The structure of $C(aa)$

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Abstract

Stationary logic is the extension of first-order logic with a quantifier expressing that "almost all" countable subsets of a structure have a given property. This paper studies $C(aa)$, the smallest model of ZF containing the ordinal numbers and closed under the satisfaction predicate for stationary logic. This model was constructed by Kennedy–Magidor–Vanaanen [11] as a generalization of Gödel's constructible universe L, obtained by iterating definability in stationary logic rather than first-order logic. Unlike L, however, $C(aa)$ can contain large cardinals far beyond a measurable cardinal. We show in this paper that nevertheless, assuming large cardinals in V, the inner model $C(aa)$ shares many of the nice properties of L. In particular, we prove that $C(aa)$ satisfies the Generalized Continuum Hypothesis, the Ultrapower Axiom, and the axiom $V = \text{HOD}$.

1 Introduction

One of the great triumphs of modern mathematical logic is Gödel's 1941 theorem that one can freely assume the Axiom of Choice (AC) and the Generalized Continuum Hypothesis (GCH) without fear of contradiction, as long as the other axioms of Zermelo-Fraenkel set theory (ZF) are themselves consistent. This result largely put to rest what had been the greatest controversy in the history of mathematics, the debate over the nonconstructive aspects of Cantor's set theory. From the perspective of contemporary set theory, however, the true breakthrough in Gödel's 1941 work is the discovery of the constructible universe.

The constructible universe, denoted by L, is the smallest model of the ZF axioms that contains every ordinal number. It is far from clear that this minimum model exists since the intersection of two models of ZF need not be a model of ZF. Instead, Gödel built the constructible universe up from below by iterating the *definable powerset operation*, which assigns to a structure $\mathcal{M}$ the set $\text{def}(\mathcal{M})$ of all subsets of $\mathcal{M}$ that are definable over $\mathcal{M}$ from parameters. This leads to the *constructible hierarchy*:

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= \text{def}(L_\alpha, \in) \\ L_\lambda &= \bigcup_{\alpha<\lambda} L_\alpha \text{ if } \lambda \text{ is a limit ordinal} \end{aligned}$$

A set is *constructible* if it appears at some level of the constructible hierarchy; intuitively, the constructible sets are those that can be constructed in a transfinite recursive process that uses only the basic set theoretic operations. The *constructible universe* L is the class of all constructible sets.

The theory of arbitrary sets (of natural numbers or real numbers, for example) is notorious for its incompleteness: many of the basic questions — most famously Cantor's continuum hypothesis — cannot be answered on the basis of the ZF axioms, or in fact on the basis of any commonly accepted axiomatic system. But remarkably, the constructible sets admit a complete analysis, in the sense that all the classical questions of set theory can be resolved if one replaces each quantifier over arbitrary sets with a quantifier over the constructible ones. For example, Gödel showed that L satisfies the ZF axioms along with AC and GCH. This yields the celebrated consistency result discussed in the first paragraph. Later on, Jensen showed that Suslin's hypothesis is false in L. Later still, Shelah showed that Whitehead's problem has a positive answer in L.

One of the main lines of research in set theory is the study of strong axioms of infinity, or large cardinal hypotheses. It is possible to gain a better understanding of these principles by examining their behavior within the constructible universe. In this context, one might hope to obtain a complete picture of the structure of large cardinals and their relationships with other set theoretic principles. Unfortunately, this can only be carried out for large cardinals that are relatively small by the standards of modern set theory. The problem is that many large cardinal hypotheses — for example, measurable cardinals — are actually *false* in the constructible universe. In order to understand these stronger hypotheses, set theorists have defined a hierarchy of generalizations of L that contain measurable cardinals and much more and yet still admit a complete analysis in the sense that L does. The study of these models is known as inner model theory.

The subject of this paper is a generalization of the constructible universe that is different from the kind studied by inner model theorists. Introduced by Kennedy-Magidor-Vanaanen [11], this generalization is obtained by replacing the use of first-order definable subsets in the definition of the constructible hierarchy with subsets definable in stronger logics. In particular, for each strong logic $\mathcal{L}$, Kennedy-Magidor-Vanaanen produce an inner model $C(\mathcal{L})$ by iterating the $\mathcal{L}$-definable powerset operation.

