd W$, then the closure of M_W under X-sequences implies $j_U^{M_W} = j_U \restriction M_W$, so $j_U \restriction M_W$ is an internal ultrapower embedding. □

Eventually, we will prove that under UA, if U is a countably complete uniform ultrafilter on X and $U \lhd W$, then M_W is closed under X-sequences (Theorem 8.3.29). Assuming ZFC alone, however, it is consistent that $U \lhd W$ but $U \not\sqsubset W$, as we now show.

Proposition 5.5.5. *Suppose κ is 2^κ-supercompact and $2^\kappa = 2^{(\kappa^+)}$. Then there is a normal ultrafilter D on κ and a κ-complete normal fine ultrafilter $\mathcal{U}$ on $P_\kappa(\kappa^+)$ such that $\mathcal{U} \lhd D$.*

Sketch. Since κ is κ^+-supercompact, there is a κ-complete normal fine ultrafilter $\mathcal{U}$ on $P_\kappa(\kappa^+)$. By Solovay's theorem on SCH above a strongly compact cardinal (Theorem 7.2.16), $|P_\kappa(\kappa^+)| = \kappa^+$. By Solovay's ultrafilter-capturing theorem (Theorem 6.3.7), for any set A of hereditary cardinality at most 2^κ, there is a normal ultrafilter D on κ such that $A \in M_D$. But $\mathcal{U} \subseteq P(P_\kappa(\kappa^+))$ has hereditary cardinality $2^{\kappa^+} = 2^\kappa$. Thus there is a normal ultrafilter D on κ such that $\mathcal{U} \in M_D$, or in other words, $\mathcal{U} \lhd D$. □

Thus given a failure of the weak GCH at a supercompact, one must have a rather unusual situation in which $\mathcal{U} \lhd D$ even though $\lambda_\mathcal{U} > \lambda_D$. On the other hand, the internal relation does not hold between these ultrafilters:

Proposition 5.5.6. *Assume D is a κ-complete uniform ultrafilter on κ and $\mathcal{U}$ is a κ-complete normal fine ultrafilter on $P_\kappa(\kappa^+)$. Then $\mathcal{U} \not\sqsubset D$.*

Proof. Suppose towards a contradiction that $\mathcal{U} \sqsubset D$. Then $j_\mathcal{U}(M_D) \subseteq M_D$ since $j_\mathcal{U} \restriction M_D$ is an internal ultrapower embedding of M_D. But $j_\mathcal{U}(M_D) = (M_{j_\mathcal{U}(D)})^{M_\mathcal{U}}$. Since $j_\mathcal{U}(D)$ is $j_\mathcal{U}(\kappa)$-complete in $M_\mathcal{U}$,

$$\mathrm{Ord}^{(\kappa^+)} \subseteq \mathrm{Ord}^{j_\mathcal{U}(\kappa)} \cap M_\mathcal{U} \subseteq (M_{j_\mathcal{U}(D)})^{M_\mathcal{U}} = j_\mathcal{U}(M_D) \subseteq M_D$$

It follows that j_D is κ^+-supercompact, contradicting the bound on the supercompactness of the ultrapower associated to an ultrafilter on κ (Proposition 4.2.30). □

It is not clear in general whether a uniform ultrafilter on κ^+ can be internal to a uniform ultrafilter on κ, although this turns out to be impossible for countably complete ultrafilters assuming UA.

We have not checked that the implication from $U \lhd W$ to $U \sqsubset W$ can fail under the Generalized Continuum Hypothesis, but we are confident that it can. Under UA, however, this implication is a theorem:

Theorem 8.3.29 (UA)**.** *Suppose U and W are countably complete ultrafilters. If $U \lhd W$, then $U \sqsubset W$.* □

5.5.3 Basic theory of the internal relation

The true motivation for the definition of the internal relation comes from the theory of ultrapower comparisons:

Lemma 5.5.7. *Suppose U and W are countably complete ultrafilters. Then*

$$(j_U(j_W), j_U \restriction M_W) : (M_U, M_W) \to j_U(M_W)$$

is a left-internal minimal comparison of (j_U, j_W).

$$\begin{tikzcd}
 & j_U(M_W) & \\
M_U \arrow[ur, "j_U(j_W)"] & & M_W \arrow[ul, "j_U\restriction M_W"'] \\
 & V \arrow[ul, "j_U"] \arrow[ur, "j_W"'] &
\end{tikzcd}$$

Figure 5.8: Comparison and the internal relation

Proof. The fact that $(j_U(j_W), j_U \restriction M_W)$ is a comparison of (j_U, j_W) is immediate from the standard application-composition identity:

$$j_U(j_W) \circ j_U = (j_U \restriction M_W) \circ j_W$$

Since j_W is an internal ultrapower embedding of V, $j_U(j_W)$ is an internal ultrapower embedding of M_U by the elementarity of j_U, and so in particular, $(j_U(j_W), j_U \restriction M_W)$ is left-internal.

We now show that $(j_U(j_W), j_U \restriction M_W)$ is a minimal comparison of (j_U, j_W), or in other words that

$$j_U(M_W) = H^{j_U(M_W)}(j_U(j_W)[M_U] \cup j_U[M_W])$$

The proof begins with the fact that $M_W = H^{M_W}(j_W[V] \cup \{\text{id}_W\})$. Applying j_U to both sides of the equation, we obtain:

$$j_U(M_W) = H^{j_U(M_W)}(j_U(j_W)[M_U] \cup \{j_U(\text{id}_W)\})$$

Since $j_U(\text{id}_W) \in j_U[M_W]$,

$$H^{j_U(M_W)}(j_U(j_W)[M_U] \cup \{j_U(\text{id}_W)\}) \subseteq H^{j_U(M_W)}(j_U(j_W)[M_U] \cup j_U[M_W])$$

This yields that $j_U(M_W) \subseteq H^{j_U(M_W)}(j_U(j_W)[M_U] \cup j_U[M_W])$, which of course implies that equality holds, as desired. $\square$

Combining Lemma 5.5.7 with the fact that minimal comparisons of ultrapowers are ultrapower comparisons (Lemma 5.4.10), we obtain the following lemma:

Lemma 5.5.8. *Suppose U and W are countably complete ultrafilters. Then $j_U \restriction M_W$ is an ultrapower embedding of M_W.*

Of course, we do not mean that $j_U \restriction M_W$ is necessarily an *internal* ultrapower embedding of M_W, just that there is a point $a \in j_U(M_W)$ such that $j_U(M_W) = H^{j_U(M_W}(j_U[M_W] \cup \{a\})$. Note, however, that this point a need not be id_U itself.

Corollary 5.5.9. *Suppose U and W are countably complete ultrafilters. Then the following are equivalent:*

(1) $U \sqsubset W$.

(2) $j_U \restriction M_W$ *is definable from parameters over* M_W.

(3) $j_U \restriction M_W$ *is close to* M_W. □

Inspecting the proof of Lemma 5.4.10 in the context of the minimal comparison $(j_U(j_W), j_U \restriction M_W)$, one can extract a specific M_W-ultrafilter giving rise to the embedding $j_U \restriction M_W$:

Definition 5.5.10. Suppose U and W are countably complete ultrafilters. Let X be the underlying set of U. Then the *pushforward of U into M_W* is the M_W-ultrafilter $s_W(U)$ on $j_W(X)$ defined as follows: if $A \subseteq j_W(X)$ and $A \in M_W$,

$$A \in s_W(U) \iff j_W^{-1}[A] \in U$$

The reason we call $s_W(U)$ a pushforward is that it is literally equal to the pushforward of U by j_W intersected with M_W, or more precisely to $f_*(U) \cap M_W$ where $f : X \to j_W(X)$ is given by $f = j_W \restriction X$.

For the reader's convenience, let us chase through all the lemmas and prove that $s_W(U)$ behaves as it should:

Lemma 5.5.11. *Suppose U and W are countably complete ultrafilters on X and Y. Then $s_W(U)$ is the M_W-ultrafilter on $j_W(X)$ derived from $j_U \restriction M_W$ using $j_U(j_W)(\mathrm{id}_U)$. Moreover,*

$$j^{M_W}_{s_W(U)} = j_U \restriction M_W$$

Thus $U \sqsubset W$ if and only if $s_W(U) \in M_W$.

Proof. Let $f = j_W \restriction X$. Then $f_*(U)$ is the ultrafilter derived from j_U using $j_U(f)(\mathrm{id}_U)$ by the basic theory of the Rudin-Keisler order (Lemma 3.2.16). Thus $f_*(U) \cap M_W$ is the M_W-ultrafilter derived from $j_U \restriction M_W$ using $j_U(f)(\mathrm{id}_U) = j_U(j_W)(\mathrm{id}_U)$. But $f_*(U) \cap M_W = s_W(U)$, so $s_W(U)$ is the M_W-ultrafilter on $j_W(X)$ derived from $j_U \restriction M_W$ using $j_U(j_W)(\mathrm{id}_U)$.

We finish by proving $j^{M_W}_{s_W(U)} = j_U \restriction M_W$. Since $s_U(W)$ is derived from $j_U \restriction M_W$ using $j_U(j_W)(\mathrm{id}_U)$, there is a factor embedding $k : M^{M_W}_{s_W(U)} \to j_U(M_W)$ with $k \circ j^{M_W}_{s_W(U)} = j_U \restriction M_W$ and $k(\mathrm{id}_{s_W(U)}) = j_U(j_W)(\mathrm{id}_U)$. Since $(j_U(j_W), j_U \restriction M_W) : (M_U, M_W) \to j_U(M_W)$ is a minimal comparison of (j_U, j_W), Lemma 5.4.10 yields:

$$j_U(M_W) = H^{j_U(M_W)}(j_U[M_W] \cup \{j_U(j_W)(\mathrm{id}_U)\})$$

But $H^{j_U(M_W)}(j_U[M_W] \cup \{j_U(j_W)(\mathrm{id}_U)\}) \subseteq k[M^{M_W}_{s_W(U)}]$. In other words, k is a surjection. It follows that $M^{M_W}_{s_W(U)} = j_U(M_W)$ and k is the identity. Therefore $j^{M_W}_{s_W(U)} = k \circ j^{M_W}_{s_W(U)} = j_U \restriction M_W$ as desired. □

As a corollary, one can characterize the internal relation in terms of amenability of ultrafilters.

Lemma 5.5.12. *Suppose U and W are countably complete ultrafilters. Then the following are equivalent:*

(1) $U \sqsubset W$.

(2) For all $U' \leq_{\text{RK}} U$, $U' \cap M_W \in M_W$.

(3) For all $U' \equiv_{\text{RK}} U$, $U' \cap M_W \in M_W$.

Proof. (1) implies (2): Suppose $U' \leq_{\text{RK}} U \sqsubset W$. Fix a set X and a point $a \in M_U$ such that U' is the ultrafilter on X derived from j_U using a. If $X \cap M_W \notin U'$, then $U' \cap M_W = \emptyset$, and so $U' \cap M_W \in M_W$ vacuously. Therefore assume $X \cap M_W \in U'$. In other words, $a \in j_U(X \cap M_W)$, so $a \in j_U(M_W)$. Then $U' \cap M_W$ is the ultrafilter derived from $j_U \restriction M_W$ using a, so since $j_U \restriction M_W$ is an internal ultrapower embedding of M_W, $U' \cap M_W \in M_W$.

(2) implies (3): Trivial.

(3) implies (1): Let X be the underlying set of U. Let $f : X \to j_W(X)$ be the restriction $f = j_W \restriction X$. Since j_W is injective, $f_*(U) \equiv_{\text{RK}} U$. Moreover $f_*(U) \cap M_W = s_W(U)$, so if $f_*(U) \cap M_W \in M_W$, then $U \sqsubset W$ by Lemma 5.5.11. □

This has the following corollary, which is perhaps not immediately obvious:

Corollary 5.5.13. *Suppose U, W, and Z are countably complete ultrafilters and*

$$Z \leq_{\text{RK}} U \sqsubset W$$

Then $Z \sqsubset W$.

Proof. By Lemma 5.5.12, for all $U' \leq_{\text{RK}} U$, $U' \sqsubset W$. In particular (by the transitivity of the Rudin-Keisler order), for all $U' \leq_{\text{RK}} Z$, $U' \sqsubset W$. Applying Lemma 5.5.12 again, $Z \sqsubset W$, as desired. □

There is also an obvious relationship in the other direction between the Rudin-Frolík order and the internal relation:

Proposition 5.5.14. *Suppose U, W, and Z are countably complete ultrafilters and*

$$U \leq_{\text{RF}} W \sqsupset Z$$

Then $Z \sqsubset U$.

Proof. Since $Z \sqsubset W$, Lemma 5.5.11 implies $s_W(Z) \in M_W$. Since $U \leq_{\text{RF}} W$, there is an internal ultrapower embedding $h : M_U \to M_W$. We claim that $h^{-1}[s_W(Z)] = s_U(Z)$. Let X be the underlying set of Z. If $A \in j_U(P(X))$,

$$\begin{aligned} A \in h^{-1}[s_W(Z)] &\iff h(A) \in s_W(Z) \\ &\iff j_W^{-1}[h(A)] \in Z \\ &\iff (h \circ j_U)^{-1}[h(A)] \in Z \\ &\iff j_U^{-1}[A] \in Z \\ &\iff A \in s_U(Z) \end{aligned}$$

Since h is definable over M_U and $s_W(Z) \in M_W \subseteq M_U$, $s_U(Z) = h^{-1}[s_W(Z)] \in M_U$. Hence $Z \sqsubset U$ by Lemma 5.5.11, as desired. □

The key to understanding the internal relation under UA is the following theorem, which takes advantage of the theory of pushouts and translations (Section 5.4.2 and Section 5.4.4):

Lemma 5.5.15 (UA)**.** *Suppose U and W are countably complete ultrafilters. Then the following are equivalent:*

(1) $U \sqsubset W$.

(2) $(j_U(j_W), j_U \restriction M_W)$ is the pushout of (j_W, j_U).

(3) $t_U(W) = j_U(W)$.

(4) $t_W(U) = s_W(U)$.

If the underlying set of W is an ordinal, we can add to the list:

(5) $M_U \vDash j_U(W) \leq_{\Bbbk} t_U(W)$.

Proof. (1) implies (2): Since $U \sqsubset W$, $(j_U(j_W), j_U \restriction M_W)$ is a minimal internal ultrapower comparison of (j_U, j_W). Therefore by Lemma 5.4.15, $(j_U(j_W), j_U \restriction M_W)$ is the pushout of (j_U, j_W), so (2) holds.

(2) implies (3): Let X be the underlying set of W. By the definition of $t_U(W)$, $t_U(W)$ is the M_U-ultrafilter on $j_U(X)$ derived from k using $h(\text{id}_W)$ where $(k, h) : (M_U, M_W) \to N$ is the pushout of (j_U, j_W). By (2), $(k, h) = (j_U(j_W), j_U \restriction M_W)$, and hence $t_U(W)$ is the M_U-ultrafilter on $j_U(X)$ derived from $j_U(j_W)$ using $j_U(\text{id}_W)$. Since W is the ultrafilter on X derived from j_W using id_W, by the elementarity of j_U, $j_U(W)$ is the ultrafilter on $j_U(X)$ derived from $j_U(j_W)$ using $j_U(\text{id}_W)$. This yields that $t_U(W) = j_U(W)$, so (3) holds.

(3) implies (4): Let $(k, h) : (M_U, M_W) \to N$ be the pushout of (j_U, j_W). Since $t_U(W) = j_U(W)$, Lemma 5.4.36 implies $k = j_U(j_W)$ and $h(\text{id}_W) = \text{id}_{j_U(W)} = j_U(\text{id}_W)$.

We claim that $h = j_U \restriction M_W$. Note that $h \restriction j_W[V] = j_U \restriction j_W[V]$ since $h \circ j_W = k \circ j_U = j_U(j_W) \circ j_U = j_U \circ j_W$. Moreover $h(\text{id}_W) = j_U(\text{id}_W)$, so

$$h \restriction j_W[V] \cup \{\text{id}_W\} = j_U \restriction j_W[V] \cup \{\text{id}_W\}$$

Since $M_W = H^{M_W}(j_W[V] \cup \{\text{id}_W\})$ it follows that $h = j_U \restriction M_W$, as claimed.

Now $t_W(U)$ is the M_W-ultrafilter derived from $h = j_U \restriction M_W$ using $k(\text{id}_U) = j_U(j_W)(\text{id}_U)$. By Lemma 5.5.11, $t_W(U) = s_W(U)$.

(4) implies (1): Since $t_W(U) = s_W(U)$, $s_W(U) \in M_W$. By Lemma 5.5.11, $U \sqsubset W$.

Finally, assume that the underlying set of W is an ordinal δ, and we will show the equivalence of (3) and (5). Clearly (3) implies (5), so let us prove the converse. Assume (5) holds. By Corollary 5.4.43, $t_U(W) \leq_{\Bbbk} j_U(W)$ in M_U. Thus $t_U(W) \leq_{\Bbbk} j_U(W)$ and $j_U(W) \leq_{\Bbbk} t_U(W)$ in M_U, so $j_U(W) = t_U(W)$ since the Ketonen order is antisymmetric. □

5.5.4 Commuting ultrapowers and wellfoundedness

Like the Mitchell order (Lemma 4.2.38), the internal relation is irreflexive:

Corollary 5.5.16. *If U is a nonprincipal countably complete ultrafilter, then $U \not\sqsubset U$.*

Proof. Proposition 4.2.30 implies $j_U \restriction \lambda_U^+$ does not belong to M_U. □

Unlike the Mitchell order, however, the internal relation is not a strict relation. In fact, it has 2-cycles, which typically come from the phenomenon of *commuting ultrafilters*:

Definition 5.5.17. Suppose U and W are countably complete ultrafilters. Then U and W *commute* if $j_U(j_W) = j_W \restriction M_U$ and $j_W(j_U) = j_U \restriction M_W$.

Clearly if U and W commute, then $U \sqsubset W$ and $W \sqsubset U$. Let us provide some obvious combinatorial characterizations of commuting ultrafilters. Below, if U and W are ultrafilters, then $U \otimes W$ denotes the sum U-$\sum_{i\in I} W$. Also, we write $Ux\,\varphi(x)$ to denote that $\varphi(x)$ holds for U-almost all x.

Lemma 5.5.18. *Suppose U and W are countably complete ultrafilters on sets X and Y. The following are equivalent:*

(1) U and W commute.

(2) For all binary relations $R \subseteq X \times Y$, $Ux\,Wy\,R(x,y) \iff Wy\,Ux\,R(x,y)$.

(3) $\text{flip}_*(U \otimes W) = W \otimes U$, where $\text{flip}(x,y) = (y,x)$ □

Somewhat surprisingly, there are nontrivial examples of commuting ultrafilters:

Theorem 5.5.19 (Kunen)**.** *Suppose U and W are countably complete ultrafilters and $U \in V_\kappa$ where $\kappa = \text{crit}(j_W)$. Then $j_W(j_U) = j_U \restriction M_W$ and $j_U(j_W) = j_W \restriction M_U$.*

Let us give our pet proof of Theorem 5.5.19, which uses the following reformulation of commutativity:

$$\begin{tikzcd}
& N & \\
M_U \arrow[ur, bend left, "j_W \restriction M_U"] \arrow[ur, "j_U(j_W)"'] & & M_W \arrow[ul, "j_U \restriction M_W = j_W(j_U)"'] \\
& V \arrow[ul, "j_U"] \arrow[ur, "j_W"'] &
\end{tikzcd}$$

Figure 5.9: The proof of Proposition 5.5.20

Proposition 5.5.20. *Suppose U and W are countably complete ultrafilters such that $j_W(j_U) = j_U \restriction M_W$. Then U and W commute.*

Proof. To show U and W commute, we must show that $j_W \restriction M_U = j_U(j_W)$. By Lemma 5.5.7, $(j_W \restriction M_U, j_W(j_U))$ and $(j_U(j_W), j_U \restriction M_W)$ are left-internal and right-internal minimal comparisons of (j_U, j_W). Since $j_W(j_U) = j_U \restriction M_W$, we can conclude that

$$(j_W \restriction M_U) \circ j_U = j_U(j_W) \circ j_U$$

In particular, $j_W \restriction M_U$ and $j_U(j_W)$ are elementary embeddings of M_U with the same target model, which we will denote by

$$N = j_W(M_U) = j_U(j_W)(M_U) = j_U(M_W) = j_W(j_U)(M_W)$$

Let ξ be the least ordinal such that $M_U = H^{M_U}(j_U[V] \cup \{\xi\})$. We claim that

$$j_W(\xi) = j_U(j_W)(\xi)$$

First, by the strictness of the seed order (Proposition 3.3.9), $j_U(j_W)(\xi) \leq j_W(\xi)$: otherwise the right-internal comparison $(j_U(j_W), j_W \restriction M_U) : (M_U, M_U) \to N$ witnesses $U <_{\Bbbk} U$ by Lemma 3.3.4.

In the other direction, by elementarity, $j_W(\xi)$ is the least ordinal α with $N = H^N(j_W(j_U)[M_W] \cup \{\alpha\})$. On the other hand, since $(j_U(j_W), j_U \restriction M_W)$ is a minimal comparison of (j_U, j_W) (Lemma 5.5.7), $N = H^N(j_U[M_W] \cup \{j_U(j_W)(\xi)\})$ (Lemma 5.4.10). Since $j_U \restriction M_W = j_W(j_U) \restriction M_W$, this yields

$$N = H^N(j_W(j_U)[M_W] \cup \{j_U(j_W)(\xi)\})$$

By the minimality of $j_W(\xi)$, $j_W(\xi) \leq j_U(j_W)(\xi)$, as desired.

Thus $j_U(j_W)$ and $j_W \restriction M_U$ coincide on $j_U[V] \cup \{\xi\}$, and so since $M_U = H^{M_U}(j_U[V] \cup \{\xi\})$, it follows that $j_U(j_W) = j_W \restriction M_U$, as desired. □

Given the reliance of the above proof on the existence of a minimal seed for U, it is a natural question whether the result holds when U is not countably complete.

Proof of Theorem 5.5.19. It is trivial to see that $j_W(j_U) = j_U \restriction M_W$. Hence by Proposition 5.5.20, U and W commute. □

Under UA, the only counterexamples to the strictness of the internal relation are commuting ultrafilters:

Theorem 5.5.21 (UA). *Suppose $U \sqsubset W$ and $W \sqsubset U$. Then U and W commute.*

Proof. Since $U \sqsubset W$, $t_U(W) = j_U(W)$. Since $W \sqsubset U$, $t_U(W) = s_U(W)$. Therefore $j_U(W) = s_U(W)$. It follows that $j_U(j_W) = j^{M_U}_{j_U(W)} = j^{M_U}_{s_U(W)} = j_W \restriction M_U$ by Lemma 5.5.11. Similarly, $j_W(j_U) = j_U \restriction M_W$. In other words, U and W commute, as desired. □

This raises an interesting technical question:

Question 5.5.22 (ZFC). *Suppose U and W are countably complete ultrafilters such that $U \sqsubset W$ and $W \sqsubset U$. Do U and W commute?*

This question is answered positively (for arbitrary ultrafilters) in [**?**] assuming GCH.

The supercompactness analysis of Chapter 7 occasionally requires a partial converse to Theorem 5.5.19: the only way certain nice pairs of ultrafilters can commute is if one lies below the completeness of the other.

Definition 5.5.23. Suppose λ is a cardinal. A countably complete ultrafilter W is *λ-internal* if $U \sqsubset W$ for all U such that $\lambda_U < \lambda$.

Proposition 5.5.24. *Suppose U and W are countably complete hereditarily uniform ultrafilters. Assume U is λ_U-internal and W is λ_W-internal. Let $\kappa_U = \mathrm{crit}(j_U)$ and $\kappa_W = \mathrm{crit}(j_W)$. Then the following are equivalent:*

(1) U and W commute.

(2) Either $U \in V_{\kappa_W}$ or $W \in V_{\kappa_U}$.

One can also state Proposition 5.5.24 avoiding the notion of hereditary uniformity: if U is λ_U-internal and W is λ_W-internal, then U and W commute if and only if $\lambda_U < \kappa_W$ or $\lambda_W < \kappa_U$.

The proof of Proposition 5.5.24 requires a number of lemmas. The first allows us to approximate an arbitrary ultrapower embedding by a small ultrafilter:

Lemma 5.5.25. *Suppose $j : V \to M$ is an ultrapower embedding. Then for any cardinal λ, there is a countably complete ultrafilter D with $\lambda_D \leq 2^\lambda$ such that there is an elementary embedding $k : M_D \to M$ with $k \circ j_D = j$ and* $\text{crit}(k) > \lambda$.

Proof. Suppose γ is an ordinal. We will find an ultrafilter D on γ^γ such that there is an elementary embedding $k : M_D \to M$ with $k \circ j_D = j$ and $\text{crit}(k) \geq \gamma$. Taking $\gamma = \lambda + 1$ proves the lemma.

Fix $a \in M$ such that $M = H^M(j[V] \cup \{a\})$ and X such that $a \in j(X)$. Fix functions $\langle f_\alpha : \alpha < \gamma \rangle$ on X such that $\alpha = j(f_\alpha)(a)$. Define a function $g : X \to \gamma^\gamma$ by letting $g(x)$ be the function with $g(x)(\alpha) = f_\alpha(x)$ for all $\alpha < \gamma$.

Let D be the ultrafilter on γ^γ derived from j using $j(g)(a)$. Let $k : M_D \to M$ be the factor embedding such that $k \circ j_D = j$ and $k(\text{id}_D) = j(g)(a)$.

We claim that $\text{crit}(k) \geq \gamma$. It suffices to show that $\gamma \subseteq k[M_D] = H^M(j[V] \cup \{j(g)(a)\})$. Fix $\alpha < \gamma$. Then

$$\alpha = j(f_\alpha)(a) = j(f)_{j(\alpha)}(a) = j(g)(a)(j(\alpha))$$

Thus α is definable in M from $j(g)(a)$ and $j(\alpha)$. Thus $\alpha \in H^M(j[V] \cup \{j(g)(a)\})$, as desired. $\square$

The coarseness of the bound 2^λ actually causes a number of problems down the line. An argument due to Silver (which appears as Theorem 7.5.26) provides a major improvement in a special case, and is instrumental in our analysis of the linearity of the Mitchell order on normal fine ultrafilters under UA without GCH assumptions. Further improvements could potentially solve the problems concerning so-called isolated cardinals discussed in Section 7.5.

Using Lemma 5.5.25, we prove the following lemma, which can be seen as a version of the Kunen Inconsistency Theorem (Theorem 4.2.35) that replaces the strength requirement of that theorem with a requirement involving the internal relation:

Lemma 5.5.26. *Suppose U is a countably complete ultrafilter and κ is a strong limit cardinal. Then the following are equivalent:*

(1) U is κ-internal and $\sup j_U[\kappa] \subseteq \kappa$.

(2) U is κ-complete.

Proof. (1) implies (2). Let $j : V \to M$ be the ultrapower of the universe by U. We first show that j is $<\kappa$-supercompact. Fix $\gamma < \kappa$, and we will prove that $j \restriction \gamma \in M$. Let $\lambda = j(\gamma)$, so $\lambda < \kappa$ by the assumption that $j[\kappa] \subseteq \kappa$. By Lemma 5.5.25, one can find a countably complete ultrafilter D with $\lambda_D \leq 2^\lambda < \kappa$ and an elementary embedding $k : M_D \to M$ with $k \circ j_D = j$ and $\text{crit}(k) > \lambda = j(\gamma)$. In particular $j_D \restriction \gamma = j \restriction \gamma$. Moreover since $\lambda_D < \kappa$, $D \sqsubset U$. Therefore $j \restriction \gamma = j_D \restriction \gamma \in M$, as desired.

Now j is $<\kappa$-supercompact and $j[\kappa] \subseteq \kappa$. Since j is an ultrapower embedding, if κ is singular, then j is κ-supercompact. Therefore the Kunen Inconsistency

Theorem (Theorem 4.2.35 or Theorem 4.4.32) implies $\text{crit}(j) \geq \kappa$, so U is κ-complete.

(2) implies (1). Trivial. □

Lemma 5.5.27. *Suppose U and W are nonprincipal countably complete ultrafilters. Let $\kappa_U = \text{crit}(j_U)$ and $\kappa_W = \text{crit}(j_W)$. Assume U is κ_W-internal and W is κ_U-internal. Then either $j_U(\kappa_W) > \kappa_W$ or $j_W(\kappa_U) > \kappa_U$.*

Proof. Assume towards a contradiction that $j_U(\kappa_W) = \kappa_W$ and $j_W(\kappa_U) = \kappa_U$. Since U is κ_W-internal and $j_U[\kappa_W] \subseteq \kappa_W$, U is κ_W-complete. Therefore $\kappa_U \geq \kappa_W$. By symmetry, $\kappa_W \geq \kappa_U$. Thus $\kappa_U = \kappa_W$. This contradicts that $j_U(\kappa_W) = \kappa_W$ while $j_U(\kappa_U) > \kappa_U$ by the definition of a critical point. □

We can finally prove Proposition 5.5.24:

Proof of Proposition 5.5.24. (1) implies (2): Since U and W commute, $j_U(\kappa_W) = \kappa_W$ and $j_W(\kappa_U) = \kappa_U$. By Lemma 5.5.27, either U is not κ_W-internal or W is not κ_U-internal. Therefore either $\lambda_U < \kappa_W$ or $\lambda_W < \kappa_U$.

Assume first that $\lambda_U < \kappa_W$. Since U is hereditarily uniform, the underlying set of U has hereditary cardinality λ_U, and hence $U \in V_{\kappa_W}$ since κ_W is inaccessible.

If instead $\lambda_W < \kappa_U$, then $W \in V_{\kappa_U}$ by a similar argument.

(2) implies (1): Immediate from Theorem 5.5.19. □

Chapter 6

V = HOD and GCH from UA

6.1 Introduction

6.1.1 The universe above a supercompact cardinal

In this short chapter, we exposit two results that show that something when UA is combined with very large cardinal hypotheses, instead of simply proving structural results for countably complete ultrafilters, the axiom can resolve classical questions independent from the usual axioms of set theory.

Since UA is preserved by forcing to add a Cohen real, UA does not imply $V = \text{HOD}$, no matter what large cardinals one assumes in addition to UA. But it turns out it is possible to prove that forcing is the only obstruction:

Theorem 6.2.8 (UA)**.** *Assume there is a supercompact cardinal. Then V is a generic extension of* HOD*.*

Similarly, UA is preserved by forcing to change the value of the continuum, so UA does not imply the Continuum Hypothesis. But UA implies that for sufficiently large cardinals λ, $2^\lambda = \lambda^+$:

Theorem 6.1.1 (UA)**.** *Assume κ is supercompact. Then for all cardinals $\lambda \geq \kappa$, $2^\lambda = \lambda^+$.*

UA has many other combinatorial consequences above the least supercompact cardinal κ: for example, $\diamondsuit(S^{\delta^{++}}_{\delta^+})$ holds for all cardinals δ of cofinality at least κ. In fact, there are no small forcing invariant statements known to be independent of UA plus large cardinals, and there is no known technique that could establish such an independence result. We close this section with some precise conjectures expressing the intuition that UA decides all structural questions about the universe above the least supercompact.

6.1.2 Outline of Chapter 6

We now outline the rest of the chapter.

SECTION 6.2. We prove the results on ordinal definability under UA and large cardinals. This is quite straightforward, but many open questions remain. For example, we prove that if κ is supercompact and UA holds, then V is a generic extension of HOD. How small is the forcing? The best upper bound we know is κ^{++}, which comes from Section 6.3.8 below.

SECTION 6.3. We prove the results on GCH under UA and large cardinals. We begin in Section 6.3.2 by discussing two results of Solovay relating large cardinals to the Generalized Continuum Problem. The first (Theorem 7.2.19) is his very well-known theorem that SCH holds above a supercompact cardinal [19]. The second (Theorem 6.3.3) is his little-known observation that the linearity of the Mitchell order on normal ultrafilters on a cardinal κ of maximal Mitchell order implies that $2^{2^\kappa} = (2^\kappa)^+$. We use a version of this to prove an easy result indicating that UA should prove GCH above a supercompact from UA: if κ is supercompact and $\lambda \geq \kappa$ is a singular strong limit cardinal of cofinality at least κ, then $2^\gamma = \gamma^+$ for all cardinals γ such that $\lambda \leq \gamma \leq \lambda^{+\omega}$.[1] In Section 6.3.3, we introduce a generalization of the Habič-Honzík local capturing property, and show that in certain contexts, this property can lead to instances of GCH. In Section 6.3.5, we prove a result regarding the Mitchell order and supercompactness that shows that under UA, if D and U are ultrafilters with λ_D below the supercompactness of U, then $D \lhd U$. This is immediate given GCH, but proving this using UA alone is a little bit subtle. Combined with the results on the local capturing, this yields the theorem on GCH above a supercompact and certain more local facts. Finally we use the cardinal arithmetic results established here to

6.2 Ordinal definability

The proof that V is a generic extension of HOD assuming UA plus a supercompact relies on the following simple fact:

Proposition 6.2.1 (UA). *Every countably complete ultrafilter on an ordinal is ordinal definable.*

Proof. Suppose δ is an ordinal. Then the ordinal definable set $\mathbf{UF}(\delta)$ of all countably complete ultrafilters on δ is wellordered by the Ketonen order. Thus every element of $\mathbf{UF}(\delta)$ is ordinal definable from its rank in the Ketonen order. $\square$

Corollary 6.2.2 (UA). *For any set of ordinals X and any ultrapower embedding $j : V \to M$:*

(1) $j(\mathrm{OD}_X) \subseteq \mathrm{OD}_X$.

(2) $j(\mathrm{HOD}_X) \subseteq \mathrm{HOD}_X$.

[1] This was one of the earliest consequences of UA, proved, like Theorem 4.4.2, before the axiom itself had even been isolated.

(3) *For any* $Y \in \mathrm{HOD}_X$, $j \restriction Y \in \mathrm{HOD}_X$.

Proof. We first prove (1). We have $j(\mathrm{OD}_X) = \mathrm{OD}^M_{j(X)}$. Fix a countably complete ultrafilter U on an ordinal such that $j = j_U$. Then since M is definable from U and $U \in \mathrm{OD}$ by Proposition 6.2.1, $\mathrm{OD}^M_{j(X)} \subseteq \mathrm{OD}_{j(X)}$. Moreover $j(X) = j_U(X)$ is definable from X and U, so $j(X) \in \mathrm{OD}_X$. Hence $\mathrm{OD}^M_{j(X)} \subseteq \mathrm{OD}_{j(X)} \subseteq \mathrm{OD}_X$.

For (2), note that $j(\mathrm{HOD}_X)$ is the class of sets that are hereditarily $j(\mathrm{OD}_X)$, and this is contained in the class of sets that are hereditarily OD_X by (1).

For (3), clearly $j \restriction Y \in \mathrm{OD}_Y \subseteq \mathrm{OD}_X$. But moreover by (2), $j \restriction Y \subseteq \mathrm{HOD}_X$. Therefore $j \restriction Y \in \mathrm{HOD}_X$. □

The following lemma should be compared with the theorem of Shelah that if λ is a singular strong limit cardinal of uncountable cofinality, then for any X such that $P(\alpha) \subseteq \mathrm{HOD}_X$ for all $\alpha < \lambda$, in fact $P(\lambda) \subseteq \mathrm{HOD}_X$.

Lemma 6.2.3 (UA)**.** *Suppose κ is λ-supercompact and $X \subseteq \kappa$ is such that $V_\kappa \subseteq \mathrm{HOD}_X$. Then $P(\lambda) \subseteq \mathrm{HOD}_X$.*

Proof. Fix a λ-supercompact ultrapower embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. Then

$$P(\lambda) \subseteq j(V_\kappa) \subseteq j(\mathrm{HOD}_X) \subseteq \mathrm{HOD}_X$$

The final inclusion follows from Corollary 6.2.2. □

Theorem 6.2.4 (UA)**.** *Suppose κ is supercompact. Then $V = \mathrm{HOD}_X$ for some $X \subseteq \kappa$.*

Proof. Fix $X \subseteq \kappa$ such that $V_\kappa \subseteq \mathrm{HOD}_X$.[2] Since κ is supercompact, Lemma 6.2.3 implies that for all $\lambda \geq \kappa$, $P(\lambda) \subseteq \mathrm{HOD}_X$, and therefore $V = \mathrm{HOD}_X$. □

To connect this to generic extensions of HOD, we use Vopěnka's Theorem.

Definition 6.2.5. Suppose X is a set such that and $X \cup \{X\} \subseteq \mathrm{OD}$. The OD-*cardinality of* X, denoted $|X|^{\mathrm{OD}}$, is the least ordinal λ such that there is an OD bijection between λ and X.

The OD cardinality of X is defined for all X with $X \cup \{X\} \subseteq \mathrm{OD}$. It is always a HOD-cardinal. In fact OD cardinality satisfies all the usual properties of cardinality; for example, $|X|^{\mathrm{OD}}$ is the least ordinal that ordinal definably surjects onto X and the least ordinal into which X ordinal definably injects.

Definition 6.2.6. Suppose κ is an ordinal. Let $\mathbb{A}_\kappa$ be the Boolean algebra $P(P(\kappa)) \cap \mathrm{OD}$ and let $\lambda = |\mathbb{A}_\kappa|^{\mathrm{OD}}$. Fix an OD bijection $\pi_\kappa : \lambda \to \mathbb{A}_\kappa$. Then $\mathbb{V}_\kappa$ is the Boolean algebra on λ given by pulling back the operations on $\mathbb{A}_\kappa$ under π_κ.

[2]To obtain such a set X, let E be a binary relation on κ such that $(V_\kappa, \in) \cong (\kappa, E)$ using the fact that $|V_\kappa| = \kappa$. Code E as a subset of κ using a pairing function $\kappa \to \kappa \times \kappa$.

Note that $\mathbb{V}_\kappa \in \mathrm{HOD}$. The Boolean algebra $\mathbb{V}_\kappa$ is called the *Vopěnka algebra at κ.*

Theorem 6.2.7 (Vopěnka). *If κ is an ordinal, then $\mathbb{V}_\kappa$ is a complete Boolean algebra in* HOD *and for any $X \subseteq \kappa$, there is a* HOD*-generic ultrafilter $G \subseteq \mathbb{V}_\kappa$ such that* $\mathrm{HOD}[G] = \mathrm{HOD}_X$.

Proof. The fact that $\mathbb{V}_\kappa$ is HOD-complete follows immediately from the fact that $\mathbb{A}_\kappa = P(P(\kappa)) \cap \mathrm{OD}$ is closed under ordinal definable unions. Let $H = \{S \in \mathbb{A}_\kappa : X \in S\}$, and let $G = \pi_\kappa^{-1}[H]$. The filter H is obviously closed under ordinal definable intersections, and so since $\pi_\kappa : \mathbb{A}_\kappa \to \mathbb{V}_\kappa$ is an ordinal definable isomorphism of Boolean algebras, G is closed under ordinal definable meets. This implies that G is a HOD-generic ultrafilter on $\mathbb{V}_\kappa$. Obviously $\mathrm{HOD}[G] \subseteq \mathrm{HOD}_X$ since G is OD_X. Note also that if $S \subseteq \mathrm{Ord}$ is OD_X, then there is an ordinal β and a Σ_2-formula $\varphi(v_0, v_1, v_2)$ such that $\alpha \in S$ if and only if $\varphi(\alpha, \beta, X)$. For $\alpha \leq \gamma = \sup S$, $A_\alpha = \{Y \subseteq X : \varphi(\alpha, \beta, Y)\} \in \mathbb{A}_\kappa$. Then $\langle A_\alpha : \alpha < \gamma \rangle \in \mathrm{OD}$, so letting $p_\alpha = \pi_\kappa^{-1}(A_\alpha)$, $\langle p_\alpha : \alpha < \gamma \rangle \in \mathrm{HOD}$, and $\alpha \in S$ if and only if $p_\alpha \in G$, so $S \in \mathrm{HOD}[G]$, as desired. $\square$

This yields a proof of our main theorem on HOD:

Theorem 6.2.8 (UA). *Assume there is a supercompact cardinal. Then V is a generic extension of* HOD.

Proof. Let κ be the least supercompact cardinal. By Theorem 6.2.4, $V = \mathrm{HOD}_X$ for some $X \subseteq \kappa$, so by Theorem 6.2.7, $V = \mathrm{HOD}[G]$ for some generic $G \subseteq \mathbb{V}_\kappa$. $\square$

Question 6.2.9 (UA). *Let κ be the least supercompact cardinal.*

- *Is $V = \mathrm{HOD}[X]$ for some $X \subseteq \kappa$?*
- *Is $V = \mathrm{HOD}[G]$ for $G \subseteq \kappa$ generic for a partial order $\mathbb{P} \in \mathrm{HOD}$ such that $|\mathbb{P}| \leq \kappa$? What about a κ-cc Boolean algebra?*
- *Is $V = \mathrm{HOD}_{V_\kappa}$?*

Assuming UA, one can actually calculate the cardinality of $\mathbb{V}_\kappa$ precisely:

Theorem 6.3.36 (UA). *If κ is κ^{++}-supercompact then $|\mathbb{V}_\kappa|^{\mathrm{HOD}} = \kappa^{++}$ and $\mathbb{V}_\kappa$ is κ^{++}-cc in* HOD.

Thus if κ is supercompact, then $V = \mathrm{HOD}[A]$ for some $A \subseteq \kappa^{++}$. As an immediate consequence, we have that HOD is very close to V:

Corollary 6.2.10 (UA). *Let κ be the least supercompact cardinal. Then for all cardinals $\lambda \geq \kappa^{++}$:*

(1) $\lambda^{+\mathrm{HOD}} = \lambda^+$.

(2) $(2^\lambda)^{\mathrm{HOD}} = 2^\lambda$.

Moreover if $\delta \geq \kappa^{++}$ is regular, then HOD *is correct about stationary subsets of δ.* $\square$

Given Corollary 6.2.10, the structure of HOD at κ itself becomes an interesting question.

Question 6.2.11 (UA)**.** *Assume κ is supercompact. Is $\kappa^{+\mathrm{HOD}} = \kappa^+$?*

By the Lévy-Solovay Theorem [1], HOD is also close to V in the sense that it absorbs large cardinals above κ. In fact, it turns out that itself κ is supercompact in HOD.

Definition 6.2.12. If N is an inner model and S is a set, we say S is *amenable to N* if $S \cap N \in N$.

Definition 6.2.13. Suppose κ is supercompact. An inner model N is a *weak extender model at κ* if for all ordinals $\lambda \geq \kappa$, there is a normal fine κ-complete ultrafilter on $P_\kappa(\lambda)$ that concentrates on N and is amenable to N.

Lemma 6.2.14. *Suppose N is an inner model and κ is supercompact. Then the following are equivalent:*

(1) N is a weak extender model at κ.

(2) For arbitrarily large $\delta \geq \kappa$, there is a normal fine κ-complete ultrafilter on $P_\kappa(\delta)$ that concentrates on N and is amenable to N.

Proof. (1) implies (2): Trivial.

(2) implies (1): Fix $\lambda \geq \kappa$. We will show that there is a normal fine κ-complete ultrafilter on $P_\kappa(\lambda)$ that concentrates on N and is amenable to N. By (2), there is some $\delta \geq \lambda$ such that there is a normal fine κ-complete ultrafilter $\mathcal{U}$ on $P_\kappa(\delta)$ that concentrates on N and is amenable to N. Let $\mathcal{W} = f_*(\mathcal{U})$ where $f : P_\kappa(\delta) \to P_\kappa(\lambda)$ is defined by $f(\sigma) = \sigma \cap \lambda$. Easily $\mathcal{W}$ is a normal fine ultrafilter. Moreover $f^{-1}[P_\kappa(\lambda) \cap M] = P_\kappa(\delta) \cap M \in \mathcal{U}$, so $P_\kappa(\lambda) \cap M \in \mathcal{W}$. Thus $\mathcal{W}$ concentrates on M. Finally, letting $g = f \restriction M$, clearly $g \in M$ and hence $\mathcal{W} \cap M = f_*(\mathcal{U}) \cap M = g_*(\mathcal{U} \cap M) \in M$ since $\mathcal{U} \cap M \in M$. Thus $\mathcal{W}$ is amenable to M. $\square$

Theorem 6.2.15 (UA)**.** *Let κ be the least supercompact cardinal. Then* HOD *is a weak extender model at κ.*

Proof. First note that every normal fine ultrafilter on an ordinal definable set is ordinal definable. We will prove this using the fact that Rudin-Keisler equivalent normal fine ultrafilters on the same set are equal (Lemma 4.4.11). Suppose $\mathcal{U}$ is a normal fine ultrafilter on $Y \in \mathrm{OD}$, and let U be a countably complete ultrafilter on an ordinal Rudin-Keisler equivalent to $\mathcal{U}$; then by Lemma 4.4.11, $\mathcal{U}$ is the *unique* normal fine ultrafilter on Y Rudin-Keisler equivalent to U, and hence $\mathcal{U} \in \mathrm{OD}_{Z,U} = \mathrm{OD}_U = \mathrm{OD}$, with the final equality coming from Proposition 6.2.1.

In particular, for all $\lambda \geq \kappa$, every normal fine ultrafilter $\mathcal{U}$ on $P_\kappa(\lambda)$ is amenable to HOD. The issue is to show that there are such $\mathcal{U}$ concentrating on HOD.

Fix a regular cardinal $\delta \geq \kappa^{++}$. Then by Corollary 6.2.10, HOD is correct about stationary subsets of δ. Let $\langle S_\alpha : \alpha < \delta\rangle \in \text{HOD}$ be a partition of S^δ_ω into stationary subsets. Let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \delta$ and $j[\delta] \in M$. We claim that $j[\delta] \in \text{HOD}^M$.

By Corollary 4.4.31,

$$j[\delta] = \{\alpha < j(\delta) : M \vDash j(S)_\alpha \text{ is stationary in } \sup j[\delta]\}$$

Thus

$$j[\delta] \in \text{HOD}^M_{j(\langle S_\alpha:\alpha<\delta\rangle)}$$

But since $\langle S_\alpha : \alpha < \delta\rangle$ is in HOD, $j(\langle S_\alpha : \alpha < \delta\rangle) \in \text{HOD}^M$. Thus $j[\delta] \in \text{HOD}^M$.

Let $\mathcal{U}$ be the ultrafilter on $P_\kappa(\delta)$ derived from j using $j[\delta]$. Since $j[\delta] \in \text{HOD}^M = j(\text{HOD})$, $\mathcal{U}$ concentrates on HOD by Łoś's Theorem. Thus $\mathcal{U}$ is a normal fine κ-complete ultrafilter on $P_\kappa(\delta)$ that concentrates on HOD and is amenable to HOD.

This shows that for unboundedly many cardinals δ, there is a normal fine κ-complete ultrafilter on $P_\kappa(\delta)$ that concentrates on HOD and is amenable to HOD. Therefore by Lemma 6.2.14, HOD is a weak extender model at κ. □

As a consequence of theorems of Woodin [12], this implies that a version of Jensen's Covering Lemma is true for HOD:

Corollary 6.2.16 (UA). *Every set $A \subseteq \text{HOD}$ is contained in a set $B \in \text{HOD}$ such that $|B| \leq |A| + \gamma$ for some γ less than the least supercompact cardinal.*

We omit the proof. Of course one has a much stronger covering results above κ^{++} as a consequence of Theorem 6.3.36.

One also obtains the κ-approximation property for HOD:

Corollary 6.2.17 (UA). *Suppose $A \subseteq \text{HOD}$ has the property that $A \cap \tau \in \text{HOD}$ for all $\tau \in \text{HOD}$ of cardinality less than κ. Then $A \in \text{HOD}$.*

The theorems stated above ignore the local nature of the definability phenomena under UA:

Theorem 6.2.18 (UA). *Assume $\kappa \leq \lambda$ are cardinals, $\text{cf}(\lambda) \geq \kappa$, and every κ-complete filter on λ extends to a κ-complete ultrafilter. Then there is a definable wellorder of $H((2^\lambda)^+)$.*

Sketch. It suffices to show that there is a wellorder of $P(2^\lambda)$ that is definable over $H((2^\lambda)^+)$. Fix a κ-independent family $\langle S_\alpha : \alpha < 2^\lambda\rangle$ of subsets S_α of λ (see Definition 7.3.26). For $A \subseteq 2^\lambda$, let $\mathscr{B} = \{S_\alpha : \alpha \in A\} \cup \{\lambda \setminus S_\alpha : \alpha \in \lambda^+ \setminus A\}$ and let U_A be the $<_\Bbbk$-least ultrafilter on λ extending $\mathscr{B}$. Then wellorder $P(2^\lambda)$ by setting $A_0 \preceq A_1$ if $U_{A_0} \leq_\Bbbk U_{A_1}$. □

Similarly, one can show:

Theorem 6.2.19 (UA)**.** *If $\kappa < \lambda$ are cardinals, λ is a limit cardinal, and κ is δ-supercompact for all $\delta < \lambda$, then $H(\lambda)$ is definably wellordered.* □

A theorem of Shelah (which follows from the proof of [27, Theorem 4.6], but see [28]) shows that $H(\lambda^+)$ is definably wellordered if λ is a strong limit singular cardinal of uncountable cofinality. This leaves open the following questions:

Question 6.2.20 (UA)**.** *Let κ be the least supercompact cardinal.*

- *Suppose $\lambda \geq \kappa$ is inaccessible. Is $H(\lambda^+)$ definably wellordered?*
- *Suppose $\lambda \geq \kappa$ and $\text{cf}(\lambda) < \kappa$. Is $H(\lambda^{++})$ definably wellordered?*
- *Suppose $\lambda \geq \kappa$ is singular and the Axiom of Choice is false in $L(H(\lambda^+))$. Is there an elementary embedding from $L(H(\lambda^+))$ to $L(H(\lambda^+))$ with critical point less than λ?*

The point of the last question is that the large cardinal axiom $I_0(\lambda)$ asserting the existence of an elementary embedding from $L(H(\lambda^+))$ to $L(H(\lambda^+))$ with critical point less than λ does imply the failure of AC in $L(H(\lambda^+))$. The limitations on the locality of the wellorders definable by our methods are therefore to some extent welcome, in the sense that *some* limitations must exist if UA is to be compatible with the strongest large cardinal hypotheses.

6.3 The Generalized Continuum Hypothesis

6.3.1 Introduction

In this section, we prove that GCH holds above the least supercompact assuming UA. We actually prove a local version of this theorem that requires some extra work that is not actually necessary for the global result. We will apply this sharper result at various points in Chapters 7 and 8 to eliminate cardinal arithmetic hypotheses from the statements of various theorems. For example, Theorem 7.5.42 exploits this result (among others) to remove the cardinal arithmetic hypothesis from the proof of the linearity of the Mitchell order on normal fine ultrafilters (Theorem 4.4.2).

6.3.2 The number of supercompactness measures

The original motivation for this work comes from two remarkable theorems of Solovay. The first is his theorem that SCH holds above a strongly compact cardinal [19], which is proved in Section 7.2 in this more local form:

Theorem 7.2.19 (Solovay)**.** *Suppose $\kappa \leq \lambda$ are cardinals and κ is λ-supercompact.*

(1) If $\text{cf}(\lambda) < \kappa$, then $\lambda^{<\kappa} = \lambda^+$.

(2) If $\mathrm{cf}(\lambda) \geq \kappa$ *then* $\lambda^{<\kappa} = \lambda$.

Second, and less well-known, is his remarkable observation that the linearity of the Mitchell order implies instances of GCH.

Definition 6.3.1. Suppose $\kappa \leq \lambda$ are cardinals. Then $\mathscr{N}(\kappa,\lambda)$ denotes the set of normal fine κ-complete ultrafilters on $P_\kappa(\lambda)$.

Under sufficient large cardinal assumptions, Solovay [2] showed that $P_\kappa(\lambda)$ carries the maximum possible number of supercompactness measures. Note that if $\eta = \lambda^{<\kappa} = |P_\kappa(\lambda)|$, the cardinality of $\mathscr{N}(\kappa,\lambda)$ is bounded by 2^{2^η}, since $\mathscr{N}(\kappa,\lambda)$ is contained in the double powerset of $P_\kappa(\lambda)$. Solovay showed that this bound is achieved:

Theorem 6.3.2 (Solovay)**.** *Suppose $\kappa \leq \lambda$ are cardinals. Let $\eta = \lambda^{<\kappa}$, and assume κ is 2^η-supercompact. Then $|\mathscr{N}(\kappa,\lambda)| = 2^{2^\eta}$.* □

As a corollary, Solovay proved instances of GCH from the linearity of the Mitchell order.

Theorem 6.3.3 (Solovay)**.** *Suppose $\kappa \leq \lambda$ are cardinals,* $\mathrm{cf}(\lambda) \geq \kappa$*, and $|\mathscr{N}(\kappa,\lambda)| = 2^{2^\lambda}$. Assume the Mitchell order is linear on $\mathscr{N}(\kappa,\lambda)$. Then $2^{2^\lambda} = (2^\lambda)^+$.*

Proof. Since $(\mathscr{N}(\kappa,\lambda), \lhd)$ is a wellorder,

$$2^{2^\lambda} = |\mathscr{N}(\kappa,\lambda)| \leq \mathrm{ot}(\mathscr{N}(\kappa,\lambda), \lhd)$$

It therefore suffices to show that $\mathrm{ot}(\mathscr{N}(\kappa,\lambda), \lhd) \leq (2^\lambda)^+$. To accomplish this, we show that any $\mathcal{U} \in \mathscr{N}(\kappa,\lambda)$ has at most 2^λ-many predecessors in $(\mathscr{N}(\kappa,\lambda), \lhd)$. Note that the set of predecessors of $\mathcal{U}$ in $(\mathscr{N}(\kappa,\lambda), \lhd)$ is equal to $\mathscr{N}(\kappa,\lambda) \cap M_\mathcal{U}$. But

$$\mathscr{N}(\kappa,\lambda) \cap M_\mathcal{U} \subseteq j_\mathcal{U}(V_\kappa) = (V_\kappa)^{P_\kappa(\lambda)}/\mathcal{U}$$

so $|\mathscr{N}(\kappa,\lambda) \cap M_\mathcal{U}| \leq |(V_\kappa)^{P_\kappa(\lambda)}| = \kappa^\lambda = 2^\lambda$. This calculation uses that $|P_\kappa(\lambda)| = \lambda$, which is a consequence of Theorem 7.2.19. □

Thus under UA, if κ is 2^κ-supercompact, then GCH holds at 2^κ. More generally, we have the following consequence of UA:

Corollary 6.3.4 (UA)**.** *Suppose $\kappa \leq \lambda$ are cardinals,* $\mathrm{cf}(\lambda) \geq \kappa$*, and $2^{<\lambda} = \lambda$. If κ is 2^λ-supercompact, then $2^{2^\lambda} = (2^\lambda)^+$.*

Proof. Since $2^{<\lambda} = \lambda$, Theorem 4.4.2 implies that the Mitchell order is linear on $\mathscr{N}(\kappa,\lambda)$. By Theorem 6.3.2, $|\mathscr{N}(\kappa,\lambda)| = 2^{2^\lambda}$. We can therefore apply Theorem 6.3.3. □

As a corollary, we obtain a result that strongly suggests that UA plus a supercompact cardinal implies the eventual GCH:

Corollary 6.3.5 (UA). *Suppose κ is supercompact. Let $\lambda \geq \kappa$ be a strong limit singular cardinal with* $\text{cf}(\lambda) \geq \kappa$. *Then for all $n \leq \omega$, $2^{(\lambda^{+n})} = \lambda^{+n+1}$.*

Proof. We first claim that for all $n < \omega$, $2^{(\lambda^{+n})} = \lambda^{+n+1}$. The proof is by induction. For the base case $n = 0$, we have $2^\lambda = \lambda^+$ by Solovay's Theorem [19] since λ is a singular strong limit cardinal above a supercompact cardinal. Now suppose that the claim is true for $n \leq k$, and we will show it is true when $n = k + 1$. By our induction hypothesis (or if $k = 0$, by the fact that λ is a strong limit cardinal), $2^{<\lambda^{+k}} = \lambda^{+k}$, Corollary 6.3.4 implies

$$2^{(\lambda^{+k+1})} = 2^{2^{(\lambda^{+k})}} = (2^{(\lambda^{+k})})^+ = \lambda^{+k+2}$$

The final equality follows from our induction hypothesis that $2^{(\lambda^{+k})} = \lambda^{+k+1}$.

To finish, we show that $2^{(\lambda^{+\omega})} = \lambda^{+\omega+1}$. The previous paragraph implies that $\lambda^{+\omega}$ is a singular strong limit cardinal. Thus $2^{(\lambda^{+\omega})} = \lambda^{+\omega+1}$ by Solovay's Theorem. $\square$

The proof breaks down when one tries to show that $2^{\lambda^{+\omega+1}} = \lambda^{+\omega+2}$. Moreover, the argument yields no insight into the value of 2^κ itself. To handle these cases, we must take a closer look at the proof of Theorem 6.3.2.

6.3.3 The Local Capturing Property

Habič-Honzík [29] define a generalization of the Mitchell order, extracted from the proof of Theorem 6.3.2, that describes the relationship between ultrafilters and powersets:

Definition 6.3.6 (Local Capturing Property). Suppose $\mathscr{S}$ is a set of countably complete ultrafilters and λ is a cardinal. Then the powerset of λ is *locally captured by $\mathscr{S}$*, denoted $\text{LCP}(\lambda, \mathscr{S})$, if every subset of λ belongs to the ultrapower of the universe by an ultrafilter in $\mathscr{S}$.

This capturing is local in the sense that there need not be a single ultrafilter U such that $P(\lambda) \subseteq M_U$.

The proof of Theorem 6.3.2 shows that the Local Capturing Property holds for supercompactness measures:

Theorem 6.3.7 (Solovay). *Suppose $\kappa \leq \gamma$ are cardinals and $j : V \to M$ is an elementary embedding witnessing that κ is γ-supercompact. Let $\mathcal{U}$ be the ultrafilter on $P_\kappa(\gamma)$ derived from j using $j[\gamma]$.*

- *If $\mathcal{U} \in M$, then* $\text{LCP}(2^\gamma, \mathcal{N}(\kappa, \gamma))$ *holds in M.*
- *Therefore if $\mathcal{U} \in M$, $\lambda \leq 2^\gamma$, and $P(\lambda) \subseteq M$, then* $\text{LCP}(\lambda, \mathcal{N}(\kappa, \gamma))$ *holds.* $\square$

We will consider the statement $\mathrm{LCP}(\lambda, \mathbf{HU}(\delta))$ where $\mathbf{HU}(\delta)$ denotes the set of hereditarily uniform ultrafilters of size less than δ (Definition 4.2.11). This is equivalent to local capturing by uniform ultrafilters (except that there is a proper class of uniform ultrafilters of size less than δ).

Since $\mathscr{N}(\kappa, \gamma) \subseteq \mathbf{HU}(\eta)$ where $\eta = (\gamma^{<\kappa})^+$, we have the following implication:

Proposition 6.3.8. $\mathrm{LCP}(\lambda, \mathscr{N}(\kappa, \gamma))$ *implies* $\mathrm{LCP}(\lambda, \mathbf{HU}(\eta))$ *where* $\eta = (\gamma^{<\kappa})^+$. □

It will be convenient to use the following self-improvement of $\mathrm{LCP}(\lambda, \mathbf{HU}(\delta))$:

Lemma 6.3.9. *Suppose δ and λ are cardinals such that* $\mathrm{LCP}(\lambda, \mathbf{HU}(\delta))$ *holds. Then for any $A \subseteq \lambda$, there is a cardinal $\gamma < \delta$ and a countably complete uniform ultrafilter D on γ such that $A \in j_D(P(\gamma))$.*

Proof. We start with an observation. Let $\eta \leq \lambda$ be the least cardinal such that $2^\eta > \lambda$. Fix a sequence $\langle X_\alpha : \alpha < \lambda \rangle$ of distinct subsets of η. Fix a set $B \subseteq \lambda$ such that $P(\alpha) \subseteq L[B]$ for all $\alpha < \eta$. Suppose $D \in \mathbf{HU}(\delta)$ has the property that $\langle X_\alpha : \alpha < \lambda \rangle$ and B belong to M_D. Then $H(\eta) \subseteq M_D$ and $(2^\eta)^{M_D} \geq \lambda$. By the Kunen Inconsistency Theorem (Theorem 4.2.35), $\kappa_\omega(j_D) \geq \eta$, and since M_D is closed under ω-sequences, in fact, $\kappa_{n+1}(j_D) > \eta$ for some η. For the least such n, $\kappa_n(j_D) \leq \lambda_D$ is measurable and hence

$$j_D(\lambda_D) \geq \kappa_{n+1}(j_D) > (2^\eta)^{M_D} \geq \lambda$$

Now suppose $A \subseteq \lambda$, and we will find a countably complete uniform ultrafilter D on a cardinal $\gamma < \delta$ such that $A \in j_D(P(\gamma))$. By $\mathrm{LCP}(\lambda, \mathbf{HU}(\delta))$, there is a countably complete uniform ultrafilter $D \in \mathbf{HU}(\delta)$ such that $\langle X_\alpha : \alpha < \lambda \rangle$, B, and A belong to M_D. Let $\gamma = \lambda_D$. We may assume without loss of generality that γ is the underlying set of D. Since $\langle X_\alpha : \alpha < \lambda \rangle$ and B belong to M_D, $j_D(\gamma) \geq \lambda$ by the previous paragraph. Thus $A \in P(\lambda) \cap M_D \subseteq j_D(P(\gamma))$. □

6.3.4 λ-Mitchell ultrafilters

The key concept in our proof of GCH is that of a λ-Mitchell ultrafilter:

Definition 6.3.10. Suppose λ is a cardinal. A countably complete ultrafilter U is λ-*Mitchell* if every countably complete uniform ultrafilter on a cardinal less than λ belongs to M_U.

Lemma 6.3.11. *Suppose λ is a cardinal and U is a λ-Mitchell ultrafilter. Then $\mathbf{HU}(\lambda) \subseteq M_U$.*

Proof. This is immediate from Lemma 4.2.13, which asserts the invariance of the Mitchell order on hereditarily uniform ultrafilters under Rudin-Keisler equivalence. □

Assuming $2^{<\lambda} = \lambda$, any countably complete ultrafilter U such that $P(\lambda) \subseteq M_U$ is λ-Mitchell. Under UA, we can get away without the cardinal arithmetic hypothesis:

Theorem 6.3.16 (UA). *Suppose λ is a cardinal and U is a countably complete ultrafilter such that M_U is closed under λ-sequences. Then U is λ-Mitchell.*

This theorem will be the engine for our results on cardinal arithmetic under UA. In this subsection, let us show how we will use it:

Theorem 6.3.12. *Suppose $\kappa \leq \gamma$ are cardinals with $\text{cf}(\gamma) \geq \kappa$. Assume the following hold:*

- *There is a γ^+-Mitchell ultrafilter on a set of cardinality γ^+*
- *There is a γ^{++}-Mitchell ultrafilter on a set of cardinality γ^{++}.*
- *There is an elementary embedding $j : V \to M$ with the following properties:*
  - *j witnesses that κ is γ-supercompact.*
  - *The normal fine ultrafilter on $P_\kappa(\gamma)$ derived from j using $j[\gamma]$ belongs to M.*
  - *$P(\gamma^{++}) \subseteq M$.*

Then $2^\gamma = \gamma^+$.

Theorem 6.3.16 below implies that all the conditions of Theorem 6.3.12 follow under UA from the assumption that κ is γ^{++}-supercompact. This immediately yields GCH above a supercompact (Corollary 6.3.26) and more.

The proof of Theorem 6.3.12 requires one or two interesting lemmas which are motivated by a theorem of Cummings [30], which states that it is is consistent that there is a normal ultrafilter U on a cardinal κ with the property that $P(\kappa^+) \subseteq M_U$, or in other words (abusing notation slightly), $\text{LCP}(\kappa^+, U)$. Since $P(2^\kappa)$ is never contained in M_U, $\text{LCP}(\kappa^+, U)$ implies $2^\kappa > \kappa^+$.

First, we need the following fact, recalling the notation for "iterated ultrafilters" from Definition 3.5.8.

Lemma 6.3.13. *Suppose U is a countably complete ultrafilter on a set X, Y is a transitive set, and Z is an M_U-ultrafilter on $j_U(Y)$. Then U and Z belong to $L(P(X \times Y), [U, Z])$.*

Proof. Note that $U = (\pi_0)_*([U, Z])$ where $\pi_0 : X \times Y \to X$ is the projection. This easily implies that $U \in L(P(X \times Y), [U, Z])$.

On the other hand, $Z = \{[x \mapsto A_x]_U : A \in [U, Z]\}$ where $A_x = \{y \in Y : (x, y) \in A\}$. Since $P(X \times Y) \subseteq L(P(X \times Y), [U, Z])$, so is $P(Y)^X$. Therefore the map from $\Phi : P(Y)^X \to P(Y)^X/U$ defined by $\Phi(f) = [f]_{[U,Z]}$ is in $L(P(X \times Y), [U, Z])$. Since Y is transitive, $P(Y)$ is transitive, and so the transitive collapse of $P(Y)^X/U$ yields an isomorphism $\pi : P(Y)^X/U \to j_U(P(Y))$

that belongs to $L(P(X \times Y), [U, Z])$. Therefore $Z = \{\pi(\Phi(A)) : A \in [U, Z]\}$ belongs to $L(P(X \times Y), [U, Z])$. $\square$

The key lemma on the way to Theorem 6.3.12 implies that the existence of a κ^+-Mitchell ultrafilter on a set of size κ^+ refutes $\text{LCP}(\kappa^+, U)$ for all ultrafilters U on κ:

Lemma 6.3.14. *Suppose there is a nonprincipal λ-Mitchell ultrafilter on a set of cardinality λ. Suppose U is a countably complete ultrafilter such that $P(\lambda) \subseteq M_U$. Then $\lambda_U \geq \lambda$.*

Proof. We may assume without loss of generality that the underlying set of U is the cardinal λ_U, which we denote by γ. Assume towards a contradiction that $\gamma < \lambda$. Since $P(\lambda) \subseteq M_U$, we must have $j_U(\gamma) > \lambda$ by Lemma 4.2.36. Let W be a λ-Mitchell ultrafilter on λ. Let Z be the M_U-ultrafilter on $j_U(\gamma)$ projecting to W: in other words,

$$Z = \{A \subseteq j_U(\gamma) : A \in M_U \text{ and } A \cap \lambda \in W\}$$

Consider the ultrafilter $[U, Z]$ on $\gamma \times \gamma$. As a consequence of Lemma 3.5.9, $[U, Z]$ is a countably complete ultrafilter, and it is easy to see that $[U, Z]$ is hereditarily uniform with $\lambda_{[U,Z]} = \gamma$. Thus $[U, Z] \in \mathbf{HU}(\lambda)$. Since W is λ-Mitchell, Lemma 6.3.11 implies that $[U, Z] \lhd W$. In other words, $[U, Z] \in M_W$. But $[U, Z]$ codes Z (Lemma 6.3.13), so $Z \in M_W$. Hence $W \in M_W$: indeed $W = \{A \cap \lambda : A \in Z\}$. No countably complete nonprincipal ultrafilter belongs to its own ultrapower, so this is a contradiction. $\square$

We also need the following lemma:

Lemma 6.3.15. *Suppose δ and λ are cardinals. Assume* $\text{LCP}(\lambda, \mathbf{HU}(\delta))$ *holds. Then for any δ-Mitchell ultrafilter U, $P(\lambda) \subseteq M_U$.*

Proof. Fix $A \subseteq \lambda$, and we will show $A \in M_U$. By Lemma 6.3.9, there is a countably complete uniform ultrafilter D on a cardinal $\gamma < \delta$ such that $A \in j_D(P(\gamma))$. Since U is δ-Mitchell, $D \lhd U$. Therefore in particular $P(\gamma) \in M_U$, so $j_D(P(\gamma)) = P(\gamma)^\gamma/D \in M_U$. Since $A \in j_D(P(\gamma))$, it follows that $A \in M_U$. $\square$

This yields the proof of Theorem 6.3.12:

Proof of Theorem 6.3.12. Assume towards a contradiction that $2^\gamma > \gamma^+$. Then Theorem 6.3.7 combined with the fact that $\gamma^{++} \leq 2^\gamma$ yields $\text{LCP}(\gamma^{++}, \mathscr{N}(\kappa, \gamma))$. By Theorem 7.2.19, $\gamma^{<\kappa} = \gamma$. Therefore $\mathscr{N}(\kappa, \gamma) \subseteq \mathbf{HU}(\gamma^+)$, so $\text{LCP}(\gamma^{++}, \mathscr{N}(\kappa, \gamma))$ implies $\text{LCP}(\gamma^{++}, \mathbf{HU}(\gamma^+))$.