For certain logics $\mathcal{L}$, the structure of $C(\mathcal{L})$ is subject to the same sort of incompleteness phenomena that arise for arbitrary sets. For example, if $\mathcal{L}$ is second-order logic, then $C(\mathcal{L}) = \text{HOD}$, an inner model whose structure is infamously difficult to pin down. Surprisingly, however, under large cardinal hypotheses, there are natural examples of logics $\mathcal{L}$ such that $C(\mathcal{L})$ is a proper extension of L that contains measurable cardinals and admits a fairly detailed analysis. This paper concerns one such example, which is arguably the most interesting instance of the Kennedy-Magidor-Vanaanen construction that has emerged so far: the inner model $C(aa)$, constructed using iterated definability in *stationary logic*.

Stationary logic is a strong logic originally proposed by Shelah [9, p. 356] and then developed in detail by Barwise–Kaufmann–Makkai [1]. It is obtained from first-order logic by adding a generalized quantifier, denoted aa, intended to express that *almost all* countable subsets of a given structure have a certain property. (Formally, a property is said to hold of almost all countable subsets if there is a closed unbounded family of countable sets with the property.) Stationary logic is remarkable because although it is strictly more expressive than first-order logic, it still admits a complete proof system [1]. The insight of Kennedy–Magidor–Vanaanen [5] is that the Barwise–Kaufmann–Makkai completeness theorem is closely related to the key comparison lemma in inner model theory [6]. This connection between classical inner model theory and strong logics leads to a comparison lemma for the inner model $C(aa)$ ([5, Theorem ??]), a corollary of which is that the continuum hypothesis holds in $C(aa)$ ([5, Theorem ??]).

The purpose of this paper is to develop the theory of $C(aa)$ using the Kennedy–

Magidor–Vanaanen comparison lemma and ideas from inner model theory. The main issue encountered by Kennedy–Magidor–Vanaanen is that it is unclear how to use the comparison lemma to understand the structure of $C(aa)$ beyond $\aleph_1^V$. We overcome this difficulty by applying techniques from the theory of the Ultrapower Axiom [3] for extracting information from comparison even when at the outset, one has very litte information about the embeddings of comparison. This is used to answer the main question left open by Kennedy–Magidor–Vanaanen: does the Generalized Continuum Hypothesis hold in $C(aa)$? Theorem 4.3 shows that the answer is yes. Notably, the proof of this theorem uses core model theory to take advantage of the *smallness* of $C(aa)$, namely the fact that $C(aa)$ does not contain an inner model with a Woodin cardinal. This idea, which comes up in Theorem 4.1, is the second author's main contribution to this paper.

Besides the GCH, we establish several other structural properties of $C(aa)$. For example, we show that $C(aa)$ has an internally definable well-ordering; in other words, $C(aa)$ is a model of $V = \text{HOD}$. Regarding the large cardinal structure of $C(aa)$, we show that $C(aa)$ is a model of the Ultrapower Axiom (Theorem 3.12) and that $\aleph_1^V$ is the least measurable cardinal of $C(aa)$ (Corollary 3.16), which answers another question from [5].

2 Preliminaries

2.1 Stationary logic

The syntax of stationary logic is that of monadic second-order logic. The logic therefore has variables of two sorts: the first-order variables, which range over elements of a structure, and the second-order variables, which range over countable subsets of a structure. For each second-order variable X and each first-order term t, it is stipulated that $t \in X$ is a well-formed formula.

Since the interpretation of the second-order quantifiers will not be the usual one, we use the symbol *aa* to denote the second-order universal quantifier and the symbol *stat* to denote the second-order existential quantifier. The *aa* quantifier will be interpreted to mean "for almost all countable subsets," while *stat* means "for stationary many countable subsets." The *aa*-satisfaction relation for a structure M is defined by recursion in the usual way, except for the nonstandard interpretation of the second-order quantifiers. Let us now say more precisely what this interpretation is.

Suppose $\varphi(\overline{x}, \overline{X}, Y)$ is a formula in *aa*-logic and, by induction, that for all $\overline{a} \in M^{<\omega}$, $\overline{\sigma} \in ([M]^{\aleph_0})^{<\omega}$, and $\tau \in [M]^{\aleph_0}$, we have defined whether $M \vDash \varphi(\overline{a}, \overline{\sigma}, \tau)$. We then define $M \vDash aaY\, \varphi(\overline{a}, \overline{\sigma}, Y)$ if the set of $\tau \in [M]^{\aleph_0}$ such that $M \vDash \varphi(\overline{a}, \overline{\sigma}, \tau)$ contains a closed unbounded subset of $[M]^{\aleph_0}$. We define $stat\, Y\, \varphi$ to have the same truth value as $\neg(aaY(\neg\varphi))$.[1] The other steps in the recursive definition of satisfaction are as expected.