Now let U be a γ^+-Mitchell ultrafilter on γ^+. By Lemma 6.3.15, since $\text{LCP}(\gamma^{++}, \mathbf{HU}(\gamma^+))$ holds, $P(\gamma^{++}) \subseteq M_U$. This contradicts Lemma 6.3.14: since γ^{++} carries a γ^{++}-Mitchell ultrafilter, no countably complete ultrafilter D on γ^+ can satisfy $P(\gamma^{++}) \subseteq M_D$. $\square$

6.3.5 λ-Mitchell ultrafilters from UA

The main theorem of this section shows that assuming the Ultrapower Axiom, every λ-supercompact ultrafilter is λ-Mitchell.

Theorem 6.3.16 (UA)**.** *Suppose W is a countably complete ultrafilter such that M_W is closed under λ-sequences. Then W is λ-Mitchell.*

Interestingly, in the context of UA, this theorem turns out to be an easy consequence of a folklore theorem about good points in the Stone space of a complete Boolean algebra [31, Proposition 4.8]. (The main difficulty is translating the topological notions into the language of elementary embeddings.) Here we will give a somewhat different proof.

The first step in the proof is a straightforward fact about the relationship between supercompactness and the Mitchell order:

Proposition 6.3.17. *Suppose γ is an ordinal, U is a countably complete ultrafilter on γ, and W is a countably complete ultrafilter such that M_W is closed under γ-sequences. Then the following are equivalent:*

(1) $U \lhd W$.

(2) There is a right-internal ultrapower comparison $(k, h) : (M_U, M_W) \to N$ of (j_U, j_W) such that $k([\mathrm{id}]_U) \in h(j_W[\gamma])$.

Proof. (1) implies (2): This is immediate from Proposition 5.5.4 and Lemma 5.5.7.

(2) implies (1): We will show that U is definable over M_W, and hence $U \in M_W$. In fact, we will prove:

$$U = \{A \subseteq \gamma : k([\mathrm{id}]_U) \in h(j_W[A])\} \tag{6.1}$$

Since $j_W \restriction \gamma \in M_W$, the function on $P(\gamma)$ given by $A \mapsto j_W[A]$ belongs to M_W. Moreover, h is an internal ultrapower embedding of M_W, and so in particular, h is a definable subclass of M_W. Thus (6.1) implies that U is definable over M_W.

To finish, we prove (6.1). For any $A \subseteq \gamma$:

$$\begin{aligned} A \in U &\iff [\mathrm{id}]_U \in j_U(A) \\ &\iff k([\mathrm{id}]_U) \in k(j_U(A)) \\ &\iff k([\mathrm{id}]_U) \in h(j_W(A)) \\ &\iff k([\mathrm{id}]_U) \in h(j_W(A)) \cap h(j_W[\gamma]) \end{aligned}$$

For the final equivalence, we use that $k([\mathrm{id}]_U) \in h(j_W[\gamma])$. Note that

$$h(j_W(A)) \cap h(j_W[\gamma]) = h(j_W(A) \cap j_W[\gamma]) = h(j_W[A])$$

This yields (6.1). $\square$

Remark 6.3.18. In the context of Proposition 6.3.17, the statement that $k([\mathrm{id}]_U) \in h(j_W[\gamma])$ is actually equivalent to the a priori weaker statement that $k([\mathrm{id}]_U) \in h(j_W[\beta])$ for some ordinal $\beta \geq \gamma$ such that $j_W[\beta] \in M_W$.

To see this, suppose $k([\mathrm{id}]_U) \in h(j_W[\beta])$. Since $[\mathrm{id}]_U < j_U(\gamma)$,

$$k([\mathrm{id}]_U) < k(j_U(\gamma)) = h(j_W(\gamma)$$

Therefore $k([\mathrm{id}]_U) \in h(j_W[\beta]) \cap h(j_W(\gamma)) = h(j_W[\beta] \cap j_W(\gamma))$. Finally, $j_W[\beta] \cap j_W(\gamma) = j_W[\gamma]$, so we have $k([\mathrm{id}]_U) \in h(j_W[\gamma])$, as desired.

To prove Theorem 6.3.16, it now suffices to prove the following fact:

Lemma 6.3.19. *Suppose γ is an ordinal, U is a countably complete ultrafilter on γ, and W is a countably complete ultrafilter whose ultrapower M_W is closed under γ^+-sequences. If (k, h) is a left-internal ultrapower comparison of (j_U, j_W), then $k([\mathrm{id}]_U) \in h(j_W[\gamma])$.*

Proof of Theorem 6.3.16. By the invariance of the Mitchell order on hereditarily uniform ultrafilters under Rudin-Keisler equivalence (Lemma 4.2.13), it suffices to show that for any countably complete ultrafilter U on an ordinal $\gamma < \lambda$, $U \lhd W$. Fix such an ultrafilter U. By UA, there is an internal ultrapower comparison (k, h) of (j_U, j_W). By Lemma 6.3.19, this implies $U \lhd W$. □

If one drops the assumption that k is an internal ultrapower embedding of M_U, then the conclusion of Lemma 6.3.19 that $k([\mathrm{id}]_U) \in h(j_W[\gamma])$ can easily fail. Thus the argument must make use of the fact that k is an internal ultrapower embedding.

The proof uses the following concepts which are essentially part of the theory of strongly compact cardinals:

Definition 6.3.20. Suppose $j : V \to M$ is an elementary embedding and λ is a cardinal. A set $A \subseteq j(\lambda)$ is a *cover of* $j[\lambda]$ if $j[\lambda] \subseteq A$. A cover of $j[\lambda]$ is *j-closed* if for any $f : \lambda \to \lambda$, $j(f)[A] \subseteq A$.

We need three general lemmas regarding closed covers. The first concerns the interaction of closed covers with compositions:

Lemma 6.3.21. *Suppose $V \xrightarrow{j} M \xrightarrow{k} N$ are elementary embeddings and λ is a cardinal.*

- *If B is a $k \circ j$-closed cover of $k \circ j[\lambda]$, then $k^{-1}[B]$ is a j-closed cover of $j[\lambda]$.*
- *If $A \in M$ is a j-closed cover of $j[\lambda]$, then $k(A)$ is a $k \circ j$-closed cover of $k \circ j[\lambda]$.* □

Ultrafilters on small sets cannot have small covers:

Lemma 6.3.22. *Suppose U is a countably complete ultrafilter and $\lambda > \lambda_U$ is a regular cardinal. Suppose $A \in M_U$ is a cover of $j_U[\lambda]$. Then $|A|^{M_U} = j_U(\lambda)$.*

Proof. Since $\lambda > \lambda_U$ is regular, $j_U(\lambda) = \sup j_U[\lambda]$. Thus $j_U[\lambda]$ is cofinal in $j_U(\lambda)$. It follows that A is cofinal in $j_U(\lambda)$ Since $j_U(\lambda)$ is a regular cardinal of M_U, $|A|^{M_U} = j_U(\lambda)$. □

Combined with Lemma 6.3.22, the following lemma shows that closed covers past λ_U are highly constrained:

Lemma 6.3.23. *Suppose U is a countably complete ultrafilter, $\lambda \geq \lambda_U$ is a cardinal, and A is a j_U-closed cover of $j_U[\lambda]$ such that $A \in M_U$ and $|A|^{M_U} = j_U(\lambda)$. Then $j_U(\lambda) \subseteq A$.*

Proof. We may assume without loss of generality that U is a uniform ultrafilter on λ_U. Fix $f : \lambda_U \to \lambda$, and we will show that $[f]_U \in A$.

Since $A \in M_U$, there is a sequence $\langle A_\alpha : \alpha < \lambda_U\rangle$ of subsets of λ with $A = [\langle A_\alpha : \alpha < \lambda_U\rangle]_U$. Since $|A|^{M_U} = j_U(\lambda)$, we may assume by Łoś's Theorem that $|A_\alpha| = \lambda$ for all $\alpha < \lambda$. Therefore there is an injective function $g : \lambda_U \to \lambda$ such that $g(\alpha) \in A_\alpha$ for all $\alpha < \lambda_U$. Let $h : \lambda \to \lambda$ be a function such that $h \circ g = f$. (Such a function necessarily exists because g is injective.) Then

$$j_U(h)([g]_U) = j_U(h)(j_U(g)([\mathrm{id}]_U)) = j_U(h \circ g)([\mathrm{id}]_U) = j_U(f)([\mathrm{id}]_U) = [f]_U$$

Since $[g]_U \in A$ and $j_U(h)[A] \subseteq A$, it follows that $[f]_U \in A$, as desired. □

Lemma 6.3.24. *Suppose U and W are countably complete ultrafilters. Suppose $\lambda > \lambda_U$ is a regular cardinal and $B \in M_W$ is a j_W-closed cover of $j_W[\lambda]$. Suppose $(k, h) : (M_U, M_W) \to N$ is a left-internal ultrapower comparison of (j_U, j_W). Then $k[j_U(\lambda)] \subseteq h(B)$.*

Proof. Since $B \in M_W$ is a j_W-closed cover of $j_W[\lambda]$, $h(B)$ is a $h \circ j_W$-closed cover of $h \circ j_W[\lambda]$ by Lemma 6.3.21. Since $h \circ j_W = k \circ j_U$, it follows that $h(B)$ is a $k \circ j_U$-closed cover of $k \circ j_U[\lambda]$. Therefore $k^{-1}[h(B)]$ is a j_U-closed cover of $j_U[\lambda]$ by Lemma 6.3.21.

Let $A = k^{-1}[h(B)]$. Since k is an internal ultrapower embedding of M_U, $A \in M_U$. Since $A \in M_U$ is a cover of $j_U[\lambda]$ and $\lambda_U < \lambda$, by Lemma 6.3.22, $|A|^{M_U} \geq j_U(\lambda)$. By Lemma 6.3.23, $j_U(\lambda) \subseteq A$. Thus $j_U(\lambda) \subseteq k^{-1}[h(B)]$, or in other words, $k[j_U(\lambda)] \subseteq h(B)$. □

As an immediate consequence, we have proved Lemma 6.3.19:

Proof of Lemma 6.3.19. Trivially, $j_W[\gamma^+]$ is a j_W-closed cover of $j_W[\gamma^+]$. Since $j_W[\gamma^+] \in M_W$, applying Lemma 6.3.24 with $\lambda = \gamma^+$ and $B = j_W[\gamma^+]$ yields that $k[j_U(\gamma^+)] \subseteq h(j_W[\gamma^+])$. In particular, $k([\mathrm{id}]_U) \in h(j_W[\gamma^+])$. By Remark 6.3.18, this is equivalent to the statement that $k([\mathrm{id}]_U) \in h(j_W[\gamma])$. By Proposition 6.3.17, we can conclude that $U \lhd W$. □

6.3.6 GCH from UA

Our main theorems follow at once from Theorem 7.2.19, Theorem 6.3.12, and Theorem 6.3.16.

Theorem 6.3.25 (UA). *Suppose $\kappa \leq \lambda$ are cardinals and κ is λ^+-supercompact. Then for any cardinal γ with $\kappa \leq \gamma < \lambda$, $2^\gamma = \gamma^+$.*

Proof. There are two cases. Suppose first that $\text{cf}(\gamma) \geq \kappa$. We claim that the hypotheses of Theorem 6.3.12 are satisfied.

We first show that there is a γ^+-Mitchell ultrafilter on a set of cardinality γ^+. Let $\mathcal{U}$ be a normal fine ultrafilter on $P_\kappa(\gamma^+)$, which exists by Lemma 4.4.10 since κ is γ^+-supercompact. Note that $|P_\kappa(\gamma^+)| = \gamma^+$ by Theorem 7.2.19. By Lemma 4.4.10, $M_\mathcal{U}$ is closed under γ^+-sequences, so by Theorem 6.3.16, $\mathcal{U}$ is γ^+-Mitchell.

Let $\mathcal{W}$ be a normal fine ultrafilter on $P_\kappa(\gamma^{++})$. As in the previous paragraph, $\mathcal{W}$ is a γ^{++}-Mitchell ultrafilter on a set of cardinality γ^{++}.

Finally, consider the elementary embedding $j_\mathcal{W} : V \to M_\mathcal{W}$. Let $\mathcal{D}$ be the normal fine ultrafilter on $P_\kappa(\gamma)$ derived from $j_\mathcal{W}$ using $M_\mathcal{W}$. Then $\mathcal{D} \lhd \mathcal{W}$ since $\mathcal{W}$ is γ^{++}-Mitchell and $\mathcal{D} \in \mathbf{HU}(\gamma^{++})$. In other words $\mathcal{D} \in M_\mathcal{W}$. By Lemma 4.4.10, $M_\mathcal{W}$ is closed under γ^{++}-sequences, and as a consequence $M_\mathcal{W}$ is closed under γ-sequences and $P(\gamma^{++}) \subseteq M_\mathcal{W}$.

This verifies that the hypotheses of Theorem 6.3.12 are satisfied with $j = j_\mathcal{W}$, so $2^\gamma = \gamma^+$.

This leaves us with the case that $\text{cf}(\gamma) < \kappa$. Note that γ is a limit of regular cardinals, and by the previous case, GCH holds at all of them. In particular, $2^{<\gamma} = \gamma$. Thus $2^\gamma = (2^{<\gamma})^{\text{cf}(\gamma)} \leq \gamma^{<\kappa} = \gamma^+$ by Theorem 7.2.19. ☐

Corollary 6.3.26 (UA). *If κ is supercompact, then $2^\lambda = \lambda^+$ for all $\lambda \geq \kappa$.* ☐

One can actually prove two more local instances of GCH by incorporating the argument of Corollary 6.3.5:

Theorem 6.3.27 (UA). *Suppose κ is λ^{++}-supercompact and $\text{cf}(\lambda) \geq \kappa$. Then for any cardinal γ such that $\kappa \leq \gamma \leq \lambda^{++}$, $2^\gamma = \gamma^+$.*

Proof. By Theorem 6.3.25, $2^\gamma = \gamma^+$ for any cardinal $\gamma \in [\kappa, \lambda]$. It therefore suffices to show that $2^{(\lambda^+)} = \lambda^{++}$ and $2^{(\lambda^{++})} = \lambda^{+++}$.

We begin by showing $2^{(\lambda^+)} = \lambda^{++}$. Since $2^{<\lambda} = \lambda$, Corollary 6.3.4 implies $2^{2^\lambda} = (2^\lambda)^+$. In other words, $2^{(\lambda^+)} = \lambda^{++}$, as desired.

We continue by showing $2^{(\lambda^{++})} = \lambda^{+++}$. Since $2^\lambda = \lambda^+$, Corollary 6.3.4 implies $2^{2^{(\lambda^+)}} = (2^{(\lambda^+)})^+$. Since $2^{(\lambda^+)} = \lambda^{++}$ by the previous paragraph, this yields $2^{(\lambda^{++})} = \lambda^{+++}$, as desired. ☐

6.3.7 $\diamondsuit$ on the critical cofinality

Our final result shows that UA implies instances of Jensen's $\diamondsuit$ Principle above a supercompact cardinal. Results of Shelah generalizing Jensen's Theorem that CH does not imply $\diamondsuit_{\omega_1}$ show that under GCH, $\diamondsuit(S^{\kappa^+}_\kappa)$ may also fail for κ a regular uncountable cardinal.

Theorem 6.3.28 (UA). *Suppose κ is δ^{++}-supercompact where $\text{cf}(\delta) \geq \kappa$. Then $\diamondsuit(S^{\delta^{++}}_{\delta^+})$ holds.*

Recall that $S^{\delta^{++}}_{\delta^+} = \{\alpha < \delta^{++} : \text{cf}(\alpha) = \delta^+\}$.
For the proof, we need a theorem of Kunen.

Definition 6.3.29. Suppose λ is a regular uncountable cardinal and $S \subseteq \lambda$ is a stationary set. Suppose $\langle \mathscr{A}_\alpha : \alpha \in S\rangle$ is a sequence of sets with $\mathscr{A}_\alpha \subseteq P(\alpha)$ and $|\mathscr{A}_\alpha| \leq \alpha$ for all $\alpha < \lambda$. Then $\langle \mathscr{A}_\alpha : \alpha \in S\rangle$ is a $\diamondsuit^-(S)$-sequence if for all $X \subseteq \lambda$, $\{\alpha \in S : X \cap \alpha \in \mathscr{A}_\alpha\}$ is stationary.

Definition 6.3.30. $\diamondsuit^-(S)$ is the assertion that there is a $\diamondsuit^-(S)$-sequence.

The usual principle $\diamondsuit(S)$ therefore asserts the existence of a $\diamondsuit^-(S)$-sequence $\langle \mathscr{A}_\alpha : \alpha \in S\rangle$ such that $|\mathscr{A}_\alpha| = 1$ for all $\alpha < \lambda$. Somewhat surprisingly, these two statements are equivalent:

Theorem 6.3.31 (Kunen, [32]). *Suppose λ is a regular uncountable cardinal and $S \subseteq \lambda$ is a stationary set. Then $\diamondsuit^-(S)$ is equivalent to $\diamondsuit(S)$.* $\square$

Proof of Theorem 6.3.28. By Theorem 6.3.27, GCH holds on the interval $[\kappa, \delta^{++}]$, and we will use this without further comment.

For each $\alpha < \delta^{++}$, let $\mathscr{U}_\alpha$ be the unique ultrafilter of rank α in the wellorder $(\mathscr{N}(\kappa,\delta), \lhd)$. (The linearity of the Mitchell order on normal fine ultrafilters on $P_\kappa(\delta)$ is a consequence of Theorem 4.4.2 which applies in this context since $2^{<\delta} = \delta$.) Let $\mathscr{A}_\alpha = P(\alpha) \cap M_{\mathscr{U}_\alpha}$. Note that $|\mathscr{A}_\alpha| \leq \kappa^\delta = \delta^+$. Let

$$\vec{\mathscr{A}} = \langle \mathscr{A}_\alpha : \alpha < \delta^{++}\rangle$$

Note that $\vec{\mathscr{A}}$ is definable in $H_{\delta^{++}}$ without parameters.

Claim 1. *$\vec{\mathscr{A}}$ is a $\diamondsuit^-(S^{\delta^{++}}_{\delta^+})$-sequence.*

Proof. Suppose towards a contradiction that $\vec{\mathscr{A}}$ is not a $\diamondsuit^-(S^{\delta^{++}}_{\delta^+})$-sequence. Let $\mathscr{W}$ be a normal fine ultrafilter on $P_\kappa(\delta^{++})$. Then in $M_\mathscr{W}$, $\vec{\mathscr{A}}$ is not a $\diamondsuit^-(S^{\delta^{++}}_{\delta^+})$-sequence. Let $\mathscr{U}$ be the normal fine ultrafilter on $P_\kappa(\delta)$ derived from $\mathscr{W}$ and let $k : M_\mathscr{U} \to M_\mathscr{W}$ be the factor embedding. Let $\gamma = \textsc{crt}(k) = \delta^{++M_\mathscr{U}}$.

Since $\vec{\mathscr{A}}$ is definable in $H_{\delta^{++}}$ without parameters, $\vec{\mathscr{A}} \in \text{ran}(k)$. Therefore $k^{-1}(\vec{\mathscr{A}}) = \vec{\mathscr{A}} \restriction \gamma$ is not a $\diamondsuit^-(S^\gamma_{\delta^+})$-sequence in $M_\mathscr{U}$. Fix a witness $A \in P(\gamma) \cap M_\mathscr{U}$ and a closed unbounded set $C \in P(\gamma) \cap M_\mathscr{U}$ such that for all $\alpha \in C \cap S^\gamma_{\delta^+}$, $A \cap \alpha \notin \mathscr{A}_\alpha$. By elementarity, for all $\alpha \in k(C) \cap S^{\delta^{++}}_{\delta^+}$, $k(A) \cap \alpha \notin \mathscr{A}_\alpha$. Since

$M_{\mathcal{U}}$ is closed under δ-sequences, $\text{cf}(\gamma) = \delta^+$, and so in particular $k(A) \cap \gamma \notin \mathscr{A}_\gamma$. Since $\gamma = \textsc{crt}(k)$, this means $A \notin \mathscr{A}_\gamma$.

Note however that $\mathcal{U}$ has Mitchell rank $\delta^{++M_{\mathcal{U}}} = \gamma$, so $\mathcal{U} = \mathcal{U}_\gamma$. Therefore $\mathscr{A}_\gamma = P(\gamma) \cap M_{\mathcal{U}}$, so $A \in \mathscr{A}_\gamma$ by choice of A. This is a contradiction. □

By Theorem 6.3.31, this completes the proof. □

6.3.8 The size and saturation of the Vopěnka algebra

Theorem 6.3.32 (UA). *Suppose κ is an inaccessible cardinal such that every $A \subseteq P(\kappa)$ belongs to M_U for some countably complete ultrafilter U on κ. Then $|\mathbb{V}_\kappa|^{\text{HOD}} = (2^\kappa)^+$.*

Proof. Let $\lambda = |\mathbb{V}_\kappa|^{\text{HOD}}$. Note that $\lambda = |P(P(\kappa)) \cap \text{OD}|^{\text{OD}}$.

Recall that $\mathbf{UF}(\kappa)$ denotes the set of countably complete ultrafilters on κ. As in Theorem 6.3.3, $|\mathbf{UF}(\kappa)| = 2^{2^\kappa}$.

We claim that in fact $|\mathbf{UF}(\kappa)| = (2^\kappa)^+$. It suffices to show the upper bound $\mathbf{UF}(\kappa) \leq 2^{2^\kappa}$. For this, we show that every initial segment of the Ketonen order has cardinality 2^κ.

Since κ is inaccessible, for any $\alpha < \kappa$, the set $\mathbf{UF}(\kappa, \alpha)$ of countably complete ultrafilters on κ that concentrate on α has cardinality less than κ. Thus for any $U \in \mathbf{UF}(\kappa)$, U has at most $2^\kappa \cdot \prod_{\alpha<\kappa} |S_\alpha| = 2^\kappa$ predecessors in the Ketonen order, since if $W <_\Bbbk U$, then

$$W = U\text{-}\lim_{\alpha \in I} W_\alpha$$

for some $I \in U$ and some sequence $\langle W_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathbf{UF}(\kappa, \alpha)$.

Therefore let $\langle U_\alpha : \alpha < (2^\kappa)^+ \rangle$ be the $<_\Bbbk$-increasing enumeration of $\mathbf{UF}(\kappa)$.

For the lower bound $(2^\kappa)^+ \leq \lambda$, we apply the fact that every countably complete ultrafilter on an ordinal is OD (Proposition 6.2.1) to obtain $\mathbf{UF}(\kappa) \subseteq P(P(\kappa)) \cap \text{OD}$, so in fact $\lambda \geq |\mathbf{UF}(\kappa)| = 2^{2^\kappa} = (2^\kappa)^+$.

We now turn to the upper bound.

Suppose $U \in \mathbf{UF}(\kappa)$. Then $|P(P(\kappa)) \cap M_U| \leq |j_U(V_\kappa)| \leq |(V_\kappa)^\kappa| = 2^\kappa$. Let

$$\mathscr{A}_U = P(P(\kappa)) \cap M_U \cap \text{OD}$$

Note that $P(P(\kappa) \cap M_U \cap \text{OD}$ is an ordinal definable subset of OD, so let $\gamma_U = |\mathscr{A}_U|^{\text{OD}}$ and let $\pi_U : \gamma_U \to \mathscr{A}_U$ be the OD-least bijection. Note that $|\mathscr{A}_U| \leq 2^\kappa$ so $\gamma_U < (2^\kappa)^+$.

Let $\lambda_0 = \sup\{\gamma_U : U \in S\}$, so $\lambda_0 \leq (2^\kappa)^+$. Define $\pi : (2^\kappa)^+ \times \lambda_0 \to P(P(\kappa)) \cap \text{OD}$ by

$$\pi(\alpha, \beta) = \pi_{f(\alpha)}(\beta)$$

Then our large cardinal assumption on κ implies that π is a surjection and π is ordinal definable, so $\lambda \leq (2^\kappa)^+ \cdot \lambda_0 = (2^\kappa)^+$. □

Next, we calculate the chain condition of $\mathbb{V}_\kappa$.

Theorem 6.3.33. *Suppose κ is a cardinal. Then $\mathbb{V}_\kappa$ is $(2^\kappa)^+$-cc in* HOD.

For this, we will just cite a remarkable theorem of Bukovsky, which requires the following definition:

Definition 6.3.34. An inner model M has the δ-uniform cover property if for all ordinals γ, for any function $f : \gamma \to P_\delta(\mathrm{Ord})$, there is a function $F : \gamma \to P_\delta(\mathrm{Ord})$ in M such that $f(\alpha) \subseteq F(\alpha)$ for all $\alpha < \gamma$.

Theorem 6.3.35 (Bukovsky)**.** *An inner model M has the δ-uniform cover property if and only if there is a δ-cc forcing $\mathbb{P}$ of M and an M-generic filter $G \subseteq \mathbb{P}$ such that $V = M[G]$.*

Proof of Theorem 6.3.33. Let $\delta = (2^\kappa)^+$. By Theorem 6.3.35, it suffices to show that HOD has the δ-uniform cover property in HOD_X for any $X \subseteq \kappa$. (This is because whenever a nonzero condition $p \in \mathbb{V}_\kappa$ forces a statement, this statement is true in HOD_X for any $X \in \pi_\kappa(p)$.) Let $f : \lambda \to P_\delta(\mathrm{Ord})$ be function in HOD_X. Then there is an ordinal definable function $H : P(\kappa) \times \gamma \to P_\delta(\mathrm{Ord})$ such that $f(\alpha) = H(X, \alpha)$ for all $\alpha < \gamma$. For $\alpha < \gamma$, let $F(\alpha) = \bigcup_{Y \in P(\kappa)} H(Y, \alpha)$. Clearly F is in HOD, and moreover, $|F(\alpha)| \leq 2^\kappa \cdot \sup_{Y \in P(\kappa)} |H(Y, \alpha)| = 2^\kappa$. In other words, $F : \gamma \to P_\delta(\mathrm{Ord})$, as desired. □

We finally prove Theorem 6.3.36, the fact that under UA, if κ is supercompact then V is a generic extension of HOD for a forcing of size κ^{++}.

Theorem 6.3.36 (UA)**.** *If κ is κ^{++}-supercompact then $|\mathbb{V}_\kappa|^{\mathrm{HOD}} = \kappa^{++}$ and $\mathbb{V}_\kappa$ has the κ^{++}-chain condition in* HOD.

Proof. Note that since κ is κ^{++}-supercompact, by Theorem 6.3.27, $(2^\kappa)^+ = \kappa^{++}$. In particular, κ is 2^κ-supercompact, so the hypotheses of Theorem 6.3.32 hold by Theorem 6.3.3. Thus $|\mathbb{V}_\kappa|^{\mathrm{HOD}} = (2^\kappa)^+ = \kappa^{++}$. Similarly, Theorem 6.3.33 can be applied to obtain the κ^{++}-chain condition for $\mathbb{V}_\kappa$. □

Chapter 7

The Least Supercompact Cardinal

7.1 Introduction

7.1.1 The identity crisis

How large is the least strongly compact cardinal? A precise form of this question was first posed by Tarski shortly after he introduced the notion of strong compactness: *is the least strongly compact cardinal larger than the least measurable cardinal?* About a decade later, Solovay mounted the first serious attack on this problem. Fusing Scott's elementary embedding analysis of measurability with his own combinatorial characterization of strong compactness, he defined the notion of a supercompact cardinal, which have since become one of the most important points in the large cardinal hierarchy. He then conjectured that every strongly compact cardinal is supercompact. This is certainly a natural conjecture to make given the many analogies between the theories of supercompact and strongly compact cardinals. (See Section 7.2 and especially Section 7.2.2.) Since one can easily show that the least supercompact cardinal is (much) larger than the least measurable cardinal, Solovay's conjecture would have implied a positive answer to Tarski's question.

Telis Menas, then a graduate student under Solovay at UC Berkeley, was the first to realize that Solovay's conjecture is false. To find a strongly compact cardinal that is not supercompact, Menas resorted to somewhat larger cardinals:

Theorem 8.1.1 (Menas)**.** *The least strongly compact limit of strongly compact cardinals is not supercompact.*

This theorem closed off Solovay's approach to Tarski's question while leaving the question itself wide open. The fundamental breakthrough occurred mere months after Menas's discovery, with Magidor's landmark independence result [33]:

Theorem (Magidor). *If κ is strongly compact, then in a forcing extension, κ remains strongly compact but becomes the least measurable cardinal.* $\square$

Theorem (Magidor). *If κ is supercompact, then in a forcing extension, κ remains supercompact but becomes the least strongly compact cardinal.* $\square$

Thus the ZFC axioms are insufficient to answer Tarski's question. Magidor described this peculiar situation as an "identity crisis" for the least strongly compact cardinal.

The main result of this chapter is that the Ultrapower Axiom resolves this crisis:

Theorem 7.4.23 (UA). *The least strongly compact cardinal is supercompact.*

We will prove much stronger results than this, explaining exactly why the least strongly compact cardinal is supercompact and identifying much weaker notions than strong compactness that are equivalent to supercompactness under UA. We defer the analysis of larger strongly compact cardinals until the final chapter, in which we generalize Theorem 7.4.23 to the second strongly compact cardinal and beyond.

7.1.2 Outline of Chapter 7

We now outline the rest of the chapter.

SECTION 7.2. This section exposits the basic theory of strong compactness. (None of the results are due to the author.) We use the theory of the Ketonen order to prove Ketonen's Theorem [14] that κ is strongly compact if and only if every regular cardinal carries a κ-complete ultrafilter (Theorem 7.2.15). This argument (due to Ketonen) is the basis for many of the results of this chapter. We use Ketonen's Theorem to prove the local version of Solovay's Theorem [19] on SCH above a strongly compact (Theorem 7.2.16), which we have at this point cited several times.

SECTION 7.3. This section introduces Fréchet cardinals and Ketonen ultrafilters. Under UA, each Fréchet cardinal λ carries a unique Ketonen ultrafilter $\mathscr{K}_\lambda$. For regular λ, we analyze $\mathscr{K}_\lambda$ under the assumption that some $\kappa \leq \lambda$ is λ-strongly compact (Proposition 7.4.11), showing that its associated embedding is $<\lambda$-supercompact and λ-tight (Definition 7.2.5).

SECTION 7.4. The analysis of $\mathscr{K}_\lambda$ in the previous section motivates the conjecture that if λ is a regular Fréchet cardinal, then some cardinal $\kappa \leq \lambda$ is λ-strongly compact. In this section, we come close to proving this conjecture, showing that it holds unless λ is *isolated* (Theorem 7.4.9). Isolated cardinals turn out to be rare enough that even this partial result suffices to prove the supercompactness of the least strongly compact cardinal (Theorem 7.4.23).

SECTION 7.5. We study the structure of isolated cardinals in an attempt to understand the one remaining limitation in the analysis of Fréchet cardinals from Section 7.4. Assuming GCH, we rule out the existence of pathological isolated cardinals (Proposition 7.5.4), proving that the class of isolated cardinals coincides with the class of measurable cardinals κ such that κ is not a limit of measurable cardinals and no cardinal $\delta < \kappa$ is κ-supercompact. Without GCH, assuming just UA, we are still able to give a fairly complete analysis of ultrafilters on an isolated cardinal (Section 7.5.2), which turn out to look just like the ultrafilters on the least measurable cardinal (Section 5.3). We prove that nonmeasurable isolated cardinals are associated with dramatic failures of the Generalized Continuum Hypothesis (Theorem 7.5.23 and Theorem 7.5.25). We leverage these results to prove the linearity of the Mitchell order on normal fine ultrafilters without assuming GCH (Theorem 7.5.42).

7.2 Strong compactness

7.2.1 Some characterizations of strong compactness

Strongly compact cardinals were first isolated by Tarski in the context of infinitary logic: κ is strongly compact if the logic $\mathscr{L}_{\kappa,\kappa}$ satisfies a generalized version of the Compactness Theorem. In keeping with modern large cardinal theory, the definition of strong compactness employed in this monograph is formulated in terms elementary embeddings of the universe of sets into inner models with certain closure properties. The relevant closure property is a two-cardinal version of the cover property:

Definition 7.2.1. Suppose M is an inner model, λ is a cardinal, and δ is an M-cardinal. Then M has the (λ, δ)-*cover property* if every set $A \subseteq M$ such that $|A| < \lambda$ is contained in a set $B \in M$ such that $|B|^M < \delta$.

Definition 7.2.2. A cardinal κ is strongly compact if for any cardinal $\lambda \geq \kappa$, there is an elementary embedding $j : V \to M$ such that $\text{crit}(j) = \kappa$ and M has the $(\lambda, j(\kappa))$-cover property.

Definition 7.2.3. We make the following abbreviations:

- The $({\leq}\lambda, \delta)$-*cover property* is the (λ^+, δ)-cover property.
- The $(\lambda, {\leq}\delta)$-*cover property* is the (λ, δ^{+M})-cover property.
- The $({\leq}\lambda, {\leq}\delta)$-*cover property* is the (λ^+, δ^{+M})-cover property.
- The λ-*cover property* is the (λ, λ)-cover property.
- The ${\leq}\lambda$-*cover property* is the $({\leq}\lambda, {\leq}\lambda)$-cover property.

This notation is chosen so that, for example, an inner model M has the $({\leq}\lambda, {\leq}\delta)$-cover property if every subset $A \subseteq M$ such that $|A| \leq \lambda$ is contained in a set $B \in M$ such that $|B|^M \leq \delta$.

We will be particularly interested in the following local version of strong compactness (especially when λ is regular):

Definition 7.2.4. Suppose $\kappa \leq \lambda$ are cardinals. Then κ is λ-*strongly compact* if there is an inner model M and an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ such that M has the $(\leq\lambda, j(\kappa))$-cover property.

Note that if $j : V \to M$ and M has the $(\leq\lambda, j(\kappa))$-cover property, then $j(\kappa) > \lambda$.

Theorem 7.2.10 puts down several equivalent reformulations of strong compactness. These involve the notions of *tightness* and *filter bases*, which we now define.

The concept of tightness had not been given a name before this monograph, but it plays a role analogous to that of the supercompactness of embedding (Definition 4.2.15) in the theory of supercompact cardinals:

Definition 7.2.5. Suppose M is an inner model, λ is a cardinal, and δ is an M-cardinal. An elementary embedding $j : V \to M$ is (λ, δ)-*tight* if there is a set $A \in M$ with $|A|^M \leq \delta$ such that $j[\lambda] \subseteq A$. An elementary embedding is said to be λ-*tight* if it is (λ, λ)-tight.

Thus (λ, δ)-tightness is a weakening of λ-supercompactness. Any $j : V \to M$ such that M has the $(\leq\lambda, j(\kappa))$-cover property is $(\lambda, {<}j(\kappa))$-tight. Moreover, many of the general theorems about supercompact embeddings generalize to the context of (λ, δ)-tight ones. For example, Lemma 4.2.16 generalizes:

Lemma 7.2.6. *Suppose $j : V \to M$ is an elementary embedding. The following are equivalent:*

(1) j is (λ, δ)-tight.

(2) For some X with $|X| = \lambda$, there is some $Y \in M$ with $|Y|^M \leq \delta$ such that $j[X] \subseteq Y$.

(3) For any A such that $|A| \leq \lambda$, there is some $B \in M$ with $|B|^M \leq \delta$ such that $j[A] \subseteq B$.

Proof. (1) implies (2): Trivial.

(2) implies (3): Suppose $|A| \leq \lambda$. We will find $B \in M$ with $|B|^M \leq \delta$ such that $j[A] \subseteq B$. Using (2), fix X with $|X| = \lambda$ such that for some $Y \in M$ with $|Y|^M \leq \delta$ such that $j[X] \subseteq Y$. Let $p : X \to A$ be a surjection. Then

$$j[A] = j(p)[j[X]] \subseteq j(p)[Y]$$

Let $B = j(p)[Y]$. Then $j[A] \subseteq B$, $B \in M$, and $|B|^M \leq |Y|^M \leq \delta$.

(3) implies (1): Trivial. $\square$

The relationship between the λ-supercompactness of an embedding and the closure of its target model under λ-sequences is analogous to the relationship between the (λ, δ)-tightness of an elementary embedding and the $(\leq\lambda, \leq\delta)$-cover property of its target model. For example, there is an analog of Corollary 4.2.20 regarding the closure properties of ultrapower embeddings (Definition 2.2.9).

Lemma 7.2.7. *Suppose $j : V \to M$ is a (λ, δ)-tight ultrapower embedding. Then M has the $(\leq\lambda, \leq\delta)$-cover property.*

Proof. Suppose $A \subseteq M$ with $|A| \leq \lambda$, and we will find $B \in M$ such that $|B|^M \leq \delta$ and $A \subseteq B$. Fix $a \in M$ such that $M = H^M(j[V] \cup \{a\})$. Fix a set of functions F of cardinality λ such that $A = \{j(f)(a) : f \in F\}$. By Lemma 7.2.6, fix $G \in M$ with $|G|^M \leq \delta$ and such that $j[F] \subseteq G$. Let $B = \{g(a) : g \in G\}$. Then $B \in M$, $A \subseteq B$, and $|B|^M \leq |G|^M \leq \delta$, as desired. $\square$

We now discuss an equivalent formulation of strong compactness in terms of *filter extension properties*. To state the local version of this formulation that we will need, it is convenient to work with filter bases rather than filters. Many filters (for example, the closed unbounded filter, the tail filter, and the fine filter) are most naturally presented in terms of smaller families of sets that "generate" the filter. The notion of a filter base makes this precise:

Definition 7.2.8. A *filter base* on X is a family $\mathscr{B}$ of subsets of X with the finite intersection property: for all $A_0, A_1 \in \mathscr{B}$, $A_0 \cap A_1 \neq \emptyset$. If κ is a cardinal, a filter base $\mathscr{B}$ is said to be *κ-complete* if for all $\nu < \kappa$, for all $\{A_\alpha : \alpha < \nu\} \subseteq \mathscr{B}$, $\bigcap_{\alpha<\nu} A_\alpha \neq \emptyset$.

The term "filter base" is motivated by the fact that every filter base $\mathscr{B}$ on X generates a filter.

Definition 7.2.9. Suppose $\mathscr{B}$ is a filter base. The *filter generated by $\mathscr{B}$* is the filter $F(\mathscr{B}) = \{A \subseteq X : \exists A_0, \ldots, A_{n-1} \in \mathscr{B}\ A_0 \cap \cdots \cap A_{n-1} \subseteq A\}$. If $\mathscr{B}$ is a κ-complete filter base, the *filter κ-generated by $\mathscr{B}$* is the filter

$$F_\kappa(\mathscr{B}) = \{A \subseteq X : \exists S \in P_\kappa(\mathscr{B}) \bigcap S \subseteq A\}$$

Suppose κ is singular. Then it may be the case that the filter κ-generated by a κ-complete filter base is not itself κ-complete. To see this, fix $\lambda \geq \kappa$. For $\alpha < \lambda$, let $A_\alpha = \{\sigma \in P_\kappa(\lambda) : \alpha \in \sigma\}$, and let $\mathscr{B} = \{A_\alpha : \alpha < \lambda\}$. Then $\mathscr{B}$ is obviously κ-complete, but the filter κ-generated by $\mathscr{B}$ is not. Therefore Theorem 7.2.10 (3) implies that κ is regular.

Theorem 7.2.10. *Suppose $\kappa \leq \lambda$ are uncountable cardinals. Then the following are equivalent:*

(1) κ is λ-strongly compact.

(2) There is an elementary embedding $j : V \to M$ with critical point κ that is (λ, δ)-tight for some M-cardinal $\delta < j(\kappa)$.

(3) *Every κ-complete filter base of cardinality λ extends to a κ-complete ultrafilter.*

(4) *There is a κ-complete fine ultrafilter on $P_\kappa(\lambda)$.*

(5) *There is an ultrapower embedding $j : V \to M$ with critical point κ that is (λ, δ)-tight for some M-cardinal $\delta < j(\kappa)$.*

(6) *There is an elementary embedding $j : V \to M$ with critical point κ such that M has the $(\leq\lambda, \delta)$-cover property for some M-cardinal $\delta < j(\kappa)$.*

Proof. (1) implies (2): Trivial.

(2) implies (3): Let $j : V \to M$ be an elementary embedding such that $\text{crit}(j) = \kappa$ and j is (λ, δ)-tight for some M-cardinal $\delta < j(\kappa)$. Suppose $\mathscr{B}$ is a κ-complete filter base on X of cardinality λ. By Lemma 7.2.6, there is a set $S \in M$ such that $j[\mathscr{B}] \subseteq S$ and $|S|^M < j(\kappa)$. By replacing S with $S \cap j(\mathscr{B})$, we may assume without loss of generality that $S \subseteq j(\mathscr{B})$. By the elementarity of j, since $j(\mathscr{B})$ is $j(\kappa)$-complete, the intersection $\bigcap j(S)$ is nonempty. Fix $a \in \bigcap j(S)$. Since $j[\mathscr{B}] \subseteq S$, it follows that $a \in j(A)$ for all $A \in \mathscr{B}$. Let U be the ultrafilter on X derived from j using a. Then U extends $\mathscr{B}$ and U is κ-complete since $\text{crit}(j) = \kappa$.

(3) implies (4): For any $\alpha < \lambda$, let $A_\alpha = \{\sigma \in P_\kappa(\lambda) : \alpha \in \sigma\}$, and let $\mathscr{B} = \{A_\alpha : \alpha < \lambda\}$. Then $\mathscr{B}$ a κ-complete filter base on $P_\kappa(\lambda)$, and any filter on $P_\kappa(\lambda)$ that extends $\mathscr{B}$ is fine. By (3), there is a κ-complete ultrafilter extending $\mathscr{B}$. Thus there is a κ-complete fine ultrafilter on $P_\kappa(\lambda)$, as desired.

(4) implies (5): Suppose $\mathcal{U}$ is a κ-complete fine ultrafilter on $P_\kappa(\lambda)$. Let $j : V \to M$ be the ultrapower of the universe by $\mathcal{U}$. The κ-completeness of $\mathcal{U}$ implies that $\text{crit}(j) \geq \kappa$. By Lemma 4.4.9, $j[\lambda] \subseteq \text{id}_\mathcal{U}$. Moreover $\text{id}_\mathcal{U} \in j(P_\kappa(\lambda))$, so letting $\delta = |\text{id}_\mathcal{U}|^M$, $\delta < j(\kappa)$. Therefore j is an ultrapower embedding that is (λ, δ)-tight for some $\delta < j(\kappa)$. Since $\kappa \leq \lambda$ and $\lambda \leq \text{ot}(j[\lambda]) \leq \delta^{+M} < j(\kappa)$, it follows that $j(\kappa) > \lambda$. In particular, $\text{crit}(j) = \kappa$.

(5) implies (6): This is an immediate consequence of the fact that tight ultrapowers have the cover property (Lemma 7.2.7).

(6) implies (1): Trivial. □

7.2.2 Ketonen's Theorem

The main theorem of this subsection is a famous theorem of Ketonen [14] that provides a deeper ultrafilter theoretic characterization of strong compactness:

Theorem 7.2.11 (Ketonen)**.** *A cardinal κ is strongly compact if and only if every regular cardinal $\lambda \geq \kappa$ carries a uniform κ-complete ultrafilter.*

Part of what is surprising about this theorem is that it does not even require that the ultrafilters in the hypothesis be κ^+-incomplete. Beyond this, it is not even obvious at the outset that the existence of κ-complete ultrafilters on, say, κ and κ^+ implies that κ is κ^+-strongly compact.

We begin, however, with a less famous but no less important theorem of Ketonen, which is also a key step in the proof of Theorem 7.2.11. This theorem is in a sense the strongly compact generalization of Solovay's Lemma [19]. Suppose $j : V \to M$ is an elementary embedding. For regular cardinals λ, Solovay's Lemma (or more specifically Corollary 4.4.30) yields a simple criterion for whether j is λ-supercompact solely in terms of the inner model M and the ordinal $\sup j[\lambda]$:

Theorem 4.4.30 (Solovay)**.** *Suppose $j : V \to M$ is an elementary embedding and λ is a regular cardinal. Then j is λ-supercompact if and only if M is correct about stationary subsets of* $\sup j[\lambda]$. □

Ketonen proved a remarkable analog of this theorem for strongly compact embeddings:

Theorem 7.2.12 (Ketonen)**.** *Suppose $j : V \to M$ is an elementary embedding, λ is a regular uncountable cardinal, and δ is an M-cardinal. Then j is (λ, δ)-tight if and only if* $\text{cf}^M(\sup j[\lambda]) \leq \delta$.

For example, suppose $j : V \to M$ is an ultrapower embedding. Theorem 7.2.12 implies that all that is required for M to have the $\leq\lambda$-cover property is that M correctly compute the cofinality of $\sup j[\lambda]$.

The proof of Theorem 7.2.12 we give, due to Woodin, is fundamentally different from Ketonen's original proof. (Ketonen's proof is in some ways more general.) The trick is to choose the cover first, and then choose the set whose image is being covered:

Proof of Theorem 7.2.12. First assume j is (λ, δ)-tight. Fix $A \in M$ with $j[\lambda] \subseteq A$ such that $|A|^M \leq \delta$. Then $A \cap \sup j[\lambda]$ is cofinal in $\sup j[\lambda]$, so $\sup j[\lambda]$ has cofinality at most $|A|^M$ in M.

Now we prove the converse. Assume $\text{cf}^M(\sup j[\lambda]) \leq \delta$. Let $Y \in M$ be an ω-closed cofinal subset of $\sup j[\lambda]$ of order type at most δ. Note that $j[\lambda]$ is itself an ω-closed cofinal subset of $\sup j[\lambda]$, so since $\sup j[\lambda]$ has uncountable cofinality, $Y \cap j[\lambda]$ is an ω-closed cofinal subset of λ. In particular, since $\text{cf}(\sup j[\lambda]) = \lambda$, $Y \cap j[\lambda]$ has order type at least λ. Let $X = j^{-1}[Y]$. Then $j[X] = Y \cap j[\lambda]$, so

$$\text{ot}(X) = \text{ot}(j[X]) = \text{ot}(Y \cap j[\lambda]) \geq \lambda$$

Thus $|X| = \lambda$. Since $|X| = \lambda$, $Y \in M$, $j[X] \subseteq Y$, and $|Y|^M \leq \delta$, Lemma 7.2.6 implies that j is (λ, δ)-tight. □

With Theorem 7.2.12 in hand, we turn to the proof of Ketonen's characterization of strong compactness. The key point is that the strong compactness of an elementary embedding is equivalent to an ultrafilter theoretic property:

Proposition 7.2.13. *Suppose $\kappa \leq \lambda$ are uncountable cardinals and λ is regular. Suppose M is an inner model and $j : V \to M$ is an elementary embedding. Suppose every regular cardinal in the interval $[\kappa, \lambda]$ carries a uniform κ-complete ultrafilter. Then the following are equivalent:*

(1) *j is (λ,δ)-tight for some M-cardinal $\delta < j(\kappa)$.*

(2) $\sup j[\lambda]$ *carries no $j(\kappa)$-complete fine ultrafilter.*

Proof. (1) implies (2): By Theorem 7.2.12, (1) implies $\text{cf}^M(\sup j[\lambda]) < j(\kappa)$. Therefore the tail filter on $\sup j[\lambda]$ is not $j(\kappa)$-complete in M, so $\sup j[\lambda]$ does not carry a $j(\kappa)$-complete fine ultrafilter in M.

(2) implies (1): By Lemma 3.2.10, (2) implies $\text{cf}^M(\sup j[\lambda])$ carries no uniform $j(\kappa)$-complete ultrafilter in M. By elementarity, every M-regular cardinal in the interval $j([\kappa,\lambda])$ carries a uniform κ-complete ultrafilter. Therefore $\text{cf}^M(\sup j[\lambda])$ does not lie in the interval $j([\kappa,\lambda])$. Clearly $\text{cf}^M(\sup j[\lambda]) \leq j(\lambda)$, so it follows that $\text{cf}^M(\sup j[\lambda]) < j(\kappa)$. □

Ketonen introduced the Ketonen order $<_{\Bbbk}$ (Definition 3.3.2) as a tool to prove the following theorem, generalizing a theorem of Solovay that states that every measurable cardinal carries a normal ultrafilter that concentrates on non-measurable cardinals. For the proof, recall that if U is an ultrafilter on an ordinal, then δ_U denotes the least ordinal in U.

Theorem 7.2.14 (Ketonen)**.** *Suppose λ is a regular cardinal. If λ carries a κ-complete uniform ultrafilter, then λ carries a κ-complete uniform ultrafilter U such that* $\sup j_U[\lambda]$ *carries no fine κ-complete ultrafilter in M_U. Indeed, any $<_{\Bbbk}$-minimal κ-complete uniform ultrafilter on λ has this property.*

Proof. Let U be a $<_{\Bbbk}$-minimal element of the set of uniform κ-complete ultrafilters on λ. Suppose towards a contradiction that in M_U, $\sup j_U[\lambda]$ carries a fine κ-complete ultrafilter. Equivalently, there is a κ-complete ultrafilter Z on $j_U(\lambda)$ such that $\delta_Z = \sup j_U[\lambda]$. Let $W = j_U^{-1}[Z]$. Then $\text{crit}(j_W) \geq \text{crit}(j_Z^{M_U} \circ j_U)$ (by Lemma 3.2.17), so W is κ-complete. Moreover since $\delta_Z = \sup j_U[\lambda]$, $\delta_W = \lambda$. Thus W is a κ-complete uniform ultrafilter on λ. Since Z concentrates on $\sup j_U[\lambda] \leq \text{id}_U$, $W <_{\Bbbk} U$ by the definition of the Ketonen order (Lemma 3.3.4). This contradicts the $<_{\Bbbk}$-minimality of U. □

We can now prove a local version of Ketonen's theorem, which fits into the list of reformulations of λ-strong compactness from Theorem 7.2.10:

Theorem 7.2.15 (Ketonen)**.** *Suppose $\kappa \leq \lambda$ are regular uncountable cardinals. Then the following are equivalent:*

(1) *κ is λ-strongly compact.*

(2) *Every regular cardinal in the interval $[\kappa,\lambda]$ carries a uniform κ-complete ultrafilter.*

(3) *λ carries a κ-complete ultrafilter U such that j_U is (λ,δ)-tight for some $\delta < j_U(\kappa)$.*

Proof. (1) implies (2): Note that the Fréchet filter on a regular cardinal δ is δ-complete. Thus (2) follows from (1) as an immediate consequence of the filter extension property of strongly compact cardinals (Theorem 7.2.10 (3)).

(2) implies (3): Assume (2). By Theorem 7.2.14, there is a κ-complete ultrafilter U on λ such that $\sup j_U[\lambda]$ carries no fine κ-complete ultrafilter in M_U. Therefore by Proposition 7.2.13, j_U is (λ, δ)-tight for some $\delta < j_U(\kappa)$.

(3) implies (1): See Theorem 7.2.10 (5). $\square$

7.2.3 Solovay's Theorem

In this section, we give a proof of a local version of Solovay's theorem that we use throughout this monograph.

Theorem 7.2.16 (Solovay). *Suppose $\kappa \leq \lambda$ are uncountable cardinals, λ is regular, and κ is λ-strongly compact. Then $\lambda^{<\kappa} = \lambda$.*

We need the following lemma, which is in a sense an analog of Proposition 4.2.30, though much easier:

Lemma 7.2.17. *Suppose U is a countably complete ultrafilter. Let $j : V \to M$ be the ultrapower of the universe by U. Then for any $\eta \geq \lambda_U^+$, j is not (η, δ)-tight for any M-cardinal $\delta < j(\eta)$.*

Proof. We may assume by induction that η is a successor cardinal. In particular, η is regular, so by Lemma 2.2.34, $j(\eta) = \sup j[\eta]$. Suppose towards a contradiction that $\delta < j(\eta)$ is an M-cardinal such that j is (η, δ)-tight. By Theorem 7.2.12, $\text{cf}^M(j(\eta)) = \text{cf}^M(\sup j[\eta]) \leq \delta < j(\eta)$. This contradicts that η is regular in M by elementarity. $\square$

Lemma 7.2.18. *Suppose $\kappa \leq \gamma$ are cardinals. Suppose γ is singular and*

$$\sup_{\eta<\gamma} \eta^{<\kappa} \leq \gamma \tag{7.1}$$

Suppose γ^+ carries a uniform κ-complete ultrafilter U. Then $\gamma^{<\kappa} \leq \gamma^+$.

Proof. Let $\lambda = \gamma^+$. We will prove the equivalent statement that $\lambda^{<\kappa} = \lambda$.

Let $j : V \to M$ be the ultrapower of the universe by U. Let $\delta = \text{cf}^M(\sup j[\lambda])$. Note that $\delta < j(\lambda)$, so $\delta \leq j(\gamma)$. In fact, since $j(\gamma)$ is singular in M, $\delta < j(\gamma)$. Therefore by (7.1) and the elementarity of j:

$$(\delta^{<\kappa})^M \leq (\delta^{<j(\kappa)})^M \leq j(\gamma) \tag{7.2}$$

By Theorem 7.2.12, j is (λ, δ)-tight, so we can fix $B \in M$ with $j[\lambda] \subseteq B$. Now $j[P_\kappa(\lambda)] \subseteq B^{<\kappa}$, and since M is closed under κ-sequences, $B^{<\kappa} \in M$. Lemma 7.2.6 now implies that j is $(\lambda^{<\kappa}, (\delta^{<\kappa})^M)$-tight.

Assume towards a contradiction that $\lambda^{<\kappa} \geq \lambda^+$. Then j is $(\lambda^+, (\delta^{<\kappa})^M)$-tight. Since $\lambda_U = \lambda$, it follows from Lemma 7.2.17 that $(\delta^{<\kappa})^M \geq j(\lambda^+)$, contradicting (7.2). $\square$

We now prove Solovay's theorem:

Proof of Theorem 7.2.16. Suppose κ is λ-strongly compact. Assume by induction that for all regular $\iota < \lambda$, $\iota^{<\kappa} = \iota$. Since λ is regular, every element of $P_\kappa(\lambda)$ is bounded below λ, so $P_\kappa(\lambda) = \bigcup_{\eta<\lambda} P_\kappa(\eta)$. Thus computing cardinalities:

$$\lambda^{<\kappa} = \sup_{\eta<\lambda} \eta^{<\kappa}$$

If λ is a limit cardinal, it follows immediately from our induction hypothesis that $\lambda^{<\kappa} = \lambda$. Therefore assume λ is a successor cardinal. If the cardinal predecessor of λ is a regular cardinal ι, then applying our induction hypothesis we obtain:

$$\lambda^{<\kappa} = \sup_{\eta<\lambda} \eta^{<\kappa} = \lambda \cdot \iota^{<\kappa} = \lambda$$

Therefore assume the cardinal predecessor of λ is a singular cardinal γ. Then since γ is a limit of regular cardinals, out induction hypothesis implies $\sup_{\eta<\gamma} \eta^{<\kappa} \leq \gamma$. In this case, by Lemma 7.2.18, $\lambda^{<\kappa} = \lambda$. □

Theorem 7.2.19 (Solovay)**.** *Suppose $\kappa \leq \lambda$ are cardinals and κ is λ-strongly compact.*

(1) If $\mathrm{cf}(\lambda) \geq \kappa$ *then* $\lambda^{<\kappa} = \lambda$.

(2) If $\mathrm{cf}(\lambda) < \kappa$, *then* $\lambda^{<\kappa} = \lambda^+$.

Proof. (1) is immediate from Theorem 7.2.16, while (2) is an immediate consequence of (1): by König's Theorem, $\lambda^+ \leq \lambda^{<\kappa}$, while $\lambda^{<\kappa} \leq (\lambda^+)^{<\kappa} = \lambda^+$ by (1). □

7.3 Fréchet cardinals and the least ultrafilter $\mathscr{K}_\lambda$

7.3.1 Fréchet cardinals

In this section, we begin our systematic study of strong compactness assuming UA. We will ultimately prove that UA implies that strong compactness and supercompactness coincide to the extent that this is possible. (A theorem of Menas shows that assuming sufficiently large cardinals, not all strongly compact cardinals are supercompact; see Section 8.1.2.) An oddity of the proof is that it requires a preliminary analysis of the first strongly compact cardinal. Indeed, to obtain the strongest results, one must enact a local analysis of essentially the weakest ultrafilter-theoretic forms of strong compactness.

With this in mind, we introduce the following central concept:

Definition 7.3.1. An uncountable cardinal λ is *Fréchet* if λ carries a countably complete uniform ultrafilter.

Fréchet cardinals almost certainly do not appear in the work of Fréchet. Their name derives from the fact that λ is Fréchet if and only if the Fréchet filter on λ extends to a countably complete ultrafilter.

Recall that the size λ_U of an ultrafilter U is the least cardinality of a set in U (Definition 2.2.26).

Proposition 7.3.2. *A cardinal λ is Fréchet if and only if $\lambda = \lambda_U$ for some countably complete ultrafilter U.* □

For regular cardinals λ, we have the following obvious characterizations of Fréchetness:

Proposition 7.3.3. *Suppose λ is a regular uncountable cardinal. The following are equivalent:*

(1) λ is Fréchet.

(2) There is a countably complete fine ultrafilter on λ.

(3) Some ordinal of cofinality λ carries a fine ultrafilter.

(4) Every ordinal of cofinality λ carries a fine ultrafilter.

(5) There is an elementary embedding $j : V \to M$ that is discontinuous at λ.

Proof. (1) implies (2): Since λ is a cardinal, any uniform ultrafilter on λ is fine (Lemma 3.2.10).

(2) implies (3): Trivial.

(3) implies (4): Two ordinals α and β have the same cofinality if and only if there is a monotonically increasing cofinal function $f : (\alpha, \leq) \to (\beta, \leq)$. In particular, $f_*(T_\alpha) = T_\beta$ where T_α is the tail filter on α. Thus if α carries a countably complete fine ultrafilter U, then so does β, namely $f_*(U)$.

(4) implies (5): Suppose U is a countably complete fine ultrafilter on λ. Let $j : V \to M$ be the ultrapower of the universe by U. Note that for any $\alpha < \lambda$, $j(\alpha) < \mathrm{id}_U$ since $\alpha < \delta_U$. Thus $\sup j[\lambda] \leq \mathrm{id}_U < j(\lambda)$. In other words, j is discontinuous at λ. □

Singular Fréchet cardinals are more subtle, especially when one does not assume the Generalized Continuum Hypothesis. The following fact gives a sense of how singular Fréchet cardinals should arise:

Proposition 7.3.4. *Suppose λ is a singular limit of Fréchet cardinals. Let ι be the cofinality of λ. Then λ is Fréchet if and only if ι is Fréchet.*

Proof. If λ is Fréchet, then ι is Fréchet by Proposition 7.3.3 (4), and this does not require that λ is a limit of Fréchet cardinals.

We now turn to the converse. Let $\langle \lambda_\alpha : \alpha < \iota\rangle$ be an increasing cofinal sequence of Fréchet cardinals less than λ. Let U_α be a countably complete ultrafilter on λ with $\lambda_{U_\alpha} = \lambda_\alpha$. Let D be a countably complete uniform ultrafilter on ι. Let

$$U = D\text{-}\lim_{\alpha<\iota} U_\alpha$$

Clearly U is a countably complete ultrafilter on λ. We claim that U is uniform, or in other words that every set $X \in U$ has cardinality λ. Suppose $X \subseteq \lambda$ is such a set. By the definition of ultrafilter limits, $\{\alpha < \iota : X \in U_\alpha\} \in D$. Since D is a uniform ultrafilter, the set $\{\alpha < \iota : X \in U_\alpha\}$ is unbounded in ι. Therefore $X \in U_\alpha$ for unboundedly many $\alpha < \iota$, and in particular $|X| \geq \lambda_{U_\alpha} = \lambda_\alpha$ for unboundedly many $\alpha < \iota$. Thus $|X| \geq \sup_{\alpha<\iota} \lambda_\alpha = \lambda$, as desired. Since λ carries a countably complete uniform ultrafilter, follows that λ is a Fréchet cardinal. □

Proposition 7.3.4 tells us that when λ is a singular limit of Fréchet cardinals, whether λ is Fréchet depends only on whether the regular cardinal $\text{cf}(\lambda)$ is Fréchet. One might therefore hope to reduce problems about Fréchet cardinals in general to the regular case, where we have a bit more information. It is not provable in ZFC, however, that a singular Fréchet cardinal must be a limit of Fréchet cardinals. The Fréchet cardinals where this fails are called *isolated cardinals* (Definition 7.4.7), and arise as a major barrier in our analysis of strong compactness under UA. Isolated cardinals are studied in Section 7.4 and especially Section 7.5.

7.3.2 Ketonen ultrafilters

The following definition is inspired by the proof of Theorem 7.2.15, which turned on the existence of a κ-complete ultrafilter U on λ such that $\sup j_U[\lambda]$ carries no κ-complete fine ultrafilter in M_U.

Recall from Lemma 4.4.17 that a uniform ultrafilter U on a regular cardinal λ is weakly normal if and only if letting $j : V \to M$ be the ultrapower of the universe by U, $\text{id}_U = \sup j[\lambda]$. Equivalently, U is weakly normal if it is closed under decreasing diagonal intersections.

Definition 7.3.5. If λ is a regular cardinal, an ultrafilter U on λ is a *Ketonen ultrafilter* if the following hold:

- U is countably complete and weakly normal.
- U concentrates on ordinals that carry no countably complete fine ultrafilter.

By Lemma 4.4.17 and Proposition 7.3.3, we have the following characterization of Ketonen ultrafilters on regular cardinals:

Lemma 7.3.6. *Suppose λ is a regular cardinal and U is a countably complete ultrafilter on λ. Then U is Ketonen if and only if $\text{id}_U = \sup j_U[\lambda]$ and either of the following equivalent statements holds:*

- $\sup j_U[\lambda]$ *carries no countably complete fine ultrafilter in* M_U.
- $\mathrm{cf}^{M_U}(\sup j_U[\lambda])$ *is not Fréchet in* M_U. □

In this way, the key ordinal $\mathrm{cf}^{M_U}(\sup j_U[\lambda])$ from Theorem 7.2.12 arises immediately in the study of Ketonen ultrafilters on regular cardinals.

Ketonen ultrafilters are analogous to λ-minimal ultrafilters of Section 4.4.2, except that Ketonen ultrafilters are minimal in the Ketonen order $<_\Bbbk$ (Definition 3.3.2) rather than the Rudin-Keisler order.

Lemma 7.3.7. *Suppose λ is a regular cardinal. Then U is a Ketonen ultrafilter on λ if and only if U is a $<_\Bbbk$-minimal element of the set of countably complete uniform ultrafilters on λ.*

Proof. Suppose first that U is a Ketonen ultrafilter. Let

$$\alpha = \mathrm{id}_U = \sup j_U[\lambda]$$

Suppose $W <_\Bbbk U$. We will show that $\lambda_W < \lambda$. By the definition of the Ketonen order (Lemma 3.3.4), there is an ultrafilter Z of M_U on $j_U(\lambda)$ concentrating on $\sup j_U[\lambda]$ such that $j_U^{-1}[Z] = W$. Since $\sup j_U[\lambda]$ does not carry a countably complete fine ultrafilter in M_U, there is some $\beta < \sup j_U[\lambda]$ such that Z concentrates on β. Fix $\alpha < \lambda$ such that $j_U(\alpha) \geq \beta$. Then $j_U(\alpha) \in Z$, so $\alpha \in W$. Thus $\lambda_W < \lambda$ as desired.

Conversely, assume U is a $<_\Bbbk$-minimal element of the set of uniform ultrafilters on λ. In particular, U is an $<_{\mathrm{rk}}$-minimal element of the set of uniform ultrafilters on λ, which by Lemma 4.4.20 is equivalent to being weakly normal.

Finally, fix an ultrafilter Z of M_U on $j_U(\lambda)$, and we will show that $\delta_Z < \sup j_U[\lambda]$. Let $W = j_U^{-1}[Z]$. Then $W <_\Bbbk U$ by the definition of the Ketonen order. It follows from the minimality of U that $\delta_W < \lambda$, so for some $\alpha < \lambda$, $\alpha \in W$. Now $j_U(\alpha) \in Z$, so $\delta_Z \leq j_U(\alpha) < \sup j_U[\lambda]$, as desired.

It follows that $\sup j_U[\lambda]$ does not carry a countably complete fine ultrafilter in M_U, so U is Ketonen by Lemma 7.3.6. □

Reflecting on Lemma 7.3.7, we obtain a definition of Ketonen ultrafilters on arbitrary cardinals:

Definition 7.3.8. Suppose λ is a Fréchet cardinal. An ultrafilter U on λ is *Ketonen* if U is a $<_\Bbbk$-minimal element of the set of countably complete uniform ultrafilters on λ.

The wellfoundedness of the Ketonen order (Theorem 3.3.8) immediately yields the existence of Ketonen ultrafilters:

Theorem 7.3.9. *Every Fréchet cardinal carries a Ketonen ultrafilter.* □

When λ is singular, it is important that the definition of a Ketonen ultrafilter demands minimality only among uniform ultrafilters and not among the broader class of fine ultrafilters, since an ultrafilter on λ that is minimal in this stronger sense is essentially the same thing as a Ketonen ultrafilter on $\mathrm{cf}(\lambda)$:

Lemma 7.3.10. *Suppose γ is an ordinal and U is a $<_\Bbbk$-minimal among countably complete ultrafilters W with $\delta_W = \gamma$. Let $\lambda = \text{cf}(\gamma)$ and let $f : \lambda \to \gamma$ be a continuous cofinal function. Then $U = f_*(D)$ for some Ketonen ultrafilter D on λ.*

Proof. Since U is $<_\Bbbk$-minimal among countably complete ultrafilters W with $\delta_W = \gamma$, in particular U is $<_{\text{rk}}$-minimal, so every function $g : \gamma \to \gamma$ that is regressive on a set in U is bounded on a set in U. It follows that U contains every closed cofinal $C \subseteq \gamma$: letting $A = \gamma \setminus C$ and $g(\alpha) = \sup(C \cap \alpha)$, g is regressive on A and unbounded on any cofinal subset of A.

Let $C = f[\lambda]$. Then $C \in U$. Let $g : C \to \lambda$ be the inverse of f. Let $D = g_*(U)$. Clearly $U = f_*(D)$. We must show that D is Ketonen. Suppose $W <_\Bbbk D$. We claim $f_*(W) <_\Bbbk U$. Given this, it follows that $\delta_{f_*(W)} < \gamma$ and hence $\delta_W < \lambda$. It follows that D is a $<_\Bbbk$-minimal element of the set of countably complete uniform ultrafilters on λ, so D is Ketonen.

We finally verify $f_*(W) <_\Bbbk U$. (The proof will show that if $f : \lambda \to \gamma$ is an order preserving function, then the pushforward map f_* is Ketonen order preserving.) Fix $I \in D$ and $\langle W_\alpha : \alpha \in I \rangle$ such that $W = D\text{-}\lim_{\alpha \in I} W_\alpha$ and $\delta_\alpha \leq \alpha$ for all $\alpha \in I$. Let $J = f[I]$, so $J \in U$ and moreover:

$$f_*(W) = U\text{-}\lim_{\beta \in J} f_*(W_{g(\beta)})$$

Moreover $\delta_{f_*(W_{g(\beta)})} \leq \sup f[\delta_{W_{g(\beta)}}] \leq \sup f[g(\beta)] \leq \beta$. Thus the sequence $\langle f_*(W_{g(\beta)}) : \beta \in J \rangle$ witnesses $f_*(W) <_\Bbbk U$, as desired. □

7.3.3 Introducing $\mathscr{K}_\lambda$

Under the Ultrapower Axiom, the Ketonen order is linear, so there is a canonical Ketonen ultrafilter on each Fréchet cardinal λ:

Definition 7.3.11 (UA)**.** For any Fréchet cardinal λ, the *least ultrafilter on* λ, denoted by $\mathscr{K}_\lambda$, is the unique Ketonen ultrafilter on λ.

The least ultrafilters $\mathscr{K}_\lambda$ play an outsized role in the analysis of supercompactness under UA, which proceeds by first completely analyzing the ultrafilters $\mathscr{K}_\lambda$ and then propagating the structure of $\mathscr{K}_\lambda$ to all ultrafilters. This is the main reason that the analysis of the first strongly compact cardinal is separate from that of the other strongly compact cardinals.

Let us begin with some simple examples. Let κ_0 be the least measurable cardinal. Then without assuming UA, it is easy to prove that an ultrafilter on κ_0 is Ketonen if and only if it is normal. Assuming UA, $\mathscr{K}_{\kappa_0}$ is the unique normal ultrafilter on κ_0.