We will consider two-sorted structures $\mathcal{M}$ with a universe of first-order objects M and a distinguished collection $P^{\mathcal{M}}$ of countable subsets of M. (Often $P = \emptyset$.) For such a structure $\mathcal{M}$, we let $\mathcal{L}_{aa}(\mathcal{M})$ denote the set of *aa*-formulas in the language of $\mathcal{M}$ allowing parameters from M and $P^{\mathcal{M}}$. The *aa-satisfaction predicate of* $\mathcal{M}$ is then defined to be the set

$$\text{Sat}_{aa}(\mathcal{M}) = \{\psi \in \mathcal{L}_{aa}(\mathcal{M}) : \mathcal{M} \vDash \psi\}$$

The role of $P^{\mathcal{M}}$ here is to restrict the second-order parameters under consideration; the interpretation of quantifiers is unchanged. Our convention is that if we have not specified $P^{\mathcal{M}}$, then $P^{\mathcal{M}} = \emptyset$. If moreover $\mathcal{M}$ is a transitive structure containing V_ω

[1] In fact, we will never use the "stationary many" quantifier in this paper.

and $\mathcal{M}$ closed under the Quine-Rosser pairing function,[2] we will identify $\mathrm{Sat}_{aa}(\mathcal{M})$ with a subset of $\mathcal{M}$.

2.2 The inner model $C(aa)$

We now define the main object of study of this paper: $C(aa)$, the smallest inner model of ZF that is closed under the aa-satisfaction predicate.

By transfinite recursion, we define for each infinite ordinal α a structure $\mathcal{C}_\alpha = (C_\alpha, \in, T_\alpha)$ where C_α is a transitive set and T_α is a binary predicate on C_α. This sequence of structures is the *$C(aa)$ hierarchy.*

We begin by letting $C_\omega = V_\omega$. Once the structure $\mathcal{C}_\alpha$ has been defined, we will let $C_{\alpha+1}$ be the set of subsets of C_α definable over $\mathcal{C}_\alpha$ in aa-logic. In general, if $\gamma > \omega$ is any infinite limit ordinal and C_β has been defined for all $\beta < \gamma$, we will let $C_\gamma = \bigcup_{\omega \leq \beta < \gamma} C_\beta$.

Finally, let us explain how to define the structure $\mathcal{C}_\alpha$, having already defined the set C_α and the structures $\mathcal{C}_\beta$ for every infinite $\beta < \alpha$. We simply set

$$T_\alpha = \{(\beta, \psi) : \beta < \alpha \text{ and } \psi \in \mathrm{Sat}_{aa}(\mathcal{C}_\beta)\}$$

and then $\mathcal{C}_\alpha = (C_\alpha, \in, T_\alpha)$.

The inner model $C(aa)$ is the proper class union of the sets C_α for all infinite ordinals α. By construction, for any structure $\mathcal{M} \in C(aa)$, $\mathrm{Sat}_{aa}(\mathcal{M}) \in C(aa)$. On the other hand, suppose that N is an inner model such that for all structures $\mathcal{M} \in N$, $\mathrm{Sat}_{aa}(\mathcal{M})$ belongs to N, and let us show that $C(aa) \subseteq N$.

This is proved by transfinite induction. Suppose ξ is an infinite ordinal and for all infinite $\beta < \xi$, $\mathcal{C}_\beta \in N$. We will show that $\mathcal{C}_\xi \in N$. If $\xi = \alpha+1$ is a successor ordinal, this is easy: C_ξ is the set of aa-definable subsets of $\mathcal{C}_\alpha$, which belongs to N, and $T_{\alpha+1}$ can be computed in N from $\mathrm{Sat}_{aa}(\mathcal{C}_\alpha)$, which belongs to N. So $\mathcal{C}_\xi = (C_{\alpha+1}, \in, T_{\alpha+1})$ belongs to N. If ξ is a limit ordinal, we automatically have that for all $\beta < \xi$, $\mathcal{C}_\beta \in V_\xi \cap N$. Let $M = (V_\xi \cap N, \in)$. Then $\mathcal{C}_\xi$ can be computed in N using $\mathrm{Sat}_{aa}(M)$, and hence $\mathcal{C}_\xi \in N$.

2.3 Club determinacy

The key assumption behind the results of this paper is called *club determinacy.* A structure M is *club determined* if for all formulas $\varphi \in \mathcal{L}(M)$, either $M \vDash aaX\, \varphi$ or $M \vDash aaX\, \neg\varphi$. In other words, every subset of $[M]^{\aleph_0}$ definable in aa-logic is measurable with respect to the closed unbounded filter.

The statement that a structure is club determined can be expressed as a scheme in aa-logic, which we call the *club determinacy scheme.* The *club determinacy principle* states that for all ordinals α, the structure $\mathcal{C}_\alpha$, the α-th level of the $C(aa)$ hierarchy, is club determined. This principle permits the analysis of $C(aa)$ we will carry out in this paper. Fortunately, it follows from large cardinal hypotheses:

Theorem 2.1 (Kennedy-Magidor-Vanaanen)**.** *Assume there is a proper class of Woodin cardinals. Then club determinacy holds.*

2.4 Potential aa-premice

A *potential aa-premouse* is a transitive structure $\mathcal{M} = (M, \in, T, P, S)$ where M is a transitive set, P is a family of subsets of M, T is a binary relation on M, and S is a

[2]The details of the pairing function are not so important. Though we will suppress the details, what is important is that the function is uniformly Δ_1-definable over transitive structures and for sets x, y of rank less at most α, $\langle x, y\rangle$ has rank at most $1 + \alpha$.

complete consistent theory in aa-logic extending the elementary diagram of $(M, \in, T, P)$ and containing the club determinacy scheme.[3] Given such a potential aa-premouse $\mathcal{M}$, we will denote its predicates by $T^{\mathcal{M}}$, $P^{\mathcal{M}}$, and $S^{\mathcal{M}}$.

We establish some conventions to make the theory of aa-mice smoother. The *language of aa-mice*, denoted $\mathcal{L}_{aa}$, is the set of formulas in aa-logic in the vocabulary of set theory with an additional binary predicate. If $\mathcal{M}$ is a potential aa-premouse, $\varphi \in \mathcal{L}_{aa}$, and $\overline{a} \subseteq \mathcal{M}$ and $\overline{\sigma} \subseteq P^{\mathcal{M}}$ are finite sequences of parameters, we write

$$\mathcal{M} \vDash \varphi(\overline{a}, \overline{\sigma})$$

to abbreviate the statement that $\varphi(\overline{a}, \overline{\sigma}) \in S^{\mathcal{M}}$. We will write $\mathcal{L}_{aa}(\mathcal{M})$ to denote the set of formulas in $\mathcal{L}_{aa}$ with parameters from $\mathcal{M}$ and $P^{\mathcal{M}}$.

If $\mathcal{M}$ is a potential aa-premouse, an embedding $\pi : \mathcal{M} \to \mathcal{N}$ is an *aa-elementary embedding* if for all formulas φ in the language of aa-mice, for all finite sequences $\overline{a} \subseteq \mathcal{M}$ and $\overline{\sigma} \in P^{\mathcal{M}}$, $\mathcal{M} \vDash \varphi(\overline{a}, \overline{\sigma}) \iff \mathcal{N} \vDash \varphi(\pi(\overline{a}), \pi(\overline{\sigma}))$.

2.5 aa-premice

For each potential aa-premouse $\mathcal{N} \in \mathcal{M}$, define

$$\text{Sat}^{\mathcal{M}}_{aa}(\mathcal{N}) = \{\psi \in \mathcal{L}_{aa}(\mathcal{N}) : \mathcal{M} \vDash \psi^{\mathcal{N}}\}$$

One can run the construction of Section 2.2 but replacing the aa-satisfaction predicate with S. This produces a sequence of structures $\langle \mathcal{C}^{\mathcal{M}}_\alpha : \alpha < \nu \rangle$ for some ordinal $\nu \leq o(M)$. The recursive definition terminates as soon as $\mathcal{C}^{\mathcal{M}}_\alpha \notin M$. We say $\mathcal{M}$ is an *aa-premouse* if $\nu = o(M)$, $M = \bigcup_{\alpha<\nu} C^{\mathcal{M}}_\alpha$, and

$$T = \{(\beta, \psi) : \beta < o(\mathcal{M}) \text{ and } \psi \in \text{Sat}^{\mathcal{M}}_{aa}(\mathcal{C}^{\mathcal{M}}_\beta)\}$$

Note that there is a recursive set of axioms $A \subseteq \mathcal{L}_{aa}$ such a potential aa-premouse $\mathcal{M}$ is an aa-premouse if and only if $\mathcal{M} \vDash A$.