Moving up to the second measurable cardinal κ_1, it is *not* provable in ZFC that the Ketonen ultrafilters on κ_1 are normal, or even that there is a normal Ketonen ultrafilter on κ_1. This is because it is consistent that κ_0 is κ_1-strongly compact. Under this assumption, if U is a normal ultrafilter on κ_1, κ_0 is $j_U(\kappa_1)$-strongly compact in M_U, and hence U concentrates on ordinals that carry κ_0-complete uniform ultrafilters. In fact, under this hypothesis, if W is a Ketonen ultrafilter on κ_1, then j_W is (κ_1, δ)-tight for some $\delta < j_W(\kappa_0)$, and hence W witnesses the κ_1-strong compactness of κ_0.

Of course, under UA, κ_0 is not κ_1-strongly compact, since by Theorem 5.3.18, every countably complete ultrafilter in V_{κ_1} is Rudin-Keisler equivalent to $\mathscr{K}_{\kappa_0}^n$ for some $n < \omega$. In fact, once again $\mathscr{K}_{\kappa_1}$ is the unique normal ultrafilter on κ_1. To see this, one can apply the fact that irreducible ultrafilters below the least measurable cardinal κ of Mitchell order 2^{2^κ} are normal (Theorem 5.3.8) and the following lemma. Recall here that an ultrafilter U is irreducible if it cannot be decomposed as an iterated ultrapower (Definition 5.1.3).

Lemma 7.3.12 (UA)**.** *For any regular cardinal λ, $\mathscr{K}_\lambda$ is an irreducible ultrafilter.*

Proof. Suppose $D <_{\text{RF}} \mathscr{K}_\lambda$. Then since $D <_{\text{RK}} \mathscr{K}_\lambda$ and $\mathscr{K}_\lambda$ is weakly normal, $\lambda_D < \lambda$. Therefore by Lemma 2.2.34,

$$j_D(\lambda) = \sup j_D[\lambda]$$

Assume towards a contradiction that D is nonprincipal. Then by Proposition 5.4.5, $t_D(\mathscr{K}_\lambda) <_{\Bbbk} j_D(\mathscr{K}_\lambda)$, so $\delta_{t_D(\mathscr{K}_\lambda)} < j_D(\lambda)$ by Lemma 7.3.7 applied in M_D. But $\mathscr{K}_\lambda = j_D^{-1}[t_D(\mathscr{K}_\lambda)]$, so

$$\delta_{\mathscr{K}_\lambda} = \min\{\delta : j_D(\delta) > \delta_{t_D(\mathscr{K}_\lambda)}\} < \lambda$$

This contradicts that $\mathscr{K}_\lambda$ is a uniform ultrafilter on λ. □

We do not know whether this lemma is provable in ZFC, although it does follow from Theorem 5.3.14.

If λ is singular, then $\mathscr{K}_\lambda$ is not necessarily irreducible. (In fact, we will show under UA that for strong limit singular cardinals λ, $\mathscr{K}_\lambda$ is *never* irreducible.) For example, suppose λ_0 is the least singular cardinal that carries a uniform countably complete ultrafilter. Of course, assuming just ZFC, one cannot prove much about λ_0: it is consistent that $\lambda_0 = \kappa_0^{+\kappa_0}$, or that λ_0 is not a limit of regular cardinals that carry uniform countably complete ultrafilters.

Assuming UA, it is not hard to give a complete analysis of λ_0 and $\mathscr{K}_{\lambda_0}$. Let $\langle \kappa_\alpha : \alpha < \kappa_0 \rangle$ enumerate the first κ_0 measurable cardinals in increasing order. Then $\lambda_0 = \sup_{\alpha<\kappa_0} \kappa_\alpha$, and

$$\mathscr{K}_{\lambda_0} = \mathscr{K}_{\kappa_0}\text{-}\lim_{\alpha<\kappa_0} (\mathscr{K}_{\kappa_\alpha} \mid \lambda_0)$$

(Recall the notation $U \mid Y$ for the projection of the ultrafilter U to a set Y such that $P(Y) \cap U \neq \emptyset$; see Definition 3.2.1.) The sets $A_\alpha = \kappa_\alpha \setminus \sup_{\beta<\alpha} \kappa_\beta$ witness that the sequence $\langle \mathscr{K}_{\kappa_\alpha} \mid \lambda_0 : \alpha < \kappa_0 \rangle$ is discrete, so $\mathscr{K}_{\kappa_0} <_{\text{RF}} \mathscr{K}_{\lambda_0}$ by Definition 5.2.2. In other words, $\mathscr{K}_{\lambda_0}$ is the iterated ultrapower $[\mathscr{K}_\kappa, \mathscr{K}_{\lambda_0}^{M_{\mathscr{K}_\kappa}}]$. This is closely related to Proposition 7.3.4.

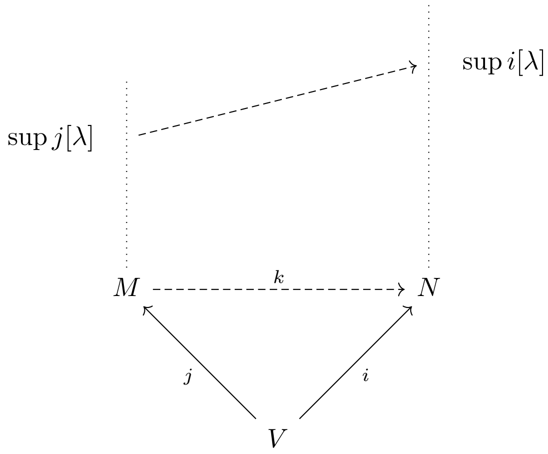

Figure 7.1: The universal property of $\mathscr{K}_\lambda$

For singular cardinals λ, $\mathscr{K}_\lambda$ is of greatest interest if λ is not a limit of Fréchet cardinals, since in this case $\mathscr{K}_\lambda$ cannot be represented in terms of ultrafilters on smaller cardinals. Such a cardinal is *isolated* in the sense of Definition 7.4.7. It is open whether UA rules out singular isolated cardinals altogether, but it certainly places major constraints on their structure (Section 7.5.2).

7.3.4 The universal property of $\mathscr{K}_\lambda$

The main result of this section is a universal property of the least ultrafilter $\mathscr{K}_\lambda$ on a regular Fréchet cardinal:

Theorem 7.3.13 (UA)**.** *Suppose λ is a regular Fréchet cardinal. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Suppose $i : V \to N$ is an ultrapower embedding. Then the following are equivalent:*

(1) There is an internal ultrapower embedding $k : M \to N$ such that $k \circ j = i$.

(2) $\sup i[\lambda]$ *carries no fine ultrafilter in N.*

(3) $\text{cf}^N(\sup i[\lambda])$ *is not Fréchet in N.*

While the proof is quite simple, the result has profound consequences for the structure of the ultrafilters $\mathscr{K}_\lambda$. In fact, this universal property is ultimately responsible for all of our results on supercompactness under UA. The supercompactness of the least strongly compact cardinal, for example, can be proved directly from the conclusion of Theorem 7.3.13 without assuming UA.

Before proving Theorem 7.3.13, let us show how it can be used to give a complete analysis of the internal ultrapower embeddings of $M_{\mathscr{K}_\lambda}$ when λ is regular.

Theorem 7.3.14 (UA). *Suppose λ is a regular Fréchet cardinal. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Suppose $k : M \to N$ is an ultrapower embedding. Then the following are equivalent:*

(1) k is an internal ultrapower embedding.

(2) k is continuous at $\sup j[\lambda]$.

(3) k is continuous at $\text{cf}^M(\sup j[\lambda])$.

Proof. (1) implies (2): Since $\sup j[\lambda]$ carries no fine countably complete ultrafilter in M, every elementary embedding of M that is close to M is continuous at $\sup j[\lambda]$. In particularly, every internal ultrapower embedding of M is continuous at $\sup j[\lambda]$.

(2) implies (1): Let $i = k \circ j$. Then $\sup i[\lambda] = \sup k[\sup j[\lambda]] = k(\sup j[\lambda])$ since k is continuous at $\sup j[\lambda]$. It follows that $\sup i[\lambda]$ carries no fine countably complete ultrafilter in N. Therefore by Theorem 7.3.13, there is an internal ultrapower embedding $k' : M \to N$ such that $k' \circ j = i$.

We claim $k' = k$. First of all, $k' \circ j = k \circ j$. In other words, $k' \restriction j[V] = k \restriction j[V]$. Moreover since k' is M-internal $k'(\sup j[\lambda]) = \sup i[\lambda] = k(\sup j[\lambda])$. But $M = H^M(j[V] \cup \{\text{id}_{\mathscr{K}_\lambda}\}) = H^M(j[V] \cup \{\sup j[\lambda]\})$ since $\mathscr{K}_\lambda$ is weakly normal. Since we have shown $k' \restriction j[V] \cup \{\sup j[\lambda]\} = k \restriction j[V] \cup \{\sup j[\lambda]\}$, it follows that $k' = k$.

Since k' is an internal ultrapower embedding, so is k, as desired.

The equivalence of (2) and (3) is trivial (and does not require UA). ☐

Indecomposable ultrafilters were introducted by Príkry in order to study the large cardinal properties of countably incomplete ultrafilters. Although countably incomplete ultrafilters barely arise in this monograph, the concept of indecomposability turns out to be relevant to the structure of Fréchet cardinals.

Definition 7.3.15. Suppose M is a transitive model of ZFC and U is an M-ultrafilter on X. Suppose δ is an M-cardinal. Then U is *δ-indecomposable* if for any partition $\langle X_\alpha : \alpha < \delta\rangle \in M$ of X, there is some $S \subseteq \delta$ in M with $|S|^M < \delta$ and $\bigcup_{\alpha \in S} X_\alpha \in U$.

In this section, we are only concerned with regular Fréchet cardinals, and for regular cardinals, indecomposability has a simpler characterization, which we leave to the reader to verify:

Lemma 7.3.16. *If M is a transitive model of* ZFC*, δ is an M-regular cardinal, and U is an M-ultrafilter. Then the following are equivalent:*

(1) U is δ-indecomposable.

(2) For any decreasing sequence $\langle A_\alpha\rangle_{\alpha<\delta} \in M$ of U-large sets, $\bigcap_{\alpha<\delta} A_\alpha \in U$.

(3) $j^M_U(\delta) = \sup j^M_U[\delta]$. ☐

As a corollary of Theorem 7.3.14, every λ-indecomposable ultrafilter is internal to $\mathscr{K}_\lambda$:

Corollary 7.3.17 (UA). *Suppose λ is a regular Fréchet cardinal. Then every countably complete λ-indecomposable ultrafilter is internal to $\mathscr{K}_\lambda$. In particular, if D is a countably complete ultrafilter such that $\lambda_D < \lambda$, then $D \sqsubset \mathscr{K}_\lambda$.*

Proof. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. To show $D \sqsubset \mathscr{K}_\lambda$, we need to show that $j_D \restriction M$ is an internal ultrapower embedding of M_U. By Lemma 5.5.11, $j_D \restriction M$ is an ultrapower embedding. Since D is λ-indecomposable, j_D is continuous at all ordinals of cofinality λ, and in particular, j_D is continuous at $\sup j[\lambda]$. Thus $j_D \restriction M$ is an ultrapower embedding of M that is continuous at $\sup j[\lambda]$, and it follows from Theorem 7.3.14 that $j_D \restriction M$ is an internal ultrapower embedding of M, as desired. ☐

This fact is reminiscent of Corollary 4.3.28, the theorem that analyzes which ultrafilters lie Mitchell below a Dodd sound ultrafilter. In fact, we will show that $\mathscr{K}_\lambda$ gives rise to a supercompact ultrapower precisely by leveraging the fact that so many ultrapower embeddings are internal to it. (See Section 7.3.5, Section 7.3.6, and especially Proposition 7.3.33.)

Lemma 7.3.16 (3), we therefore obtain the following combinatorial characterization of the countably complete M-ultrafilters that belong to M when M is the ultrapower of the universe by a Ketonen ultrafilter on a regular cardinal:

Theorem 7.3.18 (UA). *Suppose λ is a regular Fréchet cardinal. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Let $\delta = \mathrm{cf}^M(\sup j[\lambda])$. Suppose U is a countably complete M-ultrafilter. Then U belongs to M if and only if U is δ-indecomposable. In particular, if U is a countably complete M-ultrafilter on a cardinal $\gamma < \delta$, then $U \in M$.* ☐

We finally prove Theorem 7.3.13:

Proof of Theorem 7.3.13. (1) implies (2): First, $k(\sup j[\lambda])$ carries no fine countably complete ultrafilter in N by elementarity, since $\sup j[\lambda]$ carries no fine countably complete ultrafilter in M. Note also that $k : M \to N$ is continuous at $\sup j[\lambda]$ since $\sup j[\lambda]$ carries no countably complete fine ultrafilter in M. Therefore $k(\sup j[\lambda]) = \sup k \circ j[\lambda] = \sup i[\lambda]$. Hence $\sup i[\lambda]$ carries no fine countably complete ultrafilter in N.

(2) implies (1): Let $(e, h) : (M, N) \to P$ be an internal ultrapower comparison of (j, i). Then

$$e(\sup j[\lambda]) = \sup e \circ j[\lambda] = \sup h \circ i[\lambda] = h(\sup i[\lambda])$$

The theorem is now an immediate consequence of Corollary 5.2.8: since $\mathrm{id}_U = \sup j[\lambda]$ and $e(\sup j[\lambda]) \in h[N]$, there is an internal ultrapower embedding $k : M \to N$ such that $k \circ j = i$.

The equivalence of (2) and (3) is trivial (and does not require UA). ☐

7.3.5 Independent families and the Hamkins properties

Can one force to create new large cardinals? The Lévy-Solovay Theorem [1] establishes roughly that no forcing of cardinality less than κ can alter the large cardinal properties of κ. For larger forcings, the question becomes quite interesting. The earliest example of a forcing that creates large cardinals is due to Kunen, who showed that it is consistent that there is a forcing that makes a measurable cardinal out of a cardinal that is not even weakly compact. Woodin's Σ_2-Resurrection Theorem ([10], Theorem 2.5.10) yields even more striking examples: for example, if there is a proper class of Woodin cardinals and a single huge cardinal, then arbitrarily large cardinals can be forced to be huge cardinals.

In connection with this question, Hamkins isolated two closure properties that ensure that an inner model M inherits all large cardinal properties of the universe of sets: the *approximation and cover properties*, or collectively the *Hamkins properties*. For many forcing extensions $V[G]$, the universe V satisfies the Hamkins properties inside $V[G]$, and therefore the large cardinals of $V[G]$ must already exist in V.

Somewhat unexpectedly, the Hamkins properties have turned out to be relevant outside forcing, in the domain of inner model theory. Woodin [12] showed that any inner model that inherits a supercompact cardinal κ from the universe of sets in a natural way necessarily satisfies the Hamkins properties at κ, and therefore inherits all large cardinals from the ambient universe. Such inner models are called *weak extender models* for the supercompactness of κ. If there is a canonical inner model with a supercompact cardinal, it seems likely to be a weak extender model, and therefore Woodin conjectures that this canonical model is the *ultimate inner model*, a canonical inner model that satisfies all true large cardinal axioms.

As it turns out, the Hamkins properties arise in the analysis of supercompactness under UA. We are concerned here with whether certain ultrapowers of the universe satisfy local instances of the Hamkins properties. Our goal is to show that the ultrapower of the universe by $\mathscr{K}_\lambda$ is closed under λ-sequences. All we know so far is that this ultrapower absorbs many countably complete ultrafilters (Theorem 7.3.18). We prove a converse to Hamkins and Woodin's absoluteness theorems for models with the Hamkins properties that implies that any inner model that inherits enough ultrafilters from the ambient universe must satisfy the Hamkins properties. In our context, this will lead to a proof that the ultrapower of the universe by $\mathscr{K}_\lambda$ is closed under λ-sequences, at least when λ is a successor cardinal.

The ultrapowers we consider do not satisfy the (relevant) Hamkins properties in full, but rather satisfy local versions of these properties:

Definition 7.3.19. Suppose M is an inner model, κ is a cardinal, and λ is an ordinal.

- M has the *κ-cover property at λ* if every $\sigma \in P_\kappa(\lambda)$ there is some $\tau \in P_\kappa(\lambda) \cap M$ with $\sigma \subseteq \tau$.

- M has the *κ-approximation property at λ* if any $A \subseteq \lambda$ with $A \cap \sigma \in M$ for all $\sigma \in P_\kappa(\lambda) \cap M$ is an element of M.

We say M has the *κ-cover property* if M has the κ-cover property at every ordinal, and M has the *κ-approximation property* if M has the κ-approximation property at every ordinal.

In this section, we identify necessary and sufficient conditions for the κ-cover and approximation properties that involve the absorption of filters. We are working in slightly more generality than we will need, but we think the results are quite interesting and hopefully lead to a clearer exposition than would arise by working in a more specific case.

The condition equivalent to the cover property essentially comes from Woodin's proof of the cover property for weak extender models:

Proposition 7.3.20. *Suppose M is an inner model, κ is a regular cardinal, and λ is an ordinal. Then M has the κ-cover property at λ if and only if there is a κ-complete fine filter on $P_\kappa(\lambda)$ that concentrates on M.*

Proof. First assume there is a κ-complete fine filter $\mathscr{F}$ on $P_\kappa(\lambda)$ that concentrates on M. Fix $\sigma \in P_\kappa(\lambda)$, and we will find $\tau \in P_\kappa(\lambda) \cap M$ such that $\sigma \subseteq \tau$. For each $\alpha < \lambda$, let

$$A_\alpha = \{\tau \in P_\kappa(\lambda) : \alpha \in \tau\}$$

so that $A_\alpha \in \mathscr{F}$ by the definition of a fine filter. Then suppose $\sigma \in P_\kappa(\lambda)$. The set

$$\{\tau \in P_\kappa(\lambda) : \sigma \subseteq \tau\} = \bigcap_{\alpha \in \sigma} \{A_\alpha : \alpha \in \sigma\} \in \mathscr{F}$$

since $\mathscr{F}$ is κ-complete. Since $\mathscr{F}$ concentrates on M, $\{\tau \in P_\kappa(\lambda) : \sigma \subseteq \tau\} \cap M \in \mathscr{F}$, and in particular this set is nonempty. Any τ that belongs to this set satisfies $\tau \in P_\kappa(\lambda) \cap M$ and $\sigma \subseteq \tau$, as desired.

Conversely, assume M has the κ-cover property at λ. Let $\mathscr{B} = \{A_\alpha \cap M : \alpha < \lambda\}$. Then $\mathscr{B}$ is a κ-complete filter base: for any $S \subseteq \mathscr{B}$ with $|S| < \kappa$, we have $S = \{A_\alpha \cap M : \alpha \in \sigma\}$ for some $\sigma \in P_\kappa(\lambda)$, and so fixing $\tau \in P_\kappa(\lambda) \cap M$ such that $\sigma \subseteq \tau$, we have $\tau \in \bigcap_{\alpha \in \sigma}(A_\alpha \cap M) = \bigcap S$. Therefore $\mathscr{B}$ extends to a κ-complete filter $\mathscr{G}$. Let

$$\mathscr{F} = \mathscr{G} \restriction P_\kappa(\lambda) = \{A \subseteq P_\kappa(\lambda) : A \cap M \in \mathscr{G}\}$$

be the canonical extension of $\mathscr{G}$ to an filter on $P_\kappa(\lambda)$. Then $\mathscr{F}$ is κ-complete and concentrates on M. Moreover, $A_\alpha \in \mathscr{F}$ for all $\alpha < \lambda$, so $\mathscr{F}$ is fine. Thus we have produced a κ-complete fine filter on $P_\kappa(\lambda)$ that concentrates on M, as desired. $\square$

Before stating our general characterization of the approximation property, we state its most important corollary:

Theorem 7.3.21. *Suppose κ is strongly compact and M is an inner model with the κ-cover property. Then M has the κ-approximation property if and only if every κ-complete M-ultrafilter belongs to M.*

We will actually prove a local version of this theorem that requires no large cardinal assumptions. The locality of this theorem is important in our analysis of the ultrafilters $\mathscr{K}_\lambda$. For the statement, we need use the following definition:

Definition 7.3.22. Suppose X is a set and Σ is an algebra of subsets of X. A set $U \subseteq \Sigma$ is said to be an *ultrafilter over* Σ if U is closed under intersections and for any $A \in \Sigma$, $A \in U$ if and only if $X \setminus A \notin U$. An ultrafilter U over Σ is said to be κ-complete if for any $\sigma \in P_\kappa(U)$, $\bigcap \sigma \neq \emptyset$.

What we call an ultrafilter over Σ is commonly referred to as an *ultrafilter on the Boolean algebra* Σ, but we are being a bit pedantic: we do not want to confuse this with an ultrafilter with underlying set Σ, which in our terminology is a family of subsets of Σ rather than a subset of Σ. Notice that for us a κ-complete ultrafilter over Σ is the same thing as an ultrafilter over Σ that is a κ-complete filter base. (It is not the same thing as being a κ-complete ultrafilter on the Boolean algebra Σ.)

Theorem 7.3.23. *Suppose M is an inner model, κ is a regular cardinal, λ is an M-cardinal, and M has the κ-cover property at λ. Then the following are equivalent:*

(1) M has the κ-approximation property at λ.

(2) Suppose $\Sigma \in M$ is an algebra of sets of M-cardinality λ. Then every κ-complete ultrafilter over Σ belongs to M.

To simplify notation, we use the following lemma (analogous to Lemma 7.2.6) characterizing the approximation property at λ:

Lemma 7.3.24. *Suppose M is an inner model, κ is a cardinal, and λ is an M-cardinal. Then the following are equivalent:*

(1) M has the κ-approximation property at λ

(2) For all $\Sigma \in M$ such that $|\Sigma|^M \leq \lambda$, for all $B \subseteq \Sigma$ such that $B \cap \sigma \in M$ for all $\sigma \in P_\kappa(\Sigma) \cap M$, $B \in M$.

(3) For some $\Sigma \in M$ such that $|\Sigma|^M = \lambda$, for all $B \subseteq \Sigma$ such that $B \cap \sigma \in M$ for all $\sigma \in P_\kappa(\Sigma) \cap M$, $B \in M$. □

The following notation will be convenient (although of course it is a bit ambiguous):

Definition 7.3.25. Suppose X is a set and σ is a family of subsets of X. Then the *dual of σ in X* is the family $\sigma^* = \{X \setminus A : A \in \sigma\}$.

Despite the notation, σ^* depends implicitly on the underlying set X of σ.

Definition 7.3.26. Suppose κ is a cardinal and X is a set. A family Γ of subsets of X is *κ-independent* if for any disjoint sets $\tau_0, \tau_1 \in P_\kappa(\Gamma)$, $\bigcap \tau_0 \cap \bigcap \tau_1^* \neq \emptyset$.

Equivalently, Γ is κ-independent if for any disjoint sets $X, Y \subseteq \Gamma$, the collection $X \cup Y^*$ is a κ-complete filter base. Note that a κ-complete family of subsets of X is never an algebra of sets, since if $A \in \Gamma$, then $X \setminus A \notin \Gamma$.

Theorem 7.3.27 (Hausdorff)**.** *Suppose κ and λ are cardinals. Then there is a κ-independent family of subsets of $X = \{(\sigma, s) : \sigma \in P_\kappa(\lambda)$ and $s \in P_\kappa(P(\sigma))\}$ of cardinality 2^λ.*

Proof. Define $f : P(\lambda) \to P(X)$ by

$$f(A) = \{(\sigma, s) \in X : \sigma \cap A \in s\}$$

Let $\Gamma = \text{ran}(f)$. Suppose $\tau_0, \tau_1 \in P_\kappa(P(\lambda))$ are disjoint. We will prove that the set

$$S = \bigcap f[\tau_0] \cap \bigcap f[\tau_1]^*$$

is nonempty. This simultaneously shows that f is injective and Γ is κ-independent. Therefore Γ is a κ-independent family of cardinality 2^λ.

Let $\sigma \in P_\kappa(\lambda)$ be large enough that $\sigma \cap A_0 \neq \sigma \cap A_1$ for any $A_0 \in \tau_0$ and $A_1 \in \tau_1$. Let

$$s = \{\sigma \cap A : A \in \tau_0\}$$

We claim that $(\sigma, s) \in S$.

First we show that $(\sigma, s) \in \bigcap f[\tau_0]$. Suppose $A \in \tau_0$. We will show that $(\sigma, s) \in f(A)$. By the definition of s, since $A \in \tau_0$, $\sigma \cap A \in s$. Therefore by the definition of f, $(\sigma, s) \in f(A)$, as desired. This shows $(\sigma, s) \in \bigcap f[\tau_0]$.

Next we show that $(\sigma, s) \in \bigcap f[\tau_1]^*$. Suppose $B \in \tau_1$, and we will show that $(\sigma, s) \in X \setminus B$. By the choice of σ, $\sigma \cap B \neq \sigma \cap A$ for any $A \in \tau_0$. Therefore by the definition of s, $\sigma \cap B \notin s$. Finally, by the definition of f, it follows that $(\sigma, s) \notin f(B)$, or in other words, $(\sigma, s) \in X \setminus B$. Hence $(\sigma, s) \in \bigcap f[\tau_1]^*$.

Now $(\sigma, s) \in \bigcap f[\tau_0]$ and $(\sigma, s) \in \bigcap f[\tau_1]^*$ so $(\sigma, s) \in S$. Thus S is nonempty, which completes the proof. $\square$

Applying the Axiom of Choice, Hausdorff's theorem implies the existence of κ-independent sets that are as large as possible:

Corollary 7.3.28 (Hausdorff)**.** *Suppose κ and λ are cardinals such that $\lambda^{<\kappa} = \lambda$. Then there is a κ-independent family of subsets of λ of cardinality 2^λ.*

Proof. Let $X = \{(\sigma, s) : \sigma \in P_\kappa(\lambda)$ and $s \in P_\kappa(P(\sigma))\}$. In other words,

$$X = \coprod_{\sigma \in P_\kappa(\lambda)} P_\kappa(P(\sigma))$$

Thus

$$|X| = |P_\kappa(\lambda)| \cdot \sup_{\sigma \in P_\kappa(\lambda)} |P_\kappa(P(\sigma))| = \lambda^{<\kappa} \cdot (2^{<\kappa})^{<\kappa} = \lambda^{<\kappa} = \lambda$$

By Theorem 7.3.27, there is a κ-independent family of subsets of X of cardinality 2^λ, and therefore there is a κ-independent family of subsets of λ of cardinality 2^λ. □

We now establish our characterization of the approximation property.

Proof of Theorem 7.3.23. (1) implies (2): Assume (1), and we will prove (2). Suppose $\Sigma \in M$ is an algebra of subsets of X of M-cardinality λ and U is a κ-complete ultrafilter over Σ. Fix $\sigma \in P_\kappa(\Sigma) \cap M$ and we will show that $\sigma \cap U \in M$. Since U is κ-complete,

$$S = \bigcap\{A : A \in \sigma \cap U\} \cap \bigcap\{X \setminus A : A \in \sigma \setminus U\}$$

is nonempty. Therefore fix $a \in X$ with $a \in S$. By the choice of a, $\sigma \cap U = \{A \in \sigma : a \in A\}$. Thus $\sigma \cap U \in M$.

By the κ-approximation property at λ (using Lemma 7.3.24), it follows that $U \in M$.

(2) implies (1): Fix $\Gamma \in M$ such that M satisfies that Γ is a κ-independent family of subsets of some set X and $|\Gamma|^M = \lambda$. Suppose $C \subseteq \Gamma$ is such that $C \cap \sigma \in M$ for all $\sigma \in P_\kappa(\Gamma) \cap M$. We will show that $C \in M$. This verifies the condition of Lemma 7.3.24 (3), and so implies that M satisfies the κ-approximation property at λ.

Let

$$\mathscr{B} = C \cup (\Gamma \setminus C)^*$$

We claim that $\mathscr{B}$ is a κ-complete filter base on X. Suppose $\sigma \in P_\kappa(\mathscr{B})$. We must show that $\bigcap \sigma \neq \emptyset$. Using the κ-cover property at λ, fix $\tau \in P_\kappa(\Gamma) \cap M$ such that $\sigma \subseteq \tau \cup \tau^*$.

By our assumption on C, $\tau \cap C \in M$. Let $\tau_0 = \tau \cap C$ and let $\tau_1 = \tau \setminus C = \tau \setminus \tau_0 \in M$. Since $\sigma \subseteq \mathscr{B} = C \cup (\Sigma \setminus C)^*$, we have $\sigma \subseteq \tau_0 \cup \tau_1^*$. Since Γ is κ-independent in M,

$$\bigcap \tau_0 \cap \bigcap \tau_1^* \neq \emptyset$$

But $\bigcap \tau_0 \cap \bigcap \tau_1^* = \bigcap(\tau_0 \cup \tau_1) \subseteq \bigcap \sigma$, and hence $\bigcap \sigma \neq \emptyset$, as desired. This shows $\mathscr{B}$ is a κ-complete filter base.

Let Σ be the algebra on X generated by Γ and let U be the ultrafilter over Σ generated by $\mathscr{B}$. Then U is κ-complete because $\mathscr{B}$ is κ-complete. Therefore $U \in M$ by our assumption on M. But $C = \Gamma \cap \mathscr{B} = \Gamma \cap U$, so $C \in M$, as desired. Thus M has the κ-approximation property at λ. □

The proof of Theorem 7.3.23 has the following corollary, which will be important going forward:

Proposition 7.3.29. *Suppose M is an inner model, κ is a cardinal, λ is an M-cardinal, and M has the κ-cover property at λ. Then the following are equivalent:*

(1) M has the κ-approximation property at λ.

(2) There is a κ-independent family Γ of M with M-cardinality λ such that every κ-complete ultrafilter over the algebra generated by Γ belongs to M. □

7.3.6 The strength and supercompactness of $\mathscr{K}_\lambda$

Definition 7.3.30. For any Fréchet cardinal λ, κ_λ denotes the completeness of $\mathscr{K}_\lambda$.

In other words, $\kappa_\lambda = \text{crit}(j_{\mathscr{K}_\lambda})$. In Section 7.4, we will prove the following theorem:

Theorem 7.3.31 (UA)**.** *Suppose λ is a Fréchet cardinal that is either a successor cardinal or a strongly inaccessible cardinal. Then κ_λ is λ-strongly compact.*

This is one of the harder theorems of this chapter, so we will just work under this hypothesis for a while. The following theorem begins to show why it is a useful assumption:

Theorem 7.3.32 (UA)**.** *Suppose λ is a regular Fréchet cardinal and κ_λ is λ-strongly compact. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Then $P(\gamma) \subseteq M$ for all $\gamma < \lambda$.*

Because we will occasionally need to use this argument in a more general context, let us instead prove the following:

Proposition 7.3.33. *Suppose $\kappa \leq \gamma$ are cardinals, κ is γ-strongly compact, and M is an inner model that is closed under $<\kappa$-sequences. Assume every κ-complete ultrafilter on γ is amenable to M. Then $P(\gamma) \subseteq M$. Moreover if* $\text{cf}(\gamma) \geq \kappa$ *then $P(\eta) \subseteq M$ for all $\eta \leq 2^\gamma$ such that κ is η-strongly compact.*

Proof. We may assume by induction that $P(\alpha) \subseteq M$ for all ordinals $\alpha < \gamma$. Let $\nu = \text{cf}(\gamma)$.

Assume first that $\nu < \kappa$. Let $\langle \gamma_\alpha : \alpha < \nu \rangle \in M$ be an increasing sequence cofinal in γ. Suppose $A \subseteq \gamma$. Let $A_\alpha = A \cap \gamma_\alpha$, so $A_\alpha \in M$ for all $\alpha < \nu$ by our inductive assumption. Then $\langle A_\alpha : \alpha < \nu \rangle \in M$ since M is closed under $<\kappa$-sequences. Therefore $A = \bigcup_{\alpha<\nu} A_\alpha \in M$. It follows that $P(\gamma) \subseteq M$, which finishes the proof in this case.

Therefore we may assume that $\nu \geq \kappa$. We claim that κ is γ-strongly compact in M. Fix an ordinal $\alpha \in [\kappa, \gamma]$ such that $\text{cf}^M(\alpha) \geq \kappa$. Then $\text{cf}(\alpha) \geq \kappa$ since M is closed under κ-sequences. Since κ is γ-strongly compact, there is a κ-complete fine ultrafilter U on α. But $U \cap M \in M$, so in M there is a fine κ-complete ultrafilter on α. In particular, every M-regular cardinal $\iota \in [\kappa, \lambda]$ carries a κ-complete ultrafilter in M, so by Theorem 7.2.15, κ is γ-strongly compact in M.

Therefore by Theorem 7.2.19, $(\gamma^{<\kappa})^M = \gamma$, so by Corollary 7.3.28, M satisfies that there is a κ-independent family of subsets of γ of cardinality $(2^\gamma)^M$.

Let $\Gamma \in M$ be such that M satisfies that Γ is a κ-independent family of subsets of γ of cardinality γ. Let Σ be the algebra of subsets of γ generated by Γ. If U_0 is a κ-complete ultrafilter over Σ, then U_0 extends to a κ-complete ultrafilter U on γ by Theorem 7.2.10, since κ is γ-strongly compact and U_0 is a κ-complete filter base of cardinality γ. It follows from Proposition 7.3.29 that M

has the κ-approximation property at γ. Since M is closed under $<\kappa$-sequences, it follows from this that $P(\gamma) \subseteq M$.

We can now find larger independent families: since $P(\gamma) \subseteq M$, $(2^\gamma)^M \geq 2^\gamma$, and in particular, M satisfies that there is a κ-independent family of subsets of γ of cardinality $(2^\gamma)^V$.

Assume finally that $\delta \leq 2^\gamma$ is a cardinal and κ is δ-strongly compact. Then let $\Gamma \in M$ be a κ-independent family of subsets of γ in M with cardinality δ. As in the previous paragraph, any κ-complete ultrafilter over the algebra generated by Γ belongs to M, so M has the κ-approximation property at δ by Proposition 7.3.29. Since M is closed under $<\kappa$-sequences, it follows from this that $P(\delta) \subseteq M$. $\square$

Proof of Theorem 7.3.32. By Theorem 7.3.18, every countably complete M-ultrafilter U on $\gamma < \lambda$ belongs to M. Therefore if $\gamma < \lambda$, our strong compactness assumption on κ_λ implies the hypotheses of Proposition 7.3.33 hold at γ, and so $P(\gamma) \subseteq M$. $\square$

Having proved that $\mathscr{K}_\lambda$ has some strength, let us now turn to the supercompactness properties of $\mathscr{K}_\lambda$.

Theorem 7.3.34. *Suppose λ is a regular Fréchet cardinal and κ_λ is λ-strongly compact. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Then*

- *j is λ-tight.*
- *j is γ-supercompact for all $\gamma < \lambda$.*

In other words, $M^\gamma \subseteq M$ for all $\gamma < \lambda$ and M has the $\leq\lambda$-cover property.

Proof. Suppose towards a contradiction that j is not λ-tight. By Theorem 7.2.12, it follows that $\delta = \mathrm{cf}^M(\sup j[\lambda]) > \lambda$. By Theorem 7.3.18, any countably complete M-ultrafilter U on λ belongs to M. But then by Proposition 7.3.33, $P(\lambda) \subseteq M$. But then $\mathscr{K}_\lambda$ itself is a countably complete M-ultrafilter on λ, so $\mathscr{K}_\lambda \in M$. This contradicts the irreflexivity of the Mitchell order (Lemma 4.2.38).

Now that we know j is λ-tight, let us show that j is γ-supercompact for all $\gamma < \lambda$. We may assume by induction that j is $<\gamma$-supercompact. Then if γ is singular, it is easy to see that j is γ-supercompact. Therefore assume γ is regular. Let $\gamma' = \mathrm{cf}^M(\sup j[\gamma])$. Then $\gamma' \leq \lambda$ since j is λ-tight and hence j is (γ, λ)-tight. Since $\gamma < \lambda$, in fact $\gamma' < \lambda$. Thus $P(\gamma') \subseteq M$ by Theorem 7.3.32. By Theorem 7.2.12, j is (γ, γ')-tight, so fix $A \in M$ with $|A|^M = \gamma'$ and $j[\gamma] \subseteq A$. Note that since $|A|^M = \gamma'$, $P(A) \subseteq M$. Therefore $j[\gamma] \subseteq M$. Therefore j is γ-supercompact, as desired.

That $M^\gamma \subseteq M$ for all $\gamma < \lambda$ is an immediate consequence of Corollary 4.2.20. That M has the $\leq\lambda$-cover property is an immediate consequence of Lemma 7.2.7. $\square$

Finally, if λ is not a strongly inaccessible cardinal, we can show that $j_{\mathscr{K}_\lambda}$ is precisely as supercompact as it should be:

Theorem 7.3.35 (UA). *Suppose λ is a regular Fréchet cardinal and κ_λ is λ-strongly compact. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. If λ is not strongly inaccessible then j is λ-supercompact.*

Proof. Let $\kappa = \kappa_\lambda$ for ease of notation. We split into two cases:

Case 1. *For some $\gamma < \lambda$ with* $\text{cf}(\gamma) \geq \kappa$, $2^\gamma \geq \lambda$.

Proof in Case 1. Since $\gamma < \lambda$, by Theorem 7.3.18 every countably complete M-ultrafilter on γ belongs to M. Since $\text{cf}(\gamma) \geq \kappa$, $\lambda \leq 2^\gamma$, and κ is λ-strongly compact, we can therefore apply the second part of Proposition 7.3.33 to conclude that $P(\lambda) \subseteq M$.

Given that j is λ-tight by Theorem 7.3.34, it now follows easily that j is λ-supercompact: fix $A \in M$ with $|A|^M = \lambda$ and $j[\lambda] \subseteq A$; then $P(A) \subseteq M$ so $j[\lambda] \in M$, as desired. □

Case 2. *For all $\gamma < \lambda$ with* $\text{cf}(\gamma) \geq \kappa$, $2^\gamma < \lambda$.

Proof in Case 2. Since λ is not inaccessible, there is some $\eta < \lambda$ such that $2^\eta \geq \lambda$. Let $\gamma = \eta^{<\kappa}$. Then $\text{cf}(\gamma) \geq \kappa$ and $2^\gamma \geq 2^\eta \geq \lambda$. Therefore by our case hypothesis, $\lambda \leq \gamma$. By Theorem 7.3.34, j is η-supercompact. By Lemma 4.2.24, j is $\eta^{<\kappa}$-supercompact. Therefore j is λ-supercompact as desired. □

Thus in either case j is λ-supercompact, which completes the proof. □

7.4 Fréchet cardinals

7.4.1 The next Fréchet cardinal

Given the results of Section 7.3.6, to analyze $\mathscr{K}_\lambda$ when λ is a regular Fréchet cardinal, it would be enough to show that its completeness κ_λ is λ-strongly compact. The following easy generalization of Ketonen's Theorem (Theorem 7.2.15) reduces this to the analysis of Fréchet cardinals in the interval $[\kappa_\lambda, \lambda]$:

Proposition 7.4.1. *Suppose λ is a regular Fréchet cardinal. Suppose $j : V \to M$ is the ultrapower of the universe by a Ketonen ultrafilter U on λ. Suppose $\kappa \leq \lambda$ is a cardinal and every regular cardinal in the interval $[\kappa, \lambda]$ is Fréchet. Then j is (λ, δ)-tight for some $\delta < j(\kappa)$. In particular, if $\kappa = \text{crit}(j)$ then κ is λ-strongly compact.*

Proof. Since U is Ketonen, the M-cardinal $\delta = \text{cf}^M(\sup j[\lambda])$ is not Fréchet in M. Therefore by elementarity $\delta \notin j([\kappa, \lambda])$. Since $\delta < j(\lambda)$, we must have $\delta < j(\kappa)$. Theorem 7.2.12 implies that j is (λ, δ)-tight, proving the proposition. □

Suppose λ is a regular Fréchet cardinal. To obtain that every regular cardinal in the interval $[\kappa_\lambda, \lambda)$ is Fréchet, it actually suffices to show that every *successor* cardinal in the interval $(\kappa_\lambda, \lambda]$ is Fréchet. (See Corollary 7.4.5.) Our approach to this problem is as follows. Fix an ordinal $\gamma \in [\kappa_\lambda, \lambda)$. We must analyze the following cardinal:

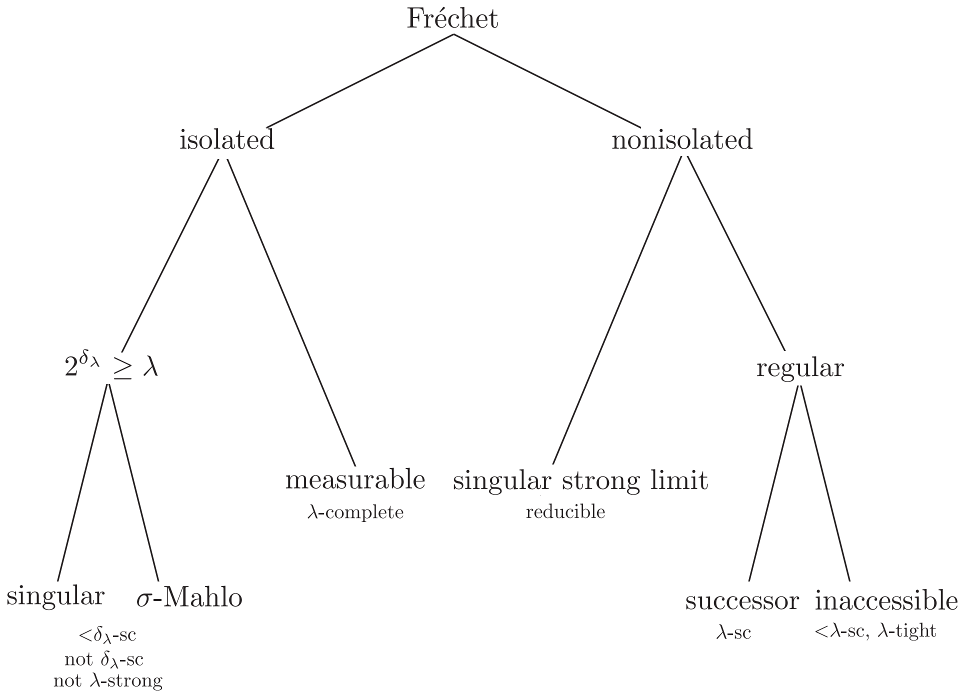

Figure 7.2: Types of Fréchet cardinals

Definition 7.4.2. Suppose γ is an ordinal. Then γ^σ denotes the least Fréchet cardinal strictly greater than γ.

We will attempt to use the fact that γ lies in the interval $[\kappa_\lambda, \lambda)$ to show that $\gamma^\sigma = \gamma^+$. Since γ^σ is Fréchet by definition, this would show γ^+ is Fréchet. In this way, we we would establish that every successor cardinal in the interval $(\kappa_\lambda, \lambda]$ is Fréchet, as desired.

The following classic result of Prikry [34] shows in particular that there is nontrivial structure to the Fréchet cardinals even if we do not assume UA:

Theorem 7.4.3 (Prikry)**.** *Suppose λ is a cardinal and U is a λ^+-decomposable ultrafilter. Either U is* $\text{cf}(\lambda)$*-decomposable or there is some $\kappa < \lambda$ such that U is (κ, λ^+)-regular.* □

A key part of our analysis of Fréchet cardinals is the following generalization of Theorem 7.4.3:

Proposition 7.4.4. *Suppose η is a cardinal such that η^+ is Fréchet. Either η is Fréchet or η is a singular cardinal and all sufficiently large regular cardinals below η are Fréchet.*

Proof. Suppose $\gamma^\sigma = \eta^+$. We will show that either η is Fréchet or η is a limit of Fréchet cardinals. Fix a countably complete uniform ultrafilter U on η^+, and let $j : V \to M$ be the ultrapower of the universe by U. Let

$$U_* = \{A \in j(P(\eta^+)) : j^{-1}[A] \in U\}$$

Thus U_* is an M-ultrafilter. Note that $\lambda_{U_*} < j(\eta^+)$ since for example $\sup j[\eta^+] \in U_*$. Thus $\lambda_{U_*} \leq j(\eta)$.

The proof now splits into two cases:

Case 1. $\lambda_{U_*} \geq \sup j[\eta]$.

Proof in Case 1. Let $\lambda = \lambda_{U_*}$. Then $\sup j[\eta] \leq \lambda \leq j(\eta)$. Let W_* be an M-ultrafilter on $j(\eta)$ that concentrates on λ and is Rudin-Keisler equivalent to U_*. In other words, there is a set $X \in U_*$ and a bijection $f : \lambda \to X$ with $f \in M$ such that $W_* = \{f^{-1}[A] : A \in U_*\}$. All we need about W_* is that $\lambda_{W_*} = \lambda \geq \sup j[\eta]$. Let

$$W = j^{-1}[W_*]$$

Then W is a countably complete ultrafilter on η.

We claim that W is uniform. Suppose $A \in W$. Then $j(A) \in W_*$ so $|j(A)|^M = \lambda$. In particular, since $\lambda \geq \sup j[\eta]$, for any cardinal $\kappa < \eta$, $|j(A)|^M > j(\kappa)$, and therefore $|A| > \kappa$. It follows that $|A| \geq \eta$. Thus W is uniform. □

Case 2. $\lambda_{U_*} < \sup j[\eta]$.

Proof in Case 2. Fix $\kappa < \eta$ and $B \in U_*$ such that letting $\delta = |B|^M$, we have $\delta < j(\kappa)$. Let $A = j^{-1}[B]$. Then $A \in U$ so $|A| = \eta^+$ since U is a uniform ultrafilter on η^+. Since $j[A] \subseteq B$, it follows that j is (η^+, δ)-tight.

We claim that j is discontinuous at every regular cardinal ι in the interval $[\kappa, \eta^+]$. To see this, note that $j(\iota) > \delta$ is a regular cardinal of M. On the other hand, $j[\iota]$ is contained in a set $C \in M$ such that $|C|^M \leq \delta$ since j is (ι, δ)-tight. Therefore C is not cofinal in $j(\iota)$, and hence neither is $j[\iota]$. It follows that j is discontinuous at ι.

Since j is discontinuous at every regular cardinal in the interval $[\kappa, \eta^+]$, which contains η, it follows that either η is a regular Fréchet cardinal or η is a singular cardinal and all sufficiently large regular cardinals below η are Fréchet. □

Thus in either case, the conclusion of the proposition holds. □

An interesting feature of Proposition 7.4.4 is that it *does not* seem to show that every η^+-decomposable ultrafilter U is either η-decomposable or ι-decomposable for all sufficiently large $\iota < \eta$. Instead the proof shows that this is true of U^2. (Under UA, we can in fact prove that every η^+-decomposable countably complete ultrafilter U is either η-decomposable or ι-decomposable for all sufficiently large $\iota < \eta$.)

Proposition 7.4.4 has two important consequences. The first is our claim above that one need only show that all successor cardinals in $[\kappa_\lambda, \lambda]$ are Fréchet to conclude that all regular cardinals in $[\kappa_\lambda, \lambda]$ are. (This is really just a consequence of Theorem 7.4.3.)

Corollary 7.4.5. *Suppose $\kappa \leq \lambda$ are cardinals and every successor cardinal in the interval $(\kappa, \lambda]$ is Fréchet. Then every regular cardinal in the interval $[\kappa, \lambda)$ is Fréchet.*

Proof. Suppose ι is a regular cardinal in the interval $[\kappa, \lambda)$. Then $\iota^+ \in (\kappa, \lambda]$, so ι^+ is a Fréchet cardinal. Therefore ι is a Fréchet cardinal by Proposition 7.4.4. □

The consequence of Proposition 7.4.4 that is ultimately most important here is a constraint on the σ-operation:

Corollary 7.4.6. *Suppose γ is an ordinal and γ^σ is a successor cardinal. Then $\gamma^\sigma = \gamma^+$.*

Proof. Suppose towards a contradiction that $\gamma^\sigma = \eta^+$ for some cardinal $\eta > \gamma$. Since η^+ is Fréchet, by Proposition 7.4.4, η is either Fréchet or a limit of Fréchet cardinals. Either way, there is a Fréchet cardinal in the interval (γ, η^+). But the definition of γ^σ implies that there are no Fréchet cardinals in (γ, γ^σ). This is a contradiction.

Thus $\gamma^\sigma = \eta^+$ for some cardinal $\eta \leq \gamma$. In other words, $\gamma^\sigma = \gamma^+$. □

The problematic cases in the analysis of the σ-operation therefore occur when γ^σ is a limit cardinal:

Definition 7.4.7. A cardinal λ is *isolated* if the following hold:

- λ is Fréchet.
- λ is a limit cardinal.
- λ is not a limit of Fréchet cardinals.

By Proposition 7.4.4, λ is isolated if and only if $\lambda = \gamma^\sigma$ for some ordinal γ such that $\gamma^+ < \lambda$. Our analysis of Fréchet cardinals would be essentially complete if we could prove the following conjecture:

Conjecture 7.4.8 (UA)**.** *A cardinal λ is isolated if and only if λ is a measurable cardinal, λ is not a limit of measurable cardinals, and no cardinal $\kappa < \lambda$ is λ-supercompact.*

Proposition 7.5.4 below shows that Conjecture 7.4.8 is a consequence of UA + GCH, so to some extent this problem is solved in the most important case. But assuming UA alone, we do not know how to rule out, for example, the existence of singular isolated cardinals. Enacting an analysis of isolated cardinals under UA that is as complete as possible allows us to prove our main results without cardinal arithmetic assumptions.

7.4.2 The strong compactness of κ_λ

In this section we will prove the following theorem:

Theorem 7.4.9 (UA). *Suppose λ is a nonisolated regular Fréchet cardinal. Then κ_λ is λ-strongly compact.*

This yields the following corollary, which gives a complete analysis of Fréchet successor cardinals:

Corollary 7.4.10 (UA). *Suppose λ is a Fréchet successor cardinal. Then κ_λ is λ-supercompact and in fact the ultrapower embedding associated to $\mathscr{K}_\lambda$ is λ-supercompact.*

Proof. This is an immediate consequence of Theorem 7.4.9 and Theorem 7.3.35. □

In general, we only obtain

Proposition 7.4.11 (UA). *Suppose λ is a nonisolated regular Fréchet cardinal. Then κ_λ is $<\lambda$-supercompact and λ-strongly compact. In fact, the ultrapower embedding associated to $\mathscr{K}_\lambda$ is $<\lambda$-supercompact and λ-tight.*

Proof. This is an immediate consequence of Theorem 7.4.9 and Theorem 7.3.34. □

As we have sketched above, the proof of Theorem 7.4.9 will follow from an analysis of Fréchet cardinals in the interval $[\kappa_\lambda, \lambda]$:

Lemma 7.4.12. *Suppose $\kappa \leq \lambda$ are cardinals and there are no isolated cardinals in the interval $(\kappa, \lambda]$. Suppose that for all $\gamma \in [\kappa, \lambda)$, there is a Fréchet cardinal in the interval $(\gamma, \lambda]$. Then every regular cardinal in the interval $[\kappa, \lambda)$ is Fréchet.*

Proof. Since λ is Fréchet, we need only show that every regular cardinal in the interval $[\kappa, \lambda)$ is Fréchet. By Corollary 7.4.5, for this it is enough to show that every successor cardinal in the interval $(\kappa, \lambda]$ is Fréchet. In other words, it suffices to show that for any ordinal $\gamma \in [\kappa, \lambda)$, γ^+ is Fréchet. Therefore fix $\gamma \in [\kappa, \lambda)$. By assumption, $\gamma^\sigma \in (\gamma, \lambda]$, so in particular γ^σ is not isolated. Therefore γ^σ is not a limit cardinal. It follows that γ^σ is a successor cardinal, so by Proposition 7.4.4, $\gamma^\sigma = \gamma^+$, as desired. □

Our goal now it to prove the following lemma:

Lemma 7.4.13 (UA). *Suppose λ is a Fréchet cardinal that is either regular or isolated. Then there are no isolated cardinals in the interval $[\kappa_\lambda, \lambda)$.*

Given this, we could complete the proof of Theorem 7.4.9 as follows:

Proof of Theorem 7.4.9 assuming Lemma 7.4.13. By Lemma 7.4.13, there are no isolated cardinals in the interval $[\kappa_\lambda, \lambda)$. Since λ is not isolated, there are no isolated cardinals in the interval $[\kappa_\lambda, \lambda]$. Therefore applying Lemma 7.4.12, every regular cardinal in the interval $[\kappa_\lambda, \lambda]$ is Fréchet. By Proposition 7.4.1, it follows that κ_λ is λ-strongly compact. □

We now proceed to the proof of Lemma 7.4.13. We will first need to improve our understanding of isolated cardinals. The first step is to provide some criteria that guarantee a cardinal's nonisolation:

Lemma 7.4.14. *Suppose η is a limit cardinal. Suppose U is a countably complete uniform ultrafilter on η. Suppose W is a countably complete ultrafilter such that j_W is discontinuous at η and $U \sqsubset W$. Then η is a limit of Fréchet cardinals.*

Proof. Let $i : V \to N$ be the ultrapower of the universe by W. Let

$$U_* = s_W(U) = \{B \in i(P(\lambda)) : i^{-1}[B] \in U\}$$

By Lemma 5.5.11, $U_* \in N$.

Case 1. $\lambda_{U_*} \geq \sup i[\eta]$

Proof in Case 1. Working in N, λ_{U_*} is a Fréchet cardinal λ with $\sup i[\eta] \leq \lambda < i(\eta)$. It follows that for any $\kappa < \eta$, N satisfies that there is a Fréchet cardinal strictly between $i(\kappa)$ and $i(\eta)$, and so by elementarity there is a Fréchet cardinal strictly between κ and η. It follows that η is a limit of Fréchet cardinals. □

Case 2. $\lambda_{U_*} < \sup i[\eta]$

Proof in Case 2. Fix $\kappa < \eta$ and $B \in U_*$ such that letting $\delta = |B|^N$, $\delta < i(\kappa)$. Let $A = i^{-1}[B]$. Then $A \in U$, so $|A| = \eta$ by the uniformity of U. Since $|A| = \eta$ and $i[A] \subseteq B$, i is (η, δ)-tight by Theorem 7.2.12. It follows that i is discontinuous at every regular cardinal in the interval $[\kappa, \eta]$. (See the proof of Proposition 7.4.4.) In particular, η is a limit of Fréchet cardinals. □

In either case, η is a limit of Fréchet cardinals, as desired. □

The second nonisolation lemma brings in a bit more of the theory of the internal relation:

Lemma 7.4.15 (UA)**.** *Suppose η is a Fréchet limit cardinal. Suppose there is a countably complete ultrafilter W such that $\mathscr{K}_\eta \sqsubset W$ but $W \not\sqsubset \mathscr{K}_\eta$. Then η is a limit of Fréchet cardinals.*

Proof. By Lemma 7.4.14, if j_W is discontinuous at η, then η is a limit of Fréchet cardinals. Therefore assume without loss of generality that j_W is continuous at η.

By the basic theory of the internal relation (Lemma 5.5.15), since $\mathscr{K}_\eta \sqsubset W$, the translation $t_W(\mathscr{K}_\eta)$ is equal to the pushforward $s_W(\mathscr{K}_\eta)$.

Since $W \not\sqsubset \mathscr{K}_\eta$, the theory of the internal relation (Lemma 5.5.15) implies that in M_W, $t_W(\mathscr{K}_\eta) <_\Bbbk j_W(\mathscr{K}_\eta)$. Since M_W satisfies that $j_W(\mathscr{K}_\eta)$ is the $<_\Bbbk$-least uniform ultrafilter on $j_W(\eta)$, it follows that

$$\lambda_{t_W(\mathscr{K}_\eta)} < j_W(\eta)$$

But $t_W(\mathscr{K}_\eta) = s_W(\mathscr{K}_\eta)$ and $j_W(\eta) = \sup j_W[\eta]$ by our assumption that j_W is continuous at η. Thus

$$\lambda_{s_W(\mathscr{K}_\eta)} < \sup j_W[\eta]$$

Fix $\kappa < \eta$ and $B \in s_W(\mathscr{K}_\eta)$ such that $\delta = |B|^{M_W} < j_W(\kappa)$. Let $A = j_W^{-1}[B]$. Then $A \in \mathscr{K}_\eta$, so $|A| = \eta$. Moreover $j_W[A] \subseteq B \in M_W$, so j_W is (η, δ)-tight. In particular, j_W is discontinuous at every regular cardinal in the interval $[\kappa, \eta]$. (See the proof of Proposition 7.4.4.) Therefore η is a limit of Fréchet cardinals. $\square$

Finally, we need a version of Theorem 7.3.14 that applies at singular cardinals.

We use a lemma that follows immediately from the ultrafilter sum construction:

Lemma 7.4.16. *Suppose U is a countably complete ultrafilter on a cardinal λ and U' is a countably complete M_U-ultrafilter with $\lambda_{U'} \leq j_U(\lambda)$. Then there is a countably complete ultrafilter W on λ such that $j_W = j_{U'}^{M_U} \circ j_U$.* $\square$

Proposition 7.4.17 (UA). *Suppose λ is an isolated cardinal. Then $\mathscr{K}_\lambda$ is λ-internal.*

Proof. Suppose D is a countably complete ultrafilter on a cardinal $\gamma < \lambda$. We will show $D \sqsubset \mathscr{K}_\lambda$. Since λ is isolated, by increasing γ, we may assume $\lambda = \gamma^\sigma$.

Assume towards a contradiction that in M_D,

$$t_D(\mathscr{K}_\lambda) <_\Bbbk j_D(\mathscr{K}_\lambda)$$

Then $\lambda_{t_D(\mathscr{K}_\lambda)} < j_D(\lambda)$, and so since $\lambda_{t_D(\mathscr{K}_\lambda)}$ is a Fréchet cardinal of M_D, $\lambda_{t_D(\mathscr{K}_\lambda)} \leq j_D(\gamma)$. Therefore, there is an ultrafilter W on γ such that

$$j_W = j_{t_D(\mathscr{K}_\lambda)}^{M_D} \circ j_D = j_{t_{\mathscr{K}_\lambda}(D)}^{M_{\mathscr{K}_\lambda}} \circ j_{\mathscr{K}_\lambda}$$

It follows from the basic theory of the Rudin-Keisler order (Lemma 3.4.4) that $\mathscr{K}_\lambda \leq_{\text{RK}} W$, which contradicts that $\lambda_{\mathscr{K}_\lambda} = \lambda > \gamma \geq \lambda_W$.

Thus our assumption was false, and in fact, $j_D(\mathscr{K}_\lambda) \leq_\Bbbk t_D(\mathscr{K}_\lambda)$ in M_D. By the theory of the internal relation (Lemma 5.5.15), this implies that $D \sqsubset \mathscr{K}_\lambda$. $\square$

In Section 7.5.2, we prove a much stronger version of this theorem that constitutes a complete generalization of Theorem 7.3.13 to isolated cardinals.

Lemma 7.4.18 (UA)**.** *Suppose $\eta < \lambda$ is are Fréchet cardinals that are regular or isolated. Then either $\eta < \kappa_\lambda$ or $\mathscr{K}_\lambda \not\sqsubset \mathscr{K}_\eta$.*

Proof. By Theorem 7.3.14 or Proposition 7.4.17, $\mathscr{K}_\eta$ and $\mathscr{K}_\lambda$ are λ-internal.

Assume $\mathscr{K}_\lambda \sqsubset \mathscr{K}_\eta$. Note that we also have $\mathscr{K}_\eta \sqsubset \mathscr{K}_\lambda$ since $\mathscr{K}_\lambda$ is λ-uniform. By Proposition 5.5.24, $\eta < \kappa_\lambda$. ☐

We can finally prove Lemma 7.4.13.

Proof of Lemma 7.4.13. Suppose towards a contradiction that $\eta \in [\kappa_\lambda, \lambda)$ is isolated. Then by Lemma 7.4.18, $\mathscr{K}_\lambda \not\sqsubset \mathscr{K}_\eta$. Therefore by Lemma 7.4.15, η is a limit of Fréchet cardinals, contrary to the assumption that η is isolated. ☐

Since we will use it repeatedly, it is worth noting that κ_λ can be characterized in terms of isolated cardinals:

Lemma 7.4.19 (UA)**.** *Suppose λ is a nonisolated regular Fréchet cardinal. Then κ_λ is the supremum of the isolated cardinals less than λ.*

Proof. Let κ be the supremum of the isolated cardinals less than λ. By Lemma 7.4.13, there are no isolated cardinals in the interval $[\kappa_\lambda, \lambda)$, so $\kappa \leq \kappa_\lambda$.

Since there are no isolated cardinals in the interval $(\kappa, \lambda]$, Lemma 7.4.12 implies that every regular cardinal in the interval $[\kappa, \lambda]$ is Fréchet. By Proposition 7.4.1, it follows that $\kappa_\lambda \leq \kappa$. Thus $\kappa_\lambda = \kappa$, as desired. ☐

7.4.3 The least supercompact cardinal

In this subsection, we show how the theory of the internal relation can be used to characterize the least supercompact cardinal (and its local instantiations).

Theorem 7.4.20 (UA)**.** *Suppose λ is a successor cardinal and κ is the least (ω_1, λ)-strongly compact cardinal. Then κ is λ-supercompact. In fact, $\kappa = \kappa_\lambda$.*

Proof. Since κ is (ω_1, λ)-strongly compact, every regular cardinal in the interval $[\kappa, \lambda]$ is Fréchet. By Proposition 7.4.1, $\kappa_\lambda \leq \kappa$. By Corollary 7.4.10, κ_λ is λ-supercompact. In particular, κ_λ is (ω_1, λ)-strongly compact. Therefore $\kappa \leq \kappa_\lambda$, and hence $\kappa = \kappa_\lambda$. Thus κ is λ-supercompact, as desired. ☐

Corollary 7.4.21 (UA)**.** *Suppose λ is a successor cardinal and κ is the least λ-strongly compact cardinal. Then κ is λ-supercompact. In fact, $\kappa = \kappa_\lambda$.* ☐

Corollary 7.4.22 (UA)**.** *The least (ω_1, Ord)-strongly compact cardinal κ is supercompact.*

Proof. No cardinal $\delta < \kappa$ is (ω_1, κ)-strongly compact. In particular, for any successor cardinal $\lambda > \kappa$, κ is the least (ω_1, λ)-strongly compact cardinal. Therefore κ is λ-supercompact by Theorem 7.4.20. ☐

Theorem 7.4.23 (UA)**.** *The least strongly compact cardinal is supercompact.*
☐

The following ordinals serve as key thresholds in the structure theory of countably complete ultrafilters:

Definition 7.4.24. The *ultrapower threshold* is the least cardinal κ such that for all α, there is an ultrapower embedding $j : V \to M$ such that $j(\kappa) > \alpha$.

Suppose γ is an ordinal. The *γ-threshold* is the least ordinal $\kappa \leq \gamma$ such that for all $\alpha < \gamma$ is an ultrapower embedding $j : V \to M$ such that $j(\kappa) > \alpha$.

The ultrapower threshold cannot be proved to exist without large cardinal assumptions, but for any ordinal γ, the γ-threshold exists and is less than or equal to γ.

Lemma 7.4.25. *Suppose κ is a cardinal. If κ is the γ-threshold for some ordinal γ, then κ is the κ-threshold.*

Proof. We may assume without loss of generality that $\kappa < \gamma$. Let $\nu \leq \kappa$ be the κ-threshold.

We claim that for any $\alpha < \gamma$, there is an ultrapower embedding $h : V \to N$ such that $h(\nu) > \alpha$. Fix $\alpha < \gamma$. Let $j : V \to M$ be such that $j(\kappa) > \alpha$. In M, $j(\nu)$ is the $j(\kappa)$-threshold, so since $\alpha < j(\kappa)$, there is an internal ultrapower embedding $i : M \to N$ such that $i(j(\nu)) > \alpha$. Let $h = i \circ j$. Then $h : V \to N$ is an ultrapower embedding such that $h(\nu) > \alpha$, as desired.

By the minimality of the γ-threshold, $\kappa \leq \nu$. Hence $\kappa = \nu$ as desired. $\square$

Theorem 7.4.26 (UA)**.** *Suppose λ is a strong limit cardinal and $\kappa < \lambda$ is the λ-threshold. Then κ is γ-supercompact for all $\gamma < \lambda$.*

The proof uses the following lemma, an often-useful approximation to Conjecture 7.4.8:

Lemma 7.4.27 (UA)**.** *Suppose λ_0 is an isolated cardinal and $\lambda_1 = (\lambda_0)^\sigma$. Then λ_1 is measurable.*

Proof. Note that $\kappa_{\lambda_1} > \lambda_0$: otherwise $\lambda_0 \in [\kappa_{\lambda_1}, \lambda_1)$ contrary to the fact that there are no isolated cardinals in the interval $[\kappa_{\lambda_1}, \lambda_1)$ by Lemma 7.4.13. Since κ_{λ_1} is measurable, κ_{λ_1} is Fréchet. Hence $\lambda_1 = (\lambda_0)^\sigma \leq \kappa_{\lambda_1}$. Obviously $\kappa_{\lambda_1} \leq \lambda_1$, so $\kappa_{\lambda_1} = \lambda_1$. Therefore λ_1 is measurable. $\square$

Proof of Theorem 7.4.26. By induction, we may assume that the theorem holds for all strong limit cardinals $\bar{\lambda} < \lambda$.

Suppose $\alpha < \lambda$. We claim that there is a countably complete ultrafilter D with $\lambda_D < \lambda$ such that $j_D(\kappa) > \alpha$. To see this, fix an ultrapower embedding $j : V \to M$ such that $j_D(\kappa) > \alpha$. Then by Lemma 5.5.25, one can find a countably complete ultrafilter D such that $\lambda_D \leq 2^{|\alpha|} < \lambda$ and an elementary embedding $k : M_D \to M$ such that $k \circ j_D = j$ and $\text{crit}(k) > \alpha$. Since $k(j_D(\kappa)) = j(\kappa) > \alpha = k(\alpha)$, the elementarity of k implies that $j_D(\kappa) > \alpha$.

Next, we show that λ is a limit of Fréchet cardinals. Suppose δ is a cardinal with $\kappa \leq \delta < \lambda$. We will find a Fréchet cardinal in the interval (δ, λ). By the previous paragraph, there is a countably complete ultrafilter D such that

$j_D(\kappa) \geq (2^\delta)^+$ and $\lambda_D < \lambda$. It follows that $\delta < \lambda_D$ since $2^\delta < |j_D(\kappa)| \leq \kappa^{\lambda_D} = 2^{\lambda_D}$. Thus λ_D is a Fréchet cardinal in the interval (δ, λ), as desired.

We claim that every regular cardinal in the interval $[\kappa, \lambda)$ is Fréchet. By Lemma 7.4.12, it suffices to show that there are no isolated cardinals in the interval $[\kappa, \lambda)$. Suppose $\lambda_0 \in [\kappa, \lambda)$ is isolated. Let $\lambda_1 = (\lambda_0)^\sigma$. Lemma 7.4.27 implies that λ_1 is measurable. Since λ is a limit of Fréchet cardinals, $\lambda_1 < \lambda$. Note that for all $\alpha < \lambda_1$, there is an ultrapower embedding $j : V \to M$ such that $j(\kappa) > \alpha$, so the λ_1-threshold κ' is less than λ_1. By our induction hypothesis, κ' is γ-supercompact for all $\gamma < \lambda_1$. This contradicts that $\lambda_1 = (\lambda_0)^\sigma$ is not a limit of Fréchet cardinals.

We finally claim that κ is δ-supercompact for any successor cardinal $\delta \in (\kappa, \lambda)$, which proves the theorem. Suppose $\delta \in (\kappa, \lambda)$ is a successor cardinal. Then κ_δ is δ-supercompact by Corollary 7.4.10. Since κ_δ is the limit of the isolated cardinals below δ (Lemma 7.4.19), $\kappa_\delta \leq \kappa$. On the other hand, by Lemma 7.4.25, κ is the κ-threshold, so in particular, no $\nu < \kappa$ is κ-supercompact. Hence $\kappa_\delta \not< \kappa$. It follows that $\kappa = \kappa_\delta$, as desired. □

7.4.4 The number of countably complete ultrafilters

We close this section with an application of the analysis of $\mathscr{K}_\lambda$ given by Theorem 7.3.14 and Proposition 7.4.17. Recall that $\mathbf{UF}(X)$ denotes the set of countably complete ultrafilters on X. The main result is a bound on the cardinality of $\mathbf{UF}(X)$:

Theorem 7.4.28 (UA). *For any set X, $|\mathbf{UF}(X)| \leq (2^{|X|})^+$.*

The theorem is proved by a generalizing Solovay's Theorem 6.3.3. To do this, we need to generalize the notion of the Mitchell rank of an ultrafilter:

Definition 7.4.29. Suppose δ is an ordinal and W is a countably complete ultrafilter on δ.

- $\mathbf{UF}_W(\delta)$ denotes the set of countably complete ultrafilters U on δ such that $U <_\Bbbk W$.
- $\sigma(W)$ denotes the rank of $(\mathbf{UF}_W(\delta), <_\Bbbk)$.
- $\sigma(\delta)$ denotes the rank of $(\mathbf{UF}(\delta), <_\Bbbk)$.
- $\sigma(<\delta) = \sup_{\alpha<\delta} \sigma(\alpha) + 1$.

Since the Ultrapower Axiom implies that the Ketonen order is linear, the rank of an ultrafilter completely determines its position in the Ketonen order:

Lemma 7.4.30 (UA). *Suppose U and W are countably complete ultrafilters on ordinals. Then $U \leq_\Bbbk W$ if and only if $\sigma(U) \leq \sigma(W)$.* □

The following lemma relates σ^V to σ^{M_U}:

Lemma 7.4.31 (UA). *Suppose U is a countably complete ultrafilter and W is a countably complete ultrafilter on an ordinal δ. Then $\sigma(W) \leq \sigma^{M_U}(t_U(W))$.*

Proof. It suffices to show that there is a Ketonen order preserving embedding from $\mathbf{UF}_W(\delta)$ to $\mathbf{UF}^{M_U}_{t_U(W)}(j_U(\delta))$. By Theorem 5.4.44, the translation function t_U restricts to such a function. $\square$

We briefly mention that a version of Lemma 7.4.31 is provable in ZFC. Suppose Z is a countably complete ultrafilter and W is an ultrafilter on an ordinal δ. If $\langle W_i : i \in I\rangle$ is sequence of countably complete ultrafilters on δ such that $W = Z\text{-}\lim_{i\in I} W_i$, then

$$\sigma(W) \leq [\langle \sigma(W_i) : i \in I\rangle]_Z$$

We omit the proof, which is an application of Lemma 3.3.10.

Corollary 7.4.32 (UA). *Suppose U is a countably complete ultrafilter and W is a countably complete ultrafilter on an ordinal. If $j_U(\sigma(W)) = \sigma(W)$ then $U \sqsubset W$.*

Proof. Assume $j_U(\sigma(W)) = \sigma(W)$. Then

$$\sigma^{M_U}(j_U(W)) = j_U(\sigma(W)) = \sigma(W) \leq \sigma^{M_U}(t_U(W))$$

For the final inequality, we use Lemma 7.4.31. By Lemma 7.4.30, it follows that $j_U(W) \leq_{\Bbbk} t_U(W)$ in M_U. By the theory of the internal relation (Lemma 5.5.15), this implies $U \sqsubset W$. $\square$

Lemma 7.4.33 (UA). *Suppose γ is an ordinal. Then for any ordinal $\xi \in [\sigma({<}\gamma), \sigma(\gamma))$, there is a countably complete fine ultrafilter U on γ with $j_U(\xi) > \xi$.*

Proof. Let U be unique element of $\mathbf{UF}(\gamma)$ with $\sigma(U) = \xi$. Since $\xi \geq \eta$, U does not concentrate on α for any $\alpha < \eta$. Therefore U is a nonprincipal fine ultrafilter on γ. Since U is nonprincipal, $U \not\sqsubset U$. Therefore $j_U(\sigma(U)) > \sigma(U)$ by Corollary 7.4.32. In other words, $j_U(\xi) > \xi$. $\square$

The following fact is ultimately equivalent to Theorem 7.5.47 below:

Lemma 7.4.34 (UA). *Suppose ξ and δ are ordinals and U is the $<_{\Bbbk}$-minimum countably complete ultrafilter on δ such that $j_U(\xi) > \xi$. Then for any countably complete ultrafilter D such that $j_D(\xi) = \xi$, $D \sqsubset U$.*

Proof. Since j_D is elementary and $j_D(\xi) = \xi$, $j_D(U)$ is the $<_{\Bbbk}^{M_D}$-minimum countably complete ultrafilter Z of M_D on $j_D(\delta)$ such that $j_Z^{M_D}(\xi) > \xi$. On the other hand, $t_D(U)$ is a countably complete ultrafilter of M_D on $j_D(\delta)$ such that

$$j^{M_D}_{t_D(U)}(\xi) = j^{M_D}_{t_D(U)}(j_D(\xi)) = j^{M_U}_{t_U(D)}(j_U(\xi)) \geq j_U(\xi) > \xi$$

Hence by the linearity of the Ketonen order, $j_D(U) \leq_{\Bbbk} t_D(U)$ in M_D. Now the basic theory of the internal relation (Lemma 5.5.15) implies that $D \sqsubset U$. $\square$

The central combinatorial argument of Theorem 7.4.28 appears in the following proposition:

Proposition 7.4.35 (UA)**.** *Suppose λ is a Fréchet cardinal. Then for any ordinal $\gamma < \lambda$, $|\mathbf{UF}(\gamma)| \leq 2^\lambda$.*

Proof. Assume towards a contradiction that λ is the least Fréchet cardinal at which the theorem fails. In particular, λ is not a limit of Fréchet cardinals, so by Theorem 7.3.14 or Proposition 7.4.17, $\mathscr{K}_\lambda$ is λ-internal. Let $\gamma < \lambda$ be the least ordinal such that $|\mathbf{UF}(\gamma)| > 2^\lambda$. Then in particular, γ is the least ordinal such that $\sigma(\gamma) \geq (2^\lambda)^+$, so $\sigma({<}\gamma) < (2^\lambda)^+$.

Let ξ be an ordinal with the following properties:

- $\sigma({<}\gamma) \leq \xi < (2^\lambda)^+$.
- For all $\alpha < \gamma$, for all $D \in \mathbf{UF}(\alpha)$, $j_D(\xi) = \xi$.
- $j_{\mathscr{K}_\lambda}(\xi) = \xi$.

To see that such an ordinal ξ exists, let $S = \bigcup_{\alpha<\gamma} \mathbf{UF}(\alpha) \cup \{\mathscr{K}_\lambda\}$. Note that $|S| \leq 2^\lambda$ by the minimality of γ. For each $D \in S$, the collection of fixed points of j_D is ω-closed unbounded in $(2^\lambda)^+$. Therefore the intersection of the fixed points of j_D for all $D \in S$ is ω-closed unbounded in $(2^\lambda)^+$.

Since $\xi \in [\sigma({<}\gamma), \sigma(\gamma))$, Lemma 7.4.33 implies that there is a countably complete fine ultrafilter U on γ with $j_U(\xi) > \xi$. Let U be the $<_\Bbbk$-least countably complete ultrafilter on γ such that $j_U(\xi) > \xi$. By Lemma 7.4.34, U is γ-internal, and moreover $\mathscr{K}_\lambda \sqsubset U$.

Since $\lambda_U < \lambda$, $U \sqsubset \mathscr{K}_\lambda$. Thus $U \sqsubset \mathscr{K}_\lambda$ and $\mathscr{K}_\lambda \sqsubset U$, so by Theorem 5.5.21, U and $\mathscr{K}_\lambda$ commute. Since U is λ_U-internal and $\mathscr{K}_\lambda$ is λ-internal, we can apply the converse to Kunen's commuting ultrapowers lemma (Proposition 5.5.24) to obtain $U \in V_{\kappa_\lambda}$. (Obviously $\mathscr{K}_\lambda$ is not in V_κ where κ is the completeness of U.) In particular $\gamma < \kappa_\lambda$. But then $|\mathbf{UF}(\gamma)|^+ < \kappa_\lambda$ since κ_λ is inaccessible. This contradicts that $\kappa_\lambda \leq \lambda < (2^\lambda)^+ \leq \sigma(\gamma) < |\mathbf{UF}(\gamma)|^+$. ☐

The proof above is a bit mysterious, and the situation can be clarified by doing a bit more work than the bare minimum required to prove the theorem. In fact one can prove the following. Suppose λ is a Fréchet cardinal that is either regular or isolated. Let ξ be the first fixed point of $j_{\mathscr{K}_\lambda}$ above κ_λ. Then for any $D <_\Bbbk \mathscr{K}_\lambda$, $j_D(\xi) = \xi$. The $<_\Bbbk$-minimum countably complete ultrafilter U on λ such that $j_U(\xi) > \xi$, if it exists, is Rudin-Keisler equivalent to the $\lhd$-least normal fine ultrafilter $\mathscr{U}$ on $P_{\kappa_\lambda}(\lambda)$ such that $\mathscr{K}_\lambda \lhd \mathscr{U}$. This is related to Proposition 8.4.20.

Incidentally, Proposition 7.4.35 yields an alternate proof of instances of GCH from UA plus large cardinals. For example, assume $|\mathbf{UF}(\kappa)| = 2^{2^\kappa}$, $|\mathbf{UF}(\kappa^+)| > 2^{(\kappa^+)}$, and κ^{++} is Fréchet. Then

$$2^{2^\kappa} = |\mathbf{UF}(\kappa)| \leq 2^{(\kappa^+)} < |\mathbf{UF}(\kappa^+)| \leq 2^{(\kappa^{++})}$$

Thus $2^{2^\kappa} < 2^{(\kappa^{++})}$, and in particular $2^\kappa < \kappa^{++}$. In other words, $2^\kappa = \kappa^+$. (This result is not as strong as Theorem 6.3.27.)

Recall Definition 3.3.1: if X is a set and A is a class, then

$$\mathbf{UF}(X, A) = \{U \in \mathbf{UF}(X) : U \text{ concentrates on } A\}$$

Theorem 7.4.36 (UA)**.** *For any Fréchet cardinal λ, for any $W \in \mathbf{UF}(\lambda)$, $|\mathbf{UF}_W(\lambda)| \leq 2^\lambda$. Hence $\sigma(\lambda) \leq (2^\lambda)^+$.*

Proof. By the definition of the Ketonen order, every element of $\mathbf{UF}_W(\lambda)$ is of the form $W\text{-}\lim_{\alpha\in I} U_\alpha$ for some $I \in W$ and $\langle U_\alpha : \alpha \in I\rangle \in \prod_{\alpha\in I} \mathbf{UF}(\lambda, \alpha)$. Thus $|\mathbf{UF}_W(\lambda)| \leq |\coprod_{I\subseteq\lambda} \prod_{\alpha\in I} \mathbf{UF}(\lambda, \alpha)|$. Since

$$\left|\coprod_{I\subseteq\lambda} \prod_{\alpha\in I} \mathbf{UF}(\lambda, \alpha)\right| = 2^\lambda \cdot \sup_{I\subseteq\lambda} \prod_{\alpha\in I} |\mathbf{UF}(\lambda, \alpha)|$$

it suffices to show that $|\mathbf{UF}(\lambda, \alpha)| \leq 2^\lambda$ for all $\alpha < \lambda$. But there is a one-to-one correspondence between $\mathbf{UF}(\lambda, \alpha)$ and $\mathbf{UF}(\alpha)$, and by Proposition 7.4.35, $|\mathbf{UF}(\alpha)| \leq \sigma(\alpha) < (2^\lambda)^+$. Thus $|\mathbf{UF}(\lambda, \alpha)| \leq 2^\lambda$, which completes the proof. □

We finally prove $|\mathbf{UF}(X)| \leq (2^{|X|})^+$:

Proof of Theorem 7.4.28. For any $A \subseteq X$ of cardinality λ,

$$|\mathbf{UF}(X, A)| = |\mathbf{UF}(A)| = |\mathbf{UF}(\lambda)|$$

Since every ultrafilter U concentrates on a set whose cardinality is a Fréchet cardinal, we have

$$\mathbf{UF}(X) = \bigcup\{\mathbf{UF}(X, A) : A \subseteq X \text{ and } |A| \text{ is Fréchet}\}$$

Hence

$$|\mathbf{UF}(X)| \leq 2^{|X|} \cdot \sup\{|\mathbf{UF}(\lambda)| : \lambda \leq |X| \text{ is Fréchet}\} \tag{7.3}$$

By Theorem 7.4.36, for any Fréchet cardinal λ such that $\lambda \leq |X|$, $|\mathbf{UF}(\lambda)| \leq (2^\lambda)^+ \leq (2^{|X|})^+$. Hence by (7.3), $|\mathbf{UF}(X)| \leq 2^{|X|} \cdot (2^{|X|})^+ = (2^{|X|})^+$, as desired. □

7.5 Isolation

In this section, we take a closer look at isolated cardinals. We are particularly interested in nonmeasurable isolated cardinals although we have stated the conjecture that there are no such cardinals (Conjecture 7.4.8). While we do not know how to prove this conjecture, we show here that there are significant constraints on the structure of nonmeasurable isolated cardinals. As an application, we prove the linearity of the Mitchell order on normal fine ultrafilters from UA without using any cardinal arithmetic assumptions (Theorem 7.5.42).

7.5.1 Isolated measurable cardinals

Recall that isolated cardinals are Fréchet limit cardinals that are not limits of Fréchet cardinals. In this section, we will provide a complete analysis of isolated strong limit cardinals in terms of their large cardinal properties, proving that Conjecture 7.4.8 holds for such cardinals.

We begin with a characterization of limits of Fréchet cardinals.

Lemma 7.5.1 (UA). *Suppose λ is a limit of Fréchet cardinals. Let κ be the supremum of the isolated cardinals less than λ, and assume $\kappa < \lambda$. Then κ is γ-supercompact for all $\gamma < \lambda$. In fact, $\kappa = \kappa_\iota$ for all regular cardinals $\iota \in [\kappa, \lambda)$.*

Proof. Since there are no isolated cardinals in the interval $[\kappa, \lambda)$, Lemma 7.4.12 implies that every regular cardinal in the interval $[\kappa, \lambda)$ is Fréchet. Assume $\iota \in [\kappa, \lambda)$ is a regular cardinal. Then ι is a nonisolated Fréchet cardinal. Since κ is the supremum of the isolated cardinals below ι, $\kappa = \kappa_\iota$ by Lemma 7.4.19. Now by Proposition 7.4.11, κ is γ-supercompact for all $\gamma < \iota$. Since ι was an arbitrary regular cardinal in $[\kappa, \lambda)$ and λ is a limit cardinal, κ is γ-supercompact for all $\gamma < \lambda$. □

Corollary 7.5.2 (UA). *Suppose λ is a cardinal. Then the following are equivalent:*

(1) λ is a limit of Fréchet cardinals.

(2) Either λ is a limit of isolated measurable cardinals or some $\kappa < \lambda$ is γ-supercompact for all $\gamma < \lambda$.

Proof. (1) implies (2): First assume λ is a limit of isolated cardinals. Then by Lemma 7.4.27, λ is a limit of isolated measurable cardinals.

Assume instead that λ is not a limit of isolated cardinals and let $\kappa < \lambda$ be the supremum of the isolated cardinals below λ. By Lemma 7.5.1, κ is γ-supercompact for all $\gamma < \lambda$.

(2) implies (1): Trivial. □

In particular, it follows that under UA, every limit of Fréchet cardinals is a strong limit cardinal: if λ is a limit of measurable cardinals, this is immediate; on the other hand, if some $\kappa < \lambda$ is γ-supercompact for all $\gamma < \lambda$, then Theorem 6.3.25 implies that for all $\gamma \in [\kappa, \lambda)$, $2^\gamma = \gamma^+$.

Lemma 7.5.3 (UA). *Suppose λ is a strong limit cardinal such that no cardinal $\kappa < \lambda$ is γ-supercompact for all $\gamma < \lambda$. Then for all ultrapower embeddings $j : V \to M$, $j[\lambda] \subseteq \lambda$. In fact, no ordinal $\kappa < \lambda$ can be mapped arbitrarily high below λ by ultrapower embeddings.*

Proof. This follows from our analysis of threshold cardinals (Theorem 7.4.26). □

Proposition 7.5.4 (UA). *Suppose λ is cardinal. Then the following are equivalent:*

(1) *λ is a strong limit isolated cardinal.*

(2) *λ is a measurable cardinal, λ is not a limit of measurable cardinals, and no cardinal $\kappa < \lambda$ is λ-supercompact.*

Proof. (1) implies (2): Since λ is not a limit of Fréchet cardinals, clearly λ is not a limit of measurable cardinals and no $\kappa < \lambda$ is λ-supercompact. It remains to show that λ is measurable. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Note that $j[\lambda] \subseteq \lambda$ by Lemma 7.5.3. By Proposition 7.4.17, $D \sqsubset \mathscr{K}_\lambda$ for all D with $\lambda_D < \lambda$. Therefore by Lemma 5.5.26, $\mathscr{K}_\lambda$ is λ-complete. Since there is a λ-complete uniform ultrafilter on λ, λ is measurable.

(2) implies (1): Since λ is measurable, λ is a strong limit cardinal. It remains to show that λ is isolated. Note that no cardinal $\kappa < \lambda$ is γ-supercompact for all $\gamma < \lambda$: since λ is measurable, this would imply κ is λ-supercompact, contrary to assumption. Corollary 7.5.2 now implies that λ is not a limit of Fréchet cardinals. Therefore λ is isolated. ☐

The main application of isolated measurable cardinals is factoring ultrapower embeddings:

Theorem 7.5.5 (UA). *Suppose κ is a strong limit cardinal that is not a limit of Fréchet cardinals. Suppose U is a countably complete ultrafilter. Then there is a countably complete ultrafilter D such that $\lambda_D < \kappa$ admitting an internal ultrapower embedding $h : M_D \to M_U$ such that $h \circ j_D = j_U$ and* $\text{crit}(h) \geq \kappa$ *if h is nontrivial.*

Proof. Fix $\gamma < \kappa$ such that $\kappa \leq \gamma^\sigma$. By Lemma 5.5.25, one can find a countably complete ultrafilter D such that $\lambda_D < \kappa$ and there is an elementary embedding $e : M_D \to M_U$ such that $\text{crit}(e) > \beth_{10}(\gamma)$ and $e \circ j_D = j_U$. Let $\lambda = \lambda_D$. We may assume without loss of generality that λ is the underlying set of D. Since $\lambda < \kappa$ is a Fréchet cardinal, $\lambda \leq \gamma$. Let $\lambda' = j_D(\lambda)$. Then $\lambda' < (2^\lambda)^+$, so $2^{2^{\lambda'}} < \beth_{10}(\gamma)$. Since $e : M_D \to M_U$ has critical point above $2^{2^{\lambda'}}$,

$$P(P(\lambda')) \cap M_D = P(P(\lambda')) \cap M_U$$

Thus the following hold where $\mathbf{UF}(X)$ denotes the set of countably complete ultrafilters on X:

- $\lambda' = j_D(\lambda) = e(\lambda') = j_U(\lambda)$.
- $\mathbf{UF}^{M_D}(\lambda') = \mathbf{UF}^{M_U}(\lambda')$.
- $\leq_{\Bbbk}^{M_D} \restriction \mathbf{UF}^{M_D}(\lambda') = \leq_{\Bbbk}^{M_U} \restriction \mathbf{UF}^{M_U}(\lambda')$
- $j_D \restriction P(\lambda) = j_U \restriction P(\lambda)$.

By Theorem 5.4.42, $t_D(D)$ is the $\leq_{\Bbbk}^{M_D}$-least element $D' \in \mathbf{UF}^{M_D}(\lambda')$ such that $j_D^{-1}[D'] = D$. By Theorem 5.4.42, $t_U(D)$ is the $\leq_{\Bbbk}^{M_U}$-least element $D' \in \mathbf{UF}^{M_U}(\lambda')$ such that $j_U^{-1}[D'] = D$. By the agreement set out in the bullet points above, it therefore follows that $t_D(D) = t_U(D)$. On the other hand, by Lemma 5.4.41, $t_D(D)$ is principal in M_D. Thus $t_U(D)$ is principal in M_U. Therefore by Lemma 5.4.41, $D \leq_{\text{RF}} U$.

Let $h : M_D \to M_U$ be the internal ultrapower embedding with $h \circ j_D = j_U$. Note that $h(\beth_{10}(\gamma)) = h(j_D(\beth_{10}(\gamma))) = e(j_D(\beth_{10}(\gamma))) = \beth_{10}(\gamma)$, and $M_D \cap H(\beth_{10}(\gamma))) = M_U \cap H(\beth_{10}(\gamma)))$ since $\text{crit}(e) > \beth_{10}(\gamma)$. Therefore by the Kunen Inconsistency Theorem (Theorem 4.2.35) applied in M_D, $\text{crit}(h) > \gamma$. Since h is an internal ultrapower embedding of M_D, if h is nontrivial then $\text{crit}(h)$ is a measurable cardinal of M_D above $j_D(\gamma)$. Since there are no measurable cardinals in the interval (γ, κ), there are no measurable cardinals of M_D in the interval $(j_D(\gamma), j_D(\kappa))$. Therefore if h is nontrivial, then $\text{crit}(h) \geq \kappa$. $\square$

7.5.2 Ultrafilters on an isolated cardinal

In this subsection, which is perhaps the most technical of this monograph, we enact a very detailed analysis of the countably complete ultrafilters on an isolated cardinal. One of the goals is to prove the following theorem:

Theorem 7.5.6. *Suppose λ is an isolated cardinal and W is a countably complete ultrafilter.*

- *$\mathscr{K}_\lambda \leq_{\text{RF}} W$ if and only if W is λ-decomposable.*
- *$W \sqsubset \mathscr{K}_\lambda$ if and only if W is λ-indecomposable.*

This should be seen as a generalization of the universal property of $\mathscr{K}_\lambda$ (Theorem 7.3.13) to isolated cardinals λ. (Theorem 7.3.13 applies to regular cardinals. Recall that Conjecture 7.4.8 implies that all isolated cardinals are regular.)

We begin with the following fact:

Theorem 7.5.7 (UA). *Suppose λ is an isolated cardinal. Then $\mathscr{K}_\lambda$ is the unique countably complete weakly normal ultrafilter on λ.*

It turns out to be easier to prove something that is a priori slightly stronger. Recall the notion of the Dodd parameter $p(j)$ of an elementary embedding j, defined in Definition 4.3.17 in the general context of elementary embeddings, and once again in Definition 5.4.26 in the more relevant special case of ultrapower embeddings.

Proposition 7.5.8 (UA). *Suppose λ is an isolated cardinal. Then $\mathscr{K}_\lambda$ is the unique countably complete incompressible ultrafilter U on λ whose Dodd parameter has cardinality 1.*

Proof. Suppose towards a contradiction that the proposition fails. Let U be the $<_\Bbbk$-least countably complete incompressible ultrafilter on λ such that $p(j_U) = 1$ and $U \neq \mathscr{K}_\lambda$. Since $\mathscr{K}_\lambda$ is the $<_\Bbbk$-least uniform ultrafilter on λ, $\mathscr{K}_\lambda <_\Bbbk U$.

Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$ and let $\nu = a_{\mathscr{K}_\lambda}$. Let $i : V \to N$ be the ultrapower of the universe by U and let $\xi = \mathrm{id}_U$. By the incompressibility of U, $p(j_U) = \{\xi\}$.

Let $(k, h) : (M, N) \to P$ be the pushout of (j, i). Since $\mathscr{K}_\lambda <_{\Bbbk} U$,

$$k(\nu) < h(\xi) \tag{7.4}$$

We claim that $h(\xi)$ is a generator of $k : M \to P$, or in other words that

$$h(\xi) \notin H^P(k[M] \cup h(\xi))$$

Since ξ is a generator of i, $h(\xi)$ is a generator of $h \circ i$ by Lemma 5.4.27. Since $k \circ j = h \circ i$, $h(\xi)$ is a generator of $k \circ j$. Since $M = H^M(j[V] \cup \{\nu\})$,

$$H^P(k[M] \cup h(\xi)) = H^P(k \circ j[V] \cup \{k(\nu)\} \cup h(\xi)) = H^P(k \circ j[V] \cup h(\xi))$$

The final equality follows from (7.4). Therefore since $h(\xi) \notin H^P(k \circ j[V] \cup h(\xi))$, $h(\xi) \notin H^P(k[M] \cup h(\xi))$, as desired.

Let $Z = t_{\mathscr{K}_\lambda}(U)$, so Z is the M-ultrafilter on $j(\lambda)$ derived from k using $h(\xi)$. Then Z is a countably complete ultrafilter on $j(\lambda)$ and $\mathrm{id}_Z = h(\xi)$ is a generator of $j_Z^M = k$.

We claim that Z is an incompressible ultrafilter on $j(\lambda)$ in M. Since $\mathrm{id}_Z = h(\xi)$ is a generator of j_Z^M, it suffices to show that Z is fine, or in other words, $\delta_Z = j(\lambda)$. Since id_Z is a generator of j_Z^M, $\delta_Z = \lambda_Z$ is a Fréchet cardinal in M. By (7.4), $\delta_Z > \mathrm{id}_{\mathscr{K}_\lambda}$. Since U is on λ, $\xi < i(\lambda)$, so $h(\xi) < h(i(\lambda)) = k(j(\lambda))$, which implies $\delta_Z \leq j(\lambda)$. Thus $\delta_Z \in (\mathrm{id}_{\mathscr{K}_\lambda}, j(\lambda)]$. Since λ is isolated, no Fréchet cardinal of M lies in the interval $[\sup j[\lambda], j(\lambda))$. Therefore $\delta_Z = j(\lambda)$, as desired.

It follows that in M, Z is a countably complete incompressible ultrafilter on $j(\lambda)$. Moreover $p(j_Z^M) = \{h(\xi)\}$ by Lemma 5.4.28, so $p(j_Z^M)$ has cardinality 1.

We claim that $Z \neq j(\mathscr{K}_\lambda)$. The reason is that $j^{-1}[Z] = U$ (since $Z = t_{\mathscr{K}_\lambda}(U)$) while $j^{-1}[j(\mathscr{K}_\lambda)] = \mathscr{K}_\lambda$.

Thus we have shown that in M, Z is a countably complete incompressible ultrafilter on $j(\lambda)$ such that $|p(j_Z^M)| = 1$ and $Z \neq j(\mathscr{K}_\lambda)$. By the definition of U and the elementarity of j, it follows that $j(U) \leq_{\Bbbk} Z$ in M. Lemma 5.5.15 now implies that $\mathscr{K}_\lambda \sqsubset U$. But j_U is discontinuous at λ since $\lambda_U = \lambda$. Thus by Lemma 7.4.14, λ is not isolated. This is a contradiction. □

Proof of Theorem 7.5.7. If U is a countably complete weakly normal ultrafilter on λ, then U is incompressible and $p(j_U) = \{\mathrm{id}_U\}$ by Proposition 4.4.23. Therefore we can apply Proposition 7.5.8. □

We now investigate the iterated ultrapowers of $\mathscr{K}_\lambda$.

Definition 7.5.9. If λ is an isolated cardinal, then the *iterated ultrapower of* $\mathscr{K}_\lambda$ is the iterated ultrapower

$$\mathscr{I}_\lambda = \langle M_n^\lambda, j_{m,n}^\lambda, U_m^\lambda : m \leq n < \omega\rangle$$

formed by setting $U_m^\lambda = j_{0,m}^\lambda(\mathscr{K}_\lambda)$ for all $m < \omega$. For $n < \omega$, let $p_\lambda^n = p(j_{0,n}^\lambda)$, and let $\mathscr{K}_\lambda^n$ be the ultrafilter on $[\lambda]^\ell$ derived from $j_{0,n}^\lambda$ using p_λ^n where $\ell = |p_\lambda^n|$.

Thus $j^\lambda_{0,n} : V \to M^\lambda_n$ is the ultrapower of the universe by $\mathscr{K}^n_\lambda$.

We will prove an analog of Proposition 7.5.8 that is most elegantly stated in terms of the natural analog of the concept of an incompressible ultrafilter (Definition 3.4.14) in the context of the parameter order (Definition 4.3.11).

Definition 7.5.10. An ultrafilter U on $[\delta]^n$ is *strongly fine* if for all $m < n$ and $\alpha < \delta$, $\{b \in [\delta]^n : \alpha < b_m\} \in U$. An ultrafilter U on $[\delta]^n$ is *incompressible* if for all $A \in U$, there is no one-to-one function $f : A \to [\text{Ord}]^{<\omega}$ such that $f(p) < p$ for all $p \in A$.

In other words, U is strongly fine if and only if $\text{id}_U \subseteq [\sup j_U[\delta], j_U(\delta))$, and U is incompressible if and only if id_U is equal to its Dodd parameter $p(j_U)$. Thus a countably complete ultrafilter W is Rudin-Keisler equivalent to a strongly fine incompressible ultrafilter on $[\delta]^n$ if and only if $p(j_W) \subseteq [\sup j_W[\delta], j_W(\delta))$.

Our analog of Proposition 7.5.8 can now be stated as follows:

Theorem 7.5.11 (UA). *Suppose λ is an isolated cardinal. Then $\mathscr{K}^n_\lambda$ is the unique strongly fine incompressible ultrafilter on $[\lambda]^n$.*

We now analyze the Dodd parameters p^n_λ of the iterated ultrapower embeddings $j^\lambda_{0,n}$. The following lemma is stated in the context of UA (so that $\mathscr{K}_\lambda$ is defined), but with a little bit of effort, it could be proved in ZFC for an arbitrary Ketonen ultrafilter on an isolated cardinal λ.

Lemma 7.5.12 (UA). *Suppose λ is an isolated cardinal and*

$$\langle M_n, j_{m,n}, U_m : m \leq n < \omega\rangle$$

is the iterated ultrapower of $\mathscr{K}_\lambda$. For $n < \omega$, let $p^n = p^n_\lambda$. Then for all $n < \omega$, $|p^n| = n$ and

$$\begin{aligned} p^{n+1} \restriction n &= p(j_{1,n+1}) \\ p^{n+1}_n &= j_{1,n+1}(\text{id}_{\mathscr{K}_\lambda}) \end{aligned} \tag{7.5}$$

Proof. Note that the conclusion of the lemma holds when $n = 0$. Assume $m \geq 1$ and that the conclusion of the lemma holds when $n = m - 1$. We will prove that the conclusion of the lemma holds when $n = m$.

Let $\nu = \text{id}_{\mathscr{K}_\lambda}$. Since $j_{0,m+1} = j_{1,m+1} \circ j_{0,1}$ and $M_1 = H^{M_1}(j_{0,1}[V] \cup \{\nu\})$, for any $x \in M_{m+1}$, an ordinal ξ is an x-generator of $j_{1,m+1}$ if and only if it is a $\langle j_{1,m+1}(\nu), x\rangle$-generator of $j_{0,m+1}$. Thus if $\xi > j_{1,m+1}(\nu)$, then ξ is an x-generator of $j_{1,m+1}$ if and only if it is an x-generator of $j_{0,m+1}$.

Using our induction hypothesis,

$$\begin{aligned} \min p(j_{1,m+1}) &= \min j_{0,1}(p^m) \\ &= j_{0,1}(\min p^m) \\ &= j_{0,1}(j_{1,m}(\nu)) \\ &= j_{2,m+1}(j_{0,1}(\nu)) \\ &> j_{2,m+1}(j_{1,2}(\nu)) \\ &= j_{1,m+1}(\nu) \end{aligned}$$

$$\begin{tikzcd}
 & p^{n+1}\restriction n & \\
 & p^{n+1}_n \arrow[u, dotted, no head] & \\
p^1 \arrow[ur] & & p^n \arrow[uul] \\
 & M_{n+1} \arrow[uu, dotted, no head] & \\
M_1 \arrow[uu, dotted, no head] \arrow[ur, "j_{0n}"] \arrow[ur, bend left, "j_{1,n+1}"] & & M_n \arrow[uu, dotted, no head] \arrow[ul, "j_{n,n+1}"'] \arrow[ul, bend right, "j_{01}"'] \\
 & V \arrow[ul, "j_{01}"] \arrow[ur, "j_{0n}"'] \arrow[uu, "j_{0,n+1}"] &
\end{tikzcd}$$

Figure 7.3: The iterated ultrapower of $\mathscr{K}_\lambda$.

Applying the recursive characterization of the Dodd parameter (Lemma 4.3.19) to $j_{1,m+1}$, for all $k < m$, $p(j_{1,m+1})_k$ is the largest $p(j_{1,m+1}) \restriction k$-generator of $j_{1,m+1}$. Since $p(j_{1,m+1})_k > j_{1,m+1}(\nu)$, it follows that $p(j_{1,m+1})_k$ is the largest $p(j_{1,m+1}) \restriction k$-generator of $j_{0,m+1}$. Therefore by another application of Lemma 4.3.19, this time to $j_{0,m+1}$, $p(j_{1,m+1})$ is an initial segment of p^{m+1}.

Clearly $j_{1,m+1}$ has no $p(j_{1,m+1})$-generators, and as a consequence, $j_{0,m+1}$ has no $\langle p(j_{1,m+1}), j_{1,m+1}(\nu)\rangle$-generators. We claim that $j_{1,m+1}(\nu)$ is a $p(j_{1,m+1})$-generator of $j_{0,m+1}$. Given this, Lemma 4.3.19 implies that $p^{m+1} = p(j_{1,m+1}) \cup \{j_{1,m+1}(\nu)\}$, completing the induction.

Since $j_{0,m+1} = j_{0,1} \circ j_{0,m}$ and $M_m = H^{M_m}(j_{0,m}[V] \cup p^m)$, an ordinal ξ is a $j_{0,1}(p^m)$-generator of $j_{0,m+1}$ if and only if ξ is a generator of $j_{0,1} \restriction M_m$. Thus to finish, we must show that $j_{0,m+1}(\nu)$ is a generator of $j_{0,1} \restriction M_m$.

Let W be the M_m-ultrafilter on $j_{0,m}(\lambda)$ derived from $j_{0,1} \restriction M_m$ using $j_{1,m+1}(\nu)$. Then by the basic theory of the internal relation (Lemma 5.5.11), since $j_{1,m+1}(\nu) = j_{0,1}(j_{0,m})(\nu)$, $W = s_{\mathscr{K}_\lambda^m}(\mathscr{K}_\lambda)$, $j_W = j_{0,1} \restriction M_n$, and $\mathrm{id}_W = j_{1,m+1}(\nu)$.

Let $\lambda' = \sup j_{0,m}[\lambda]$. We first show that $\lambda_W = \lambda'$. By the definition of $s_{\mathscr{K}_\lambda^n}(\mathscr{K}_\lambda)$, $\lambda' \in W$: note that $j_{0,m}^{-1}[\lambda'] = \lambda \in \mathscr{K}_\lambda$. It follows that $\lambda_W \leq \lambda'$. Thus we are left to show that $\lambda' \leq \lambda_W$. Assume to the contrary that there is a set $B \in W$ such that for some $\kappa < \lambda$, letting $\delta = |B|^{M_m}$, $\delta < j_{0,m}(\kappa)$. Then $j_{0,m}$ is (λ, δ)-tight, and it follows that $j_{0,m}$ is discontinuous at all regular cardinals in the interval $[\kappa, \lambda]$. (See the proof of Proposition 7.4.4.) Therefore λ is a limit of Fréchet cardinals, which contradicts that λ is isolated.

Since $\lambda_W = \lambda'$, $j_{0,1} \restriction M_m$ has a generator in the interval $[\sup j_{0,1}[\lambda'], j_{0,1}(\lambda'))$. Let ξ be the least such generator. Then

$$\xi \leq \mathrm{id}_W = j_{1,m+1}(\nu)$$

Let U be the ultrafilter on λ derived from $j_{0,m+1}$ using ξ and let $k : M_U \to M_{m+1}$ be the factor embedding with $k \circ j_U = j_{0,m+1}$ and $k(\mathrm{id}_U) = \xi$. Since ξ is a generator of $j_{0,m+1}$, id_U is a generator of j_U, and hence U is a uniform ultrafilter. Note that $(k, j_{1,m+1})$ is a right-internal comparison of $(j_U, j_{\mathscr{K}_\lambda})$. Since $\mathscr{K}_\lambda$ is Ketonen minimal among all countably complete uniform ultrafilters on λ, $k(\mathrm{id}_U) \geq j_{1,m+1}(\mathrm{id}_{\mathscr{K}_\lambda})$. In other words, $\xi \geq j_{1,m+1}(\nu)$. It follows that $\xi = j_{1,m+1}(\nu)$. Thus $j_{1,m+1}(\nu)$ is a generator of $j_{0,1} \restriction M_m$, completing the proof. □

A key parameter in the theory of Fréchet cardinals is the strict cardinal supremum of a cardinal's Fréchet predecessors:

Definition 7.5.13. For any cardinal λ, $\delta_\lambda = \sup\{\eta^+ : \eta < \lambda \text{ and } \eta \text{ is Fréchet}\}$.

If λ is a Fréchet cardinal, then λ is isolated if and only if $\delta_\lambda < \lambda$.

The following lemma is an immediate corollary of Lemma 7.5.12:

Lemma 7.5.14 (UA). *Suppose λ is an isolated cardinal and $i : V \to N$ is an ultrapower embedding of the form $i = d \circ j_{0,n}^\lambda$ where $d : M_n^\lambda \to N$ is the ultrapower* *of M_n^λ by a countably complete ultrafilter D of M_n^λ with $\lambda_D < j_{0,n}^\lambda(\delta_\lambda)$. Then $p(i) \setminus i(\delta_\lambda) = d(p_\lambda^n)$.* $\square$

The following theorem amounts to a complete analysis of the ultrafilters on an isolated cardinal:

Theorem 7.5.15 (UA)**.** *Suppose λ is an isolated cardinal. Let*

$$\langle M_n, j_{m,n}, U_m : m \leq n < \omega\rangle$$

be the iterated ultrapower of $\mathscr{K}_\lambda$. Suppose $i : V \to N$ is the ultrapower by a countably complete ultrafilter on λ. Then for some $n < \omega$, $i = d \circ j_{0,n}$ where $d : M_n \to N$ is the ultrapower of M_n by a countably complete ultrafilter D of M_n with $\lambda_D < j_{0,n}(\delta_\lambda)$.

Proof. Suppose U is a countably complete ultrafilter on λ. Assume by induction that the proposition holds when $i = j_W$ for an ultrafilter $W <_{\Bbbk} U$.

If $\lambda_U < \lambda$, then the theorem is vacuously true, taking $n = 0$. Therefore we may assume $\lambda_U = \lambda$.

Let $i : V \to N$ be the ultrapower of the universe by U, and we will show that the theorem is true for i. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Let $\nu = \text{id}_{\mathscr{K}_\lambda}$.

Let $(k, h) : (M, N) \to P$ be the pushout of (j, i). Since (k, h) is the pushout of (j, i), k is the ultrapower embedding of M associated to $t_{\mathscr{K}_\lambda}(U)$. Since $\lambda_U = \lambda$, j_U is discontinuous at λ. Hence by Lemma 7.4.14, $\mathscr{K}_\lambda \not\sqsubset U$. Therefore by Lemma 5.5.15, $t_{\mathscr{K}_\lambda}(U) <_{\Bbbk} j(U)$ in M. We can now apply our induction hypothesis, shifted by j to M, to the ultrafilter $t_{\mathscr{K}_\lambda}(U)$ of M. We conclude that for some $1 \leq n < \omega$, $k = d \circ j_{1,n}$ where $d : M_n \to P$ is the ultrapower of M_n by a countably complete ultrafilter D of M_n such that $\lambda_D < j_{1,n}(j(\delta_\lambda)) = j_{0,n}(\delta_\lambda)$. Note that

$$k \circ j = d \circ j_{0,n}$$

has the form we want to show that i has.

For all $m < \omega$, let $p^m = p_\lambda^m$. By Lemma 7.5.12,

$$p^n = j(p^{n-1}) \cup \{j_{1,n}(\nu)\} \tag{7.6}$$

Let $r = p(k) \setminus k(j(\delta_\lambda))$. By Lemma 7.5.14 applied in M, $r = d(j(p^\ell))$. Since $k \circ j = d \circ j_{0,n}$,

$$\begin{aligned} p(k \circ j) \setminus k(j(\delta_\lambda)) &= p(d \circ j_{0,n}) \setminus d(j_{0,n}(\delta_\lambda)) \\ &= d(p^n) &(7.7) \\ &= d(j(p^{n-1}) \cup \{j_{1,n}(\nu)\}) &(7.8) \\ &= r \cup \{k(\nu)\} &(7.9) \end{aligned}$$

Here (7.7) follows from Lemma 7.5.14; (7.8) follows from (7.6); and (7.9) follows from the fact that $d(j(p^{n-1})) = r$ and $d \circ j_{1,n} = k$.

Let ξ be the least generator of i such that $\sup i[\lambda] \leq \xi < i(\lambda)$.

Claim 1. $k(\nu) = h(\xi)$.

Proof of Claim 1. As in the last paragraph of the proof of Lemma 7.5.12, the Ketonen minimality of $\mathscr{K}_\lambda$ implies that $k(\nu) \leq h(\xi)$.

Assume towards a contradiction that $k(\nu) \neq h(\xi)$, so $k(\nu) < h(\xi)$.

Let $q = p(i) \setminus \sup i[\lambda]$. We claim that $h(q) = p(k) \restriction |q|$. The proof is by induction. Assume $m < |q|$ and $h(q) \restriction m = p(k) \restriction m$. By Lemma 5.4.28, q_m is the largest $q \restriction m$-generator of i. Hence $h(q_m)$ is the largest $h(q \restriction m)$-generator of $h \circ i$. Replacing like terms, $h(q_m)$ is the largest $p(k) \restriction m$-generator of $k \circ j$. Since q_m is a generator of i above $\sup i[\lambda]$, $q_m \geq \xi$. Hence $h(q_m) \geq h(\xi) > k(\nu)$ by our assumption that $h(\xi) > k(\nu)$. Therefore $h(q_m)$ is not only a $p(k) \restriction m$-generator of $k \circ j$ but also a $p(k) \restriction m \cup \{k(\nu)\}$-generator of $k \circ j$. In other words, $h(q_m)$ is a $p(k) \restriction m$-generator of k, and it must therefore be the largest $p(k) \restriction m$-generator of k. By Lemma 5.4.28, $h(q_m) = p(k)_m$.

Since q has no elements below $\sup i[\lambda]$, in particular, q has no elements below $i(\delta_\lambda)$. Therefore $h(q)$ has no elements below $h(i(\delta_\lambda)) = k(j(\delta_\lambda))$. Since $h(q) \subseteq p(k)$ by the previous paragraph, it follows that $h(q) \subseteq p(k) \setminus k(j(\delta_\lambda)) = r$.

We now claim that $k(\nu)$ is a generator of h. To show this, it suffices to show that $k(\nu)$ is a $h(p(i))$-generator of $h \circ i$. Let $s = p(i) \cap \sup i[\lambda]$. Thus $p(i) = q \cup s$. Note that $h(rs \subseteq \sup h \circ i[\lambda] = \sup k \circ j[\lambda] \leq k(\nu)$, since $\sup j[\lambda] \leq \nu$. Hence $h(s) \subseteq k(\nu)$. Thus to show that $k(\nu)$ is a $k(p(i))$-generator of $h \circ i$, it suffices to show that $k(\nu)$ is a $h(q)$-generator of $h \circ i$. Since $h(q) \subseteq r$, it suffices to show that $k(\nu)$ is a r-generator of $k \circ j$. This is an immediate consequence of (7.9): by Lemma 5.4.28, $k(\nu)$ is the largest r-generator of $k \circ j$.

Thus $k(\nu)$ is a generator of h. Let W be the fine N-ultrafilter derived from h using $k(\nu)$. Then W is an incompressible ultrafilter of N. We have $\sup i[\lambda] \leq \delta_W$ since $\sup h[\sup i[\lambda]] = \sup k \circ j[\lambda] \leq k(\nu)$. Moreover $\delta_W \leq \xi$ since $k(\nu) < h(\xi)$. Since W is incompressible, $\lambda_W = \delta_W$. But λ_W is a Fréchet cardinal of N and

$$i(\delta_\lambda) \leq \sup i[\lambda] \leq \lambda_W \leq \xi < i(\lambda)$$

This contradicts the fact that there are no Fréchet cardinals in the interval $[i(\delta_\lambda), i(\lambda)]$.

It follows that our assumption that $k(\nu) \neq h(\xi)$ was false. This proves Claim 1. $\square$

Since $k(\nu) = h(\xi)$, it follows from Corollary 5.2.7 that $\mathscr{K}_\lambda \leq_{\text{RF}} U$, and this must be witnessed by the pushout (k, h) of (j, i) in the sense that h is the identity and $k : M \to N$ is the unique internal ultrapower embedding such that $k \circ j = i$. Thus $i = k \circ j = d \circ j_{0,n}$. Since $d : M_n \to N$ is the ultrapower of M_n by a countably complete ultrafilter D of M_n with $\lambda_D < j_{0,n}(\lambda)$, this proves the proposition. $\square$

Using Theorem 7.5.15, one can fully analyze decomposability at an isolated cardinal λ.

Corollary 7.5.16 (UA)**.** *Suppose λ is an isolated cardinal. Let*

$$\langle M_n, j_{m,n}, U_m : m \leq n < \omega \rangle$$

be the iterated ultrapower of $\mathscr{K}_\lambda$. Then for any ultrapower embedding $k : V \to P$, there is some $n < \omega$ such that

$$k = h \circ d \circ j_{0,n}$$

where $M_n \xrightarrow{d} N \xrightarrow{h} P$ are ultrapower embeddings with the following properties:

- *$d : M_n \to N$ is the ultrapower of M_n by a countably complete ultrafilter D of M_n with $\lambda_D < j_{0,n}(\delta_\lambda)$.*
- *$h : N \to P$ is an internal ultrapower embedding of N with $\text{crit}(h) > d(j_{0,n}(\lambda))$ if h is nontrivial.*

Proof. We claim there is a strong limit cardinal $\kappa > \lambda$ such that there are no Fréchet cardinals in the interval (λ, κ). If there are no Fréchet cardinals above λ, let $\kappa = \beth_\omega(\lambda)$. Otherwise, let $\kappa = \lambda^\sigma$. By Lemma 7.4.27, κ is measurable, and in particular, κ is a strong limit cardinal.

By Theorem 7.5.5, there is a countably complete ultrafilter U with $\lambda_U < \kappa$ such that there is an internal ultrapower embedding $h : M_U \to P$ with $h \circ j_U = j$ and $\text{crit}(h) \geq \kappa$. Since $\lambda_U < \kappa$ is Fréchet and there are no Fréchet cardinals in the interval (λ, κ), $\lambda_U \leq \lambda$. Therefore we may assume that U is a countably complete ultrafilter on λ. In particular $\text{crit}(h) \geq \kappa > j_U(\lambda)$.

Let $i = j_U$. By Theorem 7.5.15, for some $n < \omega$, $i = d \circ j_{0,n}$ where $d : M_n \to N$ is the ultrapower of M_n by a countably complete ultrafilter D of M_n with $\lambda_D < j_{0,n}(\delta_\lambda)$. Putting everything together,

$$j = h \circ d \circ j_{0,n}$$

and this proves the corollary. □

It is not a priori obvious that p^n contains all the generators ξ of $\mathscr{K}_\lambda^n$ with $\xi \geq \sup j_{0,n}[\lambda]$. In fact this is true:

Proposition 7.5.17 (UA)**.** *Suppose λ is an isolated cardinal and $n < \omega$. Then $\mathscr{K}_\lambda^n$ is the unique countably complete ultrafilter W on $[\lambda]^n$ such that id_W is the set of generators ξ of j_W with $\xi \geq \sup j_W[\lambda]$.*

Proof. Assume by induction that the corollary is true when $n = m$, and we will prove it when $n = m + 1$.

Therefore assume W is a countably complete ultrafilter on $[\lambda]^{m+1}$ such that id_W is the set of generators ξ of j_W with $\xi \geq \sup j_W[\lambda]$. Let q be the first m generators of j_W above $\sup j_W[\lambda]$. Let U be the ultrafilter derived from j_W using q. Then by our induction hypothesis, $U = \mathscr{K}_\lambda^m$. Let $d : M_m \to M_W$ be the factor embedding with $d \circ j_{0,m} = j_W$ and $d(p^m) = q$. By Theorem 7.5.15, there is an internal ultrapower embedding $d' : M_m \to M_W$. Note that $d'(p^m)$ is a set of generators of $d' \circ j_{0,m} = d \circ j_{0,m}$, so $d'(p^m) \geq d(p^m)$. On the other hand, $d'(p^m) \leq d(p^m)$ since otherwise $\mathscr{K}_\lambda^m <_\Bbbk \mathscr{K}_\lambda^m$. Thus $d'(p^m) = d(p^m)$. Since $d' \circ j_{0,m} = d \circ j_{0,m}$, we have $d' = d$. Thus d is an internal ultrapower embedding.

Let ξ be the largest generator of j_W. Thus $d(p^m) = q \subseteq \xi$, so ξ is a $d(p^m)$-generator of j_W and hence ξ is a generator of d. Let Z be the fine M_m-ultrafilter derived from d using ξ. Then Z is an incompressible ultrafilter of M_m and $\delta_Z \in [\sup j_{0,m}[\lambda], j_{0,m}(\lambda)]$. Since $\delta_Z = \lambda_Z$ is a Fréchet cardinal of M_m, the isolation of $j_{0,m}(\lambda)$ in M_m implies $\delta_Z = j_{0,m}(\lambda)$. Therefore by Theorem 7.5.7, $Z = j_{0,m}(\mathscr{K}_\lambda)$.

Since $M_W = H^{M_W}(j_W[V] \cup q \cup \{\xi\}) = H^{M_W}(d[M_m] \cup \{\xi\})$, we have $d = j_Z^{M_m} = j_{mm+1}$. Thus $d \circ j_{0,m} = j_{0,m+1}$. Thus $j_W = j_{0,m+1}$.

Since p^{m+1} consists solely of generators of $j_{0,m+1}$ above $\sup j_{0,m+1}[\lambda]$, $p^{m+1} \subseteq \mathrm{id}_W$. Since $|\mathrm{id}_W| = |p^{m+1}|$, it follows that $p^{m+1} = \mathrm{id}_W$. Therefore $W = \mathscr{K}_\lambda^{m+1}$, as desired. □

Proposition 7.5.18 (UA)**.** *Suppose λ is an isolated cardinal. Then $\mathscr{K}_\lambda^n$ is the unique countably complete ultrafilter W on $[\lambda]^n$ such that id_W is a set of generators of j_W disjoint from $\sup j_W[\lambda]$.*

Proof. Suppose W is such an ultrafilter. Let p be the set of all generators of ξ of j_W with $\xi \geq \sup j_W[\lambda]$. Let $m = |p|$. By Proposition 7.5.17, the ultrafilter derived from j_W using p is $\mathscr{K}_\lambda^m$. It follows that $j_W = j_{0,m}$ and $p = p^m$. Therefore $p \leq \mathrm{id}_W$ by the minimality of the Dodd parameter. On the other hand, $\mathrm{id}_W \subseteq p$ since p consists of all the generators of j_W above $\sup j_W[\lambda]$. Therefore $\mathrm{id}_W = p$. Hence $m = n$ and $W = \mathscr{K}_\lambda^n$, as desired. □

Proposition 7.5.18 implies Theorem 7.5.11:

Proof of Theorem 7.5.11. By Lemma 7.5.12, $\mathscr{K}_\lambda^n$ is a strongly fine incompressible ultrafilter on $[\lambda]^n$. On the other hand, any strongly fine incompressible ultrafilter W on $[\lambda]^n$ has the property that id_W is a set of generators of j_W disjoint from $\sup j_W[\lambda]$. Thus by Proposition 7.5.18, $\mathscr{K}_\lambda$ is the only strongly fine incompressible ultrafilter on $[\lambda]^n$. □

We also have an analog of Theorem 7.3.18 at isolated cardinals:

Theorem 7.5.19 (UA)**.** *Suppose λ is an isolated cardinal. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$ and let $\nu = \mathrm{id}_{\mathscr{K}_\lambda}$. Suppose Z is a countably complete M-ultrafilter that is δ-indecomposable for all M-cardinals $\delta \in [\sup j[\lambda], \nu]$. Then $Z \in M$.*

Proof. Let $e : M \to P$ be the ultrapower of M by Z. Then $e(\nu)$ is a generator of $e \circ j$ by Lemma 5.4.27. Since Z is δ-indecomposable for all $\delta \in [\sup j[\lambda], \nu]$, e has no generators in the interval $[\sup e \circ j[\lambda], e(\nu)]$. In other words, $e \circ j$ has no $e(\nu)$-generators in the interval $[\sup e \circ j[\lambda], e(\nu)]$.

Let $k = e \circ j$. Applying Corollary 7.5.16, there is some $n < \omega$ such that

$$k = h \circ d \circ j_{0,n}$$

where $M_n \xrightarrow{d} N \xrightarrow{h} P$ are ultrapower embeddings with the following properties:

- $d : M_n \to N$ is the ultrapower of M_n by a countably complete ultrafilter D of M_n with $\lambda_D < j_{0,n}(\delta_\lambda)$.
- $h : N \to P$ is an internal ultrapower embedding of N with $\text{crit}(h) > d(j_{0,n}(\lambda))$ if h is nontrivial.

Let $e' = h \circ d \circ j_{1,n}$, so that $e' : M \to P$ is an internal ultrapower embedding with $e' \circ j = k = e \circ j$. We claim $e'(\nu) = e(\nu)$.

First of all, $e'(\nu) \leq e(\nu)$: otherwise the comparison (e, e') witnesses $\mathscr{K}_\lambda <_\Bbbk \mathscr{K}_\lambda$ contrary to Proposition 3.3.9.

Suppose towards a contradiction $e'(\nu) < e(\nu)$. Then $e'(\nu)$ is not an $e(\nu)$-generator of $e \circ j = e' \circ j$. Note that $h(d(j_{1,n}(\nu))) = e'(\nu)$ and $h(e(\nu)) = e(\nu)$, so $d(j_{1,n}(\nu))$ is not an $e(\nu)$-generator of $d \circ j_{0,n}$. But consider the ultrafilter U on $[\lambda]^2$ derived from $d \circ j_{0,n}$ using $\{d(j_{1,n}(\nu)), e(\nu)\}$. Since $d(j_{1,n}(\nu))$ and $e(\nu)$ are generators of $d \circ j_{0,n}$ disjoint from $\sup d \circ j_{0,n}[\lambda]$, id_U consists of generators j_U disjoint from $\sup j_U[\lambda]$. Thus $U = \mathscr{K}_\lambda^2$ by Proposition 7.5.18. But then by Lemma 7.5.12, $\min(\text{id}_U)$ is a $\max(\text{id}_U)$-generator of j_U. This contradicts that $d(j_{1,n}(\nu))$ is not an $e(\nu)$-generator of $d \circ j_{0,n}$.

Since $e' = e$, e is an internal ultrapower embedding of M, which implies that $Z \in M$. ☐

The main application of Theorem 7.5.19 is the following fact:

Lemma 7.5.20 (UA)**.** *Assume λ is isolated and let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Either $j[\lambda] \subseteq \lambda$ or $\mathscr{K}_\lambda \cap M \in M$.*

Proof. Assume $\sup j[\lambda] > \lambda$. Then $\mathscr{K}_\lambda \cap M$ is not γ-decomposable for any M-cardinal $\gamma \in [\sup j[\lambda], j(\lambda))$. Therefore $\mathscr{K}_\lambda \cap M \in M$. ☐

Theorem 7.5.19 gives a coarse bound on the hypermeasurability of $\mathscr{K}_\lambda$ when λ is isolated.

Proposition 7.5.21 (UA)**.** *Suppose λ is isolated and let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Then $P(\lambda) \subseteq M$ if and only if $\mathscr{K}_\lambda$ is λ-complete.*

Proof. Assume $P(\lambda) \subseteq M$. Since $\mathscr{K}_\lambda \notin M$, $\mathscr{K}_\lambda \cap M \notin M$, so by Lemma 7.5.20, $\sup j[\lambda] \subseteq \lambda$. By the Kunen Inconsistency Theorem (Theorem 4.2.35), this implies $\text{crit}(j) \geq \lambda$. In other words, $\mathscr{K}_\lambda$ is λ-complete. ☐

A natural conjecture, strengthening Proposition 7.5.21, is that if λ is isolated and $P(\delta_\lambda) \subseteq M_{\mathscr{K}_\lambda}$, then λ is measurable. Short of proving Conjecture 7.4.8, this is the best possible bound on the hypermeasurability of $\mathscr{K}_\lambda$: the $<\delta_\lambda$-supercompactness of $j_{\mathscr{K}_\lambda}$ (Proposition 7.5.22) implies that $P(\gamma) \subseteq M_{\mathscr{K}_\lambda}$ for all $\gamma < \delta_\lambda$. This conjecture can be proved assuming δ_λ is a successor cardinal, but if δ_λ is inaccessible, we do not know how to prove that λ is measurable even assuming that $P(\gamma) \subseteq M_{\mathscr{K}_\lambda}$ for every $\gamma < \lambda$.

7.5.3 Isolated cardinals and the GCH

By Proposition 7.5.4, the existence of nonmeasurable isolated cardinals implies the failure of the Generalized Continuum Hypothesis. In this section, we study precisely how GCH fails below a nonmeasurable isolated cardinal. Here the cardinal δ_λ (see Definition 7.5.13) takes the center stage.

Proposition 7.5.22 (UA)**.** *Suppose λ is an isolated cardinal that is not measurable. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Let $\kappa = \kappa_\lambda$ and $\delta = \delta_\lambda$. Then the following hold:*

(1) Every regular cardinal $\iota \in [\kappa, \delta)$ is Fréchet.

(2) j is $<\delta$-supercompact.

(3) If δ is a limit cardinal, then δ is strongly inaccessible.

(4) Otherwise δ is the successor of a cardinal γ of cofinality at least κ. In fact, no cardinal in the interval $(\mathrm{cf}(\gamma), \gamma)$ is γ-strongly compact.

Proof. We first prove (1). Let $\eta \in [\iota, \delta)$ be a Fréchet cardinal. Then for any $\gamma \in [\kappa, \eta)$, then there is a Fréchet cardinal in $(\gamma, \eta]$. By Lemma 7.4.13, there are no isolated cardinals in $[\kappa, \lambda)$. Lemma 7.4.12 implies that every regular cardinal in $[\kappa, \eta)$ is Fréchet. In particular, ι is Fréchet.

We now prove (2). Fix a regular cardinal $\iota \in [\kappa, \delta)$, and we will show that j is ι-supercompact. (This suffices since the Recall that there are no isolated cardinals in $[\kappa, \lambda)$ (Lemma 7.4.13). Thus $\kappa_\iota \leq \kappa$ as a consequence of Lemma 7.4.19. Moreover, by Theorem 7.4.9, κ_ι is ι-strongly compact. We can therefore apply our technique for converting amenability of ultrafilters into hypermeasurability (Proposition 7.3.33) to conclude that $P(\iota) \subseteq M$: κ_ι is ι-strongly compact, M is closed under κ_ι-sequences, and every countably complete ultrafilter on ι is amenable to M (Proposition 7.4.17), so $P(\iota) \subseteq M$.

By Theorem 7.3.34, $j_{\mathscr{K}_\iota}$ is ι-tight. Moreover $j_{\mathscr{K}_\iota}(\kappa) \geq j_{\mathscr{K}_\iota}(\kappa_\iota) > \iota$. By Proposition 7.4.17, $\mathscr{K}_\iota \sqsubset \mathscr{K}_\lambda$. We now use the following fact:

Lemma. *Suppose $\kappa \leq \iota$ are cardinals, U and W are countably complete ultrafilters, U is ι-tight, $j_U(\kappa) > \iota$, W is κ-complete, and $U \sqsubset W$. Then j_W is ι-tight.*

Proof. Since $j_U(W)$ is ι-complete in M_U, $\mathrm{Ord}^\iota \cap M_U \subseteq M^{M_U}_{j_U(W)} = j_U(M_W) \subseteq M_W$. (The final containment uses $U \sqsubset W$.) Therefore since M_U has the $\leq\iota$-cover property, so does M_W. Thus j_W is ι-tight. □

We can apply the fact to $U = \mathscr{K}_\iota$ and $W = \mathscr{K}_\lambda$. It follows that j is ι-tight. Since j is ι-tight and $P(\iota) \subseteq M$, j is ι-supercompact.

We now prove (3). Suppose towards a contradiction that δ is singular. Then by (2), j is δ-supercompact. If $\mathrm{cf}(\delta) \geq \kappa_\lambda$, it follows that δ is Fréchet, contrary to the definition of δ_λ. Therefore $\mathrm{cf}(\delta) < \kappa_\lambda$. But then by Lemma 4.2.24, j is δ^+-supercompact. Then δ^+ is Fréchet. The definition of δ implies that no cardinal in $[\delta, \lambda)$ is Fréchet, so it must be that $\delta^+ = \lambda$. This contradicts that λ is isolated (and in particular is a limit cardinal).

For (4), assume towards a contradiction that some cardinal ν in the interval $(\text{cf}(\gamma), \gamma)$ is γ-strongly compact. Then ν it is γ^+-strongly compact by Lemma 4.2.24. But $\gamma^+ = \delta$ is not Fréchet, and this is contradiction. $\square$

Suppose λ is an isolated cardinal, and let $\delta = \delta_\lambda$. Must $2^{<\delta} = \delta$? By Proposition 7.5.22 (3), this is true if δ is a limit cardinal, but we are unable to answer the question when δ is a successor. The following bound is sufficient for most applications:

Theorem 7.5.23. *Suppose λ is isolated and $\delta = \delta_\lambda$. Then $2^{<\delta} < \lambda$.*

Proof. Assume by induction that the theorem holds for all isolated cardinals below λ. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Then j is $<\delta$-supercompact (Proposition 7.5.22). Thus $2^{<\delta} \leq (2^{<\delta})^M$, so it suffices to show that $(2^{<\delta})^M < \lambda$.

Claim 1. $(\delta^\sigma)^M \leq \lambda$.

Proof of Claim 1. There are two cases.

First assume $\sup j[\lambda] = \lambda$. Since j is $<\delta$-supercompact, Kunen's Inconsistency Theorem (Lemma 4.2.36) implies that there is a measurable cardinal $\iota < \delta$ such that $j(\iota) > \delta$. Now $j(\iota) < \lambda$ is a measurable cardinal of M, so $(\delta^\sigma)^M \leq j(\iota) < \lambda$, as desired.

Assume instead that $\lambda < \sup j[\lambda]$. Then $\mathscr{K}_\lambda \cap M \in M$ by Theorem 7.5.19. Thus λ is Fréchet in M, so $(\delta^\sigma)^M \leq \lambda$. $\square$

If δ^{+M} is Fréchet in M, then $(2^{<\delta})^M = \delta$ by Theorem 6.3.25. Assume therefore that δ^{+M} is not Fréchet in M. Let $\eta = (\delta^\sigma)^M$. Then η is isolated in M by Proposition 7.4.4. Moreover $\eta \leq \lambda < j(\lambda)$, so our induction hypothesis shifted to M applies at η. Notice that $\delta \leq (\delta_\eta)^M$: indeed, by Proposition 7.5.22, M is correct about cardinals below δ, and by Proposition 7.4.17, all sufficiently large cardinals below δ are Fréchet in M. Thus

$$(2^{<\delta})^M \leq (2^{<\delta_\eta^M})^M < \eta \leq \lambda$$

In particular $(2^{<\delta})^M < \lambda$, as desired. $\square$

The following closely related fact can be seen as an ultrafilter-theoretic version of SCH:

Proposition 7.5.24 (UA)**.** *Suppose λ is a regular isolated cardinal. Suppose D is a countably complete ultrafilter such that $\lambda_D < \lambda$. Then $j_D(\lambda) = \lambda$.*

Proof. Suppose towards a contradiction that the theorem fails, and let λ be the least counterexample. Let $j : V \to M$ be the ultrapower by $\mathscr{K}_\lambda$. Let $\delta = \delta_\lambda$ be the strict supremum of the Fréchet cardinals below λ. By Proposition 7.5.22,

$M^{<\delta} \subseteq M$, and by Proposition 7.4.17, M satisfies that there is a countably complete ultrafilter D with $\lambda_D < \delta$ such that $j_D(\lambda) \neq \lambda$.

Suppose first that $\lambda < \sup j[\lambda]$. Then $\mathscr{K}_\lambda \cap M \in M$ by Theorem 7.5.19. Therefore λ is a regular Fréchet cardinal in M. Clearly λ is a limit cardinal in M. Since $\lambda < j(\lambda)$, λ is not a counterexample to the proposition in M. Therefore λ is not isolated in M, so λ is strongly inaccessible in M by Corollary 7.5.2. But this contradicts that there is a countably complete ultrafilter D with $\lambda_D < \delta$ such that $j_D(\lambda) \neq \lambda$.

Suppose instead that $\lambda = \sup j[\lambda]$. Let $\kappa = \kappa_\lambda$. We claim that for any countably complete ultrafilter $U \in V_\kappa$, $j_U(\lambda) = \lambda$. Fix such an ultrafilter U. Since $2^{<\delta} < \lambda$, $j_U(\delta) < \lambda$. By elementarity there are no Fréchet cardinals of M_U in the interval $[j_U(\delta), j_U(\lambda))$. But $\mathscr{K}_\lambda \sqsubset U$ (by Kunen's commuting ultrapowers lemma, Theorem 5.5.19), so $\mathscr{K}_\lambda \cap M_U \in M_U$, and hence λ is Fréchet in M_U. Thus λ is a Fréchet cardinal of M_U in the interval $[j_U(\delta), j_U(\lambda)]$, so we must have $j_U(\lambda) = \lambda$, as claimed.

Let η be the least ordinal such that for some ultrafilter D with $\lambda_D < \lambda$, $j_D(\eta) > \lambda$. (Note that η exists since λ is regular.) Fix such an ultrafilter D. We claim $j(\eta) = \eta$. Assume not. Note that if Z is an ultrafilter with $\lambda_Z < \lambda$, then $j_Z[\eta] \subseteq \eta$. (Otherwise $D \times Z$ would contradict the minimality of η as in Lemma 7.4.25.) By elementarity, M satisfies that $j_Z[j(\eta)] \subseteq j(\eta)$ for all countably complete ultrafilters Z with $\lambda_Z < j(\lambda)$. If $j(\eta) > \eta$, however, then $j(\eta) < \lambda < j_D(\eta)$, which is a contradiction since $j_D \restriction M = j_D^M$ is an internal ultrapower embedding of M.

Suppose ξ is a fixed point of j. Let γ be the least cardinal that carries a countably complete ultrafilter U such that $j_U(\eta) > \xi$. We claim $\gamma < \kappa$. Note that $\gamma < \delta$ (by assumption). Moreover, $j(\gamma) = \gamma$: M is closed under γ-sequences and contains every ultrafilter on γ, so M satisfies that there is an ultrafilter U on γ such that $j_U(\eta) > \xi$; since $j(\eta) = \eta$ and $j(\xi) = \xi$, it follows that $j(\gamma)$ is the least M cardinal carrying such an ultrafilter U, and hence $j(\gamma) = \gamma$. Since γ is a fixed point of j and $j[\gamma] \in M$, $\gamma < \kappa$ by the Kunen Inconsistency Theorem (Theorem 4.2.35).

Since j has arbitrarily large fixed points below λ, it follows that for all $\xi < \lambda$, there is an ultrafilter U on a cardinal less than κ such that $j_U(\eta) > \xi$. Since λ is regular, there must be a single ultrafilter $U \in V_\kappa$ such that $j_U(\eta) \geq \lambda$. This contradicts that for all $U \in V_\kappa$, $j_U(\lambda) = \lambda$. $\square$

Our next theorem shows that the problematic isolated cardinals λ suffer a massive failure of GCH precisely at δ_λ:

Theorem 7.5.25. *Suppose λ is a nonmeasurable isolated cardinal and $\delta = \delta_\lambda$. Then $2^\delta \geq \lambda$.*

It is not clear whether it is possible that $2^\delta = \lambda$. This of course implies that λ is regular and hence weakly Mahlo by Theorem 7.5.38 below.

This theorem requires a factorization lemma due to Silver which can be seen as an improvement of Lemma 5.5.25 in the key special case in which the ultrafilter under consideration is indecomposable.

Theorem 7.5.26 (Silver). *Suppose δ is a regular cardinal and U is a countably complete ultrafilter that is λ-indecomposable for all $\lambda \in [\delta, 2^\delta]$. Then there is an ultrafilter D with $\lambda_D < \delta$ such that there is an elementary embedding $k : M_D \to M_U$ with $j_U = k \circ j_D$ and $\text{crit}(k) > j_D((2^\delta)^+)$ if k is nontrivial.*

The proof does not really use that U is countably complete, and this was important in Silver's original work. Since we only need the theorem when U is countably complete, we make this assumption. (This is for notational convenience: the notion of the critical point of k does not really make sense if M_D is illfounded.)

We begin by describing a correspondence between partitions modulo an ultrafilter and points in the ultrapower that is implicit in Silver's proof.

Definition 7.5.27. Suppose P is a partition of a set X and A is a subset of X. Then the *restriction of P to A* is the partition $P \restriction A$ defined by

$$P \restriction A = \{A \cap S : S \in P \text{ and } A \cap S \neq \emptyset\}$$

Definition 7.5.28. Suppose U is an ultrafilter on a set X and λ is a cardinal.