The simplest example of an aa-premouse is a level $\mathcal{C}_\alpha$ of $C(aa)$. This can be viewed as an aa-premouse $(C_\alpha, \in, T_\alpha, \emptyset, S)$ where S is the aa-satisfaction predicate of $(C_\alpha, \in, T_\alpha)$. More generally, for any collection $P \subseteq [C_\alpha]^\omega$, $(C_\alpha, \in, T_\alpha, P, S)$ is an aa-premouse where S is the aa-satisfaction predicate of $(C_\alpha, \in, T_\alpha, P)$.

The completeness theorem for aa-logic [1] yields a method for embedding an aa-premouse into a level of the aa-hierarchy. This involves iterating the *aa-ultrapower construction*, which we now describe.

2.6 The aa-ultrapower construction

Suppose $\mathcal{M}$ is an aa-premouse. Let

$$p(X) = \{\psi(X) : \mathcal{M} \vDash aaX\, \psi(X)\}$$

where $\psi(X) \in \mathcal{L}_{aa}(\mathcal{M})$. Then $p(X)$ is a complete consistent type over $\mathcal{M}$, and since $\mathcal{M}$ has definable Skolem functions, $\mathcal{M}$ has a minimum extension $\mathcal{M}_*$ that realizes this type.

More precisely, there is a (possibly ill-founded) structure $\mathcal{M}_*$ such that $\mathcal{M}$ is an aa-elementary substructure of $\mathcal{M}_*$, $P^{\mathcal{M}_*} = P^{\mathcal{M}} \cup \{M\}$, and $\mathcal{M}_* \vDash p(M)$. This structure $\mathcal{M}_*$ is minimal in the sense that for any other $\mathcal{N}$ that aa-elementarily extends $\mathcal{M}$,

[3]The *elementary diagram* of $(M, \in, T, P)$ is the set of *first-order* formulas with parameters in M and P true in this structure. We do not include formulas that quantify over P.

if $\sigma \in P^{\mathcal{N}}$ realizes the type $p(X)$, then there is a unique aa-elementary embedding $\pi : \mathcal{M}_* \to \mathcal{N}$ such that $\pi \restriction M$ is the identity and $\pi(M) = \sigma$.[4]

The structure $\mathcal{M}_*$ can be constructed as a collection of "formal functions" modulo the aa-ultrafilter coded by $S^{\mathcal{M}}$. That is, for each formula $\psi(X, y) \in \mathcal{L}_{aa}(\mathcal{M})$, we introduce a formal symbol f_ψ, which we think of intuitively as denoting a partial function which takes the value y at X if and only if there is a unique $y \in M$ such that $\psi(X, y)$ holds.

We let $X = \{f_\psi : \mathcal{M} \vDash aaX\, \exists! y\, \psi(X, y)\}$. Then $M_* = X/{=^*}$ where $=^*$ is the equivalence relation on X defined by $f_\varphi = f_\psi$ if $\mathcal{M} \vDash aaX\, f_\varphi(X) = f_\psi(X)$.[5] We identify the point $[f_{y=a}] \in M_*$ with the point $a \in M$ so that $M \subseteq M_*$. The rest of the construction is straightforward; see [11]. The key thing is that $\mathcal{M}_*$ is an aa-elementary extension of $\mathcal{M}$ that realizes the type $p(X)$ in a minimal way, and moreover, that the set satisfying the type $p(X)$ in $\mathcal{M}_*$ is M itself.

This minimum structure $\mathcal{M}_*$ is called the *aa-ultrapower of* $\mathcal{M}$. If it is well-founded, we identify it with its transitive collapse, so that instead of an elementary extension, we have an aa-elementary embedding

$$\pi : \mathcal{M} \to \mathcal{M}_*$$

With this change of perspective, the new element of $P^{\mathcal{M}_*}$ is not M but $\pi[M]$. Moreover, $\mathcal{M}_*$ is itself an aa-premouse, and we can iterate the aa-ultrapower construction.