- $\mathbb{P}_U$ denotes the preorder on M_U defined by setting $x \leq y$ if x is definable in M from y and parameters in $j_U[V]$.
- $\mathbb{P}_U^\lambda \subseteq \mathbb{P}_U$ is the restriction of $\mathbb{P}_U$ to $H^{M_U}(j_U[V] \cup \sup j_U[\lambda])$.
- $\mathbb{Q}_U$ denotes the preorder on the collection of partitions of X defined by setting $P \leq Q$ if there exists some $A \in U$ such that $Q \restriction A$ refines $P \restriction A$.
- $\mathbb{Q}_U^\lambda \subseteq \mathbb{Q}_U$ consists of those P such that $|P \restriction A| < \lambda$ for some $A \in U$.

The following lemma, which is ultimately just an instance of the correspondence between partitions of X and surjective functions on X, shows that the preorders $\mathbb{Q}_U$ and $\mathbb{P}_U$ are equivalent preorders:

Lemma 7.5.29. *Suppose U is an ultrafilter on a set X. Define $\Phi : \mathbb{Q}_U \to \mathbb{P}_U$ by setting $\Phi(P)$ equal to the unique $S \in j_U(P)$ such that $\text{id}_U \in S$. Then the following hold:*

(1) Φ is order-preserving: for any $P, Q \in \mathbb{Q}_U$, $P \leq Q$ if and only if $\Phi(P) \leq \Phi(Q)$.

(2) Φ is surjective on equivalence classes: for any $x \in \mathbb{P}_U$, there is some $P \in \mathbb{Q}_U$ such that x and $\Phi(P)$ are equivalent in $\mathbb{P}_U$.

(3) For any cardinal λ, $\Phi[\mathbb{Q}_U^\lambda] \subseteq \mathbb{P}_U^\lambda$.

(4) Suppose $P \in \mathbb{Q}_U$. Let $D = \{A \subseteq P : \bigcup A \in U\}$. Then there is a unique elementary embedding $k : M_D \to M_U$ such that $k \circ j_D = j_U$ and $k(\text{id}_D) = \Phi(D)$.

Proof. Proof of (1): Suppose $P, Q \in \mathbb{Q}_U$ and $P \leq Q$. Fix $A \in U$ such that $Q \restriction A$ refines $P \restriction A$. Then $\Phi(P)$ is definable in M_U from the parameters $\Phi(Q), j_U(P), j_U(A)$ as the unique $S \in j_U(P)$ such that $\Phi(Q) \cap j_U(A) \subseteq S \cap j_U(A)$. In other words, $\Phi(P) \leq \Phi(Q)$.

Conversely suppose $\Phi(P) \leq \Phi(Q)$, so that $\Phi(P) = j_U(f)(\Phi(Q))$ for some $f : Q \to P$. Let $A \subseteq X$ consist of those $x \in X$ such that $x \in f(S)$ where S is the unique element of Q with $x \in S$. Then $A \in U$ since $\mathrm{id}_U \in j_U(f)(S)$ where $S = \Phi(Q)$ is the unique $S \in j_U(Q)$ such that $\mathrm{id}_U \in S$. Moreover for any $S \in Q$, $S \cap A \subseteq f(S) \cap A$, so $Q \restriction A$ refines $P \restriction A$. In other words, $P \leq Q$.

Proof of (2): Fix $x \in \mathbb{P}_U$. Fix $f : X \to V$ such that $x = j_U(f)(\mathrm{id}_U)$. Let

$$P = \{f^{-1}[\{y\}] : y \in \mathrm{ran}(f)\}$$

Then $\Phi(P)$ is interdefinable with x over M_U using parameters in $j_U[V]$: $\Phi(P)$ is the unique $S \in j_U(P)$ such that $x \in j_U(f)[S]$; and since $j_U(f)[\Phi(P)] = \{x\}$, $x = \bigcup j_U(f)[\Phi(P)]$.

Proof of (3): Suppose $P \in \mathbb{Q}_U^\lambda$. Fix $\delta < \lambda$ and a surjection $f : \delta \to P$. Then $\Phi(P) = j_U(f)(\xi)$ for some $\xi < j_U(\delta) \leq \sup j_U[\lambda]$. Hence $\Phi(P) \in H^{M_U}(j_U[V] \cup \sup j_U[\lambda])$, as desired.

Proof of (4): Define $g : X \to P$ by setting $g(a)$ equal to the unique $S \in P$ such that $a \in S$. Then $g_*(U) = D$ and $j_U(g)(\mathrm{id}_U) = \Phi(P)$. Therefore by the basic theory of the Rudin-Keisler order (Corollary 5.2.8), there is a unique elementary embedding $k : M_D \to M_U$ with $k \circ j_D = j_U$ and $k(\mathrm{id}_D) = \Phi(P)$. ☐

As a corollary, we obtain a useful reformulation of indecomposability in terms of partitions:

Lemma 7.5.30. *Suppose U is an ultrafilter on X and λ is a cardinal. Then U is λ-indecomposable if every partition of X into λ pieces is equivalent in $\mathbb{Q}_U$ to a partition of X into fewer than λ pieces.* ☐

We now prove Silver's theorem.

Proof of Theorem 7.5.26. Let $(\mathbb{Q}, \leq) = \mathbb{Q}_U^{(2^\delta)^+}$ be the preorder of U-refinement on the set of partitions of X of size at most 2^δ. Let $\preceq$ be the preorder of refinement on $\mathbb{Q}$, so $P \preceq Q$ implies Q refines P. Thus $(\mathbb{Q}, \leq)$ extends $(\mathbb{Q}, \preceq)$.

Note that $\preceq$ is $\leq\delta$-directed. Indeed, suppose $\mathcal{S} \subseteq \mathbb{Q}$ has cardinality δ. Then

$$P = \left\{\bigcap \mathscr{C} : \mathscr{C} \text{ meets each element of } \mathcal{S} \text{ and } \bigcap \mathscr{C} \neq \emptyset\right\}$$

refines every partition in $\mathcal{S}$, and $|P| \leq |\prod \mathcal{S}| \leq 2^\delta$. The partition P is called the *least common refinement* of $\mathcal{S}$.

We claim that $(\mathbb{Q}, \leq)$ has a maximum element (up to equivalence). Since $(\mathbb{Q}, \preceq)$ is directed, $(\mathbb{Q}, \leq)$ is directed, and thus it suffices to show that $(\mathbb{Q}, \leq)$ has a maximal element. Assume the contrary, towards a contradiction. Then since $(\mathbb{Q}, \preceq)$ is $\leq\delta$-directed, we can produce a sequence $\langle P_\alpha : \alpha \leq \delta\rangle$ of elements of $\mathbb{Q}$ such that for all $\alpha < \beta \leq \delta$, $P_\alpha \preceq P_\beta$ and $P_\beta \not\leq P_\alpha$.

Since U is λ-indecomposable for all $\lambda \in [\delta, 2^\delta]$, there is some $A \in U$ such that $|P_\delta \restriction A| < \delta$. For each $\alpha \leq \delta$, let $Q_\alpha = P_\alpha \restriction A$. We use the following general fact:

Claim. *Suppose δ is a regular cardinal, A is a set of size less than δ, and $\langle Q_\alpha : \alpha < \delta \rangle$ is a sequence of partitions of A such that for all $\alpha < \beta < \delta$, Q_β refines Q_α. Then for all sufficiently large $\alpha < \beta < \delta$, $Q_\alpha = Q_\beta$.*

Proof. Let Q be the least common refinement of $\{Q_\alpha : \alpha < \delta\}$. Suppose $S \in Q$. We claim that $S \in Q_\alpha$ for some $\alpha < \delta$. Consider the sequence $\langle S_\alpha : \alpha < \delta \rangle$ where $S_\alpha \in Q_\alpha$ is the unique element of Q_α containing S. Thus $S = \bigcap_{\alpha < \delta} S_\alpha$. Note that $\langle S_\alpha : \alpha < \delta \rangle$ is a decreasing sequence of sets, each of cardinality less than δ. Thus for all sufficiently large $\alpha < \delta$, $S_\alpha = S$, and in particular, $S \in Q_\alpha$.

For each $S \in Q$, fix $\alpha_S < \delta$ such that $S \in Q_{\alpha_S}$. Let $\gamma = \sup_{S \in Q} \alpha_S$. Then $\gamma < \delta$ since $|Q| < \delta$ and δ is regular. By definition, $Q \subseteq Q_\gamma$, so $Q_\gamma = Q$, If $\alpha \in [\gamma, \delta)$, then Q refines Q_α which refines $Q_\gamma = Q$, and hence $Q = Q_\alpha$. This proves the claim. $\square$

Thus for all sufficiently large $\alpha < \beta < \delta$, $Q_\alpha = Q_\beta$, or in other words, $P_\alpha \restriction A = P_\beta \restriction A$. It follows that $P_\beta \leq P_\alpha$, and this contradicts our choice of P_β. Thus our assumption that $(\mathbb{Q}, \leq)$ has no maximum element was false.

Let P be a maximum element of $(\mathbb{Q}, \leq)$. By the indecomposability of U, we may assume $|P| < \delta$ by replacing P with an equivalent element of $(\mathbb{Q}, \leq)$. We now apply Lemma 7.5.29. Let D be the ultrafilter corresponding to P as in Lemma 7.5.29 (4):

$$D = \{\mathscr{A} \subseteq P : \bigcup \mathscr{A} \in U\}$$

Let $k : M_D \to M_U$ be unique elementary embedding with $k \circ j_D = j_U$ and $k(\mathrm{id}_D) = \Phi(P)$. We have $\lambda_D < \delta$ since $|P| < \delta$.

Let $\eta = (2^\delta)^+$. We will show $\mathrm{crit}(k) > j_U(\eta)$ if k is nontrivial, or in other words, that $j_U(\eta) \subseteq k[M_D]$. Since P is a maximum element of $\mathbb{Q}^\eta_U$, Lemma 7.5.29 (1), (2), and (3) imply that $\Phi(P)$ is a maximum element of $\mathbb{P}^\eta_U$. In other words, if $x \in H^{M_U}(j_U[V] \cup \sup j_U[\eta])$, then x is definable in M_U from $\Phi(P)$ and parameters in $j_U[V]$, or in other words $x \in k[M_D]$. In particular, $\sup j_U[\eta] \subseteq k[M_D]$.

We finish by showing that $\sup j_U[\eta] = j_U(\eta)$. Suppose not. Then since η is regular, U is η-decomposable. Since $\eta = (2^\delta)^+$ and U is not (κ, η)-regular for any $\kappa < 2^\delta$, Theorem 7.4.3 implies that U is λ-decomposable where $\lambda = \mathrm{cf}(2^\delta)$. But by König's Theorem, $\lambda \in [\delta, 2^\delta]$, and this is a contradiction. $\square$

We can finally prove Theorem 7.5.25:

Proof of Theorem 7.5.25. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Assume $2^\delta < \lambda$. We will show that $\mathrm{crit}(j) \geq \lambda$, so $\mathscr{K}_\lambda$ is a λ-complete uniform ultrafilter on λ, and hence λ is measurable.

Since λ is isolated and $2^\delta < \lambda$, $\mathscr{K}_\lambda$ is γ-indecomposable for all cardinals in the interval $[\delta, 2^\delta]$. By Proposition 7.5.22, δ is regular. Therefore we can apply Theorem 7.5.26. Fix D with $\lambda_D < \delta$ such that there is an elementary embedding embedding $k : M_D \to M_{\mathscr{K}_\lambda}$ with $k \circ j_D = j$ and $\mathrm{crit}(k) > j_D(\delta)$ if k is nontrivial.

By Proposition 7.4.17, $D \sqsubset \mathscr{K}_\lambda$. Therefore $j_D \restriction \delta \in M$. But $j_D \restriction \delta = j \restriction \delta$ since $\text{crit}(k) > j_D(\delta)$. It follows that j is δ-supercompact. Since δ is regular and $\mathscr{K}_\lambda$ is δ-indecomposable, $j(\delta) = \sup j[\delta]$. Since j is δ-supercompact and $j(\delta) = \sup j[\delta]$, the Kunen Inconsistency (Theorem 4.4.32) implies that $\text{crit}(j) \geq \delta$. There are no measurable cardinals in the interval $[\delta, \lambda)$ since in fact there are no Fréchet cardinals in $[\delta, \lambda)$. The fact that $\text{crit}(j) \geq \delta$ therefore implies $\text{crit}(j) \geq \lambda$, as desired. $\square$

Theorem 7.5.26 can be combined with Theorem 7.4.28 to prove a strengthening of Theorem 7.5.5:

Theorem 7.5.31 (UA)**.** *Suppose δ is a regular cardinal and U is a countably complete ultrafilter that is λ-indecomposable for all $\lambda \in [\delta, 2^\delta]$. Then there is an ultrafilter D with $\lambda_D < \delta$ such that there is an internal ultrapower embedding $h : M_D \to M_U$ with $h \circ j_D = j_U$ and* $\text{crit}(h) > j_D(\delta)$ *if h is nontrivial.*

Proof. Using Silver's theorem, fix a uniform countably complete ultrafilter D on a cardinal $\eta < \delta$ such that there is an elementary embedding $k : M_D \to M_U$ with $k \circ j_D = j_U$ and $\text{crit}(k) > j_D((2^\delta)^+)$ if k is nontrivial.

Recall that $\mathbf{UF}(X)$ denotes the set of countably complete ultrafilters on X. Theorem 7.4.28 implies that $|\mathbf{UF}(\eta)| \leq (2^\eta)^+$. Thus $j_D(\mathbf{UF}(\eta))$ has cardinality less than or equal to $j_D((2^\eta)^+)$ in M_U. Since $\text{crit}(k) > j_D((2^\eta)^+)$, k restricts to an isomorphism from $j_D(\mathbf{UF}(\eta), <_\Bbbk)$ to $j_U(\mathbf{UF}(\eta), <_\Bbbk)$. Moreover, for any $Z \in j_D(\mathbf{UF}(\eta))$,

$$j_D^{-1}[Z] = j_U^{-1}[k(Z)]$$

We now use the fact that k is an isomorphism conjugating j_D^{-1} to j_U^{-1} to conclude that $k(t_D(D)) = t_U(D)$. By Theorem 5.4.42, $t_D(D)$ is the least element of $j_D(\mathbf{UF}(\eta), <_\Bbbk)$ with $j_D^{-1}[Z] = D$. Therefore since k is an order-isomorphism that conjugates j_D^{-1} to j_U^{-1}, $k(t_D(D))$ is the least element Z of $j_U(\mathbf{UF}(\eta), <_\Bbbk)$ with $j_U^{-1}[Z] = D$. But by Theorem 5.4.42, the least such Z is equal to $t_U(D)$. Thus $k(t_D(D)) = t_U(D)$.

Recall the characterization of the Rudin-Frolík order in terms of translation functions (Lemma 5.4.41): if W and Z are countably complete ultrafilters, then $W \leq_{\text{RF}} Z$ if and only if $t_Z(W)$ is principal in M_Z. Applying this characterization in one direction to $D \leq_{\text{RF}} D$, $t_D(D)$ is principal in M_D. Therefore $t_U(D) = k(t_D(D))$ is principal in M_U, so applying the characterization in the other direction, it follows that $D \leq_{\text{RF}} U$.

Let $h : M_D \to M_U$ be the unique internal ultrapower embedding such that $h \circ j_D = j_U$. By Lemma 5.4.41, $t_D(D)$ is the principal ultrafilter concentrated at id_D and $t_U(D)$ is the principal ultrafilter concentrated at $h(\text{id}_D)$. Since $k(t_D(D)) = t_U(D)$, it follows that $k(\text{id}_D) = h(\text{id}_D)$. Since $k \circ j_D = j_U$, in fact $k \restriction j_D[V] \cup \{\text{id}_D\} = h \restriction j_D[V] \cup \{\text{id}_D\}$. Thus $k = h$, since $M_D = H^{M_D}(j_D[V] \cup \{\text{id}_D\})$. It follows that $h : M_D \to M_U$ is an internal ultrapower embedding with $h \circ j_D = j_U$ and $\text{crit}(h) > j_D(\delta)$ if h is nontrivial. $\square$

Our work on isolated cardinals leads to some relatively simple criteria for the completeness of an ultrafilter in terms of a local version of irreducibility that will become important when we analyze larger supercompact cardinals:

Definition 7.5.32. Suppose λ is a cardinal and U is a countably complete ultrafilter.

- U is *λ-irreducible* if for all $D \leq_{\text{RF}} U$ with $\lambda_D < \lambda$, D is principal.
- U is *$\leq\lambda$-irreducible* if U is λ^+-irreducible.

Note that U is $\leq\lambda$-irreducible if and only if U is λ^σ-irreducible.

At isolated cardinals, we have the following fact which is often useful:

Theorem 7.5.33 (UA)**.** *Suppose λ is a cardinal and U is a countably complete ultrafilter.*

(1) If λ is a strong limit cardinal that is not a limit of Fréchet cardinals, then U is λ-irreducible if and only if U is λ-complete.

(2) If λ is a strong limit cardinal and no cardinal $\kappa < \lambda$ is γ-supercompact for all $\gamma < \lambda$, then U is λ-irreducible if and only if U is λ-complete.

(3) If λ is isolated, then U is λ^+-irreducible if and only if U is λ^+-complete.

Proof. (1) is immediate from Theorem 7.5.5.

(2) follows from (1). By Corollary 7.5.2, either λ is not a limit of Fréchet cardinals or λ is a limit of isolated cardinals. The former case is precisely (1). In the latter case, we can apply (1) at each isolated cardinal below λ. Thus we conclude that U is $\bar{\lambda}$-complete for all isolated cardinals $\bar{\lambda} < \lambda$. It follows that U is λ-complete as desired.

(3) also follows from (1). Since U is λ^+-irreducible, U is λ^σ-irreducible, and by Lemma 7.4.27, λ^σ is measurable. Thus U is λ^σ-complete by (1) and in particular, U is λ^+-complete. □

Working in a bit more generality but with a stronger irreducibility assumption, we have the following completeness result:

Theorem 7.5.34 (UA)**.** *Suppose δ is a regular cardinal such that no cardinal $\kappa \leq \delta$ is δ-supercompact. Then a countably complete ultrafilter U is δ^+-complete if and only if it is $\leq 2^\delta$-irreducible.*

Proof. The forward direction is trivial, so let us prove the converse.

Suppose that U is $\leq 2^\delta$-irreducible. We claim that U is λ-irreducible where $\lambda > \delta$ is a strong limit cardinal that is not a limit of Fréchet cardinals. An immediate consequence of the factorization theorem for isolated measurable cardinal (Theorem 7.5.5) is that any λ-irreducible ultrafilter is λ-complete, and this proves the theorem.

If δ^σ does not exist, then the $\leq\delta$-irreducibility of U implies that U itself is principal, so U is λ-irreducible and λ-complete for any cardinal λ. Thus assume δ^σ exists.

There are two cases. Suppose first that δ is a Fréchet cardinal. Let $\lambda = \delta^\sigma$. Since U is $\leq\delta$-irreducible, U is λ-irreducible. We claim that λ is an isolated measurable cardinal. First note that $\lambda > \delta^+$ since otherwise κ_{δ^+} is δ-supercompact by Corollary 7.4.10. Thus by Proposition 7.4.4, λ is isolated. Assume towards a contradiction that λ is not measurable. Then by Proposition 7.5.22, κ_λ is $<\delta_\lambda$-supercompact. But $\delta < \delta_\lambda$ since $\delta < \lambda$ is Fréchet, and hence κ_λ is δ-supercompact, a contradiction. Hence λ is measurable.

Suppose instead that δ is not a Fréchet cardinal. If δ^σ is measurable, let $\lambda = \delta^\sigma$. Suppose δ^σ is not measurable. By Theorem 7.5.25, $\delta^\sigma \leq 2^\delta$, so in particular U is $\leq\delta^\sigma$-irreducible. Let $\lambda = \delta^{\sigma\sigma}$. (If $\delta^{\sigma\sigma}$ does not exist, then again since U $\leq\delta^\sigma$-irreducible, U is principal.) By Lemma 7.4.27, λ is measurable; here, one must check that δ^σ is isolated. Since U is $\leq 2^\delta$-irreducible, U is $\leq\delta^\sigma$-irreducible, so U is $<\lambda$-irreducible. ☐

One might expect a strengthening of this theorem to be true: if U is just $\leq\delta$-irreducible and no $\kappa \leq \delta$ is δ-supercompact, then U should be δ^+-complete. The main issue is that if $\lambda = \delta^\sigma$ is a nonmeasurable isolated cardinal, then $U = \mathscr{K}_\lambda$ is a counterexample. If instead δ is measurable, then $\leq\delta$-irreducibility indeed suffices.

One might also hope that assuming $2^{<\delta} = \delta$ and that no cardinal is δ-supercompact, then δ-irreducibility implies δ-completeness. If δ is the least cardinal such that $\mathscr{K}_\delta$ exists and does not have a δ-supercompact ultrapower, then $U = \mathscr{K}_\delta$ is a counterexample. Thus each of the main problems in the UA analysis of supercompact cardinals arise as obstructions to the natural attempt to generalize Theorem 7.5.34.

A theorem similar to Theorem 7.5.34 holds for singular cardinals:

Theorem 7.5.35 (UA)**.** *Suppose U is a countably complete ultrafilter and γ is a singular cardinal such that no $\kappa \leq \gamma$ is γ^+-supercompact. Then U is γ^+-complete if and only if U is $\leq 2^\delta$-irreducible for all $\delta < \gamma$.*

Proof. Let $\delta = \sup\{\gamma^+ : \gamma < \lambda \text{ is a Fréchet cardinal}\}$.

Suppose first that δ is regular. Since δ is not Fréchet, no cardinal $\kappa \leq \delta$ is δ-supercompact. Since U is $\leq 2^\delta$-irreducible, we are in a position to apply Theorem 7.5.34. We can conclude that U is δ^+-complete. Since there are no measurable cardinals in the interval (δ, γ), it follows that U is γ^+-complete.

Suppose instead that δ is singular. If δ^σ does not exist, then it is easy to see that U is principal, and thus we are done. Therefore assume δ^σ exists, and let $\lambda = \delta^\sigma$. Note that $\lambda > \delta^+$: if $\delta < \gamma$ this follows from the fact that δ^+ is not Fréchet, while if $\delta = \gamma$, this follows from the fact that no cardinal is γ^+-supercompact. Thus λ is isolated. Note that $\delta_\lambda = \delta$ is singular. Therefore by Proposition 7.5.22, λ is measurable. Since U is $\leq\delta$-irreducible, U is $<\lambda$-irreducible, and therefore as an immediate consequence of the factorization theorem for isolated measurable cardinal (Theorem 7.5.5), U is λ-complete. ☐

Let us close this subsection with a remark about the size of regular isolated cardinals.

Definition 7.5.36. A regular cardinal κ is *σ-Mahlo* if there is a countably complete weakly normal ultrafilter on κ that concentrates on regular cardinals.

Proposition 7.5.37. *If κ is σ-Mahlo then κ is weakly Mahlo.* $\square$

In fact, σ-Mahlo cardinals are "greatly weakly Mahlo." A theorem of Gitik [35] shows that it is consistent that there is a σ-Mahlo cardinal that does not have the tree property.

Theorem 7.5.38 (UA)**.** *Suppose κ is a regular isolated cardinal. Then κ is σ-Mahlo. In fact, $\mathscr{K}_\kappa$ concentrates on regular cardinals.*

Proof. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\kappa$. Let $\kappa_* = \sup j[\kappa]$. Let $\delta = \text{cf}^M(\kappa_*)$. By Theorem 7.3.34, j is (κ, δ)-tight, so j is discontinuous at any regular cardinal $\iota \leq \kappa$ such that $\delta < j(\iota)$. Since κ is isolated, j is continuous at all sufficiently large regular cardinals less than κ. Putting these observations together, it follows that there are no regular cardinals $\iota < \kappa$ such that $j(\iota) > \delta$. In other words, $\sup j[\kappa] \leq \delta$. Thus $\kappa_* = \delta$, so κ_* is regular. Since $\mathscr{K}_\kappa$ is weakly normal, $\kappa_* = \text{id}_{\mathscr{K}_\kappa}$ by Lemma 4.4.17. Hence by Łoś's Theorem, $\mathscr{K}_\kappa$ concentrates on regular cardinals. $\square$

This fact has a converse: assuming UA, any σ-Mahlo cardinal that is not measurable is isolated. It is not clear that singular Fréchet cardinals must be very large. For example, we do not know how to rule out that the least Fréchet cardinal λ that is neither measurable nor a limit of measurables is in fact equal to $\kappa^{+\kappa}$ for some measurable $\kappa < \lambda$.

7.5.4 The linearity of the Mitchell order without GCH

Theorem 4.4.2 states that assuming UA + GCH, the Mitchell order is linear on normal fine ultrafilters on $P_{\text{bd}}(\lambda)$, the collection of bounded subsets of λ. Here we prove essentially the same theorem using UA alone. Because we may have $|P_{\text{bd}}(\lambda)| > \lambda$, however, we cannot in general consider normal fine ultrafilters on $P_{\text{bd}}(\lambda)$, which may not be uniform. This issue stems from the fact that the Mitchell order (unlike the internal relation) is not invariant under changes of space in its first argument.

One way of stating our theorem involves the notion of a hereditarily uniform ultrafilter. Recall that an ultrafilter U on a set X is *hereditarily uniform* if if the cardinality of the transitive closure of X is equal to the size of U, the minimum cardinality λ_U of a set of U-measure 1. We observed that the generalized Mitchell order is well-behaved on hereditarily uniform ultrafilters: for example it is Rudin-Keisler invariant (Lemma 4.2.13) and transitive (Proposition 4.2.42).

Theorem 7.5.39 (UA)**.** *If $\mathscr{U}$ and $\mathscr{W}$ are normal fine hereditarily uniform ultrafilters, either $\mathscr{U} \lhd \mathscr{W}$, $\mathscr{W} \lhd \mathscr{U}$, or $\mathscr{U}$ and $\mathscr{W}$ are Rudin-Keisler equivalent.*

There is also a second, more concrete way to state the main theorem of this section, which works by replacing $P_{\text{bd}}(\lambda)$ in Theorem 4.4.2 with the following slightly smaller set:

Definition 7.5.40. For any cardinal λ, let $P_*(\lambda) = \{\sigma \in P_{\text{bd}}(\lambda) : |\sigma|^+ < \lambda\}$.

There is an obvious characterization of $P_*(\lambda)$ that is often more useful than the definition above:

$$P_*(\lambda) = \begin{cases} P_{\text{bd}}(\lambda) & \text{if } \lambda \text{ is a limit cardinal} \\ P_\gamma(\lambda) & \text{if } \lambda \text{ is a successor cardinal and } \gamma \text{ is its cardinal predecessor} \end{cases}$$

Definition 7.5.41. For any cardinal λ, let $\mathscr{U}_\lambda$ denote the set of normal fine ultrafilters on $P_*(\lambda)$. Let $\mathscr{U} = \bigcup_{\lambda \in \text{Card}} \mathscr{U}_\lambda$.

Theorem 7.5.42 (UA)**.** *The class $\mathscr{U}$ is linearly ordered by the Mitchell order.*

Given the following proposition, Theorem 7.5.42 can be seen as a precise formulation of the (literally false) statement that the Mitchell order is linear on normal fine ultrafilters:

Proposition 7.5.43. *Every normal fine ultrafilter is Rudin-Keisler equivalent to a unique element of $\mathscr{U}$.*

Proof. Recall that for any cardinal λ, $\mathscr{N}_\lambda$ denotes the set of normal fine ultrafilters on $P_{\text{bd}}(\lambda)$ and $\mathscr{N} = \bigcup_{\lambda \in \text{Card}} \mathscr{N}_\lambda$. Also recall Proposition 4.4.12, which states that every normal fine ultrafilter is Rudin-Keisler equivalent to a unique element of $\mathscr{N}$. Therefore to prove the proposition, it suffices to show that there is a bijection $\phi : \mathscr{N} \to \mathscr{U}$ such that $\phi(\mathcal{U}) \equiv_{\text{RK}} \mathcal{U}$ for all $\mathcal{U} \in \mathscr{N}$.

In fact, if $\mathcal{U} \in \mathscr{N}_\lambda$, we will just set $\phi(\mathcal{U}) = \mathcal{U} \restriction P_*(\lambda)$. It is clear that ϕ is as desired as long as $P_*(\lambda) \in \mathcal{U}$. We now establish that this holds. Let $j : V \to M$ be the ultrapower of the universe by $\mathcal{U}$. Then $\text{id}_{\mathcal{U}} = j[\lambda]$ by Lemma 4.4.9. Of course $|j[\lambda]|^M = \lambda$, but note also that $\lambda^{+M} < j(\lambda)$: by Lemma 4.2.36, there is an inaccessible cardinal $\kappa \leq \lambda$ such that $\lambda < j(\kappa)$, so $\lambda^{+M} < j(\kappa) \leq j(\lambda)$. Thus $|j[\lambda]|^{+M} < j(\lambda)$. By Łoś's Theorem, it follows that $\{\sigma \in P_{\text{bd}}(\lambda) : |\sigma|^+ < \lambda\} \in \mathcal{U}$. That is, $P_*(\lambda) \in \mathcal{U}$. □

The reason we use $P_*(\lambda)$ as an underlying set rather than sticking with $P_{\text{bd}}(\lambda)$ is that without assuming GCH, we cannot prove $|P_{\text{bd}}(\lambda)| = \lambda$. Therefore $P_{\text{bd}}(\lambda)$ may be too large to use as an underlying set. On the other hand, we can prove $|P_*(\lambda)| = \lambda$ in the relevant cases:

Proposition 7.5.44 (UA)**.** *Suppose λ is a cardinal such that $\mathscr{U}_\lambda$ is nonempty. Then $|P_*(\lambda)| = \lambda$.*

Proof. Since $\mathscr{U}_\lambda$ is nonempty, there is a normal fine ultrafilter on $P_*(\lambda)$, and hence there is a cardinal $\kappa \leq \lambda$ that is λ-supercompact.

There are now two cases.

Suppose first that λ is a limit cardinal. Then $P_*(\lambda) = P_{\text{bd}}(\lambda)$. Moreover by Theorem 6.3.25, $2^{<\lambda} = \lambda$. Thus $|P_*(\lambda)| = |P_{\text{bd}}(\lambda)| = 2^{<\lambda} = \lambda$.

Suppose instead that λ is a successor cardinal. Let γ be the cardinal predecessor of λ. Then $P_*(\lambda) = P_\gamma(\lambda)$, so $|P_*(\lambda)| = \lambda^{<\gamma}$. Since λ is regular, $\lambda^{<\gamma} = \lambda \cdot \gamma^{<\gamma}$. To finish, it therefore suffices to show $\gamma^{<\gamma} \leq \lambda$. By Theorem 6.3.25, $2^{<\gamma} = \gamma$. If γ is singular, then γ is a singular strong limit cardinal, so by Solovay's Theorem on SCH above a strongly compact cardinal (Theorem 7.2.19), $\gamma^{<\gamma} \leq \gamma^\gamma = \gamma^+ = \lambda$. Otherwise, $\gamma^{<\gamma} = 2^{<\gamma} = \gamma$. □

Lemma 7.5.45 (UA). *Every ultrafilter in $\mathscr{U}$ is hereditarily uniform.*

Proof. Suppose $\mathcal{U} \in \mathscr{U}$. Fix a cardinal λ with $\mathcal{U} \in \mathscr{U}_\lambda$. Since $P_*(\lambda)$ is the underlying set of $\mathcal{U}$, to show that $\mathcal{U}$ is hereditarily uniform, we must show that $|\text{tc}(P_*(\lambda))| \leq \lambda_\mathcal{U}$. Of course, $\text{tc}(P_*(\lambda)) = P_*(\lambda) \cup \lambda$, which has cardinality λ by Proposition 7.5.44. Since $j_\mathcal{U}$ is λ-supercompact, Proposition 4.2.30 implies that $\lambda \leq \lambda_\mathcal{U}$. Thus $|\text{tc}(P_*(\lambda))| \leq \lambda_\mathcal{U}$, as desired. □

Recall that an ultrafilter U on a cardinal λ is *isonormal* if U is weakly normal and j_U is λ-supercompact. Theorem 4.4.37 states that every normal fine ultrafilter is Rudin-Keisler equivalent to an isonormal ultrafilter. Combined with the Rudin-Keisler invariance of the Mitchell order on hereditarily uniform ultrafilters, the following theorem therefore easily implies Theorem 7.5.39 and Theorem 7.5.42:

Theorem 7.5.46 (UA). *Suppose U is an isonormal ultrafilter. Then for any $D <_\Bbbk U$, $D \lhd U$. In particular, the Mitchell order is linear on isonormal ultrafilters.*

Note that a strong version of the fact that $D <_\Bbbk U$ implies $D \lhd U$ (Corollary 4.3.28) follows, without assuming UA, from GCH.

In case it is not clear, let us explain in full detail how to prove Theorem 7.5.39 and Theorem 7.5.42 from Theorem 7.5.46:

Proof of Theorem 7.5.39. Suppose $\mathcal{U}_0$ and $\mathcal{U}_1$ hereditarily uniform normal fine ultrafilters. We must show that either $\mathcal{U}_0 \equiv_{\text{RK}} \mathcal{U}_1$, $\mathcal{U}_0 \lhd \mathcal{U}_1$, or $\mathcal{U}_0 \rhd \mathcal{U}_1$. Since every normal fine ultrafilter is Rudin-Keisler equivalent to an isonormal ultrafilter (Theorem 4.4.37), there are isonormal ultrafilters U_0 and U_1 such that $U_0 \equiv_{\text{RK}} \mathcal{U}_0$ and $U_1 \equiv_{\text{RK}} \mathcal{U}_1$. By Theorem 7.5.46 and the linearity of the Ketonen order (Theorem 3.3.6), either $U_0 = U_1$, $U_0 \lhd U_1$, or $U_0 \rhd U_1$. If $U_0 = U_1$, then $\mathcal{U}_0 \equiv_{\text{RK}} \mathcal{U}_1$. If $U_0 \lhd U_1$, then since the Mitchell order is Rudin-Keisler invariant on hereditarily uniform ultrafilters (Lemma 4.2.13), $\mathcal{U}_0 \lhd \mathcal{U}_1$. Similarly, if $U_0 \rhd U_1$, then $\mathcal{U}_0 \rhd \mathcal{U}_1$. □

Proof of Theorem 7.5.42. Suppose $\mathcal{U}_0$ and $\mathcal{U}_1$ are ultrafilters in $\mathcal{U}$. We must show that either $\mathcal{U}_0 = \mathcal{U}_1$, $\mathcal{U}_0 \lhd \mathcal{U}_1$, or $\mathcal{U}_0 \rhd \mathcal{U}_1$. By Lemma 7.5.45, $\mathcal{U}_0$ and $\mathcal{U}_1$ are hereditarily uniform, and so by Theorem 7.5.39, either $\mathcal{U}_0 \equiv_{\text{RK}} \mathcal{U}_1$, $\mathcal{U}_0 \lhd \mathcal{U}_1$, or $\mathcal{U}_0 \rhd \mathcal{U}_1$. By the uniqueness clause of Proposition 7.5.43, however, if $\mathcal{U}_0 \equiv_{\text{RK}} \mathcal{U}_1$, then $\mathcal{U}_0 = \mathcal{U}_1$. □

We now proceed to the proof of Theorem 7.5.46. This requires a general fact from the theory of the internal relation which is of independent interest. Since no nonprincipal ultrafilter U satisfies $U \lhd U$, under UA there is a least W in the Ketonen order such that $W \not\lhd U$. What is the relationship between U and W? Perhaps $W \leq_{\mathrm{RF}} U$; this turns out to be equivalent to the Irreducible Ultrafilter Hypothesis (Question 4.2.51).

It turns out that one can make some headway if one considers instead the $<_{\Bbbk}$-least W such that $W \not\sqsubset U$. (Proposition 8.3.39 shows that this actually defines the same ultrafilter.)

Theorem 7.5.47 (UA)**.** *Suppose U is a nonprincipal countably complete ultrafilter and W is the $<_{\Bbbk}$-least countably complete uniform ultrafilter on an ordinal such that $W \not\sqsubset U$. Then for any $D \sqsubset U$, $D \sqsubset W$.*

If every internal ultrapower embedding $j : M_U \to N$ satisfied $j \restriction \alpha \in M_W$ for all $\alpha \in \mathrm{Ord}$, one could conclude that $W \leq_{\mathrm{RF}} U$.

To prove Theorem 7.5.47, we use the following closure property of the internal relation:

Lemma 7.5.48. *Suppose $D \sqsubset U$ is an ultrafilter on a set X and $\langle W_i : i \in X\rangle$ is a sequence of ultrafilters on a set Y such that $W_i \sqsubset U$ for all $i \in X$. Then $D\text{-}\sum_{i\in X} W_i \sqsubset U$ and $D\text{-}\lim_{i\in X} W_i \sqsubset U$.*

Proof. Since $D\text{-}\lim_{i\in X} W_i \leq_{\mathrm{RK}} D\text{-}\sum_{i\in X} W_i$, it suffices to show $D\text{-}\sum_{i\in X} W_i \sqsubset U$ by Corollary 5.5.13.

Let $j : V \to N$ be the ultrapower of the universe by D. Let $W = [\langle W_i : i \in X\rangle]_D$ and let $k : N \to P$ be the ultrapower of M by W. Thus $k \circ j$ is the ultrapower embedding associated to $D\text{-}\sum_{i\in X} W_i$, so to prove the lemma, we must show that $k \circ j \restriction M_U$ is an internal ultrapower embedding of M_U.

Since $D \sqsubset U$, j is an internal ultrapower embedding of M_U. Therefore to show $k \circ j \restriction M_U$ is an internal ultrapower embedding of M_U, it suffices to show that $k \restriction j(M_U)$ is an internal ultrapower embedding of $j(M_U)$. Note that by the elementarity of $j : V \to N$, $j(M_U) = (M_{j(U)})^N$. Since $k = (j_W)^N$, to show that $k \restriction j(M_U)$ is an internal ultrapower embedding of $j(M_U)$, it suffices to show that $W \sqsubset j(U)$ in N. But $W_i \sqsubset U$ for all $i \in X$, so $W \sqsubset j(U)$ in N by Łoś's Theorem. $\square$

Proof of Theorem 7.5.47. Suppose $D \sqsubset U$. By Lemma 5.5.15, $t_D(U) = j_D(U)$. We claim $j_D(W) \leq_{\Bbbk}^{M_D} t_D(W)$. Suppose towards a contradiction that this fails, so $t_D(W) <_{\Bbbk}^{M_D} j_D(W)$. Let X be the underlying set of D, and fix $\langle W_i : i \in I\rangle$ such that $t_D(W) = [\langle W_i : i \in X\rangle]_D$. Then since $W_i <_{\Bbbk} W$ for all $i \in X$, in fact $W_i \sqsubset U$ for all $i \in X$. Thus $D\text{-}\lim_{i\in X} W_i \sqsubset U$ by Lemma 7.5.48. But $W = D\text{-}\lim_{i\in I} W_i$, and this contradicts the definition of W. $\square$

Proof of Theorem 7.5.46. Let $\lambda = \lambda_U$. If $2^{<\lambda} = \lambda$, then U is Dodd sound (Theorem 4.4.25), so for all $D <_{\Bbbk} U$, we have $D \lhd U$ (Corollary 4.3.28), and thus we are done. We therefore assume that $2^{<\lambda} > \lambda$. Although it is not clear whether this assumption is consistent, we will not try to reach a contradiction, but rather to prove that the theorem is true even if $2^{<\lambda} > \lambda$.

Since j_U witnesses that some cardinal $\kappa \leq \lambda$ is λ-supercompact, the local version of the theorem that GCH holds above a supercompact under UA (Theorem 6.3.25) implies that $2^{<\lambda} = \lambda$ if λ is a limit cardinal. Therefore by our assumption that $2^{<\lambda} > \lambda$, λ is a successor cardinal.

Let γ be the cardinal predecessor of λ. To simplify notation, we will from now on refer to λ only as γ^+. We therefore reformulate our assumption that $2^{<\lambda} > \lambda$:

$$2^\gamma > \gamma^+ \tag{7.10}$$

Since γ^+ is Fréchet, our local result on GCH (Theorem 6.3.25) yields that $2^{<\gamma} = \gamma$. If γ is singular, then since $2^{<\gamma} = \gamma$, γ is a singular strong limit cardinal, so the fact that $2^\gamma > \gamma^+$ contradicts the local version of Solovay's Theorem that SCH holds above a strongly compact cardinal (Theorem 7.2.19). Therefore γ is regular.

Claim 1. *M_U satisfies that $2^{2^\gamma} = (2^\gamma)^+$*

Proof. Let $\mathscr{D}$ be the normal fine ultrafilter on $P_{\text{bd}}(\gamma)$ derived from j_U using $j_U[\gamma]$. Since M_U is closed under γ^+-sequences, every ultrafilter on γ belongs to M_U (Theorem 6.3.16). Therefore since $P_{\text{bd}}(\gamma)$ has hereditary cardinality $2^{<\gamma} = \gamma$, we have $\mathscr{D} \in M_U$. Therefore by a generalization of Solovay's argument that a 2^κ-supercompact cardinal carries 2^{2^κ} normal ultrafilters (Theorem 6.3.7), M_U satisfies $\text{LCP}(\gamma, \mathscr{N}_\gamma)$: every subset of $P(\gamma)$ belongs to $M_{\mathscr{W}}$ for some normal fine ultrafilter $\mathscr{W}$ on $P_{\text{bd}}(\gamma)$. Therefore γ carries 2^{2^γ}-many ultrafilters, and so by Theorem 6.3.3 applied in M_U, M_U satisfies that $2^{2^\gamma} = (2^\gamma)^+$. (Alternately one can use Theorem 7.4.28.) $\square$

Now let $\eta = ((\gamma^+)^\sigma)^{M_U}$ be the least Fréchet cardinal above γ^+ as computed in M_U. The following claim is a consequence of our analysis of isolated cardinals:

Claim 2. *η is a measurable cardinal of M_U.*

Proof. Since $P(\gamma) \subseteq M_U$, $(2^\gamma)^{M_U} \geq (2^\gamma)^V > \gamma^+$, and therefore M_U satisfies that $2^\gamma > \gamma^+$.

We now work in M_U to avoid a profusion of superscripts. We cannot have $\eta = \gamma^{++}$: otherwise γ^{++} is Fréchet and hence $2^\gamma = \gamma^+$ by Theorem 6.3.27, contradicting that $2^\gamma > \gamma^+$. Therefore $\eta > \gamma^+$ and so by Proposition 7.4.4, η is isolated.

Let $\delta = \delta_\eta$. Then since $\eta = (\gamma^+)^\sigma$, $\delta \leq \gamma^{++} \leq 2^\gamma$. The final inequality uses the fact that $2^\gamma > \gamma^+$. Thus $2^\delta \leq 2^{2^\gamma} = (2^\gamma)^+$ by Claim 1. But $2^\gamma < \eta$ by our results on the continuum function below an isolated cardinal (Theorem 7.5.23). Therefore $(2^\gamma)^+ < \eta$ since η is isolated (and therefore is a limit cardinal). It follows that $2^\delta < \eta$. Therefore η is measurable by Theorem 7.5.25. $\square$

Let W be the $<_\Bbbk$-least countably complete ultrafilter on an ordinal such that $W \not\sqsubset U$. Then $W \leq_\Bbbk U$. To prove the theorem, we must show $U = W$.

By Theorem 6.3.16, every countably complete ultrafilter on γ belongs to M_U and hence is internal to U by Proposition 4.2.28. Therefore $\lambda_W = \gamma^+$. Let

$$(k, h) : (M_W, M_U) \to N$$

be the pushout of (j_W, j_U).

Claim 3. *If h is nontrivial, then* $\text{crit}(h) \geq \eta$*.*

Proof. Let $W' = t_U(W)$, so $h : M_U \to N$ is the ultrapower of M_U by W'.

Suppose D is a countably complete ultrafilter of M_U with $\lambda_D < \eta$. We claim that $D \sqsubset W'$ in M_U. Since λ_D is a Fréchet cardinal of M_U below $\eta = ((\gamma^+)^\sigma)^{M_U}$, $\lambda_D \leq \gamma^+$. We may therefore assume that the underlying set of D is γ^+. Since j_U is γ^+-supercompact, $P(\gamma^+) \subseteq M_U$. Thus D is an ultrafilter on γ^+ (in V). Since M_U is closed under γ^+-sequences, $j_D \restriction M_U = j_D^{M_U}$, so in fact $D \sqsubset U$. By Theorem 7.5.47, $D \sqsubset W$. Thus $j_D \restriction N$ is amenable to both M_U and M_W. By our characterization of the internal ultrapower embeddings of a pushout (Theorem 5.4.20), $j_D \restriction N$ is an internal ultrapower embedding of N. Equivalently $j_D^{M_U} \restriction N$ is an internal ultrapower embedding of N, or in other words, $D \sqsubset W'$ in M_U.

By Lemma 7.5.3, $h[\eta] \subseteq \eta$. Working in M_U, η is a strong limit cardinal, $h[\eta] \subseteq \eta$, and for all D with $\lambda_D < \eta$, $D \sqsubset W'$. Applying in M_U our criterion for the completeness of an ultrafilter in terms of the internal relation (Lemma 5.5.26), it follows that W' is η-complete. Since h is the ultrapower of M_U by W', if h is nontrivial then $\text{crit}(h) \geq \eta$. □

Since U is a weakly normal ultrafilter on γ^+, $\text{id}_U = \sup j_U[\gamma^+]$ (Lemma 4.4.17). Since h is the identity or $\text{crit}(h) \geq \eta > \gamma^+$, h is continuous at ordinals of M_U-cofinality γ^+. Since M_U is closed under γ^+-sequences, $\sup j_U[\gamma^+]$ is on ordinal of M_U-cofinality γ^+. Therefore

$$h(\text{id}_U) = h(\sup j_U[\gamma^+]) = \sup h \circ j_U[\gamma^+] \leq \sup k \circ j_W[\gamma^+] \leq k(\text{id}_W)$$

The final inequality follows from the fact that $\lambda_W = \gamma^+$ and hence $\sup j_W[\gamma^+] \leq \text{id}_W$. Therefore (k, h) witnesses that $U \leq_\Bbbk W$. Since $U \leq_\Bbbk W$ and $W \leq_\Bbbk U$, $U = W$, as desired. □

Chapter 8

Higher Supercompactness

8.1 Introduction

8.1.1 Obstructions to the supercompactness analysis

The main result of Chapter 7 is that under UA, the first strongly compact is supercompact. What about the second? What about all of the other strongly compact cardinals? This chapter answers all these questions and more. In this introductory section, we explain the obstructions to generalizing the theory of Chapter 7 and the technique we will use to overcome them.

8.1.2 Menas's Theorem

The first obstruction to generalizing the results of Chapter 7 is that not every strongly compact cardinal is supercompact. This is a consequence of the following theorem of Menas:

Theorem 8.1.1 (Menas)**.** *The least strongly compact limit of strongly compact cardinals is not supercompact.*

In order to explain the proof, we introduce an auxiliary notion:

Definition 8.1.2. Suppose κ and λ are cardinals. A cardinal κ is *almost λ-strongly compact* if for any $\alpha < \kappa$, there is an elementary embedding $j : V \to M$ such that $\text{crit}(j) > \alpha$ and M has the $(\leq\lambda, <j(\kappa))$-cover property; κ is *almost strongly compact* if κ is almost λ-strongly compact for all cardinals λ.

As in Theorem 7.2.10, there is a characterization of almost strong compactness in terms of fine ultrafilters:

Lemma 8.1.3. *A cardinal κ is almost λ-strongly compact if and only if for every $\alpha < \kappa$, there is an α^+-complete fine ultrafilter on $P_\kappa(\lambda)$.* $\square$

Unlike strongly compact cardinals, it is easy to see that almost strongly compact cardinals form a closed class:

Lemma 8.1.4. *Any limit of almost λ-strongly compact cardinals is almost strongly compact. In particular, every limit of strongly compact cardinals is almost strongly compact.* $\square$

The following proposition shows that almost strongly compact cardinals really almost are strongly compact:

Proposition 8.1.5. *A cardinal κ is λ-strongly compact if and only if κ is measurable and almost λ-strongly compact.*

Proof. Since κ is measurable, there is a κ-complete uniform ultrafilter U on κ. Since κ is almost strongly compact, for each $\alpha < \kappa$, there is an α^+-complete fine ultrafilter $\mathscr{W}_\alpha$ on $P_\kappa(\lambda)$. Let

$$\mathscr{W} = U\text{-}\lim_{\gamma<\kappa} \mathscr{W}_\gamma$$

It is immediate that $\mathscr{W}$ is a fine ultrafilter on $P_\kappa(\lambda)$.

We claim that $\mathscr{W}$ is κ-complete. Suppose $\nu < \kappa$ and $\{A_i : i < \nu\} \subseteq \mathscr{W}$. For each $i < \nu$, let $S_i = \{\alpha < \kappa : A_i \in \mathscr{W}_\alpha\}$. Since $A_i \in \mathscr{W}$, $S_i \in U$ by the definition of an ultrafilter limit. Since U is κ-complete, $\bigcap_{i<\nu} S_i$ belongs to U. Since U is uniform, $\bigcap_{i<\nu} S_i \setminus \nu \in U$. Suppose $\alpha \in \bigcap_{i<\nu} S_i \setminus \nu$. Then $\{A_i : i < \nu\} \in \mathscr{W}_\alpha$. Therefore since $\mathscr{W}_\alpha$ is α^+-complete, $\bigcap_{i<\nu} A_i \in \mathscr{W}_\alpha$. Thus

$$\bigcap_{i<\nu} S_i \setminus \nu \subseteq \left\{\alpha < \kappa : \bigcap_{i<\nu} A_i \in \mathscr{W}_\alpha\right\}$$

It follows that $\{\alpha < \kappa : \bigcap_{i<\nu} A_i \in \mathscr{W}_\alpha\} \in U$. In other words, $\bigcap_{i<\nu} A_i \in \mathscr{W}$. $\square$

Corollary 8.1.6 (Menas)**.** *Every measurable limit of strongly compact cardinals is strongly compact.* $\square$

The least strongly compact limit of strongly compact cardinals is therefore in a sense accessible from below:

Lemma 8.1.7 (Menas)**.** *Let κ be the least strongly compact limit of strongly compact cardinals. Then the set of measurable cardinals below κ is nonstationary in κ. Therefore κ has Mitchell rank 1. In particular, κ is not μ-measurable, let alone $P(2^\kappa)$-hypermeasurable, let alone 2^κ-supercompact.*

Proof. Let C be the set of limits of strongly compact cardinals less than κ. Since κ is a regular limit of strongly compact cardinals, C is unbounded in κ. Moreover, C is closed by definition. We claim that C contains no measurable cardinals. Suppose $\delta \in C$ is measurable. Then by Corollary 8.1.6, δ is strongly compact. This contradicts that κ is the least strongly compact limit of strongly compact cardinals. It follows that the class of measurable cardinals is nonstationary in κ. $\square$

A strongly compact cardinal κ always carries 2^{2^κ}-many κ-complete ultrafilters. But Menas's theorem shows that the Mitchell order may be trivial on κ. Under UA, this has the following surprising consequence:

Theorem 8.1.8 (UA)**.** *The least strongly compact limit of strongly compact cardinals carries a unique normal ultrafilter.*

Proof. Let κ be the least strongly compact limit of strongly compact cardinals. By Menas's Theorem (Lemma 8.1.7), the rank of the Mitchell order on normal ultrafilters on κ is 1. By Theorem 2.3.11, the Mitchell order linearly orders these ultrafilters. Therefore κ carries exactly one normal ultrafilter. $\square$

8.1.3 Complete UA

The second obstruction to generalizing the results of Chapter 7 is much more subtle: UA alone does not seem to suffice to enact a direct generalization of the structure of the least supercompact cardinal to the higher ones. In order to shed light on the underlying issue, we introduce a principle called the Complete Ultrapower Axiom (Complete UA), which does suffice.

Definition 8.1.9. Suppose κ is an uncountable cardinal. Then $\text{UA}(\kappa)$ denotes the following statement. Suppose $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are ultrapower embeddings with $\text{crit}(j_0) \geq \kappa$ and $\text{crit}(j_1) \geq \kappa$. Then there is a comparison $(i_0, i_1) : (M_0, M_1) \to N$ of (j_0, j_1) such that $\text{crit}(i_0) \geq \kappa$ and $\text{crit}(i_1) \geq \kappa$.

Thus the usual Ultrapower Axiom is equivalent to $\text{UA}(\omega_1)$. Notice that $\text{UA}(\kappa)$ is equivalent to the assertion that the Rudin-Frolík order is directed on κ-complete ultrafilters.

Complete Ultrapower Axiom. $\text{UA}(\kappa)$ holds for all uncountable cardinals κ.

Assuming Complete UA, one can generalize all the proofs in the previous section to obtain results about the higher supercompact cardinals. In fact, one does not even need to dig into the details to see that this is possible:

Proposition 8.1.10 (Complete UA)**.** *Suppose κ is strongly compact. Either κ is supercompact or κ is a limit of supercompact cardinals.*

Sketch. Suppose first that κ is not a limit of strongly compact cardinals. We will show that κ is supercompact. Let $\delta < \kappa$ be the supremum of the strongly compact cardinals below κ. Let $G \subseteq \text{Col}(\omega, \delta)$ be V-generic. Then in $V[G]$, κ is the least strongly compact cardinal. Moreover, since $\text{UA}(\delta^+)$ holds in V, UA holds in $V[G]$. Therefore by the analysis of the least strongly compact cardinal under UA (Theorem 7.4.23), κ is supercompact in $V[G]$. It follows that κ is supercompact in V, as desired.

Suppose instead κ is a limit of strongly compact cardinals. Then by the result of the previous paragraph, every successor strongly compact cardinal below κ is supercompact, so κ is a limit of supercompact cardinals. $\square$

The issue now is that there is no inner model theoretic reason whatsoever to believe that Complete UA is consistent with very large cardinals. As it turns out, it *is* possible to show that Complete UA is consistent with the existence of a supercompact cardinal (if UA is):

Proposition 8.1.11 (UA)**.** *Suppose $j_0 : V \to M_0$ and $j_1 : V \to M_1$ witness that* Complete UA *is false and $\lambda = \min\{\text{crit}(j_0), \text{crit}(j_1)\}$. Then some cardinal $\kappa < \lambda$ is λ-supercompact.*

Sketch. Since λ is measurable, it suffices to show that some $\kappa < \lambda$ is γ-supercompact for all $\gamma < \lambda$. Assume towards a contradiction that there is no such κ. Then by Corollary 7.5.2, for any ultrapower embedding $i : V \to N$, $i[\lambda] \subseteq \lambda$. Let

$$(i_0, i_1) : (M_0, M_1) \to N$$

be the pushout of (j_0, j_1). Let W be a countably complete ultrafilter such that $M_W = N$ and $j_W = i_0 \circ j_0 = i_1 \circ j_1$. By the analysis of ultrafilters internal to a pushout (Theorem 5.4.20), W is λ-internal. Thus $j_W[\lambda] \subseteq \lambda$ and W is λ-internal, so the internal relation theoretic criterion for completeness (Lemma 5.5.26) implies that W is λ-complete. Thus $\text{crit}(i_0) \geq \text{crit}(i_0 \circ j_0) = \text{crit}(j_W) = \lambda$, and similarly $\text{crit}(i_1) \geq \lambda$. This contradicts that j_0 and j_1 witness the failure of Complete UA. □

One can do a bit better using the following fact, whose proof we omit:

Proposition 8.1.12 (UA)**.** *Suppose* Complete UA *fails. Then there are irreducible ultrafilters U_0 and U_1 such that j_{U_0} and j_{U_1} witness the failure of* Complete UA*.* □

Since UA implies the linearity of the Mitchell order on normal ultrafilters (Theorem 2.3.11), Complete UA cannot fail for a pair of normal ultrafilters, and hence the analysis of normality and irreducible ultrafilters (Theorem 5.3.9) implies that $\min\{\text{crit}(j_{U_0}), \text{crit}(j_{U_1})\}$ is a μ-measurable cardinal. One can push this quite a bit further, but not far enough to answer the following question:

Question 8.1.13. *Is the Complete Ultrapower Axiom consistent with the existence of cardinals $\kappa < \lambda$ that are both λ^+-supercompact?*

The most interesting possibility is that large cardinals refute Complete UA. In any case, unless one can prove Complete UA from UA (or Weak Comparison), it is far from well-justified. The analysis of the second strongly compact cardinal therefore requires a different approach.

8.1.4 Irreducible ultrafilters and supercompactness

Given the techniques of the previous chapter, the most obvious approach to higher supercompactness is to study the κ-complete generalizations of Fréchet cardinals and the ultrafilters $\mathscr{K}_\lambda$.

Definition 8.1.14. Suppose $\kappa \leq \lambda$ are uncountable cardinals. Then λ is κ-*Fréchet* if there is a κ-complete uniform ultrafilter on λ.

Definition 8.1.15 (UA)**.** Suppose λ is a κ-Fréchet cardinal. Then $\mathscr{K}^\kappa_\lambda$ denotes the minimum κ-complete uniform ultrafilter on λ in the Ketonen order.

Most of the key properties of $\mathscr{K}_\lambda$ do not directly generalize to $\mathscr{K}^\kappa_\lambda$: the proofs seem to require UA(κ). Essentially the only nontrivial UA result that lifts is Lemma 7.3.12, the fact that $\mathscr{K}_\lambda$ is irreducible for regular λ.

Lemma 8.1.16 (UA)**.** *Suppose $\kappa \leq \lambda$ and λ is κ-Fréchet. Then $\mathscr{K}^\kappa_\lambda$ is weakly normal.*

Proof. Recall Lemma 4.4.20, which asserts that a uniform ultrafilter U on a cardinal λ is weakly normal if and only if for all $W <_{\text{rk}} U$, $\lambda_W < \lambda$. We will show that this holds for $U = \mathscr{K}^\kappa_\lambda$. Suppose $W <_{\text{rk}} \mathscr{K}^\kappa_\lambda$. Since $W \leq_{\text{RK}} \mathscr{K}^\kappa_\lambda$, W is κ-complete, and since $W <_{\text{rk}} \mathscr{K}^\kappa_\lambda$, $W <_{\Bbbk} \mathscr{K}^\kappa_\lambda$. By the minimality of $\mathscr{K}^\kappa_\lambda$, $\lambda_W < \lambda$. ☐

Proposition 8.1.17 (UA)**.** *Suppose $\nu < \lambda$ and λ is a ν^+-Fréchet regular cardinal.*[1] *Then $\mathscr{K}^{\nu^+}_\lambda$ is irreducible.*

Proof. Let $\mathscr{K} = \mathscr{K}^{\nu^+}_\lambda$. Suppose $D <_{\text{RF}} \mathscr{K}$. We must show that D is principal. Since $\mathscr{K}$ is ν^+-complete and $D \leq_{\text{RK}} \mathscr{K}$, D is ν^+-complete, and in particular $j_D(\nu) = \nu$. Since $\mathscr{K}$ is weakly normal and $D <_{\text{RK}} \mathscr{K}$, $\lambda_D < \lambda$ by Proposition 4.4.22. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}$ and let $h : M_D \to M$ be the unique internal ultrapower embedding such that $h \circ j_D = j$. Then h is the ultrapower of M_D by $t_D(\mathscr{K})$, and $\text{crit}(h) \geq \text{crit}(j) > \nu = j_D(\nu)$. Thus $t_D(\mathscr{K})$ is $j_D(\nu^+)$-complete in M_D.

Assume towards a contradiction that D is nonprincipal. By Proposition 5.4.5, $t_D(\mathscr{K}) <_{\Bbbk} j_D(\mathscr{K})$ in M_D. Since $t_D(\mathscr{K})$ is $j_D(\nu^+)$-complete, the $<^{M_D}_{\Bbbk}$-minimality of $j_D(\mathscr{K})$ among $j_D(\nu^+)$-complete uniform ultrafilters on $j_D(\lambda)$ implies that $\lambda_{t_D(\mathscr{K})} < j_D(\lambda)$. Since $j_D(\lambda)$ is M_D-regular, it follows that $\delta_{t_D(\mathscr{K})} < j_D(\lambda)$. Since $\lambda_D < \lambda$ and λ is regular, $j_D(\lambda) = \sup j_D[\lambda]$ by Lemma 2.2.34. Therefore there is some ordinal $\alpha < \lambda$ such that $\delta_{t_D(\mathscr{K})} < j_D(\alpha)$. But $\alpha \in j_D^{-1}[t_D(\mathscr{K})] = \mathscr{K}$, contradicting that $\mathscr{K}$ is uniform. Thus our assumption was false, and in fact D is principal. This shows that $\mathscr{K}$ is irreducible, as desired. ☐

Beyond Proposition 8.1.17, the ultrafilters $\mathscr{K}^\kappa_\lambda$ turn out to be a bit of a red herring. The analysis of higher supercompact cardinals proceeds not by generalizing the theory of $\mathscr{K}_\lambda$ to the ultrafilters $\mathscr{K}^\kappa_\lambda$ but instead by propagating the λ-supercompactness of $\mathscr{K}_\lambda$ itself to arbitrary irreducible ultrafilters. Recall that an ultrafilter U is λ-irreducible if every ultrafilter $D \leq_{\text{RF}} U$ such that $\lambda_D < \lambda$ is principal. The main theorems of this chapter, to which we refer collectively

[1] It is necessary here to restrict to consideration of $\mathscr{K}^{\nu^+}_\lambda$, rather than considering $\mathscr{K}^\kappa_\lambda$ in general. In fact, by Theorem 8.3.4, $\mathscr{K}^\kappa_\lambda$ is irreducible if and only if either $\kappa = \lambda$ or there is some $\nu < \kappa$ such that $\mathscr{K}^\kappa_\lambda = \mathscr{K}^{\nu^+}_\lambda$. This is closely related to Menas's Theorem (Theorem 8.1.1).

as the *Irreducibility Theorem*, show that supercompactness and irreducibility are equivalent:

Theorem 8.2.22 (UA). *Suppose λ is a successor cardinal or a strong limit singular cardinal and U is a countably complete uniform ultrafilter on λ. Then the following are equivalent:*

(1) U is λ-irreducible.

(2) j_U is λ-supercompact.

It does not seem to be possible to generalize this to the case that λ is inaccessible, and instead we obtain the following theorem:

Theorem 8.2.23 (UA). *Suppose λ is an inaccessible cardinal and U is a countably complete ultrafilter on λ. Then the following are equivalent:*

(1) U is λ-irreducible.

(2) j_U is $<\lambda$-supercompact and λ-tight.

We will use these two theorems to give a complete characterization of strongly compact cardinals assuming UA:

Theorem 8.3.10 (UA). *Suppose κ is a strongly compact cardinal. Either κ is a supercompact cardinal or κ is a measurable limit of supercompact cardinals*

8.1.5 Outline of Chapter 8

We now outline the rest of this chapter.

Section 8.2. We prove the main structural result of the section, called the *Irreducibility Theorem*, from which all the other theorems flow. The Irreducibility Theorem refers to a cluster of results (especially Theorem 8.2.19 and Corollary 8.2.21) that show an equivalence between irreducibility and supercompactness.

Section 8.3. We use the Irreducibility Theorem to resolve the Identity Crisis for strongly compact cardinals under UA. We also use it in Section 8.3.3 to completely characterize the internal relation in terms of the Mitchell order.

Section 8.4. We discuss the relationship between UA and very large cardinals. We begin by (partially) analyzing the relationship between hugeness and non-regular ultrafilters under UA (Theorem 8.4.5). We then turn to the topic of cardinal preserving embeddings. We show that UA rules out such embeddings (Lemma 8.4.11), and more generally that local cardinal preservation hypotheses are equivalent to rank-into-rank large cardinal large cardinal axioms under UA (Theorem 8.4.13). Finally in Section 8.4.3, we discuss the structure of supercompactness at inaccessible cardinals, and in particular the prospect that the local equivalence of strong compactness and supercompactness breaks down there.

8.2 The Irreducibility Theorem

In this section, we prove the central Irreducibility Theorem (Theorem 8.2.22 and Theorem 8.2.23). We begin in Section 8.2.1 by proving the forward implication from supercompactness to irreducibility. This raises a central open question (Question 8.2.4) that will be discussed at greater length in Section 8.4.3. The next two sections are devoted to proving the preliminary lemmas necessary for the proof of the Irreducibility Theorem. In Section 8.2.2, we prove two key lemmas regarding the comparison of $\mathscr{K}_\lambda$ with an arbitrary ultrafilter. In the very short Section 8.2.3, we prove two theorems on the combinatorics of normal ultrafilters that show up in the proof of the Irreducibility Theorem. Finally, Section 8.2.4 contains the proof of the Irreducibility Theorem as well as some slightly more general theorems.

8.2.1 Tightness and irreducibility

In this short subsection, we prove the easy direction of the irreducibility theorem: λ-supercompactness implies λ-irreducibility. In fact, we will prove something slightly stronger. The following property is a priori somewhat weaker than λ-supercompactness, but already implies λ-irreducibility.

Definition 8.2.1. Suppose I is a set of cardinals. An elementary embedding $j : V \to M$ is said to be *I-tight* if j is γ-tight for every cardinal $\gamma \in I$.

We will really only be interested in I-tight embeddings where $I = [0, \lambda]\cap\text{Card}$ is a closed initial segment of the class of cardinals, and we will abuse notation slightly by calling these embeddings *$[0, \lambda]$-tight.*

Lemma 8.2.2. *An ultrapower embedding $j : V \to M$ is $[0, \lambda]$-tight if and only if M has the $\leq\gamma$-cover property for all $\gamma \leq \lambda$.*

Proof. This is an immediate consequence of the self-strengthening of tightness that holds for ultrapower embeddings (Lemma 7.2.7). □

Proposition 8.2.3. *Suppose λ is a cardinal and U is a countably complete ultrafilter. If j_U is $[0, \lambda]$-tight, then U is λ-irreducible.*

Proof. Suppose $D \leq_{\text{RF}} U$ and $\lambda_D < \lambda$. We must show that D is principal. We first show that j_D is $[0, \lambda]$-tight. Since j_U is $[0, \lambda]$-tight, M_U has the $\leq\gamma$-cover property for all $\gamma \leq \lambda$. Since $D \leq_{\text{RF}} U$, $M_U \subseteq M_D$. It follows that M_D has the $\leq\gamma$-cover property for all $\gamma \leq \lambda$: suppose $\gamma \leq \lambda$ and A is a set of ordinals of cardinality γ; then A is contained in a set $B \in M_U$ such that $|B|^{M_U} \leq \gamma$, and since $M_U \subseteq M_D$, we have $B \in M_D$ and $|B|^{M_D} \leq \gamma$, as desired. Thus j_D is $[0, \lambda]$-tight.

In particular, since $\lambda_D < \lambda$, D is λ_D^+-tight. Assume towards a contradiction that D is nonprincipal. By Lemma 4.2.31, $j_D(\lambda_D) > \lambda_D^+$. Thus D is (λ_D^+, δ)-tight where $\delta = \lambda_D^+ < j_D(\lambda_D)$. This contradicts Lemma 7.2.17, which states that if η is a cardinal and Z is a nonprincipal countably complete ultrafilter such that $\lambda_Z < \eta$, then Z is not (η, δ)-tight for any $\delta < j_Z(\eta)$. Thus D is principal, as desired. $\square$

The only known instances of $[0, \lambda]$-tight elementary embeddings that are not λ-supercompact come from large cardinal axioms at the level of rank-into-rank cardinals. Specifically, assume the axiom I_2. Thus there is a cardinal λ and an elementary embedding $j : V \to M$ such that $\text{crit}(j) < \lambda$, $j(\lambda) = \lambda$, and $V_\lambda \subseteq M$. The embedding j is not λ-supercompact by the Kunen Inconsistency Theorem, but j is trivially $[0, \lambda]$-tight since $j[\lambda] \subseteq \lambda$. In fact, j is $[0, \lambda^{+\alpha}]$-tight for all $\alpha < \text{crit}(j)$. On the other hand, there are no known examples of *ultrapower* embeddings that are $[0, \lambda]$-tight but not λ-supercompact. In fact, it is not known whether it is consistent that such an example exists:

Question 8.2.4 (ZFC)**.** *Suppose λ is a cardinal and $j : V \to M$ is a $[0, \lambda]$-tight ultrapower embedding. Must j be λ-supercompact?*

The natural inclination is to conjecture that the answer is no: typically large cardinal properties formulated in terms of covering do not imply supercompactness in ZFC. But the problem turns out to be much more subtle than one might expect.

The following question isolates the most basic instance of this problem:

Question 8.2.5. *Suppose $j : V \to M$ is an elementary embedding with critical point κ such that $\text{cf}^M(\sup j[\kappa^+]) = \kappa^+$. Must $j[\kappa^+]$ belong to M?*

These questions are considered in greater detail in Section 8.4.3.

On this topic, let us mention an interesting way in which tightness can act as a stand-in for hypermeasurability:

Lemma 8.2.6. *Suppose $j : V \to M$ is an elementary embedding, λ is a cardinal, δ is an M-cardinal, and j is (λ, δ)-tight. Then $2^\lambda \leq (2^\delta)^M$.*

Proof. Fix $B \in M$ such that $|B|^M = \delta$ and $j[\lambda] \subseteq B$. Then the map $f : P(\lambda) \to P(B) \cap M$ defined by $f(S) = j(S) \cap B$ is an injection: if $S \neq T$, then fix $\alpha \in S \triangle T$, and note that since $j[\lambda] \subseteq B$, $j(\alpha) \in (j(S) \triangle j(T)) \cap B = f(S) \triangle f(T)$. Since $|P(B) \cap M|^M = (2^\delta)^M$ it follows that $2^\lambda \leq |(2^\delta)^M| \leq (2^\delta)^M$. $\square$

As a sample application (and a brief diversion), suppose κ is a cardinal such that for all cardinals $\lambda \geq \kappa$, there is a λ-tight embedding $j : V \to M$ such that $j(\kappa) > \lambda$. Then the Generalized Continuum Hypothesis cannot fail first above κ. To see this, assume that for all cardinals $\gamma < \kappa$, $2^\gamma = \gamma^+$. Fix $\lambda \geq \kappa$. Let $j : V \to M$ be a λ-tight embedding with $j(\kappa) > \lambda$. Then in M, $2^\lambda = \lambda^+$. Therefore $2^\lambda \leq (2^\lambda)^M \leq (\lambda^+)^M \leq \lambda^+$, so $2^\lambda = \lambda^+$.

8.2.2 Translations of $\mathscr{K}_\lambda$

Suppose U is a λ-irreducible uniform ultrafilter on a successor cardinal λ. The Irreducibility Theorem asserts that j_U is λ-supercompact. The proof proceeds by analyzing the pushout comparison of $(j_{\mathscr{K}_\lambda}, j_U)$ where λ is a Fréchet successor cardinal. In this section, we will prove a number of lemmas regarding this pushout that will be incorporated into this analysis.

The universal property of $\mathscr{K}_\lambda$ (Theorem 7.3.13) identifies the pushout of $(j_{\mathscr{K}_\lambda}, j_U)$ when $\text{cf}^{M_U}(\sup j_U[\lambda])$ is not Fréchet in M_U: in fact, $\mathscr{K}_\lambda \leq_{\text{RF}} U$, so the pushout is given by the unique internal ultrapower embedding $h : M_{\mathscr{K}_\lambda} \to M_U$. It turns out that the universal property is powerful enough to yield an analysis of this comparison even when $\text{cf}^{M_U}(\sup j_U[\lambda])$ is a Fréchet cardinal of M_U. The following lemma tells us which ultrafilter is hit on the M_U-side of the comparison:

Lemma 8.2.7 (UA)**.** *Suppose λ is a regular Fréchet cardinal and U is a countably complete ultrafilter. Let $\delta = \text{cf}^{M_U}(\sup j_U[\lambda])$.*

- *Suppose δ is not Fréchet in M_U. Then $t_U(\mathscr{K}_\lambda)$ is principal in M_U.*
- *Suppose δ is Fréchet in M_U. Then $t_U(\mathscr{K}_\lambda) \equiv_{\text{RK}} (\mathscr{K}_\delta)^M$.*

Proof. The first bullet point is immediate from the universal property of $\mathscr{K}_\lambda$ (Theorem 7.3.13): we have $\mathscr{K}_\lambda \leq_{\text{RF}} U$, so by Lemma 5.4.41, $t_U(\mathscr{K}_\lambda)$ is principal in M_U. Therefore assume instead that δ is Fréchet in M_U.

Let $Z = t_U(\mathscr{K}_\lambda)$. We claim that in M_U, Z is a $<_\Bbbk$-minimal element of the set of countably complete ultrafilters W on $j_U(\lambda)$ with $\delta_W \geq \sup j_U[\lambda]$. Clearly $\delta_Z \geq \sup j_U[\lambda]$, since otherwise $\delta_{j_U^{-1}[Z]} < \lambda$ contradicting that $j_U^{-1}[Z] = \mathscr{K}_\lambda$. Suppose $W \in j_U(\mathbf{UF}(\lambda))$ and $W <_\Bbbk Z$ in M_U, and we will show $\delta_W < \sup j_U[\lambda]$. Let $\bar{W} = j_U^{-1}[W]$. Then $t_U(\bar{W}) \leq_\Bbbk W <_\Bbbk Z = t_U(\mathscr{K}_\lambda)$. By Theorem 5.4.44, it follows that $\bar{W} <_\Bbbk \mathscr{K}_\lambda$. Since $\mathscr{K}_\lambda$ is the $<_\Bbbk$-least uniform ultrafilter on the regular cardinal λ, $\delta_{\bar{W}} < \lambda$. But $j_U(\delta_{\bar{W}}) \in W$, so $\delta_W \leq j_U(\delta_{\bar{W}}) < \sup j_U[\lambda]$.

Applying the analysis of $<_\Bbbk$-minimal fine ultrafilters (Lemma 7.3.10) in M_U, it follows that in M_U, there is a Ketonen ultrafilter D on $\text{cf}^{M_U}(\sup j_U[\lambda]) = \delta$ that is Rudin-Keisler equivalent to Z. Applying UA in M_U, $D = \mathscr{K}_\delta$, the unique Ketonen ultrafilter on δ. $\square$

The analysis of the $M_{\mathscr{K}_\lambda}$-side of the comparison is much more subtle, and uses the following fact:

Lemma 8.2.8 (UA)**.** *Suppose λ is a nonisolated regular Fréchet cardinal. Let $M = M_{\mathscr{K}_\lambda}$. Suppose $i : M \to N$ is an internal ultrapower embedding. Then for some countably complete ultrafilter D of M with $\lambda_D < \lambda$, there is an internal ultrapower embedding $h : (M_D)^M \to N$ with $h \circ j_D = i$ and $\text{crit}(h) > j_D(\lambda)$.*

The proof uses an analysis of λ^σ in $M_{\mathscr{K}_\lambda}$ that is similar to Claim 2 of Theorem 7.5.46:

Lemma 8.2.9 (UA). *Suppose λ is a nonisolated regular Fréchet cardinal. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. Then $(\lambda^\sigma)^M$ is a measurable cardinal of M.*

Proof. By Theorem 7.3.34 and Theorem 7.4.9, j is λ-tight and therefore $\text{cf}^M(\sup j[\lambda]) = \lambda$. Therefore by the definition of $\mathscr{K}_\lambda$ (or more precisely, Lemma 7.3.6), λ is not Fréchet in M.

Let $\eta = (\lambda^\sigma)^M$. Assume towards a contradiction that η is not measurable. Let $i : M \to N$ be the ultrapower of M by $(\mathscr{K}_\eta)^M$ and let $a = \text{id}_{(\mathscr{K}_\eta)^M}$.

We claim that every countably complete N-ultrafilter D on λ belongs to M. For any such D, $j^N_D \circ i$ is continuous at λ: i is continuous at λ because i is internal to M and λ is not Fréchet in M, while j^N_D is continuous at $i(\lambda)$ since $i(\lambda)$ is an N-regular cardinal with $i(\lambda) > \lambda \geq \lambda_D$, and combining these observations:

$$j^N_D(i(\lambda)) = \sup j^N_D[i(\lambda)] = \sup j^N_D[\sup i[\lambda]] = \sup j^N_D \circ i[\lambda]$$

Thus by the characterization of internal ultrapower embeddings of $M_{\mathscr{K}_\lambda}$ (Theorem 7.3.14), $j^N_D \circ i$ an internal ultrapower embedding of M. Since j^N_D can be defined at a typical element of N by setting

$$j^N_D([f]_{(\mathscr{K}_\eta)^M}) = j^N_D \circ i(f)(j^N_D(a))$$

it follows that j^N_D is definable over M. Thus $D \in M$. Applying inside M the characterization of countably complete ultrafilters amenable to an isolated ultrapower (Theorem 7.5.19), we have that $D \in N$.

Proposition 7.3.33 states that if κ is λ-strongly compact and Q is a $<\kappa$-closed inner model such that every κ-complete ultrafilter U on λ is amenable to Q, then $P(\lambda) \subseteq Q$. By Theorem 7.4.9, κ_λ is λ-strongly compact. Moreover N is a $<\kappa_\lambda$-closed (indeed $<\lambda$-closed by Proposition 7.5.22) inner model, and every countably complete N-ultrafilter on λ belongs to N. It therefore follows that $P(\lambda) \subseteq N$. But then $\mathscr{K}_\lambda$ itself is an N-ultrafilter, so $\mathscr{K}_\lambda \in N$. Since $N \subseteq M$, this implies $\mathscr{K}_\lambda \in M = M_{\mathscr{K}_\lambda}$, so $\mathscr{K}_\lambda \lhd \mathscr{K}_\lambda$, contradicting the irreflexivity of the Mitchell order (Lemma 4.2.38). ☐

Proof of Lemma 8.2.8. By Lemma 8.2.9, $\eta = (\lambda^\sigma)^M$ is a measurable cardinal that is not a limit of Fréchet cardinals. The theorem follows by applying in M the fact that ultrapower embeddings can be factored across strong limit cardinals that are not limits of Fréchet cardinals (Theorem 7.5.5). ☐

Lemma 8.2.8 has the following curious and sometimes useful corollary:

Lemma 8.2.10 (UA). *Suppose λ is a strongly inaccessible cardinal such that one of the following holds:*

- *λ is Fréchet.*
- *λ^σ is measurable.*

Then every ultrapower embedding is λ-tight.

Proof. Suppose U is a countably complete ultrafilter. We will show that j_U is λ-tight.

Assume first that λ is not Fréchet. Then by assumption $\eta = \lambda^\sigma$ is measurable. By Theorem 7.5.5, there is a countably complete ultrafilter D with $\lambda_D < \eta$ such that there is an elementary embedding $k : M_D \to M_U$ with $k \circ j_D = j_U$ and $\text{crit}(k) \geq \eta$. Since $\lambda_D < \eta$, in fact $\lambda_D < \lambda$, so $j_D(\lambda) = \lambda$ since λ is inaccessible. But since $\text{crit}(k) > j_D(\lambda)$, $j_U(\lambda) = j_D(\lambda) = \lambda$. Therefore j_D is vacuously λ-tight.

Assume instead that λ is Fréchet. Let $(h, i) : (M_U, M_{\mathscr{K}_\lambda}) \to N$ be the pushout of $(j_U, j_{\mathscr{K}_\lambda})$. Applying Lemma 8.2.8, i factors in such a way that we can conclude that $i(\lambda) = \lambda$ by the argument of the previous paragraph. Since $\mathscr{K}_\lambda$ is λ-tight by Proposition 7.4.11 and i is vacuously λ-tight, $i \circ j_{\mathscr{K}_\lambda}$ is λ-tight. In other words, N has the $\leq\lambda$-cover property. Since $N \subseteq M_U$ and N has the $\leq\lambda$-cover property, M_U has the $\leq\lambda$-cover property. Therefore j_U is λ-tight, as desired. $\square$

8.2.3 Elementary embeddings and normal filters

In this short subsection, we prove some combinatorial constraints on comparisons involving normal filters. Suppose U and W are countably complete ultrafilters on a cardinal κ. A question that often arises in the context of UA is what sort of M_W-ultrafilters Z on $j_W(\kappa)$ pull back to U in the sense that $U = j_W^{-1}[Z]$. Such M_W-ultrafilters arise from any comparison of (j_U, j_W). Focusing on a more specific question, assume U is normal, and suppose Z is a fine M_W-ultrafilter on $j_W(\kappa)$ with $j_W^{-1}[Z] = U$. Must $Z = j_W(U)$? The following lemma, which has almost certainly been discovered before in some form, tells us that the answer is yes:

Lemma 8.2.11. *Suppose $\mathscr{F}$ is a normal fine filter on a set Y, and W is an ultrafilter on $X = \bigcup Y$. Then $j_W(\mathscr{F})$ is the unique M-filter on $j_W(Y)$ that extends $j_W[F]$ and concentrates on $\{\sigma \in j_W(Y) : \text{id}_W \in \sigma\}$. In particular, $j_W(\mathscr{F})$ is the unique fine M-filter on $j_W(Y)$ extending $j_W[F]$.*

Proof. Suppose $A \in j_W(\mathscr{F})$. We will find $B \in \mathscr{F}$ such that

$$j_W(B) \cap \{\sigma \in j_W(Y) : \text{id}_W \in \sigma\} \subseteq A$$

Fix a function $G : X \to \mathscr{F}$ such that $A = j_W(G)(\text{id}_W)$. Let

$$B = \triangle_{x \in X} G(x)$$

Suppose $\tau \in j_W(B) \cap \{\sigma \in j_W(Y) : \text{id}_W \in \sigma\}$. We will show that $\tau \in A$. Since τ belongs to $j_W(B) = \triangle_{x \in j_W(X)} j_W(G)(x)$, the definition of the diagonal intersection operation implies that $\tau \in j_W(G)(x)$ for all $x \in \tau$. But $\text{id}_W \in \tau$, and hence $\tau \in j_W(G)(\text{id}_W) = A$. $\square$

In general, one must adjoin the set $\{\sigma \in j(Y) : \text{id}_W \in \sigma\}$ in order to generate all of $\mathscr{F}$. Suppose $\mathscr{F}$ is a normal fine ultrafilter on Y and W is an ultrafilter on

$X = \bigcup Y$. Then $j_W[\mathcal{F}]$ generates $j_W(\mathcal{F})$ if and only if there is some $\tau \in Y$ such that W concentrates on τ and $\mathcal{F}$ concentrates on $\{\sigma \in Y : \tau \subseteq \sigma\}$.

To better explain how this lemma is related to UA, we offer a sample corollary:

Corollary 8.2.12 (UA). *Suppose F is a normal filter on a cardinal κ. Let U be the $<_{\Bbbk}$-least countably complete ultrafilter on κ that extends F. Then for all $D <_{\Bbbk} U$, $D \sqsubset U$.*

Proof. Suppose $D <_{\Bbbk} U$. We claim $j_D(U) \leq_{\Bbbk} t_D(U)$ in M_D, which implies $D \sqsubset U$ by the theory of the internal relation (Lemma 5.5.15). Since $j_D(U)$ is the $<_{\Bbbk}^{M_D}$-least countably complete ultrafilter of M_D that extends $j_D(F)$, it suffices to show that $j_D(F) \subseteq t_D(U)$. Of course $j_D[F] \subseteq t_D(U)$ since $j_D^{-1}[t_D(U)] = U$. Moreover since $D <_{\Bbbk} U$, we must have that $t_D(U)$ concentrates on ordinals greater than id_D (since otherwise $t_D(U)$ witnesses $U \leq_{\Bbbk} D$). In other words, $\{\alpha < j_D(\kappa) : \mathrm{id}_D \in \alpha\} \in t_D(U)$. Therefore by Lemma 8.2.11, $j_D(F) \subseteq t_D(U)$, as desired. □

Here is an intriguing consequence of Corollary 8.2.12. Suppose κ is a regular cardinal and F is the ω-club filter on κ. Suppose F extends to a countably complete ultrafilter. Mitchell [36] showed that this hypothesis is equiconsistent with a measurable cardinal of Mitchell order ω, but assuming UA, it implies that there is a μ-measurable cardinal and quite a bit more. The reason is that Corollary 8.2.12 shows that the $<_{\Bbbk}$-least extension of F is irreducible; clearly it is not normal, so we can apply the dichotomy between normal ultrafilters and μ-measurability (Theorem 5.3.8). If F extends to a Dodd sound ultrafilter U, then κ is a limit of superstrong cardinals: indeed, $j_U^{\mathrm{id}_U}$ witnesses that κ is superstrong in M_U. Thus in Jensen's canonical inner models at the level of subcompact cardinals, if F extends to a countably complete ultrafilter, then κ is a limit of superstrong cardinals (because in these models, every irreducible ultrafilter is Dodd sound).

Question 8.2.13 (UA). *Suppose there is a regular cardinal δ that carries a countably complete ultrafilter extending the ω-closed unbounded filter. Must there be a superstrong cardinal? Must there be an inner model with a superstrong cardinal?*

As a corollary of Lemma 8.2.11, we have a similar unique extension theorem for isonormal ultrafilters on regular cardinals. We begin with a corollary of Solovay's Lemma (Theorem 4.4.27) that explains the statement of Lemma 8.2.15:

Lemma 8.2.14. *Suppose λ is a regular cardinal and W is a countably complete weakly normal ultrafilter on λ. Suppose $\langle S_\xi : \xi < \lambda\rangle$ is a partition of S^λ_ω into stationary sets. Then for any $\xi < \lambda$, W concentrates on the set of $\alpha < \lambda$ such that S_ξ is stationary in α.*

Proof. Let $j : V \to M$ be the ultrapower of the universe by W. Then since W is weakly normal, $\text{id}_W = \sup j[\lambda]$. Let $\langle T_\xi : \xi < j(\lambda)\rangle = j(\langle S_\alpha : \alpha < \lambda\rangle)$. By Solovay's Lemma (Lemma 4.4.29),

$$j[\lambda] = \{\xi < j(\lambda) : T_\xi \text{ is stationary in } \sup j[\lambda]\}$$

In particular, if $\xi < \lambda$, then M satisfies that $T_{j(\xi)}$ is stationary in id_W, and so by Łoś's Theorem, W concentrates on the set of $\alpha < \lambda$ such that S_ξ is stationary in α. □

Lemma 8.2.15. *Suppose λ is a regular cardinal, W is an isonormal ultrafilter on λ, and D is a countably complete ultrafilter on λ. Let $\langle S_\xi : \xi < \lambda\rangle$ be a partition of S^λ_ω into stationary sets, and let $\langle T_\xi : \xi < j_D(\lambda)\rangle = j_D(\langle S_\xi : \xi < \lambda\rangle)$. Let*

$$A = \{\alpha < j_D(\lambda) : M_D \vDash T_{\text{id}_D} \textit{ is stationary in } \alpha\}$$

Then $j_D(W)$ is the unique M_D-filter on $j_D(\lambda)$ that extends $j_D[W]$ and concentrates on A.

Proof. Let $\mathcal{U}$ be the normal fine ultrafilter on $P(\lambda)$ Rudin-Keisler equivalent to W. Let $g : P(\lambda) \to \lambda + 1$ be the sup function

$$g(\sigma) = \sup \sigma$$

By Solovay's Lemma (Corollary 4.4.28), $g_*(\mathcal{U}) = W$. Let $f : \lambda \to P(\lambda)$ be the function defined by

$$f(\alpha) = \{\xi < \lambda : S_\xi \text{ is stationary in } \alpha\}$$

By the proof of Solovay's Lemma, for any $A \subseteq \lambda$, $f[A]$ and $g^{-1}[A]$ are equal modulo $\mathcal{U}$. Thus since $W = g_*(\mathcal{U})$,

$$W = \{A \subseteq \lambda : f[A] \in \mathcal{U}\}$$

By Lemma 8.2.11, $j_D(\mathcal{U})$ is the unique M_D-filter on $j_D(Y)$ that extends $j_D[\mathcal{U}]$ and concentrates on $\{\sigma \in j_D(P(\lambda)) : \text{id}_D \in \sigma\}$. Since $j_D(W) = \{A \subseteq \lambda : j_D(f)[A] \in j_D(\mathcal{U})\}$, it follows that $j_D(W)$ is the unique M_D-filter on $j_D(\lambda)$ that extends $\{A : j_D(f)[A] \in j_D[\mathcal{U}]\}$ and concentrates on

$$\{\alpha < \lambda : \text{id}_D \in j_D(f)(\alpha)\} = \{\alpha < \lambda : M_D \vDash T_{\text{id}_D} \text{ is stationary in } \alpha\}$$

In other words, $j_D(W)$ is the unique M_D-filter on $j_D(\lambda)$ that extends $j_D[W]$ and concentrates on A, as desired. □

Let us include one more useful combinatorial fact, this time about pullbacks of weakly normal ultrafilters. To state the lemma in the generality we will need, we introduce a relativized version of the notion of a weakly normal ultrafilter.

Definition 8.2.16. Suppose M is a transitive model of ZFC, λ is an M-regular cardinal, and F is an M-filter on λ. Then F is *weakly normal* if for all sequences $\langle A_\alpha : \alpha < \lambda\rangle \in M$ of subsets of λ such that $A_\alpha \in F$ for all $\alpha < \lambda$ and $A_\alpha \supseteq A_\beta$ for all $\alpha \leq \beta < \lambda$, the diagonal intersection $\triangle_{\alpha<\lambda} A_\alpha$ belongs to F.

We will really only need this notion for M-ultrafilters, in which case it has the following familiar formulation:

Lemma 8.2.17. *If M is a transitive model of* ZFC*, λ is an M-regular cardinal, and U is an M-ultrafilter on λ, then U is weakly normal if and only if* $\text{id}_U = \sup j_U^M[\lambda]$. □

Lemma 8.2.18. *Suppose λ is a regular cardinal and $j : V \to M$ is an elementary embedding that is continuous at λ. Suppose F is a weakly normal M-filter on $j(\lambda)$. Then $j^{-1}[F]$ is a weakly normal filter on λ.*

Proof. Suppose $\langle A_\alpha : \alpha < \lambda\rangle$ is a decreasing sequence of subsets of λ such that $A_\alpha \in j^{-1}[F]$ for all $\alpha < \lambda$. We must show that $\triangle_{\alpha<\lambda} A_\alpha \in j^{-1}[F]$. Let $\langle B_\beta : \beta < j(\lambda)\rangle = j(\langle A_\alpha : \alpha < \lambda\rangle)$. Since $j(\triangle_{\alpha<\lambda} A_\alpha) = \triangle_{\beta<j(\lambda)} B_\beta$, it suffices to show that $\triangle_{\beta<j(\lambda)} B_\beta \in F$.

By the elementarity of j, $\langle B_\beta : \beta < j(\lambda)\rangle$ is a decreasing sequence of subsets of $j(\lambda)$. We claim that for all $\beta < j(\lambda)$, $B_\beta \in F$. To see this, fix $\beta < j(\lambda)$. Since j is continuous at λ, there is some $\alpha < \lambda$ such that $\beta \leq j(\alpha)$. Now $B_{j(\alpha)} = j(A_\alpha) \in F$ since $A_\alpha \in j^{-1}[F]$. But $B_{j(\alpha)} \subseteq B_\beta$ since $\beta \leq j(\alpha)$ and $\langle B_\beta : \beta < j(\lambda)\rangle$ is a decreasing sequence. Therefore $B_\beta \in F$, as claimed. Since F is weakly normal, it follows that $\triangle_{\beta<j(\lambda)} B_\beta \in F$. □

8.2.4 A proof of the Irreducibility Theorem

We will obtain the Irreducibility Theorem as an immediate consequence of the following slightly more general fact:

Theorem 8.2.19 (UA)**.** *Suppose U is a countably complete ultrafilter and λ is a Fréchet successor cardinal. Then there is a countably complete ultrafilter D with $\lambda_D < \lambda$ and an internal ultrapower embedding $e : M_D \to M_U$ that is $j_D(\lambda)$-supercompact in M_D.*

Proof. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$ and let $i : V \to N$ be the ultrapower of the universe of by U. Let

$$(i_*, j_*) : (M, N) \to P$$

be the pushout of (j, i). *Note that i_* denotes the embedding on the M-side of the comparison and j_* denotes the embedding on the N-side of the comparison.* The proof amounts to an analysis of (i_*, j_*).

We first characterize j_*. By definition (Lemma 5.4.36), j_* is the ultrapower of N by $t_U(\mathscr{K}_\lambda)$. Let

$$\delta = \text{cf}^N(\sup i[\lambda])$$

$$\begin{tikzcd}
 & & P & & \\
 & (M_D)^M \arrow[ur, "h"] & & (M_{t_U(D)})^N \arrow[ul, "k"'] & \\
M \arrow[ur, "D"] \arrow[uurr, bend left, "i_*"] & & M_D \arrow[ul, "j_D(\mathscr{K}_\lambda)"'] \arrow[ur, "t_D(U)"] & & N \arrow[ul, "t_U(D)"'] \arrow[uull, bend right, "(\mathscr{K}_\delta)^N"'] \\
 & & V \arrow[ull, "\mathscr{K}_\lambda"] \arrow[u, "D"] \arrow[urr, "U"'] & &
\end{tikzcd}$$

Figure 8.1: The Irreducibility Theorem

By the analysis of translations of $\mathscr{K}_\lambda$ (Lemma 8.2.7), one of the following holds in N:

- δ is not Fréchet and $t_U(\mathscr{K}_\lambda)$ is principal.
- δ is Fréchet and $t_U(\mathscr{K}_\lambda)$ is Rudin-Keisler equivalent to $(\mathscr{K}_\delta)^N$.

The hard part of the proof is the analysis of i_*, the embedding on the M-side of the comparison of (j, i). Let η be the least measurable cardinal of M above λ. Applying Lemma 8.2.8, let D be a countably complete ultrafilter of M with $\lambda_D < \lambda$ such that there is an internal ultrapower embedding $h : (M_D)^M \to P$ with $\text{crit}(h) \geq \eta$ and $i_* = h \circ j_D^M$. We may assume without loss of generality that the underlying set of D is the cardinal λ_D. Recall Corollary 7.4.10, which states that $M^\lambda \subseteq M$. In particular, $P(\gamma) \subseteq M$, so D truly is an ultrafilter.

The following are the two key claims:

Claim 1. $\delta = j_D(\lambda)$ *and* $\text{Ord}^{j_D(\lambda)} \cap N = \text{Ord}^{j_D(\lambda)} \cap P = \text{Ord}^{j_D(\lambda)} \cap M_D$.

Claim 2. $D \leq_{\text{RF}} U$.

Assuming these claims, the conclusion of the theorem is immediate: by Claim 2, let $e : M_D \to N$ be the unique internal ultrapower embedding such that $e \circ j_D = i$; then e is $j_D(\lambda)$-supercompact in M_D since $\text{Ord}^{j_D(\lambda)} \cap N = \text{Ord}^{j_D(\lambda)} \cap M_D$ by Claim 1.

We therefore focus on proving these two claims.

Proof of Claim 1. We begin by showing $\mathrm{Ord}^{j_D(\lambda)} \cap M_D = \mathrm{Ord}^{j_D(\lambda)} \cap P$. Since $j : V \to M$ is a λ-supercompact ultrapower embedding, $\mathrm{Ord}^\lambda = \mathrm{Ord}^\lambda \cap M$. By the elementarity of j_D,

$$\mathrm{Ord}^{j_D(\lambda)} \cap M_D = \mathrm{Ord}^{j_D(\lambda)} \cap j_D(M) = \mathrm{Ord}^{j_D(\lambda)} \cap (M_D)^M$$

The final equality follows from the fact that M is closed under λ-sequences and hence correctly computes the ultrapower of M by D. But $h : (M_D)^M \to P$ is an internal ultrapower embedding such that $\mathrm{crit}(h) \geq \eta > j_D(\lambda)$. Hence $\mathrm{Ord}^{j_D(\lambda)} \cap (M_D)^M = \mathrm{Ord}^{j_D(\lambda)} \cap P$. Putting all this together, we have shown

$$\mathrm{Ord}^{j_D(\lambda)} \cap M_D = \mathrm{Ord}^{j_D(\lambda)} \cap P$$

One consequence of the agreement between M_D and P, which we set down now for future use, is that $j_D(\lambda)$ is a successor cardinal of P: λ is a successor cardinal, so by elementarity, $j_D(\lambda)$ is a successor cardinal of M_D, and therefore since $\mathrm{Ord}^{j_D(\lambda)} \cap M_D = \mathrm{Ord}^{j_D(\lambda)} \cap P$, $j_D(\lambda)$ is a successor cardinal of P.

Next, we show that $\delta = j_D(\lambda)$. To do this, we calculate the P-cofinality of the ordinal $\sup j_* \circ i[\lambda]$ in two different ways.

On the one hand, we claim

$$\mathrm{cf}^P(\sup j_* \circ i[\lambda]) = j_D(\lambda) \tag{8.1}$$

We have that $j_* \circ i = h \circ j_D \circ j = h \circ j_D(j) \circ j_D$. Since $\lambda_D < \lambda$, and λ is regular, $j_D(\lambda) = \sup j_D[\lambda]$ (Lemma 2.2.34). Now we calculate:

$$\begin{aligned}\mathrm{cf}^P(\sup h \circ j_D(j) \circ j_D[\lambda]) &= \mathrm{cf}^P(\sup h \circ j_D(j)[\sup j_D[\lambda]])\\ &= \mathrm{cf}^P(\sup h \circ j_D(j)[j_D(\lambda)])\\ &= \mathrm{cf}^{M_D}(\sup h \circ j_D(j)[j_D(\lambda)])\\ &= j_D(\lambda)\end{aligned}$$

The second-to-last equality uses the fact that $\mathrm{Ord}^{j_D(\lambda)} \cap P = \mathrm{Ord}^{j_D(\lambda)} \cap M_D$. The final equality uses the fact that $h \circ j_D(j)$ is increasing and definable over M_D and $j_D(\lambda)$ is regular in M_D. Putting everything together, $\mathrm{cf}^P(\sup j_* \circ i[\lambda]) = j_D(\lambda)$, as claimed.

On the other hand, we claim

$$\mathrm{cf}^P(\sup j_* \circ i[\lambda]) = \mathrm{cf}^P(\sup j_*[\delta]) \tag{8.2}$$

Since $\delta = \mathrm{cf}^N(\sup i[\lambda])$, there is an increasing cofinal function $f : \delta \to \sup i[\lambda]$ with $f \in N$. Now $\sup j_* \circ i[\lambda] = \sup j_*[\sup f[\delta]] = \sup j_*(f)[\sup j_*[\delta]]$. Thus $j_*(f) \in P$ restricts to an increasing cofinal function from $\sup j_*[\delta]$ to $\sup j_* \circ i[\lambda]$. It follows that $\mathrm{cf}^P(\sup j_* \circ i[\lambda]) = \mathrm{cf}^P(\sup j_*[\delta])$, as desired.

Combining (8.1) and (8.2), we have shown $\mathrm{cf}^P(\sup j_*[\delta]) = j_D(\lambda)$. To show $\delta = j_D(\lambda)$, we must show $\mathrm{cf}^P(\sup j_*[\delta]) = \delta$. In other words (applying the easy direction of Theorem 7.3.34), we must show j_* is δ-tight.

Recall that j_* is the ultrapower of N by $t_U(\mathscr{K}_\lambda)$. If $t_U(\mathscr{K}_\lambda)$ is principal, then trivially j_* is δ-tight. Therefore assume $t_U(\mathscr{K}_\lambda)$ is nonprincipal. By the second paragraph of this proof, N satisfies that δ is Fréchet and $t_U(\mathscr{K}_\lambda)$ is Rudin-Keisler equivalent to $(\mathscr{K}_\delta)^N$.

It suffices to show that δ is not isolated in N. Then applying in N the analysis of $\mathscr{K}_\delta$ at nonisolated cardinals δ (Proposition 7.4.11), j_* is δ-tight.

Thus assume towards a contradiction that δ is isolated in N. In particular, δ is a regular limit cardinal in N. Moreover, by Theorem 7.5.38, $(\mathscr{K}_\delta)^N$ concentrates on N-regular cardinals, so by Łoś's Theorem, $\text{id}_{(\mathscr{K}_\delta)^N} = \sup j_*[\delta]$ is regular in P. Thus by (8.2), $\text{cf}^P(\sup j_* \circ i[\lambda]) = \sup j_*[\delta]$, and so by (8.1), $\sup j_*[\delta] = j_D(\lambda)$. Since δ is a limit cardinal of N, $\sup j_*[\delta]$ is a limit cardinal of P. This contradicts the fact (noted above) that $j_D(\lambda)$ is a successor cardinal of P. Thus our assumption that δ is isolated in N was false. It follows that δ is not isolated and hence j_* is δ-tight, and hence $\text{cf}^P(\sup j_*[\delta]) = \delta$, and hence by (8.1) and (8.2), $j_D(\lambda) = \delta$.

We finally show that $\text{Ord}^\delta \cap N = \text{Ord}^\delta \cap P$. If $t_U(\mathscr{K}_\lambda)$ is principal then $P = N$, so this is obvious. If not, then $j_* : N \to P$ is the ultrapower embedding associated to $(\mathscr{K}_\delta)^N$. Note that $\delta = j_D(\lambda)$ is a successor cardinal of P, and so since $P \subseteq N$, δ is a successor cardinal of N. Thus by the analysis of Ketonen ultrafilters on successor cardinals (Corollary 7.4.10) applied in N, j_* is δ-supercompact. In particular, $\text{Ord}^\delta \cap N = \text{Ord}^\delta \cap P$. □

We now turn to the proof that $D \leq_{\text{RF}} U$.

Proof of Claim 2. To show $D \leq_{\text{RF}} U$, it suffices (by the definition of translation functions, or Lemma 5.4.41) to show that $t_U(D)$ is principal in N.

Let us first show that

$$t_U(D) \leq_{\text{RF}} t_U(\mathscr{K}_\lambda)$$

in N. Note that

$$(h \circ j_D(j), j_*) : (M_D, N) \to P$$

is an internal ultrapower comparison of (j_D, i). Since

$$(j^{M_D}_{t_D(U)}, j^{M_U}_{t_U(D)}) : (M_D, N) \to (M_{t_U(D)})^N$$

is the pushout of (j_D, i) (by Lemma 5.4.36), it follows that there is an internal ultrapower embedding $k : (M_{t_U(D)})^N \to P$ such that $k \circ j^{M_D}_{t_D(U)} = h \circ j_D(j)$ and $k \circ j^{M_U}_{t_U(D)} = j_* = j^N_{t_U(\mathscr{K}_\lambda)}$. The latter equation is equivalent to the statement that $t_U(D) \leq_{\text{RF}} t_U(\mathscr{K}_\lambda)$ in N.

Since $t_U(\mathscr{K}_\lambda)$ is either principal or Rudin-Keisler equivalent to the ultrafilter $(\mathscr{K}_{j_D(\lambda)})^N$, which is irreducible by Lemma 7.3.12, one of the following must hold:

(1) $j_D(\lambda)$ is Fréchet in N and $N \vDash t_U(D) \equiv_{\text{RK}} (\mathscr{K}_{j_D(\lambda)})^N$.

(2) $t_U(D)$ is principal in N.

Our goal is to show that (2) holds, so to finish the proof of the claim, it suffices to show that (1) fails. Towards this, we will prove the following subclaim:

Subclaim 1. *Assume $j_D(\lambda)$ is Fréchet in N. Then $(\mathscr{K}_{j_D(\lambda)})^N = j_D(\mathscr{K}_\lambda)$.*

Proof of Subclaim 1. We plan to prove the claim by applying our unique extension lemma for isonormal ultrafilters (Lemma 8.2.15). By Corollary 7.4.10, $\mathscr{K}_\lambda$ is an isonormal ultrafilter on λ. By Claim 1, $(\mathscr{K}_{j_D(\lambda)})^N$ is an M_D-filter on $j_D(\lambda)$. Let $\langle S_\xi : \xi < \lambda\rangle$ be a partition of S^λ_ω into stationary sets. Let $\langle T_\xi : \xi < j_D(\lambda)\rangle = j_D(\langle S_\xi : \xi < \lambda\rangle)$. By Lemma 8.2.15, to show that $j_D(\mathscr{K}_\lambda) = (\mathscr{K}_{j_D(\lambda)})^N$, it suffices to show that the following hold:

(i) $\{\alpha < j_D(\lambda) : M_D \vDash T_{\mathrm{id}_D} \text{ is stationary in } \alpha\} \in (\mathscr{K}_{j_D(\lambda)})^N$.

(ii) $j_D[\mathscr{K}_\lambda] \subseteq (\mathscr{K}_{j_D(\lambda)})^N$.

(i) will be proved by applying Lemma 8.2.14. Note that $\langle T_\xi : \xi < j_D(\lambda)\rangle$ belongs to N and N satisfies that $\langle T_\xi : \xi < j_D(\lambda)\rangle$ is a stationary partition of $S^{j_D(\lambda)}_\omega$: this follows from the fact that $P(j_D(\lambda))\cap N = P(j_D(\lambda))\cap M_D$ by Claim 1 and $\langle T_\xi : \xi < j_D(\lambda)\rangle$ is a stationary partition of $S^{j_D(\lambda)}_\omega$ in M_D. Since $(\mathscr{K}_{j_D(\lambda)})^N$ is a countably complete weakly normal ultrafilter of N, Lemma 8.2.14 implies that $(\mathscr{K}_{j_D(\lambda)})^N$ concentrates on $\{\alpha < j_D(\lambda) : M_D \vDash T_\xi \text{ is stationary in } \alpha\}$ for any $\xi < j_D(\lambda)$, and in particular $\{\alpha < j_D(\lambda) : M_D \vDash T_{\mathrm{id}_D} \text{ is stationary in } \alpha\} \in (\mathscr{K}_{j_D(\lambda)})^N$, as desired.

Towards (ii), let $W = j_D^{-1}[(\mathscr{K}_{j_D(\lambda)})^N]$. It suffices to show that $W = \mathscr{K}_\lambda$. It is clear that W is a countably complete uniform ultrafilter on λ. Recall that $\mathscr{K}_\lambda$ is the unique Ketonen ultrafilter on λ. Let A be the set of ordinals below λ that carry no countably complete fine ultrafilter. By the definition of a Ketonen ultrafilter on a regular cardinal (Definition 7.3.5), to show $W = \mathscr{K}_\lambda$, it suffices to show that the following hold:

- $A \in W$.
- W is weakly normal.

Let us show that $A \in W$. In other words, we must show that $j_D(A) \in (\mathscr{K}_\delta)^N$. Note that $j_D(A)$ is the set of ordinals less than $j_D(\lambda) = \delta$ that carry no countably complete fine ultrafilter in M_D. By the definition of a Ketonen ultrafilter on a regular cardinal (Definition 7.3.5) applied in N, $(\mathscr{K}_\delta)^N$ concentrates on the set of ordinals less than δ that carry no countably complete fine ultrafilter in N. Thus to show that $j_D(A) \in (\mathscr{K}_\delta)^N$, it suffices to show that if an ordinal less than δ carries no countably complete fine ultrafilter in N, then it carries no countably complete fine ultrafilter in M_D. In fact we will show that for any ordinal $\alpha < \delta$,

$$\mathbf{UF}^{M_D}(\alpha) = \mathbf{UF}^N(\alpha)$$

where $\mathbf{UF}(X)$ denotes the set of countably complete ultrafilters on X.

This is an application Theorem 6.3.16, which asserts that if γ is a cardinal and Q is an ultrapower of the universe that is closed under γ-sequences, then for any ordinal $\alpha < \gamma$, $\mathbf{UF}(\alpha) = \mathbf{UF}^Q(\alpha)$. Fix an ordinal $\alpha < \delta$. Applying Theorem 6.3.16 in M_D to the ultrapower P of M_D, which satisfies $\text{Ord}^\delta \cap P = \text{Ord}^\delta \cap M_D$ by Claim 1,

$$\mathbf{UF}^{M_D}(\alpha) = \mathbf{UF}^P(\alpha)$$

Similarly, applying Theorem 6.3.16 and Claim 1 in N to P,

$$\mathbf{UF}^N(\alpha) = \mathbf{UF}^P(\alpha)$$

Hence $\mathbf{UF}^{M_D}(\alpha) = \mathbf{UF}^N(\alpha)$, as desired. This shows $A \in W$.

We now show that W is weakly normal. We do this by applying Lemma 8.2.18. Note that $(\mathscr{K}_{j_D(\lambda)})^N$ is a weakly normal M_D-ultrafilter since it is a weakly normal ultrafilter of N and $P(j_D(\lambda)) \cap M_D = P(j_D(\lambda)) \cap N$. Therefore since $j_D : V \to M_D$ is continuous at λ, Lemma 8.2.18 implies that $j_D^{-1}[(\mathscr{K}_{j_D(\lambda)})]$ is weakly normal. In other words, W is weakly normal.

Thus we have shown that W is a Ketonen ultrafilter on λ, so $W = \mathscr{K}_\lambda$. This implies (ii).

As we explained above, (i), (ii), and Lemma 8.2.15 together imply $(\mathscr{K}_{j_D(\lambda)})^N = j_D(\mathscr{K}_\lambda)$, which proves the subclaim. □

Using Subclaim 1, we show that (1) above does not hold. If $j_D(\lambda)$ is not Fréchet in N, then obviously (1) does not hold, so assume instead that $j_D(\lambda)$ is Fréchet in N. Let $\mathscr{K} = (\mathscr{K}_{j_D(\lambda)})^N = j_D(\mathscr{K}_\lambda)$. Thus $\mathscr{K} \in M_D \cap N$.

Recall that $M^N_{t_U(D)}$ is the target model of the pushout of (j_D, i). Thus by the analysis of ultrafilters amenable to a pushout (Theorem 5.4.20), $\mathscr{K} \cap M^N_{t_U(D)} \in M^N_{t_U(D)}$. On the other hand, we will show that $\mathscr{K} \cap P \notin P$. By the strictness of the Mitchell order on nonprincipal ultrafilters (Lemma 4.2.38),

$$\mathscr{K} \notin M^{M_D}_{\mathscr{K}} = j_D(M_{\mathscr{K}_\lambda}) = j_D(M)$$

Recall that $h : j_D(M) \to P$ is an internal ultrapower embedding, so in particular $P \subseteq j_D(M)$, and hence $\mathscr{K} \notin P$ since $\mathscr{K} \notin j_D(M)$. Since $P(j_D(\lambda)) \cap N = P(j_D(\lambda)) \cap P$ by Claim 1, it follows that $\mathscr{K} = \mathscr{K} \cap P$, and so $\mathscr{K} \cap P \notin P$.

We have $\mathscr{K} \cap M^N_{t_U(D)} \in M^N_{t_U(D)}$ and $\mathscr{K} \cap P \notin P$, so $M^N_{t_U(D)} \neq P$. Since $P = M^N_{\mathscr{K}}$, it follows that $t_U(D)$ and $\mathscr{K}$ are not Rudin-Keisler equivalent in N: they have different ultrapowers. In other words, (1) above fails.

Thus (2) holds, which proves $D \leq_{\text{RF}} U$, establishing the claim. □

Having proved Claim 1 and Claim 2, the theorem follows, as we explained after the statement of Claim 2. □

An immediate corollary of Theorem 8.2.19 is the following fact, which will imply the Irreducibility Theorem:

Corollary 8.2.20 (UA). *Suppose λ is a Fréchet successor cardinal and U is a λ-irreducible ultrafilter. Then j_U is λ-supercompact.*

Proof. By Theorem 8.2.19, there is an ultrafilter D with $\lambda_D < \lambda$ such that there is an internal ultrapower embedding $e : M_D \to M_U$ with $e \circ j_D = j_U$ that is $j_D(\lambda)$-supercompact in M_D. Since U is λ-irreducible, D is principal, and hence $j_U = e \circ j_D = e$ is λ-supercompact as desired. ☐

Corollary 8.2.21 (UA). *Suppose λ is a strong limit cardinal and U is a λ-irreducible ultrafilter. Then j_U is $<\lambda$-supercompact. If λ is singular, then j_U is λ-supercompact. If λ is regular and Fréchet, then j_U is λ-tight.*

Proof. We start by showing that j_U is $<\lambda$-supercompact. Fix a successor cardinal $\delta < \lambda$. If δ is Fréchet, then j_U is δ-supercompact by Corollary 8.2.20. If δ is not Fréchet, then we can apply Theorem 7.5.34: no cardinal $\kappa \leq \delta$ is δ-supercompact and U is $\leq 2^\delta$-irreducible, so U is δ^+-complete and j_U is vacuously δ-supercompact. Thus j_U is $<\lambda$-supercompact.

Since j_U is a $<\lambda$-supercompact ultrapower embedding, $(M_U)^{<\lambda} \subseteq M_U$. If λ is singular, this immediately implies $(M_U)^\lambda \subseteq M_U$. Therefore j_U is λ-supercompact.

If λ is regular and Fréchet, we can apply Lemma 8.2.10 to conclude that j_U is λ-tight. ☐

As a corollary, we can finally prove the Irreducibility Theorem.

Theorem 8.2.22 (UA). *Suppose λ is a successor cardinal or a strong limit singular cardinal and U is a countably complete uniform ultrafilter on λ. Then the following are equivalent:*

(1) U is λ-irreducible.

(2) j_U is λ-supercompact.

Proof. (1) implies (2): Follows from Theorem 8.2.19 and Corollary 8.2.21.

(2) implies (1): Follows from Proposition 8.2.3. ☐

Theorem 8.2.23 (UA). *Suppose λ is an inaccessible cardinal and U is a countably complete ultrafilter on λ. Then the following are equivalent:*

(1) U is λ-irreducible.

(2) j_U is $<\lambda$-supercompact and λ-tight.

Proof. (1) implies (2): Follows from Corollary 8.2.21 and Lemma 8.2.10.

(2) implies (1): Follows from Proposition 8.2.3. ☐

It is sometimes easier to use a version of the Irreducibility Theorem in the form of Theorem 8.2.19. This follows from Corollary 8.2.21 using the structure of the Rudin-Frolík order (Theorem 5.3.14).

Lemma 8.2.24 (UA). *Suppose U is a countably complete ultrafilter and λ is a cardinal. Then there is a countably complete ultrafilter $D \leq_{\text{RF}} U$ with $\lambda_D < \lambda$ such that $t_D(U)$ is λ_*-irreducible in M_D where $\lambda_* = \sup j_D[\lambda]$.*

Proof. By the local ascending chain condition for the Rudin-Frolík order (Theorem 5.3.14), there is an $\leq_{\text{RF}}$-maximal $D \leq_{\text{RF}} U$ such that $\lambda_D < \lambda$. Let $i : M_D \to M_U$ be the unique internal ultrapower embedding such that $i \circ j_D = j_U$. Then i is the ultrapower of M_D by $t_D(U)$.

Suppose towards a contradiction that $t_D(U)$ is not λ_*-irreducible in M_D. Fix a cardinal $\gamma < \lambda$ and a countably complete ultrafilter Z of M_D on $j_D(\gamma)$ such that $Z \leq_{\text{RF}} t_D(U)$. Then the iteration $\langle D, W\rangle$ is given by an ultrafilter D' on $\lambda_D \cdot \gamma$. Now $\lambda_{D'} \leq \lambda_D \cdot \gamma < \lambda$ but $D <_{\text{RF}} D' \leq_{\text{RF}} U$. This contradicts the maximality of D. $\square$

This combined with the Irreducibility Theorem yields the following fact:

Corollary 8.2.25 (UA). *Suppose U is a countably complete ultrafilter.*

- *If λ is a Fréchet successor cardinal, then there is an ultrafilter $D \leq_{\text{RF}} U$ with $\lambda_D < \lambda$ such that the unique internal ultrapower embedding $h : M_D \to M_U$ with $h \circ j_D = j_U$ is $j_D(\lambda)$-supercompact in M_D.*
- *If λ is a Fréchet inaccessible cardinal, then there is an ultrafilter $D \leq_{\text{RF}} U$ with $\lambda_D < \lambda$ such that the unique internal ultrapower embedding $h : M_D \to M_U$ with $h \circ j_D = j_U$ is $<\lambda$-supercompact and λ-tight in M_D.*
- *If λ is a strong limit singular cardinal, then there is an ultrafilter $D \leq_{\text{RF}} U$ with $\lambda_D < \lambda$ such that the unique internal ultrapower embedding $h : M_D \to M_U$ with $h \circ j_D = j_U$ is λ_*-supercompact in M_D where $\lambda_* = \sup j_D[\lambda]$.* $\square$

8.3 Resolving the identity crisis

In this section, we characterize strong compactness in terms of supercompactness under UA. This begins with an analysis of the κ-complete analog of $\mathscr{K}_\lambda$, denoted $\mathscr{K}_\lambda^\kappa$.

8.3.1 The equivalence of strong compactness and supercompactness

Recall that if λ is κ-Fréchet, then $\mathscr{K}_\lambda^\kappa$ is the $<_{\Bbbk}$-least κ-complete uniform ultrafilter on λ. Applying the Irreducibility Theorem, Proposition 8.1.17 yields a generalization of our analysis of $\mathscr{K}_\lambda$ for successor λ (Corollary 7.4.10) to these more complete ultrafilters:

Corollary 8.3.1 (UA). *Suppose $\kappa < \lambda$ and λ is a κ^+-Fréchet successor cardinal. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda^{\kappa^+}$. Then $M^\lambda \subseteq M$.*

Proof. By Proposition 8.1.17, $\mathscr{K} = \mathscr{K}_\lambda^{\kappa^+}$ is irreducible. Since $\lambda_{\mathscr{K}} = \lambda$, $\mathscr{K}$ is λ-irreducible. Therefore by the Irreducibility Theorem (Corollary 8.2.20), $M^\lambda \subseteq M$. $\square$

Corollary 8.3.2 (UA). *Suppose $\kappa < \lambda$ and λ is a κ^+-Fréchet successor cardinal. Then there is a λ-supercompact cardinal δ such that $\kappa < \delta < \lambda$.* □

As in the case of the least supercompact cardinal, if λ is strongly inaccessible, it is not clear whether $\mathscr{K}_\lambda^{\kappa^+}$ witnesses full λ-supercompactness:

Corollary 8.3.3 (UA). *Suppose $\kappa < \lambda$ and λ is a κ^+-Fréchet inaccessible cardinal. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda^{\kappa^+}$. Then $M^{<\lambda} \subseteq M$ and M has the $\leq\lambda$-cover property.*

Proof. By Proposition 8.1.17, $\mathscr{K} = \mathscr{K}_\lambda^{\kappa^+}$ is irreducible. Since $\lambda_{\mathscr{K}} = \lambda$, $\mathscr{K}$ is λ-irreducible. Therefore by the Irreducibility Theorem (Corollary 8.2.21), $M^{<\lambda} \subseteq M$ and M has the $\leq\lambda$-cover property. □

Let us now analyze $\mathscr{K}_\lambda^\kappa$ for general κ:

Theorem 8.3.4 (UA). *Suppose $\kappa \leq \lambda$ and λ is a κ-Fréchet regular cardinal. Let $\mathscr{K} = \mathscr{K}_\lambda^\kappa$.*

(1) Suppose κ is not a measurable limit of λ-strongly compact cardinals. Then $\mathscr{K}$ is irreducible.

(2) Suppose κ is a measurable limit of λ-strongly compact cardinals. Let D be the $\lhd$-least normal ultrafilter on κ. Then $D \leq_{\mathrm{RF}} \mathscr{K}$.

Proof. Proof of (1): Assume first that κ is not measurable. Since $\mathscr{K}$ is κ-complete, it is κ^+-complete. Hence $\mathscr{K} = \mathscr{K}_\lambda^{\kappa^+}$ is irreducible by Proposition 8.1.17.

Assume instead that κ is not a limit of λ-strongly compact cardinals. Let $\nu < \kappa$ be the supremum of the λ-strongly compact cardinals below κ. Note that λ is ν^+-Fréchet since λ is κ-Fréchet and $\nu \leq \kappa$. Moreover, $\mathscr{K}$ is ν^+-complete since $\nu^+ \leq \kappa$. Since $\mathscr{K}_\lambda^{\nu^+}$ is the $<_\Bbbk$-least ν^+-complete uniform ultrafilter on λ, $\mathscr{K}_\lambda^{\nu^+} \leq_\Bbbk \mathscr{K}_\lambda^\kappa$. On the other hand, $\mathscr{K}_\lambda^{\nu^+}$ is κ-complete: by Corollary 8.3.1 and Corollary 8.3.3, the completeness of $\mathscr{K}_\lambda^{\nu^+}$ is a λ-strongly compact cardinal in the interval (ν, λ), and by choice of ν, the completeness is at least κ. Since $\mathscr{K}_\lambda^{\nu^+}$ is a κ-complete uniform ultrafilter on λ and $\mathscr{K} = \mathscr{K}_\lambda^\kappa$ is the $<_\Bbbk$-least such ultrafilter, $\mathscr{K} \leq_\Bbbk \mathscr{K}_\lambda^{\nu^+}$. By the antisymmetry of the Ketonen order, $\mathscr{K} = \mathscr{K}_\lambda^{\nu^+}$, and in particular $\mathscr{K}$ is irreducible by Proposition 8.1.17.

Proof of (2): Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}$.

We first claim that κ is not measurable in M. Since $\mathscr{K}$ is κ-complete, $\mathrm{crit}(j) \geq \kappa$. Therefore if $\delta < \kappa$ is λ-strongly compact, then δ is $j(\lambda)$-strongly compact in M. Suppose towards a contradiction that κ is measurable in M. Then κ is a measurable limit of $j(\lambda)$-strongly compact cardinals in M, so κ is $j(\lambda)$-strongly compact in M by Menas's Theorem (Corollary 8.1.6). But by the minimality of $\mathscr{K}_\lambda^\kappa$ (see Theorem 7.2.14), $\mathrm{cf}^M(\sup j[\lambda])$ is not κ-Fréchet in M, contradicting that κ is $\mathrm{cf}^M(\sup j[\lambda])$-strongly compact in M. Thus our assumption was false and so κ is not measurable in M.

Since κ is measurable in V but not in M, it follows that $\text{crit}(j) \leq \kappa$, so $\text{crit}(j) = \kappa$. Let D be the ultrafilter on κ derived from j using κ. Since D is a normal ultrafilter and κ is not measurable in M_D, D is the $\lhd$-least ultrafilter on κ (by the linearity of the Mitchell order, Theorem 2.3.11). Recall that our analysis of derived normal ultrafilters (Theorem 5.3.9) implies that either $D \lhd \mathscr{K}$ or $D \leq_{\text{RF}} \mathscr{K}$. Since κ is not measurable in $M = M_{\mathscr{K}}$, it cannot be that $D \lhd \mathscr{K}$, and therefore we can conclude that $D \leq_{\text{RF}} \mathscr{K}$. $\square$

It is not hard to show that in the situation of Theorem 8.3.4 (2), in fact $\mathscr{K}_\kappa^\lambda$ is one of the ultrafilters defined in the proof of Menas's Theorem (Corollary 8.1.6):

$$\mathscr{K}_\kappa^\lambda = D\text{-}\lim_{\alpha<\kappa} \mathscr{K}_\lambda^{\alpha^+}$$

Moreover, there is a set $I \in D$ such that the sequence $\langle \mathscr{K}_\lambda^{\alpha^+} : \alpha \in I\rangle$ is discrete, which explains why $D \leq_{\text{RF}} \mathscr{K}_\kappa^\lambda$.

We now characterize the critical point of $\mathscr{K}_\lambda^\nu$.

Definition 8.3.5. Suppose $\nu \leq \lambda$ are uncountable cardinals and λ is ν-Fréchet. Then κ_λ^ν denotes the completeness of $\mathscr{K}_\lambda^\nu$.

To analyze κ_λ^ν, we use a generalization of Proposition 7.4.1 that involves a multi-parameter strong compactness principle introduced by Bagaria-Magidor:

Definition 8.3.6. An ordinal κ is (ν, λ)-*strongly compact* if there is an elementary embedding $j : V \to M$ with $\text{crit}(j) \geq \nu$ such that $j[\lambda] \subseteq \sigma$ for some $\sigma \in M$ of M-cardinality less than $j(\kappa)$. If κ is (ν, λ)-strongly compact for all λ, we say κ is (ν, ∞)-strongly compact.

Notice that if κ is (δ, λ)-strongly compact cardinal and $\kappa' \geq \kappa$, then κ' is (δ, λ)-strongly compact. For this reason, only the least (δ, λ)-strongly compact cardinal is of particular interest.

Lemma 8.3.7. *Suppose $\nu \leq \lambda$ are cardinals and λ is regular. Suppose $\kappa \leq \lambda$ is the least (ν, λ)-strongly compact cardinal. Suppose $j : V \to M$ is an elementary embedding such that $\text{cf}^M(\sup j[\lambda])$ is not $j(\nu)$-Fréchet in M. Then j is (λ, δ)-tight for some M-cardinal $\delta < j(\kappa)$.*

Proof. Since κ is (ν, λ)-strongly compact, every cardinal in the interval $[\kappa, \lambda]$ is ν-Fréchet. Thus in M, every cardinal in the interval $j([\kappa, \lambda])$ is $j(\nu)$-Fréchet. Let $\delta = \text{cf}^M(\sup j[\lambda])$. By Theorem 7.2.12, j is (λ, δ)-tight. Moreover $\delta \leq \sup j[\lambda] \leq j(\lambda)$ and $\delta \notin j([\kappa, \lambda])$ since δ is not $j(\nu)$-Fréchet. Thus $\delta < j(\kappa)$. This proves the lemma. $\square$

The following proposition shows that under UA, the Magidor-Bagaria generalizations of strong compactness collapse to the classical notion:

Proposition 8.3.8 (UA). *Suppose $\nu \leq \kappa \leq \lambda$ are cardinals, λ is a regular cardinal, and κ is the least (ν, λ)-strongly compact cardinal. Then $\kappa = \kappa_\lambda^\nu$ and κ is λ-strongly compact.*

Proof. Since there is a (ν, λ)-strongly compact cardinal $\kappa \leq \lambda$, there is some cardinal below λ that is λ-supercompact. Thus if λ is a limit cardinal then λ is strongly inaccessible by our results on GCH (Theorem 6.3.27). In particular, we are in a position to apply the Irreducibility Theorem.

By Theorem 8.3.4, either κ_λ^ν is a measurable limit of λ-strongly compact cardinals or $\mathscr{K}_\lambda^\nu$ is irreducible. In the former case κ_λ^ν is λ-strongly compact by Theorem 8.1.1. In the latter case, $\mathscr{K}_\lambda^\nu$ witnesses that κ_λ^ν is $<\lambda$-supercompact and λ-strongly compact by the Irreducibility Theorem (Corollary 8.2.20 and Corollary 8.2.21).

In particular, κ_λ^ν is (ν, λ)-strongly compact, so $\kappa \leq \kappa_\lambda^\nu$.

On the other hand, by Lemma 8.3.7, $\kappa \leq \kappa_\lambda^\nu$.

Thus $\kappa = \kappa_\lambda^\nu$, and in particular κ is λ-strongly compact. □

Corollary 8.3.9 (UA). *Suppose $\kappa \leq \lambda$ are cardinals, λ is a successor cardinal, and κ is λ-strongly compact. Then either κ is λ-supercompact or κ is a measurable limit of λ-supercompact cardinals.*

Proof. Assume by induction that the theorem is true for $\bar{\kappa} < \kappa$. By Proposition 8.3.8, $\kappa = \kappa_\lambda^\kappa$. By Theorem 8.3.4, either κ is a measurable limit of λ-strongly compact cardinals or $\mathscr{K}_\lambda^\kappa$ is irreducible. If κ is a limit of λ-strongly compact cardinals, then by our induction hypothesis, κ is a measurable limit of λ-supercompact cardinals. If instead $\mathscr{K}_\lambda^\kappa$ is irreducible, then by Theorem 8.2.19, $\mathscr{K}_\lambda^\kappa$ witnesses that κ is λ-supercompact. □

This implies our converse to Menas's Theorem, stating that under UA, a strongly compact cardinal is either a supercompact cardinal or a measurable limit of supercompact cardinals:

Theorem 8.3.10 (UA). *Suppose κ is a strongly compact cardinal. Either κ is a supercompact cardinal or κ is a measurable limit of supercompact cardinals.*

Proof. Suppose κ is strongly compact. By the Pigeonhole Principle, there is a cardinal $\gamma \geq \kappa$ such that a cardinal $\bar{\kappa} \leq \kappa$ is supercompact if and only if $\bar{\kappa}$ is γ-supercompact. Since κ is γ^+-strongly compact, Corollary 8.3.9 implies that either κ is γ^+-supercompact or κ is a limit of γ^+-supercompact cardinals. By our choice of γ, it follows that either κ is supercompact or κ is a limit of supercompact cardinals, as desired. □

The use of the Pigeonhole Principle is unnecessary here, since the cardinal γ turns out to equal κ; a more careful argument appears in the proof of Corollary 8.3.17.

Next, we generalize our result on ultrapower thresholds (Theorem 7.4.26). The result we prove here is actually a bit stronger than was possible for the ultrapower threshold itself since our stronger large cardinal assumption puts us in a local GCH context:

Lemma 8.3.11 (UA). *Suppose λ is a regular Fréchet cardinal. Suppose λ is also κ_λ^+-Fréchet.*[2] *Then for all cardinals $\gamma \in [\kappa_\lambda, \lambda]$, $2^\gamma = \gamma^+$.*

Proof. Let $\kappa = \kappa_\lambda$. Let $\mathscr{K}_0 = \mathscr{K}_\lambda$ and $\mathscr{K}_1 = \mathscr{K}_\lambda^{\kappa^+}$. Then $\mathscr{K}_1$ is λ-decomposable yet since $\mathscr{K}_1$ is κ^+-complete, $\mathscr{K}_0 \not\leq_{\text{RF}} \mathscr{K}_1$. Therefore Theorem 7.5.15 implies that λ is not isolated. It follows that κ is $<\lambda$-supercompact. In particular, applying our results on GCH (namely Theorem 6.3.25), either λ is a successor cardinal or λ is a strongly inaccessible cardinal. Thus we are in a position to apply Corollary 8.3.1 and Corollary 8.3.3.

A weak consequence of the conjunction of these two theorems is that there is an elementary embedding $j : V \to M$ such that $\text{crit}(j) > \kappa$, $j(\lambda) > \lambda^{++M}$, and j is $[0, \lambda]$-tight (or in other words, j is γ-tight for all $\gamma \leq \lambda$). Since $j(\kappa) = \kappa$ and $j(\lambda) > \lambda^{++M}$, κ is λ^{++M}-supercompact in M. Thus by our results on GCH (Theorem 6.3.25) applied in M, M satisfies that for all $\gamma \in [\kappa, \lambda]$, $2^\gamma = \gamma^+$. But for all $\gamma \leq \lambda$, the γ-tightness of j implies that $2^\gamma \leq (2^\gamma)^M$ (by Lemma 8.2.6), and hence

$$2^\gamma \leq (2^\gamma)^M \leq \gamma^{+M} \leq \gamma^+$$

as desired. $\square$

Definition 8.3.12. Suppose $\nu \leq \lambda$ are ordinals. The (ν, λ)-*threshold* is the least ordinal κ such that for all $\alpha < \lambda$, there is an ultrapower embedding $j : V \to M$ such that $\text{crit}(j) \geq \nu$ and $j(\kappa) \geq \alpha$.

The following theorem is proved in ZFC and has nothing to do with UA.

Theorem 8.3.13. *Suppose $\kappa \leq \lambda$ are cardinals, λ is regular, and κ is the $(\nu, \lambda^+ + 1)$-threshold. Assume $2^\gamma = \gamma^+$ for all cardinals $\gamma \in [\kappa, \lambda]$. Then κ is (ν, λ)-strongly compact.*

Proof. Let U be a ν-complete ultrafilter such that $j_U(\kappa) \geq \lambda^+$. Suppose γ is a regular cardinal in the interval $[\kappa, \lambda]$. Suppose towards a contradiction that U is γ-indecomposable and γ^+-indecomposable. Since $2^\gamma = \gamma^+$, we can apply Silver's Theorem (Theorem 7.5.26). This yields an ultrafilter D with $\lambda_D < \gamma$ such that there is an elementary embedding $k : M_D \to M_U$ with $k \circ j_D = j_U$ and $\text{crit}(k) > j_D(\gamma^+)$. Since $j_D(\kappa) \leq j_D(\gamma^+)$,

$$j_D(\kappa) = k(j_D(\kappa)) = j_U(\kappa) \geq \lambda^+$$

But $j_D(\kappa) < (\kappa^{\lambda_D})^+ \leq (\gamma^{<\gamma})^+ = \gamma^+ \leq \lambda^+$, which is a contradiction.

Therefore U is either γ-decomposable or γ^+-decomposable. But if U is γ^+-decomposable, then since γ is regular, in fact, U is γ-decomposable (by Prikry's Theorem [34], or the proof of Proposition 7.4.4). In particular, every regular cardinal in the interval $[\kappa, \lambda]$ carries a ν-complete uniform ultrafilter, which implies that κ is (ν, λ)-strongly compact. $\square$

[2]By the proof of the lemma, this hypothesis can be reformulated as the statement that there are distinct λ-strongly compact cardinals.

The converse to Theorem 8.3.13 is also true. In fact, if $\kappa = 2^{<\kappa}$ and κ is (ν,λ)-strongly compact, then any $j : V \to M$ witnessing this satisfies $j(\kappa) \geq 2^\lambda$: indeed, j is $(\lambda, {\leq}\delta)$-tight for some $\delta < j(\kappa)$, and so by Lemma 8.2.6, $2^\lambda \leq (2^\delta)^M \leq j(\kappa)$.

Theorem 8.3.13 answers a question of Hamkins [37] assuming GCH.

Definition 8.3.14 (Hamkins)**.** A cardinal κ is *strongly tall* if κ can be mapped arbitrarily high by ultrapower embeddings with critical point κ.

Hamkins asked whether strongly tall cardinals must be strongly compact.

Theorem 8.3.15 (GCH)**.** *If κ is strongly tall, then κ is strongly compact.*

Proof. Since κ is strongly tall, κ is the (κ,λ)-threshold for all $\lambda \geq \kappa$. Applying Theorem 8.3.13, κ is strongly compact. □

Under UA, the cardinal arithmetic hypothesis of Theorem 8.3.13 can be eliminated, at least above κ_λ:

Theorem 8.3.16 (UA)**.** *Suppose λ is a regular Fréchet cardinal. Suppose $\kappa \leq \lambda$ is the $(\nu, \lambda^+ + 1)$-threshold for some $\nu > \kappa_\lambda$. Then κ is λ-strongly compact.*

Proof. The following is the main claim:

Claim. *λ is ν-Fréchet.*

Sketch. We first claim that there is some ν-Fréchet cardinal in the interval $[\lambda, 2^\lambda]$. Assume towards a contradiction that this fails. Fix a ν-complete ultrafilter U such that $j_U(\kappa) \geq \lambda^+$. By Silver's Theorem (Theorem 7.5.26), there is an ultrafilter D with $\lambda_D < \lambda$ such that there is an elementary embedding $k : M_D \to M_U$ with $\text{crit}(k) > j_D((2^\lambda)^+)$. In particular, $j_D(\kappa) \geq \lambda^+$, and it follows that λ is not isolated by Proposition 7.5.24. Let $\gamma = \lambda_D$. We claim that $2^\gamma = \gamma^+$. If γ is singular, this follows from Theorem 6.3.25: note that $\gamma \in [\kappa_\lambda, \lambda]$ so some cardinal is γ-supercompact by Theorem 7.4.9, and hence $2^\gamma = \gamma^+$ by Theorem 6.3.25. If γ is regular, then this follows from Lemma 8.3.11 since by Lemma 7.4.19, $\kappa_\gamma \leq \kappa_\lambda \leq \nu$. Thus $2^\gamma = \gamma^+$ in either case. From this (and Theorem 6.3.25) it follows that $\lambda^\gamma = \lambda$. This contradicts that $j_D(\lambda) \geq \lambda^+$. Thus our assumption was false, so there is a ν-Fréchet cardinal in the interval $[\lambda, 2^\lambda]$.

Now let λ' be the least ν-Fréchet cardinal greater than or equal to λ. Suppose towards a contradiction that $\lambda' > \lambda$.

We claim λ' is an isolated cardinal. Clearly λ' is Fréchet. By the proof of Proposition 7.4.4, λ' is a limit cardinal. Finally, λ' is not a limit of Fréchet cardinals: otherwise by Corollary 7.5.2, λ' is a strong limit cardinal, contradicting that $\lambda < \lambda' \leq 2^\lambda$. Thus λ' is isolated, as claimed.