2.7 aa-mice

By starting with $\mathcal{M}_0 = \mathcal{M}$ and taking aa-ultrapowers at successor stages and direct limits at limit stages, we produce a sequence of structures $\langle \mathcal{M}_\alpha : \alpha \leq \gamma \rangle$ and a commuting system of elementary embeddings $\pi_{\alpha\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$. Here γ denotes the least countable ordinal such that $\mathcal{M}_\gamma$ is ill-founded if such an ordinal exists, or else $\gamma = \omega_1$. If $\gamma = \omega_1$, the aa-premouse $\mathcal{M}$ is said to be *iterable*; an iterable aa-premouse is called an *aa-mouse*.

The key result on aa-mice that will be exploited in this paper is the following aa-comparison theorem [11]:

Theorem 2.2 (Kennedy-Magidor-Vanaanen)**.** *If $\mathcal{M}$ is a countable aa-mouse and*

$$\mathcal{M}_{\omega_1} = (M_{\omega_1}, \in, T_{\omega_1}, P_{\omega_1}, S_{\omega_1})$$

is the direct limit of its aa-iteration, then for some infinite ordinal α, $(M_{\omega_1}, \in, T_{\omega_1}) = \mathcal{C}_\alpha$ and $S_{\omega_1} = \mathrm{Sat}_{aa}(\mathcal{C}_\alpha, P_{\omega_1})$. □

3 Countably complete ultrafilters

For each aa-premouse $\mathcal{M}$, let $\kappa_{\mathcal{M}}$ denote the critical point of the aa-ultrapower of $\mathcal{M}$. This is the least ordinal $\alpha \leq o(\mathcal{M})$ such that $\mathcal{M} \vDash aa\sigma\, \alpha \nsubseteq \sigma$. The following is a slight improvement of [11, Lemma 5.29].

Lemma 3.1. *If $\mathcal{M}$ is an aa-premouse and $\mathcal{N}$ is its aa-ultrapower, then for all ordinals $\gamma < \kappa_{\mathcal{M}}$, $P^{\mathcal{M}}(\gamma) = P^{\mathcal{N}}(\gamma)$.*

Proof. Fix a formula $\varphi(X, v)$, allowing first-order and second-order parameters from $\mathcal{M}$ such that $\mathcal{M} \vDash aaX\, \exists! v\, \varphi(X, v)$ and $\mathcal{M} \vDash aaX\, \forall v(\varphi(X, v) \to v \subseteq \gamma)$. Let $f = f_\varphi$. We will show that $[f] \in \mathcal{M}$.

[4]Here we must extend our conventions on aa-premice to the possibly ill-founded structures $\mathcal{M}_*$ and $\mathcal{N}$.

[5]Formally this means $\mathcal{M} \vDash aaX\, \forall y\, \varphi(X, y) \leftrightarrow \psi(X, y)$.

First, let us show that $\mathcal{M} \vDash aaX\, aaY\, f(X) = f(Y)$. Suppose not. Then

$$\mathcal{M} \vDash aaX\, aaY\, \exists \nu < \gamma\, f(X)(\nu) \neq f(Y)(\nu)$$

(Here we identify the set $f(X)$ with its characteristic function.) Using Fodor's lemma and club determinacy, we can fix some $\nu < \gamma$ such that

$$\mathcal{M} \vDash aaX\, aaY\, f(X)(\nu) \neq f(Y)(\nu)$$

contradicting that there is some $i \in \{0, 1\}$ such that $\mathcal{M} \vDash aaX\, f(X)(\nu) = i$.

Since $\mathcal{M} \vDash aaX\, aaY\, f(X) = f(Y)$ and $\mathcal{M} \vDash aaX\, aay f(X) \in Y$, it follows that $\mathcal{M} \vDash aaY\, f(Y) \in Y$. It follows that there is some $T \in \mathcal{M}$ such that $\mathcal{M} \vDash aaY\, f(Y) = T$. Hence $[f] = T \in \mathcal{M}$, as desired. □

Corollary 3.2. *If $\mathcal{M}$ is an aa-premouse and $\mathcal{N}$ is an aa-iterate of $\mathcal{M}$, then for all ordinals $\gamma < \kappa_{\mathcal{M}}$, $P^{\mathcal{M}}(\gamma) = P^{\mathcal{N}}(\gamma)$.* □

Proposition 3.3. *Suppose $\mathcal{M}_0$ and $\mathcal{M}_1$ are finitely generated aa-mice whose underlying sets are models of* ZFC. *Either $\mathcal{M}_0 \in \mathcal{M}_1$, $\mathcal{M}_1 \in \mathcal{M}_0$, or $\mathcal{M}_0$ and $\mathcal{M}_1$ iterate into the same level of $C(aa)$.*

Proof. Let $\pi_0 : \mathcal{M}_0 \to \mathcal{C}_{\nu_0}$ and $\pi_1 : \mathcal{M}_1 \to \mathcal{C}_{\nu_1}$ denote the *aa*-iterations of $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively. By symmetry, it suffices to show that if $\nu_0 < \nu_1$, then $\mathcal{M}_0 \in \mathcal{M}_1$.