Theorem 7.5.15 implies $\mathscr{K}_{\lambda'} \leq_{\text{RF}} \mathscr{K}_{\lambda'}^\nu$, which implies that $\mathscr{K}_{\lambda'}$ is ν-complete, or in other words $\kappa_{\lambda'} \geq \nu$. Since $\lambda \geq \kappa_{\lambda'}$, Lemma 7.4.18 implies $\mathscr{K}_{\lambda'} \not\sqsubset \mathscr{K}_\lambda$. By the characterization of internal ultrapower embeddings of $M_{\mathscr{K}_\lambda}$ (Theorem 7.3.14), $\mathscr{K}_{\lambda'}$ must be discontinuous at λ. But this implies λ is $\kappa_{\lambda'}$-Fréchet, and hence λ is ν-Fréchet. This contradicts our assumption that $\lambda' > \lambda$ is the least ν-Fréchet cardinal greater than or equal to λ. □

Since λ is ν-Fréchet and $\nu > \kappa_\lambda$, we are in the situation of Lemma 8.3.11. Therefore for all cardinals $\gamma \in [\kappa_\lambda, \lambda]$, $2^\lambda = \lambda^+$. This yields the cardinal arithmetic hypothesis of Theorem 8.3.13, so we can conclude that κ is the least (ν, λ)-strongly compact cardinal. By Proposition 8.3.8, it follows that κ is λ-strongly compact. □

Of course, if one works below a strong limit cardinal, one obtains a complete generalization of Theorem 7.4.26:

Corollary 8.3.17 (UA)**.** *If λ is a strong limit cardinal and $\kappa < \lambda$ is the (ν, λ)-threshold, then κ is γ-strongly compact for all $\gamma < \lambda$. Therefore one of the following holds:*

- *κ is γ-supercompact for all $\gamma < \lambda$.*
- *κ is a measurable limit of cardinals that are γ-supercompact for all $\gamma < \lambda$.*

Proof. Let κ_0 be the λ-threshold. By Theorem 7.4.26, κ_0 is $<\lambda$-supercompact. If $\nu \leq \kappa_0$, then κ_0 is the (ν, λ)-threshold, so $\kappa = \kappa_0$, which proves the corollary.

Therefore assume $\nu > \kappa$. Suppose $\delta \in [\kappa, \lambda]$ is a regular cardinal. By the proof of Theorem 7.4.26, $\kappa_0 = \kappa_\delta$. Moreover κ is the (ν, δ^+)-threshold by Lemma 7.4.25. Therefore we can apply Theorem 8.3.16 to obtain that κ is δ-strongly compact.

The final two bullet points are immediate from Corollary 8.3.9. Suppose κ is not δ-supercompact for some $\delta < \lambda$. By Corollary 8.3.9, κ is a measurable limit of γ-supercompact cardinals for all $\gamma \in [\delta, \lambda)$. Now suppose $\kappa_0 < \kappa$ is κ-supercompact. We claim κ_0 is γ-supercompact for all $\gamma < \lambda$. Fix $\gamma < \lambda$. There is some $\kappa_1 \in (\kappa_0, \kappa]$ that is γ-supercompact. But κ_0 is κ_1-supercompact, so in fact, κ_0 is γ-supercompact, as desired. □

8.3.2 Level-by-level equivalence at singular cardinals

A well-known theorem of Apter-Shelah [38] shows the consistency of *level-by-level equivalence* of strong compactness and supercompactness: it is consistent with very large cardinals that for all regular λ, a cardinal κ is λ-strongly compact if and only if it is λ-supercompact or a measurable limit of λ-supercompact cardinals. (By Corollary 8.1.6, this is best possible.) We showed this is a consequence of UA for successor cardinals λ (Corollary 8.3.9); for inaccessible cardinals λ, this is an open problem, discussed further in Section 8.4.3.

When λ is singular, level-by-level equivalence is in general false. This is a consequence of the following observation:

Lemma 8.3.18. *Suppose $\kappa \leq \lambda$ are cardinals.*

- *If $\mathrm{cf}(\lambda) < \kappa$, then κ is λ-strongly compact if and only if κ is λ^+-strongly compact.*

- *If $\kappa \leq \text{cf}(\lambda) < \lambda$, then κ is λ-strongly compact if and only if κ is $<\lambda$-strongly compact.* $\square$

The first bullet point shows that if level-by-level equivalence holds at successor cardinals, it also holds at singular cardinals of small cofinality. But by the second bullet point, it need not hold at singular cardinals of larger cofinality:

Proposition 8.3.19. *Suppose κ is the least cardinal δ that is $\beth_\delta(\delta)$-strongly compact. Then κ is not $\beth_\kappa(\kappa)$-supercompact.*

Proof. In fact, if δ is $\beth_\delta(\delta)$-supercompact, then δ is a limit of cardinals $\bar{\delta} < \delta$ that are $\beth_{\bar{\delta}}(\bar{\delta})$-strongly compact. To see this, let $j : V \to M$ be an elementary embedding such that $\text{crit}(j) = \delta$, $j(\delta) > \beth_\delta(\delta)$, and $M^{\beth_\delta(\delta)} \subseteq M$. Then δ is $<\beth_\delta(\delta)$-supercompact in M. It follows from Lemma 8.3.18 that δ is $\beth_\delta(\delta)$-strongly compact in M. Therefore by the usual reflection argument, δ is a limit of cardinals $\bar{\delta} < \delta$ that are $\beth_{\bar{\delta}}(\bar{\delta})$-strongly compact. $\square$

Upon further thought, however, Proposition 8.3.19 *does not* rule out that a version of level-by-level equivalence that holds at singular cardinals, but rather shows that the conventional local formulation of strong compactness degenerates at singular cardinals of large cofinality. We therefore introduce a slightly stronger notion:

Definition 8.3.20. A cardinal κ is λ-*club compact* if there is a κ-complete ultrafilter on $P_\kappa(\lambda)$ that extends the closed unbounded filter.

If κ is λ-supercompact, then κ is λ-club compact: a normal fine ultrafilter always extends the closed unbounded filter. On the other hand, if every κ-complete filter on $P_\kappa(\lambda)$ extends to a κ-complete ultrafilter, then in particular, the closed unbounded filter on $P_\kappa(\lambda)$ extends to a κ-complete ultrafilter, so κ is λ-club compact.

Question 8.3.21 (ZFC)**.** *Suppose λ is a regular cardinal and κ is λ-strongly compact. Must κ be λ-club compact?*

Gitik [39, Theorem 7] answers this question positively under the assumption that $2^\lambda = \lambda^+$. (The statement of Gitik's theorem omits the hypothesis that λ is regular, which is presumably a typo.) There is a much simpler proof that this question has a positive answer when $2^{<\lambda} = \lambda$:

Proposition 8.3.22. *Suppose λ is a regular cardinal such that $2^{<\lambda} = \lambda$. Then any λ-strongly compact cardinal is λ-club compact.*

Proof. Let $j : V \to M$ be an elementary embedding such that M has the $(\leq\lambda, j(\kappa))$-cover property. Let S be the set of functions f such that $f : [\alpha]^{<\omega} \to \lambda$ for some ordinal $\alpha < \lambda$. Our cardinal arithmetic hypothesis implies that $|S| = \lambda^{<\lambda} = \lambda$. (This is where we use that λ is regular.) Since M has the $(\leq\lambda, j(\kappa))$-cover property, there is some $T \in M$ such that $|T|^M < j(\kappa)$ and

$j[S] \subseteq T$. We may assume without loss of generality that T is a set of functions g such that $\text{dom}(g) : [\alpha]^{<\omega} \to \sup j[\lambda]$ for some $\alpha < \sup j[\lambda]$.

Fix $A \in M$ such that $j[\lambda] \subseteq A \subseteq \sup j[\lambda]$ and $|A|^M < j(\kappa)$. Let $B \subseteq \sup j[\lambda]$ be the smallest set with the following properties:

- $A \subseteq B$.
- For any $g \in T$, if $s \in [B]^{<\omega} \cap \text{dom}(g)$ then $g(s) \in B$.
- $B \cap j(\kappa) \in j(\kappa)$.

Clearly $B \in M$ and $|B|^M < j(\kappa)$. Let $\mathcal{U}$ be the ultrafilter on $P_\kappa(\lambda)$ derived from j using B.

We claim that for any function $f : [\lambda]^{<\omega} \to \lambda$, the set

$$\text{cl}(f) = \{\sigma \in P_\kappa(\lambda) : f[[\sigma]^{<\omega}] \subseteq \sigma\}$$

belongs to $\mathcal{U}$. To see this, it suffices to show that

$$j(f)[[B]^{<\omega}] \subseteq [B]^{<\omega}$$

Suppose $s \in [B]^{<\omega}$. Fix $\alpha < \lambda$ such that $s \subseteq j(\alpha)$. Let $g = j(f \restriction [\alpha]^{<\omega})$. Then $g \in j[S] \subseteq T$. Since B is closed under functions in T, $j(g)(s) \in B$, and so since $j(f)(s) = j(g)(s)$, $j(f)(s) \in B$. This proves our claim.

Since we demanded that $B \cap j(\kappa) \in j(\kappa)$, the set

$$C = \{\sigma \in P_\kappa(\lambda) : \sigma \cap \kappa \in \kappa\}$$

belongs to $\mathcal{U}$. By [40, Lemma 3.7], the closed unbounded filter on $P_\kappa(\lambda)$ is generated by C along with the sets $\text{cl}(f)$ for $f : [\lambda]^{<\omega} \to \lambda$. □

In particular, Question 8.3.21 has a positive answer assuming UA: if λ is a successor cardinal, one can use the equivalence between strong compactness and supercompactness at successors (Corollary 8.3.9), while if λ is a limit cardinal, then one can apply the theorem that UA implies GCH (Theorem 6.3.25) to obtain that $2^{<\lambda} = \lambda$, so that Proposition 8.3.22 can be applied.

Following Bagaria-Magidor [41], we introduce a multi-parameter version of club compactness as well:

Definition 8.3.23. A cardinal κ is (ν, λ)-*club compact* if there is a ν-complete ultrafilter on $P_\kappa(\lambda)$ that extends the closed unbounded filter, and κ is *almost* λ-*club compact* if κ is (ν, λ)-club compact for all $\nu < \kappa$.

As is typical in the Bagaria-Magidor notation, if κ is (ν, λ)-club compact, then every cardinal greater than κ is (ν, λ)-club compact.

Menas's Theorem (Corollary 8.1.6) carries over to club compactness:

Lemma 8.3.24. *Suppose λ is a cardinal. Any limit of λ-club compact cardinals is almost λ-club compact. An almost λ-club compact cardinal is λ-club compact if and only if it is measurable. Thus every measurable limit of λ-club compact cardinals is λ-club compact.* □

Given Proposition 8.3.22, λ-club compactness is most interesting when λ is a singular cardinal. In contrast to Proposition 8.3.19, it is consistent with ZFC that the least cardinal κ that is $\beth_\kappa(\kappa)$-strongly compact is actually $\beth_\kappa(\kappa)$-club compact: for example, this is true if the least measurable cardinal is strongly compact. The main theorem of this section is that under UA, level-by-level equivalence holds for club compactness at singular cardinals.

Theorem 8.3.25 (UA). *Suppose $\kappa \leq \lambda$ are cardinals and λ is singular. Then the following are equivalent:*

(1) κ is λ-club compact.

(2) κ is the least (ν, λ)-club compact cardinal for some $\nu \leq \kappa$.

(3) κ is λ-supercompact or a measurable limit of λ-supercompact cardinals.

For the proof, we prove a much more general lemma whose statement involves a variant of the Rudin-Keisler order on filters.

Definition 8.3.26. The *Katětov order* is defined on filters F and G by setting $F \leq_{\text{Kat}} G$ if there is a function f on a set in G such that $F \subseteq f_*(G)$.

Thus $F \leq_{\text{Kat}} G$ if and only if there is an extension F' of F below G in the Rudin-Keisler order.

Lemma 8.3.27 (UA). *Suppose $\nu < \lambda$ are cardinals. Suppose $\mathscr{F}$ is a normal fine filter on a set Y such that $\lambda \subseteq Y \subseteq P(\lambda)$. Suppose A is a set of ordinals and U is the $<_\Bbbk$-least ν^+-complete ultrafilter on A such that $\mathscr{F} \leq_{\text{Kat}} U$. Then U is λ-irreducible.*

Proof. Suppose $D \leq_{\text{RF}} U$ and $\lambda_D < \lambda$. We must show that D is principal. To do this, we will show that $j_D(U) \leq_\Bbbk t_D(U)$ in M_D. By Proposition 5.4.5, it then follows that D is principal. As usual, to show $j_D(U) \leq_\Bbbk t_D(U)$ in M_D, we verify that the properties for which U is minimal hold for $t_D(U)$ with the parameters shifted by j_D. In other words, we show that M_D satisfies the following:

- $t_D(U)$ is a $j_D(\nu^+)$-complete ultrafilter on $j_D(A)$.
- $j_D(\mathscr{F}) \leq_{\text{Kat}} t_D(U)$.

The first bullet point is quite easy. By definition, $t_D(U)$ is an ultrafilter on $j_D(A)$. Moreover, $t_D(U)$ is $j_D(\nu^+)$-complete in M_D since

$$\text{crit}(k) \geq \text{crit}(j) > \nu = j(\nu) \geq j_D(\nu)$$

The second bullet point is a bit more subtle. Since $\mathscr{F} \leq_{\text{Kat}} U$, there is some $B \in M_U$ such that $\mathscr{F}$ is contained in the ultrafilter derived from j_U using B. In other words, for all $S \in \mathscr{F}$, $B \in j_U(S)$. Note that for any $f : \lambda \to \lambda$, B is closed under $j_U(f)$: by normality, $\{\sigma \in Y : \sigma \text{ is closed under } f\} \in \mathscr{F}$, and hence $B \in j_U(\{\sigma \in Y : \sigma \text{ is closed under } f\})$, or in other words, B is closed under $j_U(f)$. We will use this fact in an application of Lemma 6.3.24.

Let $k : M_D \to M$ be the unique internal ultrapower embedding with $k \circ j_D = j_U$. Thus k is the ultrapower of M_D by $t_D(U)$. Let

$$\mathscr{W} = \{S \in j_D(P(Y)) : B \in k(S)\}$$

Thus $\mathscr{W}$ is the M_D-ultrafilter on $j_D(Y)$ derived from k using B. In particular, $\mathscr{W} \leq_{\text{RK}} t_D(U)$ by the characterization of the Rudin-Keisler order in terms of derived embeddings (Lemma 3.4.4). We claim that $j_D(\mathscr{F}) \subseteq \mathscr{W}$. Clearly $j_D[\mathscr{F}] \subseteq \mathscr{W}$. The key point is that by Lemma 6.3.24, $k(\text{id}_D) \in B$. In other words,

$$\{\sigma \in j_D(Y) : \text{id}_D \in \sigma\} \in \mathscr{W}$$

Therefore by our unique extension lemma for normal filters (Lemma 8.2.11), $j_D(\mathscr{F}) \subseteq \mathscr{W}$, as desired.

Now $j_D(\mathscr{F}) \subseteq \mathscr{W} \leq_{\text{RK}} t_D(U)$, or in other words $j_D(\mathscr{F}) \leq_{\text{Kat}} t_D(U)$. □

Proof of Theorem 8.3.25. (1) implies (2): Trivial.

(2) implies (3): Clearly λ is a limit of Fréchet cardinals, so by Corollary 7.5.2, λ is a strong limit cardinal.

We first handle the case in which there is some $\nu < \kappa$ such that κ is the least (ν, λ)-club compact cardinal. Note that ν is either not measurable or not almost λ-club compact, since otherwise ν would be the least (ν, λ)-club compact cardinal. If ν is not almost λ-club compact, then there is some $\bar{\nu} < \nu$ such that κ is the least $(\bar{\nu}^+, \lambda)$-club compact cardinal. If ν is not measurable, then κ is the least (ν^+, λ)-club compact cardinal. In either case, we can fix $\eta < \kappa$ such that κ is the least (η^+, λ)-club compact cardinal.

Let $\mathscr{F}$ be the closed unbounded filter on $P_\kappa(\lambda)$. Let U be the least η^+-complete ultrafilter on an ordinal such that $\mathscr{F} \leq_{\text{Kat}} U$. Then U is λ-irreducible. Since λ is a singular strong limit cardinal, by Corollary 8.2.21, $(M_U)^\lambda \subseteq M_U$. Thus $\text{crit}(j_U)$ is λ-supercompact. Note that $\text{crit}(j_U) \leq \kappa$ since $\mathscr{F} \leq_{\text{Kat}} U$ and $\mathscr{F}$ is not κ^+-complete. On the other hand $\text{crit}(j_U) > \eta$, so $\text{crit}(j_U)$ is an (η^+, λ)-club compact cardinal, and hence $\text{crit}(j_U) \leq \kappa$. Thus $\kappa = \text{crit}(j_U)$ is λ-supercompact.

We now handle the case in which κ is (κ, λ)-club compact but there is no $\nu < \kappa$ such that κ is the least (ν, λ)-club compact cardinal. Since κ is (ν, λ)-club compact for all $\nu < \kappa$, it follows that for each $\nu < \lambda$, the least (ν, λ)-club compact cardinal lies strictly below κ. Thus by the previous case, κ is a limit of λ-supercompact cardinals. Moreover, κ is measurable since κ is (κ, λ)-club compact. Thus κ is a measurable limit of λ-club compact cardinals, as desired.

(3) implies (1): This follows from Lemma 8.3.24. □

8.3.3 The Mitchell order, the internal relation, and coherence

Assume UA and suppose U is a normal ultrafilter on κ. Can $P(\kappa^+) \subseteq M_U$? The question remains open in general, but the following theorem shows that if κ^+ is Fréchet, this cannot occur:

Theorem 8.3.28 (UA). *Suppose λ is a Fréchet cardinal. Suppose U is a countably complete ultrafilter such that $P(\lambda) \subseteq M_U$. Then $(M_U)^\lambda \subseteq M_U$.*

Proof. Assume by induction that the theorem holds for cardinals below λ. If λ is a limit of Fréchet cardinals, we then have $(M_U)^{<\lambda} \subseteq M_U$. In particular, if λ is a singular limit of Fréchet cardinals, then $(M_U)^\lambda \subseteq M_U$. Thus we may assume that λ is either regular or isolated. This puts the analysis of $\mathscr{K}_\lambda$ (especially Theorem 7.3.14 and Proposition 7.4.17) at our disposal.

We first show that U is λ-irreducible. Suppose towards a contradiction that there is a uniform ultrafilter $D \leq_{\text{RF}} U$ on an infinite cardinal $\gamma < \lambda$. Note that $P(\lambda) \subseteq M_U \subseteq M_D$, so the bound on the hypermeasurability of ultrapowers (Lemma 4.2.39) implies that

$$\lambda < j_D(\gamma)$$

Assume first that λ is isolated. By Proposition 7.4.17, $D \sqsubset \mathscr{K}_\lambda$, and by Proposition 7.5.22, $P(\gamma) \subseteq M_{\mathscr{K}_\lambda}$. Thus

$$P(\lambda) \subseteq j_D(P(\gamma)) \subseteq M_{\mathscr{K}_\lambda}$$

Therefore by our bound on the strength of $j_{\mathscr{K}_\lambda}$ for nonmeasurable isolated cardinals λ (Proposition 7.5.21), λ is measurable. Since λ is a strong limit, $D \in H(\lambda) \subseteq M_D$, and this is a contradiction.

Assume instead that λ is a nonisolated regular cardinal. We use an argument similar to the one from the local proof of GCH (Theorem 6.3.25). Let $M = M_{\mathscr{K}_\lambda}$ and let $N = (M_D)^M$. Consider the embedding $j^N_{\mathscr{K}_\lambda} \circ j^M_D$. (Note: $j^N_{\mathscr{K}_\lambda}$ denotes the ultrapower formed by using functions in N modulo the N-ultrafilter $\mathscr{K}_\lambda$, *not* the ultrafilter $(\mathscr{K}_\lambda)^N$, which we have not proved to exist.) This is an ultrapower embedding from M, and we claim that it is internal to M. By our analysis of internal ultrapower embeddings of M (Theorem 7.3.14), it suffices to show that $j^N_{\mathscr{K}_\lambda} \circ j^M_D$ is continuous at $\text{cf}^M(\sup j[\lambda]) = \lambda$. (To compute the cofinality of $\sup j[\lambda]$ in M, we use Proposition 7.4.11.) Clearly $j^M_D(\lambda) = \sup j^M_D[\lambda]$ since λ is regular and D lies on $\gamma < \lambda$. Moreover $j^M_D(\lambda)$ is regular in N and is larger than λ since $j^M_D(\gamma) = j_D(\gamma) > \lambda$. Thus $j^N_{\mathscr{K}_\lambda}(j^M_D(\lambda)) = \sup j^N_{\mathscr{K}_\lambda}[j^M_D(\lambda)]$. Putting it all together,

$$j^N_{\mathscr{K}_\lambda} \circ j^M_D(\lambda) = \sup j^N_{\mathscr{K}_\lambda}[j^M_D(\lambda)] = \sup j^N_{\mathscr{K}_\lambda}[\sup j^M_D[\lambda]] = \sup j^N_{\mathscr{K}_\lambda} \circ j^M_D[\lambda]$$

Thus $j^N_{\mathscr{K}_\lambda} \circ j^M_D$ is an internal ultrapower embedding of M.

In fact, $j^N_{\mathscr{K}_\lambda}$ itself is definable over M: for any $f \in M^\gamma$,

$$j^N_{\mathscr{K}_\lambda}([f]^M_D) = j^N_{\mathscr{K}_\lambda} \circ j^M_D(f)(\text{id}^N_{\mathscr{K}_\lambda})$$

Thus $j^N_{\mathscr{K}_\lambda}$ is definable over M. Since $P(\lambda) \subseteq N$, we have $\mathscr{K}_\lambda = \{A \subseteq \lambda : \text{id}^N_{\mathscr{K}_\lambda} \in j^N_{\mathscr{K}_\lambda}(A)\}$. Thus $\mathscr{K}_\lambda$ is definable over M, and it follows that $\mathscr{K}_\lambda \in M$, or in other words, $\mathscr{K}_\lambda \lhd \mathscr{K}_\lambda$. This is a contradiction.

Thus our assumption was false, and in fact U is λ-irreducible.

To finish the proof, we break once again into cases.

Suppose first that λ is a nonmeasurable isolated cardinal. We will show that U is λ^+-complete. We claim that $\mathscr{K}_\lambda \not\leq_{\text{RF}} U$: otherwise, $P(\lambda) \subseteq M_U \subseteq M_{\mathscr{K}_\lambda}$, and hence $\mathscr{K}_\lambda$ is λ-complete by Proposition 7.5.21, contradicting that λ is not measurable. Since $\mathscr{K}_\lambda \not\leq_{\text{RF}} U$, our factorization theorem for isolated cardinals (Theorem 7.5.15) implies that U is λ^+-irreducible. Therefore by Theorem 7.5.33, U is λ^+-complete, as claimed.

If λ is not a nonmeasurable isolated cardinal, then λ is either a Fréchet successor cardinal or a Fréchet inaccessible cardinal. Since U is λ-irreducible, the Irreducibility Theorem (Corollary 8.2.20 and Corollary 8.2.21) implies that $j_U[\lambda]$ is contained in a set $A \in M_U$ such that $|A|^{M_U} = \lambda$. Since $P(\lambda) \subseteq M_U$ and $|A|^{M_U} = \lambda$, in fact $P(A) \subseteq M_U$. In particular, the subset $j_U[\lambda] \subseteq A$ belongs to M_U, so j_U is λ-supercompact, and hence $(M_U)^\lambda \subseteq M_U$. □

Theorem 8.3.29 (UA)**.** *Suppose U and W are countably complete ultrafilters such that $U \lhd W$. Then $(j_U)^{M_W} = j_U \restriction M_W$. In fact, $(M_W)^{\lambda_U} \subseteq M_W$.*

Proof. Let $\lambda = \lambda_U$. Fix $A \in U$ with $|A| = \lambda$. Since $U \in M_W$, $P(A) \subseteq M_W$, and hence $P(\lambda) \subseteq M_W$. Since $\lambda = \lambda_U$, λ is Fréchet. Hence $(M_W)^\lambda \subseteq M_W$ by Theorem 8.3.28. By Proposition 4.2.28, this implies $(j_U)^{M_W} = j_U \restriction M_W$. □

As a consequence, UA implies that the internal relation and the seed order extend the Mitchell order:

Corollary 8.3.30 (UA)**.** *Suppose U and W are countably complete ultrafilters such that $U \lhd W$. Then $U \sqsubset W$. Assume moreover that λ_U is the underlying set of U and W concentrates on ordinals. Then $U <_S W$.*

Proof. By Theorem 8.3.29, $U \sqsubset W$. Moreover, j_W is λ_U-supercompact, so by Proposition 4.2.30, $\lambda_U \leq \lambda_W$. Thus if λ_U is the underlying set of U and W concentrates on ordinals, then

$$\delta_U = \lambda_U \leq \lambda_W \leq \delta_W$$

Since $U \sqsubset W$, the comparison $(j_U(j_W), j_U) : (M_U, M_W) \to j_U(M_W)$ of (j_U, j_W) is internal. Since $\delta_U \leq \delta_W$, we have $j_W(\alpha) < \text{id}_W$ for U-almost all ordinals α, and hence $j_U(j_W)(\text{id}_U) < j_U(\text{id}_W)$. Therefore $(j_U(j_W), j_U) : (M_U, M_W) \to j_U(M_W)$ witnesses $U <_S W$. □

Using the Irreducibility Theorem, we prove some converses of Corollary 8.3.30 that demystify the internal relation. This requires an argument we have seen before but which we now make explicit:

Lemma 8.3.31. *Suppose W is a countably complete ultrafilter such that j_W is $P_{\text{bd}}(\lambda)$-hypermeasurable and λ-tight.*[3] *Suppose that there is a countably complete ultrafilter U on λ such that $U \sqsubset W$ and $\sup j_U[\lambda] < j_U(\lambda)$. Then j_W is λ-supercompact.*

[3] Equivalently, j_W is $<\lambda$-supercompact and λ-tight.

Proof. We first show that $P(\lambda) \subseteq M_W$. Since W is $P_{\text{bd}}(\lambda)$-hypermeasurable, $P(\alpha) \subseteq M_W$ for all $\alpha < \lambda$. Therefore by the elementarity of j_U, M_U satisfies that $P(\sup j_U[\lambda]) \subseteq j_U(M_W)$. In other words, $P^{M_U}(\sup j_U[\lambda]) \subseteq j_W(M_U)$. Since $U \sqsubset W$, $j_U(M_W) \subseteq M_W$, and therefore $P^{M_U}(\sup j_U[\lambda]) \subseteq M_W$. Now fix $A \subseteq \lambda$. We have $j_U(A) \cap \sup j_U[\lambda] \in P^{M_U}(\sup j_U[\lambda]) \subseteq M_W$. Moreover $j_U \restriction \lambda \in M_W$ since $U \sqsubset W$. Hence

$$A = j_U^{-1}[j_U(A) \cap \sup j_U[\lambda]] \in M_W$$

This shows that $P(\lambda) \subseteq M_W$, as claimed.

Now suppose B is a subset of M_W of cardinality at most λ. Since j_W is λ-tight, there is a set $C \in M_W$ of M_W-cardinality at most λ such that $B \subseteq C$. Since $P(\lambda) \subseteq M_W$ and $|C|^{M_W} \leq \lambda$, $P(C) \subseteq M_W$. Thus $B \in M_W$. It follows that j_W is λ-supercompact. $\square$

Theorem 8.3.32 (UA)**.** *Suppose W is a countably complete ultrafilter and U is a countably complete uniform ultrafilter on a set $X \subseteq M_W$. Then the following are equivalent:*

(1) $U \lhd W$.

(2) $U \sqsubset W$ and W is $|X|$-irreducible.

Proof. Let $\lambda = \lambda_U = |X|$.

(1) implies (2): Suppose $U \lhd W$. Then j_W is λ-supercompact by Theorem 8.3.28, so W is λ-irreducible by Proposition 8.2.3. Moreover by Corollary 8.3.30, $U \sqsubset W$. This shows that (2) holds.

(2) implies (1): Suppose $U \sqsubset W$ and W is λ-irreducible.

Suppose first that λ is an isolated cardinal. We claim that W is λ^+-complete. Note that j_W must be continuous at λ by Lemma 7.4.14. It follows that W is λ^+-irreducible. Hence W is λ^σ-irreducible. But λ^σ is measurable (by Lemma 7.4.27), so by Theorem 7.5.34 it follows that W is λ^+-complete. As an immediate consequence, $U \lhd W$.

Suppose instead that λ is not isolated. We can then apply the Irreducibility Theorem (Corollary 8.2.20 and Corollary 8.2.21) to conclude that W is $<\lambda$-supercompact and λ-tight. Since $U \sqsubset W$, Lemma 8.3.31 yields that j_W is λ-supercompact. In particular, $P(\lambda) \subseteq M_W$, so $U \lhd W$, as desired. $\square$

We can reformulate Theorem 8.3.32 slightly to characterize the internal relation in terms of the Mitchell order:

Theorem 8.3.33 (UA)**.** *Suppose U and W are hereditarily uniform irreducible ultrafilters. Then the following are equivalent:*

(1) $U \sqsubset W$.

(2) Either $U \lhd W$ or $W \in V_\kappa$ where $\kappa = \text{crit}(j_U)$.

For this, we need the following theorem, which shows that the notions of λ-irreducible, λ-Mitchell, and λ-internal ultrafilters (Definition 7.5.32, Definition 6.3.10, Definition 5.5.23 respectively) coincide under UA:

Theorem 8.3.34 (UA)**.** *Suppose U is an ultrafilter and λ is a cardinal. Then the following are equivalent:*

(1) U is λ-irreducible.

(2) U is λ-Mitchell.

(3) U is λ-internal.

Proof. (1) implies (2): Assume U is λ-irreducible. We may assume by induction that for all $U' \leq_{\Bbbk} U$ and $\lambda' \leq \lambda$ with $U' <_{\Bbbk} U$ or $\lambda' < \lambda$, if U' is λ'-irreducible then U' is λ'-Mitchell. Thus U is λ'-Mitchell for all $\lambda' < \lambda$. In particular, U is automatically λ-Mitchell unless λ is a successor cardinal and and the cardinal predecessor γ of λ is Fréchet. Therefore we can assume $\lambda = \gamma^+$ where γ is a Fréchet cardinal.

We may also assume that γ^σ exists, since otherwise the λ-irreducibility of U implies U is principal, so (2) holds automatically. Let $\eta = \gamma^\sigma$.

Assume first that $\eta = \gamma^+$. Then γ^+ is Fréchet, so by the Irreducibility Theorem (Corollary 8.2.20), U is γ^+-supercompact. Therefore every countably complete ultrafilter on γ belongs to M_U by Theorem 6.3.16. In other words, U is γ^+-Mitchell.

This leaves us with the case that $\eta > \gamma^+$. In other words, by Proposition 7.4.4, η is isolated.

Assume first that $\mathscr{K}_\eta \not\leq_{\mathrm{RF}} U$. Then by Theorem 7.5.15, U is η-indecomposable, and so in particular U is η^+-irreducible. By Theorem 7.5.33 (3), U is η^+-complete, which easily implies that U is γ^+-Mitchell.

Assume finally that $\mathscr{K}_\eta \leq_{\mathrm{RF}} U$. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\eta$. Let $h : M \to M_U$ be the unique internal ultrapower embedding with $h \circ j = j_U$.

Recall that $t_{\mathscr{K}_\eta}(U)$ is the canonical ultrafilter Z of M such that $j^M_Z = h$. We claim that $t_{\mathscr{K}_\eta}(U)$ is γ^+-irreducible in M. Suppose M satisfies that D is an ultrafilter on γ with $D \leq_{\mathrm{RF}} t_{\mathscr{K}_\eta}(U)$. Let $i : (M_D)^M \to M_U$ be the unique internal ultrapower embedding such that

$$i \circ j^M_D = h$$

We will show D is principal by showing that $D \leq_{\mathrm{RF}} U$. By Proposition 7.5.22, M is closed under γ-sequences. In particular, $P(\gamma) \subseteq M$, so D really is an ultrafilter on γ, and hence the question of whether $D \leq_{\mathrm{RF}} U$ makes sense. Moreover $j_D \restriction M = j^M_D$, and so $j^M_D \circ j = j_D(j) \circ j_D$. Now

$$i \circ j_D(j) \circ j_D = i \circ j^M_D \circ j = h \circ j = j_U$$

Thus $i \circ j_D(j) : M_D \to M_U$ is an internal ultrapower embedding witnessing $D \leq_{\mathrm{RF}} U$. It follows that D is principal since U is γ^+-irreducible.

Thus $t_{\mathscr{K}_\eta}(U)$ is γ^+-irreducible in M. Moreover by Proposition 5.4.5, $t_{\mathscr{K}_\eta}(U) <_{\Bbbk} j(U)$ in M. Our induction hypothesis yields that for all $U' <_{\Bbbk} U$ and all $\lambda' \leq \gamma^+$, if U' is λ'-irreducible then U' is λ'-Mitchell. Shifting this hypothesis by the elementary embedding $j : V \to M$, we have that for all $U' <_{\Bbbk} j(U)$ and all $\lambda' \leq j(\gamma^+)$, if U' is λ'-irreducible in M then U' is λ'-Mitchell in M. Applying this with $U' = t_{\mathscr{K}_\lambda}(U)$ and $\lambda' = \gamma^+$, it follows that $t_{\mathscr{K}_\lambda}(U)$ is γ^+-Mitchell in M. Thus every countably complete ultrafilter of M on γ belongs to $(M_{t_{\mathscr{K}_\eta}(U)})^M = M_U$. But by Proposition 7.4.17 and Proposition 7.5.22, every countably complete ultrafilter on γ belongs to M. Hence every countably complete ultrafilter on γ belongs to M_U. In other words, U is γ^+-Mitchell as desired.

(2) implies (3): Immediate from Corollary 8.3.30.

(3) implies (1): Assume U is λ-internal. Suppose $D \leq_{\text{RF}} U$ and $\lambda_D < \lambda$. We will show D is principal. Since $\lambda_D < \lambda$, $D \sqsubset U$. Thus $D \leq_{\text{RF}} U \sqsupset D$, so $D \sqsubset D$ by Proposition 5.5.14. Since the internal relation is irreflexive on nonprincipal ultrafilters, D is principal. ☐

Proof of Theorem 8.3.33. (1) implies (2): Suppose $U \sqsubset W$.

Assume first that $\lambda_U \leq \lambda_W$. Then since W is irreducible, W is λ_U-irreducible. By Theorem 8.3.32, $U \lhd W$.

Assume instead that $\lambda_W < \lambda_U$. Then by Theorem 8.3.34, $W \sqsubset U$. Since $U \sqsubset W$ and $W \sqsubset U$, Theorem 5.5.21 implies that U and W are commuting ultrafilters in the sense of Kunen's commuting ultrapowers lemma (Theorem 5.5.19). Moreover, again by Theorem 8.3.34, U is λ_U-internal and W is λ_W-internal. We can therefore apply our converse to Kunen's commuting ultrapowers lemma, from which it follows that $W \in V_\kappa$ where $\kappa = \text{crit}(j_U)$.

(2) implies (1): If $U \lhd W$, then $U \sqsubset W$ by Corollary 8.3.30. If $W \in V_\kappa$ where $\kappa = \text{crit}(j_U)$, then $U \sqsubset W$ by Kunen's commuting ultrapowers lemma (Theorem 5.5.19). ☐

We now reformulate UA in terms of a form of coherence:

Definition 8.3.35. Suppose C is a class of countably complete ultrafilters.

- Suppose $\mathcal{I}$ is a finite iterated ultrapower of V.
  - A countably complete ultrafilter U is *given by* $\mathcal{I}$ if $j_U = j^{\mathcal{I}}_{0,\infty}$.
  - $\mathcal{I}$ is a *C-iteration* if $U^{\mathcal{I}}_n \in j^{\mathcal{I}}_{0,n}(C)$ for all $n < \text{length}(\mathcal{I})$.
- C is *cofinal* if the class of ultrafilters given by C-iterations is Rudin-Frolík cofinal.
- C is *coherent* if for any distinct ultrafilters U and W in C, either $U \in j_W(C)$ and $(M_W)^{\lambda_U} \subseteq M_W$, or $W \in j_U(C)$ and $(M_U)^{\lambda_W} \subseteq M_U$.

The definition of coherence given here just requires that C be wellordered by the Mitchell order in a strong sense. Given the standard definition of coherence, it would be natural to demand that C is a class of fine ultrafilters on ordinals and for all $U \in C$, $j_U(C) \cap \mathbf{UF}(\lambda_U) = \{W \in C : W <_\Bbbk U\}$. This would make no difference in the theorems proved below, but we have chosen to define coherence in terms of the Mitchell order alone.

Theorem 8.3.36. *The following are equivalent:*

(1) There is a coherent cofinal class of countably complete ultrafilters.

(2) The Ultrapower Axiom holds.

For one direction of the theorem, we show that under UA, there is a canonical coherent cofinal class of ultrafilters:

Definition 8.3.37. An ultrafilter D is a *Mitchell point* if for all uniform countably complete ultrafilters U, if $U <_\Bbbk D$, then $U \lhd D$.

Dodd sound ultrafilters are Mitchell points by Corollary 4.3.28. Under UA, isonormal ultrafilters are Mitchell points by Theorem 7.5.46. The following fact is trivial:

Lemma 8.3.38 (UA)**.** *The Mitchell points form a coherent class of ultrafilters.*

Proof. Let C be the class of Mitchell points. Since the Ketonen order is linear, C is linearly ordered by $<_\Bbbk$, and hence by the definition of a Mitchell point, C is linearly ordered by the Mitchell order. The property of being a Mitchell point is absolute, so if $U \lhd W$ are Mitchell points, then $U \in j_W(C)$. Moreover Theorem 8.3.29, $(M_W)^{\lambda_U} \subseteq M_W$. Thus C is coherent. □

We next show that under UA, the Mitchell points form a cofinal class. The first step is to give an alternate characterization in terms of the internal relation:

Proposition 8.3.39 (UA)**.** *Suppose D is a nonprincipal countably complete fine ultrafilter on an ordinal δ. The following are equivalent:*

(1) For all countably complete uniform ultrafilters U, if $U <_\Bbbk D$, then $U \sqsubset D$.

(2) D is a Mitchell point

(3) For all Mitchell points D', if $D' <_\Bbbk D$, then $D' \lhd D$.

Proof. (1) implies (2): Note that (1) implies in particular that D is δ-internal. Thus D is a uniform ultrafilter on δ. There are two cases. Suppose first that $D = \mathscr{K}_\delta$. Then Theorem 8.3.34, D is δ-Mitchell, which is what (2) asserts. Assume instead that $D \neq \mathscr{K}_\delta$, so $\mathscr{K}_\delta <_\Bbbk D$ since $\mathscr{K}_\delta$ is the least uniform ultrafilter on δ. By (1), $\mathscr{K}_\delta \sqsubset D$, and in particular by Lemma 7.4.14, δ is not isolated. By Theorem 8.3.34, D is δ-irreducible, and therefore by the Irreducibility Theorem, D is $<\delta$-supercompact and δ-tight. Since $\mathscr{K}_\delta \sqsubset D$, Lemma 8.3.31 yields that j_D is δ-supercompact. In particular, $P(\delta) \subseteq M_D$, and so for any countably complete ultrafilter U on δ with $U \sqsubset D$, $U \lhd D$. Given (1), this implies (2).

(2) implies (3): Immediate.

(3) implies (1): Let D' be the $<_{\Bbbk}$-least fine ultrafilter that is not internal to D. To show that (1) holds, we must show $D' = D$. Clearly $D' \leq_{\Bbbk} D$ (since a nonprincipal ultrafilter is never internal to itself). By Corollary 8.3.30, the internal relation extends the Mitchell order, so $D' \not\lhd D$. Theorem 7.5.47 asserts that D' has the following property: for any $U \sqsubset D$, in fact $U \sqsubset D'$. In particular, for any $U <_{\Bbbk} D'$, by the minimality of D', we have $U \sqsubset D$, and so we can conclude that $U \sqsubset D'$. Since we have shown that (1) implies (2), we can conclude that D' is a Mitchell point. Since D' is a Mitchell point and $D' \not\lhd D$, (3) implies that $D' \not<_{\Bbbk} D$. Since $D' \leq_{\Bbbk} D$, it follows that $D = D'$, as desired. □

Definition 8.3.40. For any countably complete ultrafilter W, the *Mitchell point of* W, denoted $D(W)$, is the $<_{\Bbbk}$-least fine ultrafilter D such that $D \not\lhd W$.

The proof of Proposition 8.3.39 yields the following fact:

Theorem 8.3.41 (UA)**.** *Suppose W is a nonprincipal countably complete ultrafilter and $D = D(W)$. Then the following hold:*

- *D is a Mitchell point.*
- *$\{U : U \lhd W\} = \{U : U \lhd D\}$.*
- *If U is a countably complete ultrafilter such that $U \sqsubset W$, then $U \sqsubset D$.*
- *$D \not\sqsubset W$.* □

Theorem 8.3.42 (UA)**.** *The Mitchell points form a cofinal class of ultrafilters.*

Sketch. Suppose U is a countably complete ultrafilter. We will show that there is an ultrafilter U' given by a Mitchell point iteration such that $U \leq_{\mathrm{RF}} U'$. By induction, we may assume that this statement is true for all $\bar{U} <_{\Bbbk} U$. Let $D = D(U)$. Since $D \not\sqsubset U$, $t_D(U) <_{\Bbbk} j_D(U)$ in M_D. Therefore by our induction hypothesis, M_D satisfies that there is an ultrafilter W' given by a Mitchell point iteration such that $t_D(U) \leq_{\mathrm{RF}} W'$. Let U' be such that $j_{U'} = j_{W'}^{M_D} \circ j_D$. It is easy to see that U' is given by a Mitchell point iteration and $U \leq_{\mathrm{RF}} U'$. □

We now turn to the other direction of Theorem 8.3.36. It suffices to prove the following fact:

Proposition 8.3.43. *Suppose C is a coherent class of countably complete ultrafilters. Then the restriction of the Rudin-Frolík order to the class of ultrafilters given by C-iterations is directed.*

Proof. The idea of the proof is that the internal relation yields a comparison of any two ultrafilters in C via Lemma 5.5.7, and these comparisons can be iterated to yield comparisons of any pair of ultrafilters given by C-iterations. This is quite easy to see (given the right definition of a coherent class), but we nevertheless include a very detailed proof.

We begin with a one-step claim:

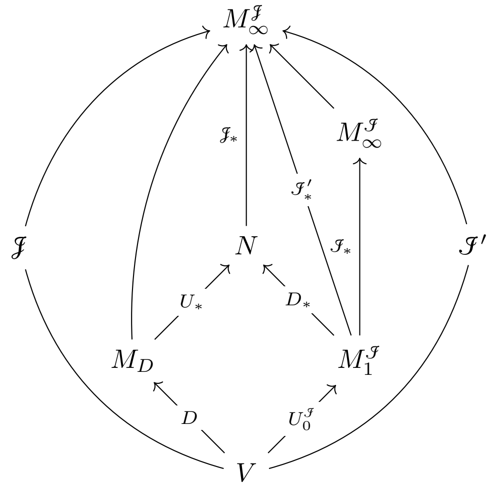

Figure 8.2: The proof of Claim 1

Claim 1. *Suppose $D \in C$. For any C-iteration $\mathcal{I}$, there exist C-iterations $\mathscr{J}$ and $\mathcal{I}'$ such that $\mathcal{I}'$ extends $\mathcal{I}$, $U_0^{\mathscr{J}} = D$, and $j_{0,\infty}^{\mathcal{I}'} = j_{0,\infty}^{\mathscr{J}}$.*

Proof of Claim 1. The proof is by induction on the length of $\mathcal{I}$.

If $U_0^{\mathcal{I}} = D$, then we can take $\mathcal{I} = \mathscr{J}$.

Therefore assume $U_0^{\mathcal{I}} \neq D$. Since C is coherent, either $D \lhd U_0^{\mathcal{I}}$ or $U_0^{\mathcal{I}} \lhd D$. Define

$$D_* = \begin{cases} j_{0,1}^{\mathcal{I}}(D) & \text{if } U_0^{\mathcal{I}} \lhd D \\ D & \text{if } D \lhd U_0^{\mathcal{I}} \end{cases}$$

and

$$U_* = \begin{cases} U_0^{\mathcal{I}} & \text{if } U_0^{\mathcal{I}} \lhd D \\ j_D(U_0^{\mathcal{I}}) & \text{if } D \lhd U_0^{\mathcal{I}} \end{cases}$$

The key point is that by the definition of a coherent class of ultrafilters, $D_* \in j_{0,1}^{\mathcal{I}}(C)$, $U_* \in j_D(C)$, and

$$j_{U_*}^{M_D} \circ j_D = j_{D_*}^{M_1^{\mathcal{I}}} \circ j_{0,1}^{\mathcal{I}}$$

Let $\mathcal{I}_* = \mathcal{I} \restriction [1, \infty)$, which is a $j_{0,1}^{\mathcal{I}}(C)$-iteration of $M_1^{\mathcal{I}}$. By our induction hypothesis applied in $M_1^{\mathcal{I}}$ to the iteration $\mathcal{I}_*$ and the ultrafilter $D_* \in j_{0,1}^{\mathcal{I}}(C)$, there exist $j_{0,1}^{\mathcal{I}}(C)$-iterations $\mathcal{I}_*'$ and $\mathscr{J}_*$ such that $\mathcal{I}_*'$ extends $\mathcal{I}_*$, $U_0^{\mathscr{J}_*} = D_*$, and $j_{0,\infty}^{\mathcal{I}_*'} = j_{0,\infty}^{\mathscr{J}_*}$.

Let $\mathcal{I}'$ be the iterated ultrapower of V given by $U_0^{\mathcal{I}}$ followed by $\mathcal{I}_*'$. Clearly $\mathcal{I}'$ is a C-iteration extending $\mathcal{I}$. Let $\ell = \text{length}(\mathscr{J}_*)$, and define a C-iteration $\mathscr{J}$

of length $\ell+1$ in terms of the ultrafilters $U_n^{\mathcal{J}}$ for $n \leq \ell$:

$$\begin{aligned}
U_0^{\mathcal{J}} &= D\\
U_1^{\mathcal{J}} &= U_*\\
U_n^{\mathcal{J}} &= U_{n-1}^{\mathcal{J}_*}
\end{aligned}$$

Then

$$j_{0,\infty}^{\mathcal{J}} = j_{1,\infty}^{\mathcal{J}_*} \circ j_{U_*}^{M_D} \circ j_D = j_{1,\infty}^{\mathcal{J}_*} \circ j_{D_*}^{M_1^{\mathcal{J}}} \circ j_{0,1}^{\mathcal{J}} = j_{0,\infty}^{\mathcal{J}_*} \circ j_{0,1}^{\mathcal{J}} = j_{0,\infty}^{\mathcal{J}'_*} \circ j_{0,1}^{\mathcal{J}} = j_{0,\infty}^{\mathcal{J}'}$$

This verifies the induction step, and proves the claim. □

We now turn to the multi-step claim:

Claim 2. *For any C-iterations $\mathcal{H}$ and $\mathcal{J}$, there are C-iterations $\mathcal{H}^*$ and $\mathcal{J}^*$ extending $\mathcal{H}$ and $\mathcal{J}$ respectively such that $j_{0,\infty}^{\mathcal{H}^*} = j_{0,\infty}^{\mathcal{J}^*}$.*

Proof of Claim 2. The proof is by induction on the length ℓ of $\mathcal{H}$: thus our induction hypothesis is that for any C-iteration $\bar{\mathcal{H}}$ of length less ℓ and any C-iteration $\bar{\mathcal{J}}$, there are C-iterations $\bar{\mathcal{H}}^*$ and $\bar{\mathcal{J}}^*$ extending $\bar{\mathcal{H}}$ and $\bar{\mathcal{J}}$ respectively such that $j_{0,\infty}^{\bar{\mathcal{H}}^*} = j_{0,\infty}^{\bar{\mathcal{J}}^*}$.

Let $D = U_0^{\mathcal{H}}$. By our first claim, there is a C-iteration $\mathcal{J}$ such that $U_0^{\mathcal{J}} = D$ and a C-iteration $\mathcal{J}'$ extending $\mathcal{J}$ such that $j_{0,\infty}^{\mathcal{J}'} = j_{0,\infty}^{\mathcal{J}}$. Now we work in M_D. Let $\bar{\mathcal{H}} = \mathcal{H} \restriction [1,\infty)$. Thus $\bar{\mathcal{H}}$ is a $j_D(C)$-iteration of M_D of length less than ℓ. Let $\bar{\mathcal{J}} = \mathcal{J} \restriction [1,\infty)$, so that $\bar{J}$ is also a $j_D(C)$-iteration of M_D.

By our induction hypothesis applied in M_D, there are $j_D(C)$-iterations $\bar{\mathcal{H}}^*$ and $\bar{\mathcal{J}}^*$ of M_D extending $\bar{\mathcal{H}}$ and $\bar{\mathcal{J}}$ respectively such that $j_{0,\infty}^{\bar{\mathcal{H}}^*} = j_{0,\infty}^{\bar{\mathcal{J}}^*}$. Define

$$\begin{aligned}
\mathcal{H}^* &= D^\frown \bar{\mathcal{H}}^*\\
\mathcal{J}^* &= \mathcal{J}'^\frown \mathcal{K}
\end{aligned}$$

where $\mathcal{K} = \bar{\mathcal{J}}^* \restriction [\text{length}(\bar{\mathcal{J}}),\infty)$.

Obviously $\mathcal{H}^*$ and $\mathcal{J}^*$ are C-iterations extending $\mathcal{H}$ and $\mathcal{J}$ respectively. Moreover

$$j_{0,\infty}^{\mathcal{H}^*} = j_{0,\infty}^{\bar{\mathcal{H}}^*} \circ j_D = j_{0,\infty}^{\bar{\mathcal{J}}^*} \circ j_D = j_{0,\infty}^{\mathcal{K}} \circ j_{0,\infty}^{\bar{\mathcal{J}}} \circ j_D = j_{0,\infty}^{\mathcal{K}} \circ j_{0,\infty}^{\mathcal{J}} = j_{0,\infty}^{\mathcal{K}} \circ j_{0,\infty}^{\mathcal{J}'} = j_{0,\infty}^{\mathcal{J}^*}$$

This proves the claim. □

It follows easily from Claim 2 that the restriction of the Rudin-Frolík order to the class of ultrafilters given by C-iterations is directed. □

We finally prove our characterization of UA in terms of coherent cofinal sequences.

Proof of Theorem 8.3.36. (1) implies (2): This is immediate from Lemma 8.3.38 and Theorem 8.3.42.

(2) implies (1): Let C be a coherent cofinal class of ultrafilters. Since C is coherent, Proposition 8.3.43 implies that the restriction of the Rudin-Frolík order to the class of ultrafilters C' given by C-iterations is directed. Since C is cofinal, C' is cofinal in the Rudin-Frolík order. Since the Rudin-Frolík order has a cofinal directed subset, the Rudin-Frolík order is itself directed. This implies that the Ultrapower Axiom holds (by Corollary 5.2.9). □

Given Theorem 8.3.36, the obvious question is whether the Ultrapower Axiom implies the existence of a coherent class of ultrafilters whose iterations represent *every* countably complete ultrafilter. The only class of ultrafilters that could possibly do the job is the class of irreducible ultrafilters. Of course, Theorem 5.3.13 implies that under UA, every countably complete ultrafilter is given by an iteration of irreducible ultrafilters. Moreover, Theorem 8.3.29 implies that under UA, the class of irreducible ultrafilters is coherent if and only if the Irreducible Ultrafilter Hypothesis holds. (The Irreducible Ultrafilter Hypothesis is stated in Section 4.2.5.) Therefore the question raised above is no more than a reformulation of Question 4.2.51: does UA imply the Irreducible Ultrafilter Hypothesis?

8.4 Very large cardinals

8.4.1 Huge cardinals

The notion of (κ, λ)-*regularity* is a two cardinal generalization of κ^+-incompleteness that has already shown up implicitly in this monograph:

Definition 8.4.1. Suppose $\kappa \leq \lambda$ are cardinals. An ultrafilter U is (κ, λ)-*regular* if there is a set $F \subseteq U$ of cardinality λ such that $\bigcap \sigma = \emptyset$ for any $\sigma \subseteq F$ of cardinality at least κ.

The combinatorial definition of (κ, λ)-regularity defined above obscures its true significance:

Lemma 8.4.2. *Suppose $\kappa \leq \lambda$ are cardinals and U is an ultrafilter. Then the following are equivalent:*

(1) U is (κ, λ)-regular.

(2) For some fine ultrafilter $\mathcal{U}$ on $P_\kappa(\lambda)$, $\mathcal{U} \leq_{\text{RK}} U$.

(3) j_U is (λ, δ)-tight for some M_U-cardinal $\delta < j_U(\kappa)$.

Proof. (1) implies (2): Fix a set $F \subseteq U$ of cardinality λ such that $\bigcap \sigma = \emptyset$ for any $\sigma \subseteq F$ of cardinality at least κ. Let X be the underlying set of U. Define $f : X \to P_\kappa(F)$ by setting $f(x) = \{A \in F : x \in A\}$. Let $\mathcal{U} = f_*(U)$. We claim $\mathcal{U}$ is a fine ultrafilter on $P_\kappa(F)$. Suppose $A \in F$. We must show

$\{\sigma \in P_\kappa(F) : A \in \sigma\} \in \mathcal{U}$. But by the definition of f, $A \in f(x)$ if and only if $x \in A$. Thus

$$f^{-1}[\{\sigma \in P_\kappa(F) : A \in \sigma\}] = A \in U$$

and so $\{\sigma \in P_\kappa(F) : A \in \sigma\} \in \mathcal{U}$.

(2) implies (3): Fix a fine ultrafilter $\mathcal{U}$ on $P_\kappa(\lambda)$ such that $\mathcal{U} \leq_{\text{RK}} U$. Let $A = \text{id}_\mathcal{U}$. Then $j_\mathcal{U}[\lambda] \subseteq A$ by Lemma 4.4.9, and $|A|^{M_\mathcal{U}} < j_\mathcal{U}(\kappa)$ by Łoś's Theorem. Let $k : M_\mathcal{U} \to M_U$ be an elementary embedding such that $k \circ j_\mathcal{U} = j_U$. Then $j_U[\lambda] = k[j_\mathcal{U}[\lambda]] \subseteq k(A)$ and $|k(A)|^{M_U} < k(j_\mathcal{U}(\kappa)) = j_U(\kappa)$. Let $\delta = |k(A)|^{M_U}$. Then $k(A)$ witnesses that j_U is (λ, δ)-tight, as desired.

(3) implies (1): Fix $A \in M_U$ such that $|A|^{M_U} < j_U(\kappa)$ and $j_U[\lambda] \subseteq A$. Let f be a function such that $A = [f]_U$. By Łoś's Theorem, there is a set $X \in U$ such that $f[X] \subseteq P_\kappa(\lambda)$. Let $S_\alpha = \{x \in X : \alpha \in f(x)\}$. Let $F = \{S_\alpha : \alpha < \lambda\}$. We claim that $\bigcap_{\alpha\in\sigma} S_\alpha = \emptyset$ for any $\sigma \subseteq \lambda$ of cardinality at least κ. Suppose towards a contradiction that $x \in \bigcap_{\alpha\in\sigma} S_\alpha$. Then $\sigma \subseteq f(x)$, so $|f(x)| \geq \kappa$, contradicting that $f(x) \in P_\kappa(\lambda)$. Thus F witnesses that U is (κ, λ)-regular. □

Another way of stating (2) above is to say that the minimum fine filter on $P_\kappa(\lambda)$ lies below U in the Katětov order.

Definition 8.4.3. If $\kappa \leq \lambda$ are cardinals, then $P^\kappa(\lambda)$ denotes the collection of subsets of λ of cardinality exactly κ.

Thus $P^\kappa(\lambda) = P_{\kappa^+}(\lambda) \setminus P_\kappa(\lambda)$.

Definition 8.4.4. A cardinal κ is *huge* if there is an elementary embedding $j : V \to M$ with critical point κ such that $M^{j(\kappa)} \subseteq M$.

A question raised in [14] is the relationship between nonregular ultrafilters and huge cardinals. Assuming UA, and restricting to countably complete ultrafilters, we can almost show an equivalence:

Theorem 8.4.5 (UA). *Suppose $\kappa < \lambda$ are cardinals and λ is regular. The following are equivalent:*

(1) There is a countably complete fine ultrafilter on $P^\kappa(\lambda)$ that cannot be pushed forward to a fine ultrafilter on $P_\kappa(\lambda)$.

(2) There is a countably complete ultrafilter that is (κ^+, λ)-regular but not (κ, λ)-regular.

(3) There is an elementary embedding $j : V \to M$ such that $j(\kappa) = \lambda$, $M^{<\lambda} \subseteq M$, and M has the $\leq\lambda$-cover property.

If λ is a successor cardinal, then we can add to the list:

(4) There is an elementary embedding $j : V \to M$ such that $j(\kappa) = \lambda$ and $M^\lambda \subseteq M$.

(5) There is a normal fine ultrafilter on $P^\kappa(\lambda)$.

Proof. The equivalence of (1) and (2) is immediate from Lemma 8.4.2. We now turn to the equivalence of (2) and (3). Before we begin, we point out that the property of being (κ^+, λ)-regular but not (κ, λ)-regular can be reformulated in terms of ultrapowers:

> U is (κ^+, λ)-regular but not (κ, λ)-regular if and only if $\text{cf}^{M_U}(\sup j_U[\lambda]) = j_U(\kappa)$.

This is an immediate consequence of Lemma 8.4.2 (3) and Ketonen's analysis of tight embeddings in terms of cofinality (Theorem 7.2.12).

(2) implies (3): Let U be the $<_{\Bbbk}$-least countably complete ultrafilter concentrating on ordinals that is (κ^+, λ)-regular but not (κ, λ)-regular.

We claim that U is λ-irreducible. (In fact, U is an irreducible weakly normal ultrafilter on λ, but this is not relevant to the proof.) Suppose $D \leq_{\text{RF}} U$ and $\lambda_D < \lambda$. We must show that D is principal. We claim $t_D(U)$ is $(j_D(\kappa^+), j_D(\lambda))$-regular but not $(j_D(\kappa), j_D(\lambda))$-regular. Let $i : M_D \to M_U$ be the unique internal ultrapower embedding with $i \circ j_D = j_U$. Thus $i : M_D \to M_U$ is the ultrapower of M_D by $t_D(U)$. Therefore to show that $t_D(U)$ is $(j_D(\kappa), j_D(\lambda))$-regular it suffices (by our remark at the beginning of the proof) to show that $\text{cf}^{M_U}(\sup i[j_D(\lambda)]) = i(j_D(\kappa))$. Since $\lambda_D < \lambda$, by Lemma 2.2.34,

$$\sup i[j_D(\lambda)] = \sup i \circ j_D[\lambda] = \sup j_U[\lambda]$$

Furthermore, since U is $(j_D(\kappa^+), j_D(\lambda))$-regular but not $(j_D(\kappa), j_D(\lambda))$-regular, applying our remark at the beginning of the proof again,

$$\text{cf}^{M_U}(\sup j_U[\lambda]) = j_U(\kappa) = i(j_D(\kappa))$$

Thus $\text{cf}^{M_U}(\sup i[j_D(\lambda)]) = i(j_D(\kappa))$, as desired.

By elementarity $j_D(U)$ is the $<_{\Bbbk}$-least ultrafilter that is $(j_D(\kappa^+), j_D(\lambda))$-regular but not $(j_D(\kappa), j_D(\lambda))$-regular. Hence $j_D(U) \leq_{\Bbbk} t_D(U)$. Recall Proposition 5.4.5, which states that if D is nonprincipal and $D \leq_{\text{RF}} U$, then $t_D(U) <_{\Bbbk} j_D(U)$. It follows that D is principal.

Since U is λ-irreducible, and now we would like to apply the Irreducibility Theorem. For this, we need that λ is either a successor cardinal or an inaccessible cardinal. Assume λ is a limit cardinal, and we will show that λ is a strong limit cardinal. Since $\kappa < \lambda$, we have $\kappa^+ < \lambda$. Since U is (κ^+, λ)-regular, U is δ-decomposable for all regular cardinals $\delta \in [\kappa^+, \lambda]$. Therefore λ is a limit of Fréchet cardinals, and hence by Corollary 7.5.2, λ is a strong limit cardinal, as desired.

To summarize, $j_U : V \to M_U$ is an elementary embedding such that $j_U(\kappa) = \lambda$, $M_U^{<\lambda} \subseteq M_U$ and M_U has the $\leq\lambda$-cover property. This shows that (3) holds.

(3) implies (2): Let U be the ultrafilter on λ derived from j using $\sup j[\lambda]$, and let $k : M_U \to M$ be the factor embedding with $k \circ j_U = j$ and $k(\text{id}_U) = \sup j[\lambda]$. Then $\text{id}_U = \sup j_U[\lambda]$, and $k(\text{cf}^{M_U}(\text{id})) = \text{cf}^M(\sup j[\lambda]) = \lambda = j(\kappa) = k(j_U(\kappa))$. By the elementarity of k,

$$\text{cf}^{M_U}(\sup j_U[\lambda]) = \text{cf}^{M_U}(\text{id}_U) = j_U(\kappa)$$

Thus by our remark at the beginning of the proof, U is (κ^+, λ)-regular but not (κ, λ)-regular. This shows that (1) holds.

Assuming λ is a successor cardinal, the argument that (2) implies (3) shows that in fact (2) implies (4), since the Irreducibility Theorem leads to full λ-supercompactness in the case that λ is a successor cardinal.

Finally, (4) and (5) are equivalent (in general) by an easy argument using derived ultrafilters and ultrapowers (Lemma 4.4.10). $\square$

We cannot show that $M^\lambda \subseteq M$ in the key case that λ is inaccessible, which blocks proving the equivalence between huge cardinals and nonregular countably complete ultrafilters.

We conclude the section with an example that shows that Theorem 8.4.5 does not generalize to the case that λ is singular.

Proposition 8.4.6. *Suppose λ is the least strong limit singular cardinal such that for some cardinal $\bar{\lambda} < \mathrm{cf}(\lambda)$, the following hold:*

- $\mathrm{cf}(\lambda)$ *is measurable with Mitchell order* 1*.*
- λ *carries a $(\bar{\lambda}^+, \lambda)$-regular ultrafilter that is not $(\bar{\lambda}, \lambda)$-regular.*

Then there is no elementary embedding $j : V \to M$ such that $M^\lambda \subseteq M$ and $j(\bar{\lambda}) = \lambda$.

Proof. Assume towards a contradiction that such an elementary embedding exists, and fix an embedding $j : V \to M$ and a cardinal $\bar{\lambda} < \mathrm{cf}(\lambda)$ such that $M^\lambda \subseteq M$ and $j(\bar{\lambda}) = \lambda$.

Let $\kappa = \mathrm{crit}(j)$. Since

$$j(\mathrm{cf}(\bar{\lambda})) = \mathrm{cf}(\lambda) > \bar{\lambda} > \mathrm{cf}(\bar{\lambda})$$

we have $\mathrm{cf}(\bar{\lambda}) \geq \kappa$, and since $\mathrm{cf}(\bar{\lambda})$ has Mitchell order 1, $\mathrm{cf}(\bar{\lambda}) \neq \kappa$. Therefore $\mathrm{cf}(\bar{\lambda}) > \kappa$.

Since λ lies below the first 2-huge cardinal, $j(\kappa) > \bar{\lambda}$.

We claim that for any cardinal γ such that $\mathrm{cf}(\lambda) \leq \gamma < \lambda$, there is a cardinal $\bar{\gamma} < j(\kappa)$ such that $P^{\bar{\gamma}}(\gamma)$ carries a normal fine ultrafilter. Assume not, towards a contradiction, and let γ_0 be the least counterexample. Since λ is a strong limit cardinal and $H(\lambda) \subseteq M$, for any cardinals $\bar{\gamma} < \gamma$ below λ, M and V agree about the set of normal fine ultrafilters on $P^{\bar{\gamma}}(\gamma)$. Therefore γ_0 is definable from λ in M. Since λ is in the range of j, it follows that γ_0 is in the range of j, and since $\gamma < \lambda = j(\bar{\lambda})$, there is in fact some $\bar{\gamma} < \bar{\lambda}$ such that $j(\bar{\gamma}) = \gamma_0$. But $\bar{\lambda} < j(\kappa)$, so $\bar{\gamma} < j(\kappa)$. By Lemma 4.4.10, the ultrafilter on $P^{\bar{\gamma}}(\gamma_0)$ derived from j using $j[\gamma_0]$ is normal and fine. This is a contradiction.

Let $\delta = \mathrm{cf}(\lambda)$ and let U be a δ-complete uniform ultrafilter on δ. Note that $j(\kappa) < \delta$ since $\kappa < \mathrm{cf}(\bar{\lambda}) = j^{-1}(\delta)$, so

$$j_U \restriction j(\kappa) + 1 = \mathrm{id}$$

In M_U, fix a normal fine ultrafilter $\mathscr{W}$ on $P^{\bar{\gamma}}(\lambda)$ where $\bar{\gamma} < j(\kappa)$. Then

$$j_{\mathscr{W}}^{M_U} \circ j_U(\bar{\gamma}) = j_{\mathscr{W}}^{M_U}(\bar{\gamma}) = \lambda$$

Consider the ultrafilter sum $Z = [U, \mathscr{W}]$. (This is just the natural ultrafilter whose associated ultrapower embedding is $j_{\mathscr{W}}^{M_U} \circ j_U$; see Definition 3.5.8 for the precise construction.) Note that Z is $(\bar{\gamma}^+, \lambda)$-regular, since $j_{\mathscr{W}}^{M_U}[\lambda] \in M_Z$, $|j_{\mathscr{W}}^{M_U}[\lambda]|^{M_Z} < \lambda^{+M_Z} = j_Z(\bar{\gamma}^+)$, and

$$j_Z[\lambda] \subseteq j_{\mathscr{W}}^{M_U}[j_U[\lambda]] \subseteq j_{\mathscr{W}}^{M_U}[\lambda]$$

Of course Z is not $(\bar{\gamma}, \lambda)$-regular, since $|j_Z[\lambda]| = \lambda = j_Z(\bar{\gamma})$.

The ultrafilter Z belongs to M since $Z = [U, \mathscr{W}]$ where $U \in M$, $H(\lambda^+) \in M$,

$$\mathscr{W} \in j_U(H(\lambda^+)) = j_U^M(H(\lambda^+)) \in M$$

Moreover the fact that Z is $(\bar{\gamma}^+, \lambda)$-regular and not $(\bar{\gamma}, \lambda)$-regular is easily seen to be downwards absolute to M. Therefore the following hold in M:

- λ is a strong limit singular cardinal.
- $\bar{\gamma} < \mathrm{cf}(\lambda)$.
- $\mathrm{cf}(\lambda)$ is a measurable cardinal of Mitchell order 1.
- λ carries a $(\bar{\gamma}^+, \lambda)$-regular ultrafilter that is not $(\bar{\gamma}, \lambda)$-regular.

By the elementarity of $j : V \to M$, however, M satisfies that $j(\lambda)$ is the least strong limit singular cardinal λ' for which there is some cardinal $\bar{\lambda}' < \mathrm{cf}(\lambda')$ such that $\mathrm{cf}(\lambda')$ is a measurable cardinal of Mitchell order 1 and λ' carries a $((\bar{\lambda}')^+, \lambda')$-regular ultrafilter that is not $(\bar{\lambda}', \lambda')$-regular. Since $\lambda < j(\lambda)$, this is a contradiction. □

8.4.2 Cardinal preserving elementary embeddings

In this section, we turn to even stronger large cardinal axioms.

Definition 8.4.7. An elementary embedding $j : V \to M$ is *weakly cardinal preserving* if whenever κ is a cardinal, $j(\kappa)$ is also a cardinal and *cardinal preserving* if M is cardinal correct.

The following question, due to Caicedo, essentially asks whether the Kunen Inconsistency Theorem can be strengthened to rule out cardinal preserving elementary embeddings:

Question 8.4.8 (Caicedo)**.** *Is it consistent that there is a nontrivial cardinal preserving elementary embedding?*

Under UA, we will show that there are no nontrivial weakly cardinal preserving embeddings.

Lemma 8.4.9 (UA)**.** *Suppose U is a countably complete uniform ultrafilter on κ^+ such that $j_U[\kappa] \subseteq \kappa$. Either κ is κ^+-supercompact or κ is a limit of κ^+-supercompact cardinals.*

Proof. By Corollary 8.2.25, there is some $D \leq_{\text{RF}} U$ with $\lambda_D < \kappa^+$ such that there is an internal ultrapower embedding $i : M_D \to M_U$ with $i \circ j_D = j_U$ that is $j_D(\kappa^+)$-supercompact in M_D. Note that $\sup j_D[\kappa] \subseteq \kappa$ and $\sup i[\kappa] \subseteq \kappa$, since both i and j_D are bounded on the ordinals by j_U.

We claim that $\text{crit}(i) \in [\kappa, j_D(\kappa)]$. To see this, note that $\sup i[\kappa] \subseteq \kappa$ and i is κ-supercompact, so by the Kunen Inconsistency Theorem (Theorem 4.2.35), $\text{crit}(i) \geq \kappa$. On the other hand, i is given by an ultrafilter on $j_D(\kappa^+)$, so $\text{crit}(i) \leq j_D(\kappa)$.

Now i witnesses that $\text{crit}(i)$ is $j_D(\kappa^+)$-supercompact in M_D. If $\text{crit}(i) = j_D(\kappa)$, then κ is κ^+-supercompact by elementarity. Otherwise $\sup j_D[\kappa] = \kappa \leq \text{crit}(i) < j_D(\kappa)$, so κ is a limit of κ^+-supercompact cardinals by a standard reflection argument. □

The following observation is due to Caicedo:

Lemma 8.4.10. *Suppose $j : V \to M$ and γ is a cardinal. If $j(\gamma^+) \neq \gamma^+$ and j is continuous at γ^+, then $j(\gamma^+)$ is not a cardinal.*

Proof. Note that $j(\gamma^+)$ is a singular ordinal since $j[\gamma^+]$ is cofinal in $j(\gamma^+)$. Moreover $j(\gamma) < j(\gamma^+) = j(\gamma)^{+M} \leq j(\gamma)^+$. There are no singular cardinals between $j(\gamma)$ and $j(\gamma)^+$, so $j(\gamma^+)$ is not a cardinal. □

Lemma 8.4.11 (UA)**.** *Suppose $j : V \to M$ is a nontrivial elementary embedding with critical point κ. Let γ be a cardinal above κ with $j(\gamma) = \gamma$. Then j is continuous at $\gamma^{+\kappa+1}$ and therefore $j(\gamma^{+\kappa+1})$ is not a cardinal.*

Proof. We begin the proof by making some general observations about the action of j on cardinals in the vicinity of γ. First, for all $\alpha < \kappa$, $j(\gamma^{+\alpha}) = (\gamma^{+\alpha})^M \leq \gamma^{+\alpha}$. It follows that $j(\gamma^{+\alpha}) = \gamma^{+\alpha}$. Hence $\sup j[\gamma^{+\kappa}] = \gamma^{+\kappa}$.

Next, we claim that $(\gamma^{+\kappa+1})^M = \gamma^{+\kappa+1}$. This is proved by following the argument of Lemma 4.2.31: fix $\alpha < \gamma^{+\kappa+1}$, and we will show that $\alpha < (\gamma^{+\kappa+1})^M$. Let $(\gamma^{+\kappa}, \prec)$ be a wellorder of order type α. Then $(\gamma^{+\kappa}, j(\prec))$ is a wellorder of $\gamma^{+\kappa}$ that belongs to M. Since $j[\gamma^{+\kappa}] \subseteq \gamma^{+\kappa}$, j embeds $(\gamma^{+\kappa}, \prec)$ into $(\gamma^{+\kappa}, j(\prec))$, so

$$\alpha \leq \text{ot}(\gamma^{+\kappa}, \prec) \leq \text{ot}(\gamma^{+\kappa}, j(\prec)) < (\gamma^{+\kappa+1})^M$$

as desired.

It follows that

$$j(\gamma^{+\kappa+1}) > j(\gamma^{+\kappa}) = (\gamma^{+j(\kappa)})^M > (\gamma^{+\kappa+1})^M = \gamma^{+\kappa+1}$$

Thus to prove $j(\gamma^{+\kappa+1})$ is not a cardinal, by Lemma 8.4.10 it suffices to show j is continuous at $\gamma^{+\kappa+1}$.

Suppose towards a contradiction that j is discontinuous at $\gamma^{+\kappa+1}$. Let U be the ultrafilter on $\gamma^{+\kappa+1}$ derived from j using $\sup j[\gamma^{+\kappa+1}]$. Then U is a countably complete uniform ultrafilter on $\gamma^{+\kappa+1}$. Moreover,

$$\sup j_U[\gamma^{+\kappa}] \leq \sup j[\gamma^{+\kappa}] = \gamma^{+\kappa}$$

Therefore by Lemma 8.4.9, $\gamma^{+\kappa}$ is either $\gamma^{+\kappa+1}$-supercompact or else a limit of $\gamma^{+\kappa+1}$-supercompact cardinals. This is impossible since there are no inaccessible cardinals in the interval $(\gamma, \gamma^{+\kappa}]$. Thus our assumption was false, and in fact j is continuous at $\gamma^{+\kappa+1}$.

Now j is continuous at $\gamma^{+\kappa+1}$ and $j(\gamma^{+\kappa+1}) > \gamma^{+\kappa+1}$. Therefore by Lemma 8.4.10, $j(\gamma^{+\kappa+1})$ is not a cardinal. □

Corollary 8.4.12 (UA). *Any weakly cardinal preserving elementary embedding of the universe is the identity.*

We now investigate the relationship between cardinal preservation and rank-into-rank axioms. Recall that Card denotes the class of cardinals.

Theorem 8.4.13 (UA). *Assume λ is an ordinal, $M \subseteq V_\lambda$ is a transitive set, and $j : V_\lambda \to M$ is an elementary embedding with critical point κ that has no fixed points above κ. Suppose that $\text{Card}^M = \text{Card} \cap \lambda$. Then $M = V_\lambda$.*

If the assumption that $\text{Card}^M \cap \lambda = \text{Card} \cap \lambda$ is weakened to the assumption that j is weakly cardinal preserving below λ (or in other words that $j[\text{Card} \cap \lambda] \subseteq \text{Card} \cap \lambda$), then the resulting statement is false. Let us provide a counterexample. Suppose $j : V \to M$ is an elementary embedding with critical point κ. Let λ be the first cardinal fixed point of j above κ. Assume $V_\lambda \subseteq M$, so j witnesses the axiom I_2. Suppose U is a κ-complete ultrafilter on κ. One can show that $j_U^M \circ j : V \to (M_U)^M$ has the property that $j_U^M \circ j \restriction \text{Ord} = j \restriction \text{Ord}$, so in particular $j_U^M \circ j[\text{Card} \cap \lambda] = j[\text{Card} \cap \lambda] \subseteq \text{Card} \cap \lambda$. But of course $(M_U)^M$ does not contain V_λ.

The assumption that j fixes no ordinals above its critical point is also necessary. To see this, suppose $k : V \to N$ is an elementary embedding with critical point κ and $\lambda \geq \kappa$ is an ordinal such that $j(\lambda) = \lambda$ and $V_\lambda \subseteq N$. Then for any $\alpha < \lambda^{+\kappa}$ such that $j(\alpha) = \alpha$, letting $k = j \restriction V_\alpha$ and $M = N \cap V_\alpha$, $k : V_\alpha \to M$ is a cardinal preserving elementary embedding, but if $\alpha > \lambda$, then $V_\alpha \not\subseteq M$ by the Kunen Inconsistency Theorem. (Note that we do not assume that M is a model of ZFC in the statement of Theorem 8.4.13.)

The key lemma in the proof of Theorem 8.4.13 is the following curiosity, a close cousin of Lemma 8.2.10:

Lemma 8.4.14 (UA). *Suppose U is a countably complete ultrafilter and δ is a successor cardinal. Then $\text{cf}^{M_U}(\sup j_U[\delta])$ is a successor cardinal of M_U.*

Proof. If $\sup j_U[\delta] = j_U(\delta)$, then $\sup j_U[\delta]$ is itself a successor cardinal of M_U, so of course its M_U-cofinality (which is again $\sup j_U[\delta]$) is a successor cardinal of M_U. We may therefore assume that $\sup j_U[\delta] < j_U(\delta)$.

Hence δ is Fréchet, and so we are in a position to apply Theorem 8.2.19. By Theorem 8.2.19, there is an ultrafilter D with $\lambda_D < \delta$ such that there is an internal ultrapower embedding $h : M_D \to M_U$ such that h is $j_D(\delta)$-supercompact in M_D. Since $\lambda_D < \delta$, $j_D(\delta) = \sup j_D[\delta]$ by Lemma 2.2.34. Thus

$$\text{cf}^{M_U}(\sup j_U[\delta]) = \text{cf}^{M_U}(\sup h[j_D(\delta)]) = \text{cf}^{M_D}(j_D(\delta)) = j_D(\delta)$$

Since $j_D(\delta)$ is a successor cardinal of M_D, and $\text{Ord}^{j_D(\delta)} \cap M_D \subseteq M_U$, $j_D(\delta)$ is a successor cardinal of M_U. $\square$

We now turn to the proof of Theorem 8.4.13.

Proof of Theorem 8.4.13. For $n < \omega$, let κ_n be the nth element of the critical sequence of j (Definition 4.2.33), and note that $\lambda = \sup_{n<\omega} \kappa_n$ since j has no fixed points above κ.

Let us make some preliminary remarks about the interaction between ultrapowers and the structure V_λ. Suppose that U is a countably complete ultrafilter on a set $X \in V_\lambda$. Then any function $f : X \to V_\lambda$ is bounded on a set in U. In particular,

$$j_U(V_\lambda) = \{[f]_U : f \in V_\lambda \text{ and } \text{dom}(f) = X\}$$

In other words, V_λ correctly computes the ultrapower by U. We will go to great lengths, however, not to work inside V_λ, which we have not yet proved to be a model of ZFC.

Suppose $X \in V_\lambda$, $a \in j(X)$, and U is the ultrafilter on X derived from j using a. Then U is countably complete, so the remark of the previous paragraph applies. Thus we can define a factor embedding $k : j_U(V_\lambda) \to M$ by setting $k([f]_U) = j(f)(a)$ whenever $f \in V_\lambda$ is a function on X. The usual argument shows that k is well-defined and elementary. Moreover, $k \circ (j_U \restriction V_\lambda) = j$ and $k(\text{id}_U) = a$.

Suppose $\delta < \lambda$ is a successor cardinal. Let U be the uniform ultrafilter derived from j using $\sup j[\delta]$, and let $k : j_U(V_\lambda) \to M$ be the factor embedding. We claim:

- $\text{cf}^M(\sup j[\delta]) = \delta$.
- j_U is δ-tight.
- $k(\delta) = \delta$.

By Lemma 8.4.14, $\sup j_U[\delta]$ is a successor cardinal of M_U. Thus $\sup j_U[\delta]$ is a successor cardinal of $j_U(V_\lambda)$, so $k(\sup j_U[\delta]) = \text{cf}^M(\sup j[\delta])$ is a successor cardinal of M. Since M is correct about cardinals below λ, $\text{cf}^M(\sup j[\delta])$ is a successor cardinal (in V). In particular, $\text{cf}^M(\sup j[\delta])$ is regular. Thus $\text{cf}^M(\sup j[\delta]) = \text{cf}(\text{cf}^M(\sup j[\delta])) = \text{cf}(\sup j[\delta]) = \delta$, as desired.

It follows that j_U is δ-tight:

$$\text{cf}^{M_U}(\sup j_U[\delta]) = \text{cf}^{j_U(V_\lambda)}(\sup j_U[\delta]) \leq k(\text{cf}^{j_U(V_\lambda)}(\sup j_U[\delta])) = \text{cf}^M(\sup j[\delta]) = \delta$$

so $\mathrm{cf}^{M_U}(\sup j_U[\delta]) = \delta$, and hence j_U is δ-tight by Theorem 7.2.12.

Repeating the same argument, it now follows that $k(\delta) = \delta$:

$$k(\delta) = k(\mathrm{cf}^{M_U}(\sup j_U[\delta])) = k(\mathrm{cf}^{j_U(V_\lambda)}(\sup j_U[\delta])) = \mathrm{cf}^M(\sup j[\delta]) = \delta$$

Since $\mathrm{Card}^M = \mathrm{Card} \cap \lambda$, M correctly computes the $\aleph$-function, and so it is trivial to prove by induction that each κ_n is a limit of $\aleph$-fixed points of uncountable cofinality.

Suppose $\delta < \lambda$ is the successor of an $\aleph$-fixed point of uncountable cofinality. Let U be the ultrafilter derived from j using $\sup j[\delta]$. By the Irreductibility Theorem (more precisely, Corollary 8.2.25), there is an ultrafilter D on a cardinal less than δ and a $j_D(\delta)$-supercompact internal ultrapower embedding $h : M_D \to M_U$ such that $h \circ j_D = j_U$. We claim that $j_D(\delta) = \delta$. Since j_U is a δ-tight ultrapower embedding (see the comments above), j_U has the $\leq\delta$-cover property, so $j_D(\delta) = \sup j_D[\delta]$ has cofinality δ in M_U. But $M_U \subseteq M_D$ since h is an internal ultrapower embedding, so $\mathrm{cf}^{M_D}(j_D(\delta)) \leq \delta$. Since $j_D(\delta)$ is regular in M_D, it follows that $j_D(\delta) = \delta$, as claimed.

Now let $k : j_U(V_\lambda) \to M$ be the factor embedding. We have $(H_\delta)^{M_U} = (H_\delta)^{M_D}$, $j_D(\delta) = \delta$, and $k(\delta) = \delta$. Therefore the embedding $i = k \circ j_D \restriction H_\delta$ is an elementary embedding from H_δ to $(H_\delta)^M$.

We claim that i is the identity. Assume not, towards a contradiction, and let κ be the critical point of i. Let γ be the cardinal predecessor of δ. By elementarity, $i(\gamma)$ is equal to the largest cardinal of $(H_\delta)^M$, so since M is cardinal correct, $i(\gamma) = \gamma$. Since γ has uncountable cofinality, i has a fixed point ν such that $\kappa < \nu < \gamma$. Since γ is an $\aleph$-fixed point, $\nu^{+\kappa+1} < \gamma$. Let E be the extender of length γ derived from i, and let $e : V \to N$ be the extender embedding associated to E. Then $e \restriction \gamma = i \restriction \gamma$, and in particular, $\mathrm{crit}(e) = \kappa$, $e(\nu) = \nu$, and $e(\nu^{+\kappa+1}) = i(\nu^{+\kappa+1})$ is a cardinal. Hence e is a counterexample to Lemma 8.4.11. This is a contradiction, so i is the identity.

Since $i : H_\delta \to (H_\delta)^M$ is elementary, $(H_\delta)^M = H_\delta$.

Since λ is a limit of $\aleph$-fixed points γ of uncountable cofinality, and $H_{\gamma^+} \subseteq M$ for all such γ, it follows that $V_\lambda \subseteq M$. Hence $M = V_\lambda$, as desired. □

The following question remains open:

Question 8.4.15 (UA). *Suppose there is a weakly cardinal preserving elementary embedding from V_λ into a transitive set $M \subseteq V_\lambda$ that fixes no ordinals above its critical point. Must there be an elementary embedding $j : V_\lambda \to V_\lambda$?*

This cannot be entirely trivial given the example following the statement of Theorem 8.4.13.

8.4.3 Supercompactness at inaccessible cardinals

The following are probably the most interesting questions left open by our work:

Question 8.4.16 (UA). *Suppose λ is an inaccessible cardinal and κ is the least λ-strongly compact cardinal. Must κ be λ-supercompact? More generally, if κ is λ-strongly compact, must κ be λ-supercompact or a measurable limit of λ-supercompact cardinals?*

This final section consists of some inconclusive observations regarding this problem.

The whole question, it turns out, reduces to the analysis of $\mathscr{K}_\lambda$:

Lemma 8.4.17 (UA). *Assume λ is an inaccessible Fréchet cardinal. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$, and let κ be the least measurable cardinal of M above λ. Then for any λ-irreducible ultrafilter U, $\text{Ord}^\kappa \cap M \subseteq M_U$.*

Proof. Let $(k, h) : (M, M_U) \to P$ be the pushout of (j, j_U), and let W be such that $P = M_W$. By the analysis of ultrafilters internal to a pushout, for any D with $\lambda_D < \lambda$, since $D \sqsubset U$ and $D \sqsubset \mathscr{K}_\lambda$, in fact, $D \sqsubset W$. In particular, W is λ-irreducible, so $V_\lambda \subseteq M_W = P$ by Corollary 8.2.21. By our factorization lemma for embeddings of M (Lemma 8.2.8), it follows that $\text{crit}(k) \geq \kappa$. (Otherwise k would factor through an ultrapower by an ultrafilter in V_λ, contrary to the fact that $V_\lambda \subseteq P$.) Therefore $\text{Ord}^\kappa \cap M \subseteq P \subseteq M_U$, as desired. □

Corollary 8.4.18 (UA). *Suppose λ is a Fréchet inaccessible cardinal. Let M be the ultrapower of the universe by $\mathscr{K}_\lambda$, and assume M is closed under λ-sequences. Then for any λ-irreducible ultrafilter U, M_U is closed under λ-sequences.*

Proof. By Lemma 8.4.17, $\text{Ord}^\lambda = \text{Ord}^\lambda \cap M \subseteq M_U$, so M_U is closed under λ-sequences. □

We now show that the $<_\Bbbk$-second irreducible ultrafilter on an inaccessible cardinal λ always witnesses λ-supercompactness. This is a bit surprising given that we cannot prove the supercompactness of $\mathscr{K}_\lambda$.

We use the following lemma, extracted from Ketonen's proof that the Ketonen order is wellfounded on weakly normal ultrafilters.

Lemma 8.4.19. *Suppose λ is a regular cardinal. Suppose W is a countably complete ultrafilter on λ that extends the closed unbounded filter. Suppose $U <_\Bbbk W$. Then $\delta_{t_U(W)} = j_U(\lambda)$. In fact, $t_U(W)$ extends the closed unbounded filter on $j_U(\delta)$.*

Proof. Let F be the closed unbounded filter on λ. Clearly $j_U[F] \subseteq t_U(W)$. Moreover $\{\alpha < j_U(\delta) : \text{id}_U \in \alpha\} \in t_U(W)$ since

$$j^{M_U}_{t_U(W)}(\text{id}_U) < j^{M_W}_{t_W(U)}(\text{id}_W) = \text{id}_{t_U(W)}$$

Thus by Lemma 8.2.11, $j_U(F) \subseteq t_U(W)$, as claimed. □

We choose not to cite the Irreducibility Theorem in the proof of the following proposition since it predates the Irreducibility Theorem and is really much easier:

Proposition 8.4.20 (UA)**.** *Suppose λ is a regular cardinal. The following are equivalent:*

(1) λ carries two Rudin-Keisler inequivalent uniform irreducible ultrafilters.

(2) There is a countably complete ultrafilter U such that $\mathscr{K}_\lambda \not\leq_{\text{RF}} U$ and $U \not\sqsubset \mathscr{K}_\lambda$.

(3) λ carries a countably complete weakly normal ultrafilter that concentrates on ordinals that carry countably complete fine ultrafilters.

(4) λ carries distinct countably complete weakly normal ultrafilters.

(5) λ carries distinct countably complete ultrafilters extending the closed unbounded filter.

(6) There is a a normal fine κ_λ-complete ultrafilter $\mathcal{U}$ on $P_{\kappa_\lambda}(\lambda)$ such that $\mathscr{K}_\lambda \lhd \mathcal{U}$.

Proof. (1) implies (2): Suppose $U \neq \mathscr{K}_\lambda$ is an irreducible ultrafilter on λ. By irreducibility, $\mathscr{K}_\lambda \not\leq_{\text{RF}} U$. Since $\sup j_{\mathscr{K}_\lambda}[\lambda]$ carries no countably complete fine ultrafilter in $M_{\mathscr{K}_\lambda}$, $j_U \restriction M_{\mathscr{K}_\lambda}$ is not internal to $M_{\mathscr{K}_\lambda}$, since it is discontinuous at $\sup j_{\mathscr{K}_\lambda}[\lambda]$. In other words $U \not\sqsubset \mathscr{K}_\lambda$.

(2) implies (3): Suppose U is a countably complete ultrafilter such that $\mathscr{K}_\lambda \not\leq_{\text{RF}} U$ and $U \not\sqsubset \mathscr{K}_\lambda$. Since $U \not\sqsubset \mathscr{K}_\lambda$, by the characterization of internal ultrapower embeddings of $M_{\mathscr{K}_\lambda}$ (Theorem 7.3.14), j_U must be discontinuous at λ. Since $\mathscr{K}_\lambda \not\leq_{\text{RF}} U$, by the universal property of $\mathscr{K}_\lambda$, $\sup j_U[\lambda]$ carries a countably complete fine ultrafilter in M_U. Let W be the ultrafilter on λ derived from j_U using $\sup j_U[\lambda]$. Then W is weakly normal (by Corollary 4.4.18) and W concentrates on ordinals carrying countably complete fine ultrafilters by the definition of a derived ultrafilter.

(3) implies (4): If λ carries a countably complete uniform ultrafilter, then λ carries a countably complete weakly normal ultrafilter that *does not* concentrate on ordinals carrying countably complete fine ultrafilters (by Theorem 7.2.14); in the context of UA, this is $\mathscr{K}_\lambda$. Thus if (3) holds, λ carries distinct countably complete weakly normal ultrafilters.

(4) implies (5): Immediate given the fact that weakly normal ultrafilters extend the closed unbounded filter.

(5) implies (6): Assume (5) holds. Let U be the $<_{\Bbbk}$-least countably complete ultrafilter that extends the closed unbounded filter on λ and is not equal to $\mathscr{K}_\lambda$. We claim that for all $D <_{\Bbbk} U$, $D \sqsubset U$. We will verify the criterion for showing $D \sqsubset U$ given by Lemma 5.5.15 by showing that $j_D(U) \leq_{\Bbbk} t_D(U)$ in M_D.

Let $U' = t_D(U)$. By Lemma 8.4.19, U' extends the closed unbounded filter on $j_D(\lambda)$. Moreover we claim that $j_D(\mathscr{K}_\lambda) \neq U'$. To see this, note that

$$j_D^{-1}[j_D(\mathscr{K}_\lambda)] = \mathscr{K}_\lambda \neq U = j_D^{-1}[U']$$

Thus $j_D(\mathscr{K}_\lambda) \neq U'$, as claimed.

By elementarity, in M_D, $j_D(U)$ is the $\leq_{\Bbbk}$-least countably complete ultrafilter that extends the closed unbounded filter on $j_D(\lambda)$ and is not equal to $j_D(\mathscr{K}_\lambda)$. It follows that $j_D(U) \leq_{\Bbbk} U'$ in M_D. Lemma 5.5.15 now implies that $D \sqsubset U$, as claimed.

Let $\kappa = \kappa_\lambda$. Since λ is not isolated, by Lemma 7.4.19, κ is a limit of isolated cardinals. By Lemma 7.5.3, for all isolated cardinals $\gamma < \kappa$, $j_U[\gamma] \subseteq \gamma$, and hence $j_U[\kappa] \subseteq \kappa$. Lemma 5.5.26 states that if κ is a strong limit cardinal such that $j_U[\kappa] \subseteq \kappa$ and $D \sqsubset U$ for all countably complete ultrafilters D with $\lambda_D < \kappa$, then U is κ-complete. Thus U is κ-complete. In particular, $\text{Ord}^\kappa \subseteq M_U$. Since $\mathscr{K}_\lambda \sqsubset U$, $j_{\mathscr{K}_\lambda}(\text{Ord}^\kappa) = \text{Ord}^{j_{\mathscr{K}_\lambda}(\kappa)} \cap M_{\mathscr{K}_\lambda} \subseteq M_U$. As $j_{\mathscr{K}_\lambda}(\kappa) > \lambda$ by Proposition 7.4.1, it follows that $\text{Ord}^\lambda \cap M_{\mathscr{K}_\lambda} \subseteq M_U$.

Now suppose $A \in \text{Ord}^\lambda$. Then $j_{\mathscr{K}_\lambda}[A]$ is contained in a set $B \in [\text{Ord}]^\lambda \cap M_{\mathscr{K}_\lambda}$. Hence $B \in M_U$. We may assume $B \subseteq j_{\mathscr{K}_\lambda}(A)$, so that $j^{-1}_{\mathscr{K}_\lambda}[B] = A$. Since $\mathscr{K}_\lambda \sqsubset U$, $j_{\mathscr{K}_\lambda} \restriction \alpha \in M_U$ for all ordinals α. Hence $A = j^{-1}_{\mathscr{K}_\lambda}[B] \in M_U$. Thus $\text{Ord}^\lambda \subseteq M_U$.

If Z is a countably complete ultrafilter extending the closed unbounded filter on λ such that $Z \lhd U$, then $Z <_{\Bbbk} U$ by Proposition 4.3.30, and consequently by the minimality of U, $Z = \mathscr{K}_\lambda$. In particular, no cardinal less than or equal to λ can be 2^λ-supercompact in M_U. It follows that $j_U(\kappa) > \lambda$: otherwise $j_U(\kappa) \leq \lambda$ is $j_U(\lambda)$-supercompact, and since $2^\lambda < j_U(\lambda)$, we contradict the previous sentence.

Thus U is κ-complete and $j_U(\kappa) > \lambda$. Let $\mathscr{U}$ be the normal fine κ-complete ultrafilter on $P_\kappa(\lambda)$ derived from j_U using $j_U[\lambda]$. It is easy to see that $\mathscr{K}_\lambda \lhd \mathscr{U}$ (and in fact $U \equiv_{\text{RK}} \mathscr{U}$). Thus (6) holds.

(6) implies (1): By the irreflexivity of the Mitchell order (Lemma 4.2.38), $\mathscr{K}_\lambda$ and $\mathscr{U}$ are not Rudin-Keisler equivalent. The ultrafilter $\mathscr{K}_\lambda$ is by definition uniform and by Lemma 7.3.12 irreducible. Finally, $\mathscr{U}$ is irreducible by Proposition 5.3.5, and $\mathscr{U}$ is Rudin-Keisler equivalent to a uniform ultrafilter on λ by Corollary 4.4.28. □

We now turn to the questions raised in Section 8.2.1. Recall that an elementary embedding is $[0, \lambda]$-tight if it is γ-tight for all $\gamma \leq \lambda$. Our main question asked whether $[0, \lambda]$-tightness and λ-supercompactness coincide below rank-into-rank cardinals. If λ is the least cardinal where this fails, then it has the following property:

Definition 8.4.21. A cardinal λ is said to be *pathological* if there is an elementary embedding $j : V \to M$ that is $<\lambda$-supercompact and λ-tight but not λ-supercompact. The embedding j is said to *witness the pathology of* λ.

Equivalently, $j : V \to M$ witnesses the pathology of λ if $H(\lambda) \subseteq M$, $j[\lambda]$ can be covered by a set of size λ in M, and yet $j[\lambda] \notin M$. The axiom $I_2(\lambda)$ asserts that there is an elementary embedding $j : V \to M$ with critical point less than λ such that $j(\lambda) = \lambda$ and $V_\lambda \subseteq M$. By the Kunen Inconsistency Theorem (Theorem 4.2.35), $\lambda = \kappa_\omega(j)$ and $j[\lambda] \notin M$. Thus if $I_2(\lambda)$ holds, then λ is pathological of countable cofinality.

Question 8.4.22. *Suppose λ is pathological. Must* $\text{cf}(\lambda) = \omega$*? Must $I_2(\lambda)$ hold?*

Our guess is that the answer is no.

We begin by establishing a dichotomy: pathological cardinals are either regular or of countable cofinality. For the proof we use the following fact, a generalization of the Kunen Inconsistency Theorem that is a slight improvement on an observation due to Foreman [42].

Theorem 8.4.23. *Suppose λ is a cardinal. Suppose Q is a transitive set that is closed under countable sequences and satisfies* $\text{Ord} \cap Q = \lambda$*. Suppose $k : Q \to H(\lambda)$ is a nontrivial elementary embedding. Let γ be the supremum of the critical sequence of k. Then $\lambda = \gamma^+$.*

Proof. Since γ has countable cofinality and Q is closed under countable sequences, $\gamma \in Q$, and in particular $\gamma < \lambda$. The closure of Q under countable sequences also easily implies that $k(\gamma) = \gamma$.

Assume towards a contradiction that $\gamma^+ < \lambda$. We claim that $k[\gamma]$ is definable over $H(\lambda)$ from the ordinal $\sup k[\gamma^{+Q}]$ and parameters in $k[Q]$. This follows from the stationary splitting argument, which actually implies that if $\langle T_\alpha : \alpha < \gamma \rangle$ is any stationary splitting of $\{\alpha < \gamma^+ : \text{cf}(\alpha) = \omega\}$ that lies in the range of k, then $k[\gamma] = \{\alpha < \gamma : T_\alpha \cap \sup k[\gamma^{+Q}] \text{ is stationary}\}$. We omit the proof.

We now split into two cases, each of which leads to a contradiction.

Case 1. $\gamma^{+Q} = \gamma^+$

In this case, $\sup k[\gamma^{+Q}] = \gamma^+$, so $k[\gamma] = \{\alpha < \gamma : T_\alpha \text{ reflects to } \gamma^+\} = \gamma$. In other words, $k \restriction \gamma$ is the identity, contrary to the fact that $\gamma > \kappa_0 = \text{crit}(k)$. (This is just Woodin's proof of the Kunen inconsistency.) Given this contradiction, we turn to our second case.

Case 2. $\gamma^{+Q} < \gamma^+$.

In this case, we will use Solovay's argument that SCH holds above a strongly compact cardinal to show that $\gamma^\omega = \gamma^+$. This immediately leads to a contradiction: by elementarity, $\gamma^{+Q} = (\gamma^\omega)^Q$; by the closure of Q under countable sequences $(\gamma^\omega)^Q \geq \gamma^\omega$; and hence $\gamma^{+Q} \geq \gamma^\omega = \gamma^+$ (again using the closure of Q under countable sequences), contrary to our case hypothesis.

We finish by showing that $\gamma^\omega = \gamma^+$. Suppose not, towards our final contradiction.

Let U be the Q-ultrafilter on γ^{+Q} derived from k using $\sup k[\gamma^{+Q}]$. Let $j : Q \to M$ be the ultrapower embedding and $i : M \to H(\lambda)$ the factor embedding. Since $k[\gamma]$ is definable from elements of $\{\sup k[\gamma^{+Q}]\} \cup k[Q] \subseteq \text{ran}(i)$, we have that $k[\gamma] = i(S)$ for some $S \in M$. But $S = i^{-1}[k[\gamma]] = j[\gamma]$. This shows that $j[\gamma] \in M$.

Since $\gamma^+ < k(\gamma^+) < \lambda$ and $k(\gamma^+)$ is a cardinal, $\gamma^{++} < \lambda$. Therefore every subset of $P_{\omega_1}(\gamma)$ of cardinality γ^{++} belongs to $H(\lambda)$. By elementarity, we can fix a set $A \subseteq P_{\omega_1}(\gamma)$ with $A \in Q$ and $|A|^Q = \gamma^{++Q}$. Now $j[A] \in M$: indeed,

$$j[A] = \{\sigma \in j(A) : \sigma \subseteq j[\gamma]\}$$

The forwards inclusion is immediate, and the reverse inclusion follows from the fact that Q is closed under countable seqences.

Now let $f : A \to \gamma^{++Q}$ be a surjection that lies in Q. Then $j(f)[j[A]] = j[\gamma^{++Q}]$, so $j[\gamma^{++Q}] \in M$. Since $j(\gamma^{++Q}) > j(\gamma^{+Q}) \geq \gamma^{++Q}$ and $j(\gamma^{++Q}) = \gamma^{++M}$ is an M-regular cardinal, $j[\gamma^{++Q}]$ cannot be cofinal in $j(\gamma^{++Q})$. It follows that j is discontinuous at γ^{++Q}. This contradicts that j is the ultrapower embedding associated to a Q-ultrafilter on γ^{+Q}: in general, the ultrapower embedding of a model N associated to an N-ultrafilter on an ordinal δ is continuous at every N-regular cardinal above γ. □

Lemma 8.4.24. *Suppose λ is a pathological cardinal of uncountable cofinality and $j : V \to M$ witnesses the pathology of λ. Let $A \in M$ be a cover of $j[\lambda]$ of M-cardinality λ, and let $\mathcal{U}$ be the fine ultrafilter on $P(\lambda)$ derived from j using A. Let $k : M_\mathcal{U} \to M$ be the factor embedding. Then* $\text{crit}(k) > \lambda$ *and therefore $j_\mathcal{U}$ witnesses the pathology of λ.*

Proof. Let $k : M_\mathcal{U} \to M$ be the factor embedding. We must show that $\text{crit}(k) > \lambda$. Let $\bar{A} = \text{id}_\mathcal{U}$, so $k(\bar{A}) = A$. Clearly $j_\mathcal{U}[\lambda] \subseteq \bar{A}$, so $|\bar{A}|^{M_\mathcal{U}} \geq |\bar{A}| \geq \lambda$. On the other hand, $|\bar{A}|^{M_\mathcal{U}} \leq k(|\bar{A}|^{M_\mathcal{U}}) = |A|^M = \lambda$. Thus $|\bar{A}|^{M_\mathcal{U}} = \lambda$, so

$$k(\lambda) = k(|\bar{A}|^{M_\mathcal{U}}) = |A|^M = \lambda$$

Assume towards a contradiction that $\text{crit}(k) < \lambda$. Since j is $<\lambda$-supercompact, j is $P_{\text{bd}}(\lambda)$-hypermeasurable, and therefore $H(\lambda) \cap M = H(\lambda)$. Thus k restricts to a nontrivial elementary embedding $k : H(\lambda) \cap M_\mathcal{U} \to H(\lambda)$. Since $M_\mathcal{U}$ is closed under countable sequences, we can apply Foreman's Inconsistency Theorem. Since λ has uncountable cofinality and $k(\lambda) = \lambda$, k has a fixed point in the interval $(\text{crit}(k), \lambda)$. Therefore by Foreman's theorem (Theorem 8.4.23), $\lambda = \gamma^+$ where γ is the supremum of the critical sequence of k. But j is γ-supercompact and j is continuous at γ, so by Lemma 4.2.24, j is γ^+-supercompact. Since j witnesses the pathology of λ, j is not λ-supercompact. This contradicts that $\lambda = \gamma^+$.

Thus our assumption was false, and in fact $\text{crit}(k) \geq \lambda$. Since $k(\lambda) = \lambda$, it follows that $\text{crit}(k) > \lambda$. We finally show that this implies $j_\mathcal{U}$ witnesses the pathology of λ.

The set $\bar{A}$ witnesses that $j_\mathcal{U}$ is λ-tight.

Assume towards a contradiction that $j_\mathcal{U}$ is λ-supercompact. Since $\text{crit}(k) > \lambda$, $k(j_\mathcal{U}[\lambda]) = k \circ j_\mathcal{U}[\lambda] = j[\lambda]$, so j is λ-supercompact, which is a contradiction.

We finally show that $j_\mathcal{U}$ is $<\lambda$-supercompact. Since $j_\mathcal{U}$ is an ultrapower embedding, it suffices to show that $j_\mathcal{U}$ is δ-supercompact for all regular cardinals $\delta < \lambda$. To do this, it is enough to show that $j[\delta] \in k[M_\mathcal{U}]$, since then $k^{-1}(j[\delta]) = j_\mathcal{U}[\delta]$ belongs to $M_\mathcal{U}$. By Solovay's Lemma (Lemma 4.4.29), $j[\delta]$ is definable in M from $\sup j[\delta]$ and parameters in $j[V]$. Since $j[V] \subseteq k[M_\mathcal{U}]$ and $k[M_\mathcal{U}]$ is closed under definability in M, to show $j[\delta] \in k[M_\mathcal{U}]$, it suffices to show that $\sup j[\delta] \in k[M_\mathcal{U}]$. To finish, we show that $k(\sup j_\mathcal{U}[\delta]) = \sup j[\delta]$, or in other words that k is continuous at $\sup j_\mathcal{U}[\delta]$. Since $\text{crit}(k) > \lambda$, it is enough to show that $\mathrm{cf}^{M_\mathcal{U}}(\sup j_\mathcal{U}[\delta]) \leq \lambda$. Since $j_\mathcal{U}$ is λ-tight, $j_\mathcal{U}$ is (δ, λ)-tight, so by the easy direction of Theorem 7.2.12, $\mathrm{cf}^{M_\mathcal{U}}(\sup j_\mathcal{U}[\delta]) \leq \lambda$, as desired. □

As a corollary, we eliminate many pathologies which a priori might have seemed plausible:

Corollary 8.4.25. *Suppose λ is a pathological cardinal. Either λ is regular or λ has countable cofinality.*

Proof. Assume λ has uncountable cofinality, and we will show that λ is regular. By Lemma 8.4.24, the pathology of λ is witnessed by an ultrapower embedding $i : V \to N$. Since i is a $<\lambda$-supercompact ultrapower embedding, N is closed under $<\lambda$-sequences. If λ is singular, it follows that N is closed under λ-sequences, contradicting that i is not λ-supercompact. Therefore λ is regular. □

Corollary 8.4.26. *Suppose λ is a regular pathological cardinal. Suppose $j : V \to M$ witnesses the pathology of λ. Let U be the ultrafilter on λ derived from j using* $\sup j[\lambda]$*, and let $k : M_U \to M$ be the factor embedding. Then* $\mathrm{crit}(k) > \lambda$ *and j_U witnesses the pathology of λ.*

Proof. Since λ is regular and j is λ-tight, $\mathrm{cf}^M(\sup j[\lambda]) = \lambda$. Note that $\mathrm{id}_U = \sup j_U[\lambda]$, so $k(\sup j_U[\lambda]) = \sup j[\lambda]$. We have $\mathrm{cf}^{M_U}(\sup j_U[\lambda]) \geq \mathrm{cf}(\sup j_U[\lambda]) = \lambda$ on the one hand, and $\mathrm{cf}^{M_U}(\sup j_U[\lambda]) \leq k(\mathrm{cf}^{M_U}(\sup j_U[\lambda])) = \mathrm{cf}^M(\sup j[\lambda]) = \lambda$ on the other. Thus $\mathrm{cf}^{M_U}(\sup j_U[\lambda]) = \lambda$. It follows that $k(\lambda) = k(\mathrm{cf}^{M_U}(\sup j_U[\lambda])) = \mathrm{cf}^M(\sup j[\lambda]) = \lambda$.

Given that $k(\lambda) = \lambda$, one can finish the proof as in Lemma 8.4.24. Instead of redoing this proof, however, we note that the corollary follows from an *application* of Lemma 8.4.24. Using Theorem 7.2.12, fix a cover $\bar{A} \subseteq \sup j_U[\lambda]$ of $j_U[\lambda]$ of M_U-cardinality λ. Let $A = k(\bar{A})$. Thus $|A|^M = k(\lambda) = \lambda$. Moreover, it is easy to see that

$$H^M(j[V] \cup \{\sup j[\lambda]\}) = H^M(j[V] \cup \{A\})$$

The left-to-right inclusion follows from the fact that $\sup j[\lambda] = \sup A$ is definable from A in M, while the right-to-left inclusion follows from the fact that $A = k(\bar{A})$ and $k[M_U] = H^M(j[V] \cup \{\sup j[\lambda]\})$. Therefore $j_U = j_\mathcal{U}$ and the factor embeddings from $M_\mathcal{U}$ into M is equal to k. Therefore by Lemma 8.4.24, $\mathrm{crit}(k) > \lambda$ and j_U witnesses the pathology of λ. □

Pathological cardinals of countable cofinality, on the other hand, have a property that is a lot like $I_2(\lambda)$:

Proposition 8.4.27. *Suppose $j : V \to M$ witnesses the pathology of a cardinal λ of countable cofinality. Then $j = k \circ j_\mathcal{U}$ where $\mathcal{U}$ is a countably complete fine ultrafilter on $P(\lambda)$ and $k : M_\mathcal{U} \to M$ is a nontrivial elementary embedding such that $\lambda = \kappa_\omega(k)$.*

Proof. Immediate from the proof of Lemma 8.4.24. □

If the ultrafilter $\mathcal{U}$ of the previous lemma is principal, then $I_2(\lambda)$ holds. Under UA, there is a way to make this conclusion:

Theorem 8.4.28 (UA)**.** *Suppose λ is a pathological cardinal of countable cofinality. Then $I_2(\lambda)$.*

Proof. Let $j : V \to M$ witness the pathology of λ. Then j witnesses that some cardinal $\kappa < \lambda$ is γ-supercompact for all $\gamma < \lambda$. In particular, by our results on GCH (Theorem 6.3.25), λ is a strong limit cardinal.

Applying Proposition 8.4.27, fix a countably complete fine ultrafilter $\mathcal{U}$ on $P(\lambda)$ and a nontrivial elementary embedding $k : M_{\mathcal{U}} \to M$ such that $k \circ j_{\mathcal{U}} = j$ and λ is the supremum of the critical sequence of k.

By Corollary 8.2.25, fix a countably complete ultrafilter D with $\lambda_D < \lambda$ and an elementary embedding $e : M_D \to M_{\mathcal{U}}$ such that e is $j_D(\lambda)$-supercompact in M_D. Since λ is a strong limit cardinal of countable cofinality, $j_D(\lambda) = \lambda$. The supercompactness of e implies that $V_\lambda \cap M_D = V_\lambda \cap M_{\mathcal{U}}$, so k acts on $V_\lambda \cap M_D$. Since $(j_D \restriction V_\lambda) : V_\lambda \to V_\lambda \cap M_D$ and $(k \restriction V_\lambda \cap M_D) : V_\lambda \cap M_D \to V_\lambda$ are elementary embeddings,

$$i = (k \restriction V_\lambda \cap M_D) \circ (j_D \restriction V_\lambda)$$

is an elementary embedding from V_λ to V_λ. Moreover, suppose $A \subseteq V_\lambda$ is a wellfounded relation. Then $i(A) = \bigcup_{\alpha<\lambda} i(A \cap V_\alpha)$ is also wellfounded since $i(A) = k(j_D(A))$, and k and j_D preserve wellfoundedness. Thus i extends to an elementary embedding $i^* : V \to N$ where N is wellfounded, and it follows that $I_2(\lambda)$ holds. □

Under UA, regular pathological cardinals are inaccessible:

Proposition 8.4.29 (UA)**.** *Suppose λ is a regular pathological cardinal. Then λ is strongly inaccessible and $\mathscr{K}_\lambda$ witnesses the pathology of λ.*

Proof. By Lemma 8.4.24, there is a countably complete ultrafilter U such that j_U witnesses the pathology of λ. In particular, j_U is $<\lambda$-supercompact and λ-tight. It follows that j_U is $[0, \lambda]$-tight, since this just means j_U is γ-tight for all cardinals $\gamma \leq \lambda$. In particular, U is λ-irreducible by Proposition 8.2.3.

Note that j_U witnesses that λ is Fréchet. Suppose towards a contradiction that λ is a successor cardinal. Then by the Irreducibility Theorem (Corollary 8.2.20), j_U is λ-supercompact, contradicting that U witnesses the pathology of λ.

Thus λ is a limit cardinal. But j_U is $<\lambda$-supercompact, so by our results on GCH (Theorem 6.3.25), λ is a strong limit cardinal. Therefore λ is strongly inaccessible.

Finally we show that $\mathscr{K}_\lambda$ witnesses the pathology of λ. Let $j : V \to M$ be the ultrapower of the universe by $\mathscr{K}_\lambda$. It suffices to show that j is not λ-supercompact, since by Theorem 7.3.34, j is $<\lambda$-supercompact and λ-tight. Suppose towards a contradiction that j is λ-supercompact. Then by Corollary 8.4.18, *every* ultrapower by a λ-irreducible ultrafilter is λ-supercompact, contradicting that j_U is not λ-supercompact. Thus $\mathscr{K}_\lambda$ witnesses the pathology of λ. □

To summarize, under UA, if a cardinal is pathological, it is at least pathological for a good reason:

Theorem 8.4.30 (UA). *If λ is a pathological cardinal, then one of the following holds:*

- *There is an elementary $j : V \to M$ with $\text{crit}(j) < \lambda$ and $V_\lambda \subseteq M$.*
- *λ is strongly inaccessible and $\mathscr{K}_\lambda$ witnesses the pathology of λ.* $\square$

Chapter 9

Open questions

This section lists a number of open questions raised by this work.

Chapter 2

9.1.1 Weak Comparison

Question 9.1.1 (Weak Comparison)**.** *Assume $V = \mathrm{HOD}$. Suppose M is a finitely generated Σ_2-hull. Is there a maximum finitely generated Σ_2-hull N such that $N \cap P(\omega) = M \cap P(\omega)$?*

Suppose M is a countable transitive model of ZFC. The *ultrapower multiverse of M* is the smallest set $\mathscr{C}_M$ containing M such that for all internal ultrapowers $j : M_0 \to M_1$, $M_0 \in \mathscr{C}_M$ if and only if $M_1 \in \mathscr{C}_M$. Assuming Weak Comparison, if M is a finitely generated Σ_2-hull, then $\mathscr{C}_M$ consists of all finitely generated Σ_2-hulls H such that $H \cap P(\omega) = M \cap P(\omega)$. Question 9.1.1 asks whether there is some model N such that $\mathscr{C}_M$ is equal to the collection of internal ultrapowers of N. Notice that assuming UA, Weak Comparison is equivalent to the conclusion of Question 9.1.1. Since Weak Comparison does not follow from UA (even assuming $V = \mathrm{HOD}$ and large cardinals), the hypothesis of Question 9.1.1 cannot be weakened. Question 9.1.1 has a positive answer in all known canonical inner models: the model N is the Skolem hull $H^M(p)$ where p is the minimum finite sequence of ordinals such that $H^M(p) \notin M$.

Question 9.1.2 (Weak Comparison)**.** *Assume that $V = \mathrm{HOD}$ and there is a Σ_2-reflecting worldly cardinal. Must* CH *hold?*

The answer seems likely to be no. Adding reals while controlling the structure of finitely generated Σ_2-hulls seems like an interesting (though somewhat technical) forcing problem.

Question 9.1.3 (Weak Comparison)**.** *Assume that $V = \mathrm{HOD}$ and there is a Σ_2-reflecting worldly cardinal. Is every precipitous ideal atomic?*

Under these hypotheses, there can be no precipitous ideal I on $P(\kappa)$ such that $P(\kappa)/I$ has a dense countably closed suborder. More generally, no countably closed forcing can add a new countably complete V-ultrafilter.

9.1.2 The Ultrapower Axiom

The *Weak Ultrapower Axiom* states that for any ultrapowers M_0 and M_1 of V, there is an ultrapower N of V that is an internal ultrapower of both M_0 and M_1.

Question 9.1.4. *Is it consistent that the Weak Ultrapower Axiom holds but the Ultrapower Axiom does not?*

The answer seems likely to be yes, but it is not clear, for example, whether the Weak Ultrapower Axiom holds in the Kunen-Paris extension of $L[U]$.

Question 9.1.5. *Assume the Weak Ultrapower Axiom and let κ be the least supercompact cardinal. Does the κ-Complete Ultrapower Axiom hold?*

It is known that the Weak Ultrapower Axiom has some strength in the presence of very large cardinals: for example, if there is an extendible cardinal, then the Weak Ultrapower Axiom implies the linearity of the Mitchell order on sufficiently complete Dodd sound ultrafilters, and consequently V is a generic extension of HOD.

Chapter 3

9.2.3 The Ketonen order

A *normed ultrafilter* is a pair (U, f) where U is a countably complete ultrafilter on a set X and $f : X \to \text{Ord}$ is a function. The *Ketonen order on normed ultrafilters* is defined by setting $(U, f) \leq_{\Bbbk} (W, g)$ if $U = W\text{-}\lim_{y \in I} U_y$ for a sequence of countably complete ultrafilters U_y such that for W-almost all y, for U_y-almost all x, $f(x) \leq g(y)$. Under UA, the Ketonen order prewellorders the class of normed ultrafilters; this prewellorder is set-like unless there is a supercompact cardinal. *Ketonen equivalence* is defined on normed ultrafilters by setting $(U, f) =_{\Bbbk} (W, g)$ if $(U, f) \leq_{\Bbbk} (W, g)$ and $(W, g) \leq_{\Bbbk} (U, f)$. Under UA, this is equivalent to the existence of an internal ultrapower comparison $(k, h) : (M_U, M_W) \to N$ such that $k([f]_U) = h([g]_W)$.

Question 9.2.6 (UA)**.** *Suppose $(U, f) =_{\Bbbk} (W, g)$. Must there be an ultrafilter $Z \leq_{\text{RF}} U, W$ and a function h such that $(Z, h) =_{\Bbbk} (U, f)$?*

Obviously the converse is true. The natural candidate for Z is the Ketonen minimum ultrafilter D such that $(D, h) =_{\Bbbk} (U, f)$ for some h. The answer to this question is positive in all known inner models, and in fact a positive answer follows from the Irreducible Ultrafilter Hypothesis.

Question 9.2.7. *Which partial orders can be realized as the Ketonen order on a cardinal in some forcing extension of V?*

Notice that for ultrafilters on the least measurable cardinal, the Ketonen order and the revised Rudin-Keisler order coincide by the proof of Rudin's Theorem (Theorem 5.2.13).

The *κ-complete Ketonen order* is defined on ultrafilters U and W on ordinals by setting $U \leq_{\Bbbk}^{\kappa} W$ if $U = W\text{-}\lim_{\alpha\in I} U_\alpha$ where U_α is a κ-complete ultrafilter that concentrates on $\alpha + 1$. It is not clear whether the

Again, the Ketonen order and the revised Rudin-Keisler order coincide on ultrafilters on ω by the proof of Rudin's Theorem (Theorem 5.2.13). But the situation seems more complicated even for ultrafilters on countable ordinals.

Question 9.2.8. *What is the possible structure of the ω-complete Ketonen order on countable ordinals, on ω_1, on ω_ω?*

One might also consider the restriction of this order to nonregular ultrafilters, weakly normal ultrafilters, or extensions of the closed unbounded filter.

Question 9.2.9. *Must the ω-complete Ketonen order and the Ketonen order coincide on countably complete ultrafilters?*

Since the ω-complete Ketonen order extends the Ketonen order, the two orders coincide under UA.

9.2.4 The Lipschitz order

Our main question concerning the Lipschitz order is whether its linearity suffices to prove the Ultrapower Axiom:

Question 3.4.36 (UA)**.** *Assume that for all ordinals δ and all $U, W \in \mathbf{UF}(\delta)$, the Lipschitz game $G_\delta(U, W)$ is determined. Must the Ultrapower Axiom hold?*

The following is a stronger version of Question 9.2.9:

Question 9.2.10 (UA)**.** *Can the Lipschitz order on countably complete ultrafilters be different from the Ketonen order? Must the Lipschitz order on ultrafilters on ω be different from the revised Rudin-Keisler order?*

Given a positive answer to the first question, the following questions become interesting.

Question 9.2.11. *Must the Lipschitz order on countably complete ultrafilters be wellfounded? Can there be Lipschitz equivalent (countably complete) ultrafilters that are not equal?*

For any cardinal λ, the *bounded topology on $P(\lambda)$* is generated by the sets $N_{\alpha,\sigma} = \{A \subseteq \lambda : A \cap \alpha = \sigma\}$ where $\sigma \subseteq \alpha < \lambda$. The *Wadge order* is defined on sets $X, Y \subseteq P(\lambda)$ by setting $X \leq_{\mathrm{W}} Y$ if there is a continuous function $f : P(\lambda) \to P(\lambda)$ such that $f^{-1}[Y] = X$. The *countably complete Wadge order* is defined on countably complete ultrafilters $U, W \in \mathbf{UF}(\lambda)$ by setting $U \leq_{\mathbb{W}} W$ if there is a continuous countably complete homomorphism $h : P(\lambda) \to P(\lambda)$ such that $U = h^{-1}[W]$.

Under the Ultrapower Axiom, the countably complete Wadge order prewellorders the ultrafilters on any cardinal λ as an immediate consequence of the linearity of the Ketonen order. In fact, the Wadge order and the countably complete Wadge order agree on $\mathbf{UF}(\lambda)$. This latter fact is proved by establishing an analog of Wadge's Lemma, proving a strong form of determinacy for the natural Wadge game $\mathrm{W}_\lambda(U, W)$ associated to any pair of ultrafilters U and W on λ such that (j_U, j_W) admits an internal ultrapower comparison.

Question 9.2.12. *Assume that for all cardinals λ, the countably complete Wadge order on $\mathbf{UF}(\lambda)$ is a prewellorder. Must the Ultrapower Axiom hold?*

The question is open even if one assumes the strong determinacy of the games $\mathrm{W}_\lambda(U, W)$ for all cardinals λ and countably complete ultrafilters U and W.

Chapter 4

9.2.5 The generalized Mitchell order

Our main question regarding the generalized Mitchell order is whether UA implies the Irreducible Ultrafilter Hypothesis:

Question 9.2.13 (UA)**.** *Suppose U and W are Rudin-Keisler inequivalent hereditarily uniform irreducible ultrafilters. Must U and W be comparable in the Mitchell order?*

This question is discussed at length in Section 4.2.5. A closely related question is whether Corollary 4.3.28 has a converse:

Question 9.2.14 (UA)**.** *Suppose W is an ultrafilter with the property that for all $U <_{\Bbbk} W$, $U \lhd W$. Is W Dodd sound?*

A positive answer to this question seems unlikely since it turns out to imply that every incompressible irreducible ultrafilter is Dodd sound, a significantly stronger hypothesis than the Irreducible Ultrafilter Hypothesis. In the Mitchell-Steel models, the answer is yes by a theorem of Schlutzenberg ([8] or Theorem 4.3.1).

Theorem 4.4.2 states essentially that under the Ultrapower Axiom plus the GCH, the Mitchell order is linear on normal fine ultrafilters. In fact this result is *local*: under GCH, if U and W are normal fine ultrafilters admitting an internal ultrapower comparison, then U and W are Mitchell comparable. Theorem 7.5.42 establishes by a much harder argument that Theorem 4.4.2 can be proved without its GCH hypothesis. The result is very far from local, however, raising the question of whether there is a more direct proof.

Question 9.2.15. *Suppose $\mathcal{U}$ and $\mathcal{W}$ are Rudin-Keisler inequivalent normal fine hereditarily uniform ultrafilters such that $(j_{\mathcal{U}}, j_{\mathcal{W}})$ has an internal ultrapower comparison. Must $\mathcal{U}$ and $\mathcal{W}$ be comparable in the Mitchell order?*

An extender E is *downward closed* if j_E is γ-supercompact for all γ such that for some $\xi < \text{length}(E)$, E_ξ is a uniform ultrafilter on γ. In [43], the wellfoundedness of the Mitchell order is established for the class of downward closed extenders below rank-to-rank type.

Question 9.2.16 (UA)**.** *Is the Mitchell order wellfounded on all extenders below rank-to-rank type?*

The hope is that the supercompactness properties ensured by Chapter 7 might allow one to adapt Neeman's proof to the general case.

Chapter 5

9.2.6 The Rudin-Frolík order

Proposition 5.4.18 shows that assuming the Ultrapower Axiom, any countably complete ultrafilters U_0 and U_1 have a greatest lower bound in the Rudin-Frolík order. One is left to wonder, however, whether there is a more natural characterization of this ultrafilter, say in terms of the comparison $(i_0, i_1) : (M_{U_0}, M_{U_1}) \to N$ of (j_{U_0}, j_{U_1}). In particular, there is a maximal elementary embedding $j : V \to H$ for which there exist $k_0 : H \to M_{U_0}$ and $k_1 : H \to M_{U_1}$ such that $j_{U_0} = k_0 \circ j$ and $j_{U_1} = k_1 \circ j$: namely, let $X = i_0[M_{U_0}] \cap i_1[M_{U_1}]$ and let $(H, j[V])$ be the transitive collapse of $(X, i_0 \circ j_{U_0}[V])$. Is j the ultrapower embedding associated to the greatest lower bound of U_0 and U_1? Here we pose an equivalent question that is a bit more succinct:

Question 9.2.17 (UA)**.** *Suppose U_0 and U_1 are countably complete ultrafilters with no nontrivial common predecessors in the Rudin-Frolík order. Let $(i_0, i_1) : (M_{U_0}, M_{U_1}) \to N$ be the internal ultrapower comparison of (j_{U_0}, j_{U_1}). Must $i_0[M_{U_0}] \cap i_1[M_{U_1}] = i_0 \circ j_{U_0}[V]$?*

A positive answer follows from the Irreducible Ultrafilter Hypothesis.

There are also questions about the nature of upper bounds in the Rudin-Frolík order.

Question 9.2.18 (UA)**.** *Suppose U and W are countably complete ultrafilters and Z is their least upper bound in the Rudin-Frolík order. Is $M_Z = M_U \cap M_W$?*

We do not know whether the equality above is *ever* true. By Corollary 5.4.21 the class of ultrafilters of M_Z is in fact equal to the intersection of the M_Z-ultrafilters in M_U and those in M_W, which is perhaps a start.

A special case of this question is interesting in the ZFC context:

Question 9.2.19. *Suppose U and W are normal ultrafilters on distinct cardinals. Is $M_{U\times W} = M_U \cap M_W$?*

Surely this question can be resolved under the assumption that $V = L[U, W]$.

The structural properties of the Rudin-Frolík lattice, beyond its local finiteness (Section 5.4.3), are all open.

Question 9.2.20 (UA)**.** *Is the Rudin-Frolík lattice distributive?*

A special case of this question arises in the proof of the Irreducibility Theorem (Section 8.2.4), where one must rule out the existence of a particular pentagonal sublattice (Fig. 8.1) of the Rudin-Frolík order. More generally, it is open whether there is any universal sentence in the language of lattices provable of the Rudin-Frolík lattice under UA plus large cardinals that does not hold of every locally finite lattice.

9.2.7 The internal relation

For the most part, the structure of the internal relation under UA is a solved problem, or at least has been reduced to the structure of the Mitchell order by Theorem 8.3.33. The nature of the internal relation in ZFC, however, is a complete mystery. For example, under the Ultrapower Axiom, Theorem 5.5.21 characterizes the 2-cycles in the internal relation in terms of commuting ultrafilters (Definition 5.5.17). Does the same characterization hold in the context of ZFC?

Question 5.5.22. *Suppose $U \sqsubset W$ and $W \sqsubset U$. Must U and W commute?*

A related question is whether commuting ultrafilters U and W must have the stronger property that $U \times W$ is generated by sets of the form $A \times B$ with $A \in U$ and $B \in W$. This follows from UA and also from GCH, and it seems likely that a positive answer to Question 5.5.22 does as well.

Chapter 6

9.2.8 V = HOD

Many questions remain about how close HOD must be to V under UA plus a supercompact cardinal.

Question 6.2.9 (UA)**.** *Let κ be the least supercompact cardinal.*

- *Is $V = \text{HOD}[X]$ for some $X \subseteq \kappa$?*
- *Is $V = \text{HOD}[G]$ for $G \subseteq \kappa$ generic for a partial order $\mathbb{P} \in \text{HOD}$ such that $|\mathbb{P}| \leq \kappa$? What about a κ-cc Boolean algebra?*
- *Is $V = \text{HOD}_{V_\kappa}$?*

The second bullet point would imply a positive answer to the following question, which does not mention forcing:

Question 6.2.11 (UA)**.** *Assume κ is supercompact. Is $\kappa^{+\text{HOD}} = \kappa^+$?*

The next question concerns local versions of $V = \text{HOD}$, motivated by Theorem 6.2.18:

Question 6.2.20 (UA). *Let κ be the least supercompact cardinal.*

- *Suppose $\lambda \geq \kappa$ is inaccessible. Is $H(\lambda^+)$ definably wellordered?*
- *Suppose $\lambda \geq \kappa$ and $\text{cf}(\lambda) < \kappa$. Is $H(\lambda^{++})$ definably wellordered?*
- *Suppose $\lambda \geq \kappa$ is singular and the Axiom of Choice is false in $L(P(\lambda))$. Is there an elementary embedding from $L(P(\lambda))$ to $L(P(\lambda))$ with critical point less than λ?*

The final item above is a variant of a question posed by Woodin, which asked the same thing under the hypothesis $V = \text{Ultimate } L$ instead of UA plus a supercompact cardinal. It is hard to say which question looks more hopeless at this time.

9.2.9 The Generalized Continuum Hypothesis

The main open cardinal arithmetic questions under UA concern whether the large cardinal hypothesis of Theorem 6.3.25 can be improved. A special case of particular interest is the following:

Question 9.2.21 (UA). *Suppose κ is κ^+-supercompact. Must $2^\kappa = \kappa^+$?*

One can ask further whether UA implies that GCH holds at any cardinal λ such that some $\kappa \leq \lambda$ is λ-supercompact. We do not know how to refute this, but we conjecture a negative answer:

Conjecture 9.2.22. *It is consistent with* UA *that* GCH *fails at a measurable cardinal.*

Friedman-Magidor forcing [44] establishes an approximation to this conjecture: Friedman-Magidor construct a model in which the least measurable cardinal κ carries a unique normal ultrafilter and yet $2^\kappa > \kappa^+$. Assuming without loss of generality that there is just one measurable cardinal, the Ultrapower Axiom is equivalent to the statement that κ carries a unique normal ultrafilter U and moreover every κ-complete ultrafilter on κ is Rudin-Keisler equivalent to U^n for some $n < \omega$. Thus to affirm Conjecture 9.2.22, one seems to have to modify Friedman-Magidor forcing to control the structure of all κ-complete ultrafilters, rather than just the normal ones.

Question 9.2.23 (UA). *Suppose κ is a strong cardinal. Can GCH fail at κ or above?*

Again, the answer seems likely to be (consistently) positive. We would guess that it is consistent with UA that the GCH fails on an unbounded set of cardinals below the first supercompact cardinal.

It remains to be seen whether further combinatorial principles follow from UA.

Question 9.2.24 (UA). *Suppose κ is supercompact. Does $\diamondsuit(S^{\kappa^+}_\kappa)$ hold? (Note that $\diamondsuit^+(\lambda)$ is false if κ is λ-supercompact.) Does $S^{\kappa^{++}}_{\geq\kappa}$ carry a partial square?*

Certain instances of GCH at regular cardinals do not require the full strength of UA and instead only use the linearity of the Mitchell order on normal fine ultrafilters (Corollary 6.3.5). One might try to prove the eventual GCH from the linearity of the Mitchell order alone (assuming large cardinals), rather than the full Ultrapower Axiom.

Question 9.2.25. *Assume there is a supercompact cardinal and for all hereditarily uniform normal fine ultrafilters $\mathscr{U}$ and $\mathscr{W}$, either $\mathscr{U} \lhd \mathscr{W}$, $\mathscr{U} \equiv_{\text{RK}} \mathscr{W}$, or $\mathscr{W} \lhd \mathscr{U}$. Does* GCH *hold at all sufficiently large cardinals?*

9.2.10 The Ground Axiom

Recall Reitz's *Ground Axiom* [45], which states that V is not a set forcing extension of any other inner model. Usuba [46] shows that if there is an extendible cardinal, V is a set forcing extension of a unique inner model that satisfies the Ground Axiom. Usuba's Theorem cannot be proved under the weaker hypothesis that there is a supercompact cardinal, but one might expect that this large cardinal hypothesis does suffice under UA.

Question 9.2.26 (UA). *Assume there is a supercompact cardinal. Is V a set forcing extension of an inner model of the Ground Axiom?*

The Ground Axiom alone has very few consequences since it can be class forced by various coding forcings. It is hard to believe, however, that such a coding could preserve the Ultrapower Axiom in the context of a supercompact cardinal.

Question 9.2.27 (UA). *Assume the Ground Axiom and an extendible cardinal. Must the Generalized Continuum Hypothesis hold? Must $V =$* Ultimate *L?*

Chapter 7

9.2.11 Isolated cardinals

A cardinal λ is *Fréchet* if it carries a uniform countably complete ultrafilter. A limit cardinal is *isolated* if it is Fréchet but not a limit of Fréchet cardinals. The structure of isolated cardinals remains the main gap in our understanding of the local structure of strong compactness under UA.

Conjecture 7.4.8 (UA). *Every isolated cardinal is measurable.*

The question comes down to whether every isolated cardinal is a strong limit by Proposition 7.5.4. By Theorem 7.5.25, it suffices to show that $2^{\delta_\lambda} < \lambda$ where δ_λ is the supremum of all Fréchet cardinals less than λ.

Though we state the conjecture, it seems plausible that Conjecture 7.4.8 is not provable, so we include a number of weaker questions.

Question 9.2.28 (UA)**.** *Is every isolated cardinal regular?*

If even this question cannot be answered, there are a number of interesting questions related to singular isolated cardinals. The following question is motivated by Theorem 7.5.38:

Question 9.2.29. *If λ is a singular isolated cardinal, must λ be a limit of weakly inaccessible cardinals?*

The following question asks whether Proposition 7.5.24 can be generalized to all isolated cardinals:

Question 9.2.30 (UA)**.** *If λ is a singular isolated cardinal and U is an ultrafilter such that $\lambda_U < \lambda$, must $j_U[\lambda] \subseteq \lambda$?*

9.2.12 Local supercompactness

The connection between the next question, concerning ultrapower thresholds (Definition 7.4.24), is also a question about isolated cardinals in disguise:

Question 9.2.31 (UA)**.** *Suppose κ is a cardinal and there is a κ-complete ultrafilter U such that $j_U(\kappa) > (2^\kappa)^+$. Must κ be κ^+-strongly compact?*

It would perhaps be more in keeping with the methodology of Chapter 7 to assume that κ is least among all cardinals δ such that there is an ultrapower embedding $j : V \to M$ with $j(\delta) > (2^\kappa)^+$. In this case, however, given Theorem 8.3.16, it is likely to be significantly easier to answer Question 9.2.31 positively if κ is *not* the least such cardinal.

The following question appears in Chapter 8, but it is really more closely related to Chapter 7:

Question 8.2.13 (UA)**.** *Suppose there is a regular cardinal δ that carries a countably complete ultrafilter extending the ω-closed unbounded filter. Must there be a superstrong cardinal? Must there be an inner model with a superstrong cardinal?*

There is a similar question regarding the filter extension property. Suppose $\kappa \leq \lambda$ are cardinals. Then κ is λ-Π^1_1-*subcompact* if for all $A \subseteq H(\lambda)$, if $(H(\lambda), A)$ satisfies a Π^1_1-sentence φ, then for some cardinal $\bar{\lambda} < \lambda$ and some set $\bar{A} \subseteq H(\bar{\lambda})$, there is an elementary embedding $j : (H(\bar{\lambda}), \bar{A}) \to (H(\lambda), A)$ such that $j[\bar{\lambda}] \cap \kappa$ is transitive and $(H(\bar{\lambda}), \bar{A})$ satisfies φ.

Question 9.2.32 (UA)**.** *Suppose κ is a cardinal such that every κ-complete filter on λ extends to a κ-complete ultrafilter. Must κ either be Π^1_1-λ^+-subcompact or a limit of Π^1_1-λ^+-subcompact cardinals?*

Chapter 8

9.2.13 The Complete Ultrapower Axiom

Recall that the *Complete Ultrapower Axiom* asserts that for all cardinals κ, the Rudin-Frolík order is directed on κ-complete ultrafilters.

Question 8.1.13. *Is the Complete Ultrapower Axiom consistent with the existence of cardinals $\kappa < \lambda$ that are both λ^+-supercompact?*

The Complete Ultrapower Axiom follows from the Irreducible Ultrafilter Hypothesis, so a positive answer to Question 4.2.51 would yield a positive answer to Question 8.1.13.

If the Complete Ultrapower Axiom is not consistent with very large cardinals, then the Ultrapower Axiom plus large cardinals might have some very surprising consequences. The following question was posed by Doug Blue:

Question 9.2.33 (Blue)**.** *Assume the Ultrapower Axiom holds and there is a supercompact cardinal. Must the Proper Forcing Axiom fail?*

If UA refutes the κ-Complete Ultrapower Axiom for all supercompact cardinals κ except the least one, then every known way of forcing PFA over a model of UA will destroy UA.

9.2.14 Cardinal preserving embeddings

Caicedo's question on cardinal preserving embeddings is an interesting test case for lifting large cardinal theory under UA to the ZFC setting.

Question 8.4.8 (Caicedo)**.** *Is it consistent that there is a nontrivial cardinal preserving elementary embedding?*

The author has shown that one can simulate enough UA in the context of a proper class of strongly compact cardinals to run the argument of Lemma 8.4.11, obtaining:

Theorem 9.2.34. *If there is a proper class of strongly compact cardinals, there are no nontrivial weakly cardinal preserving elementary embeddings.* □

It remains open whether this theorem can be proved in ZFC, Worse, it is open whether there can be a *cofinality preserving* embedding, which is for example related to Question 8.2.5.

The following question asks whether Theorem 8.4.13 can be improved by reducing its hypothesis from cardinal preservation to weak cardinal preservation.

Question 8.4.15 (UA)**.** *Suppose there is a weakly cardinal preserving elementary embedding from V_λ into a transitive set $M \subseteq V_\lambda$ that fixes no ordinals above its critical point. Must there be an elementary embedding $j : V_\lambda \to V_\lambda$?*

9.2.15 Supercompactness at inaccessible cardinals

We finally come to the questions regarding potential failures of supercompactness at inaccessible cardinals. Recall that UA implies level-by-level equivalence between strong compactness and supercompactness except at inaccessible levels.

Question 8.4.16 (UA). *Suppose λ is an inaccessible cardinal and κ is the least λ-strongly compact cardinal. Must κ be λ-supercompact? More generally, if κ is λ-strongly compact, must κ be λ-supercompact or a measurable limit of λ-supercompact cardinals?*

This raises a ZFC question: suppose λ is a cardinal, $j : V \to M$ is an elementary embedding, M is closed under $<\lambda$-sequences, and every λ-sized subset of M is contained in an element of M of M-cardinality at most λ. Must M be closed under λ-sequences?

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