Assume therefore that $\nu_0 < \nu_1$. Then $(\mathcal{C}_{\nu_0}, \mathrm{Sat}_{aa}(\mathcal{C}_{\nu_0})) \in \mathcal{C}_{\nu_1}$. Fix $p \in \mathcal{M}_0$ such that every element of $\mathcal{M}_0$ is *aa*-definable in $\mathcal{M}_0$ from p. Then the *aa*-definable hull H of $\pi_0(p)$ in $\mathcal{C}_{\nu_0}$ belongs to $\mathcal{C}_{\nu_1}$ and is countable there. Since the underlying set of $\mathcal{C}_{\nu_1}$ satisfies ZFC, the transitive collapse of this hull, which is equal to $\mathcal{M}_0$, also belongs to $\mathcal{C}_{\nu_1}$, and it is also countable there. Since $P^{\mathcal{M}_1}(\omega) = P^{\mathcal{C}_{\nu_1}}(\omega)$ by Corollary 3.2, it follows that $\mathcal{M}_0 \in \mathcal{M}_1$, as desired. □

To establish that $C(aa)$ satisfies the Ultrapower Axiom, we will prove that certain forms of Woodin's Weak Comparison Principle hold in levels of $C(aa)$.

An elementary embedding $i : \mathcal{M} \to \mathcal{N}$ is a *close embedding* if for all $S \in \mathcal{N}$, $i^{-1}[S] \in \mathcal{M}$. An *aa*-premouse $\mathcal{M}$ is *finitely generated* if there is some $x \in \mathcal{M}$ such that every element of $\mathcal{M}$ is *aa*-definable in $\mathcal{M}$ from x.

Definition 3.4. An *aa*-premouse $\mathcal{M}$ satisfies *Weak Comparison* if for any two finitely generated *aa*-premice $\mathcal{M}_0$ and $\mathcal{M}_1$ that *aa*-elementarily embed into $\mathcal{M}$, one of the following holds:

- $\mathcal{M}_0 \in \mathcal{M}_1$.
- $\mathcal{M}_1 \in \mathcal{M}_0$.
- For some *aa*-premouse $\mathcal{N}$, there are close *aa*-elementary embeddings $j_0 : \mathcal{M}_0 \to \mathcal{N}$ and $j_1 : \mathcal{M}_1 \to \mathcal{N}$.

There is a somewhat serious issue in using Proposition 3.3 to show that Weak Comparison holds in levels of $C(aa)$. The problem is that close embeddings must be cofinal, while *aa*-iterations often are not. The fact that close embeddings are cofinal follows from the definition for a somewhat trivial reason, so it might seem that one should modify closeness in the context of *aa*-mice as follows:

Lemma 3.5. *Suppose $\mathcal{M}$ is an aa-premouse and $\pi : \mathcal{M} \to \mathcal{N}$ is an aa-iteration of $\mathcal{M}$. If $S \subseteq \mathcal{N}$ is aa-definable over $\mathcal{N}$ from parameters, then $\pi^{-1}[S]$ is aa-definable over $\mathcal{M}$ from parameters.*

Proof. It suffices to prove this when $\pi : \mathcal{M} \to \mathcal{N}$ is the *aa*-ultrapower of $\mathcal{M}$. In this case, S is *aa*-definable over $\mathcal{N}$ from the second-order parameter $\pi[M]$ and some other (first-order and second-order) parameters in the range of π. Say $S = \{a \in \mathcal{N} : \mathcal{N} \vDash \psi(a, \pi[M], \pi(\vec{p}), \pi(\vec{\tau}))\}$. For $a \in \mathcal{M}$, we will have $\pi(a) \in S$ if and only if $\mathcal{M} \vDash aaX\, \psi(a, X, \vec{p}, \vec{\tau})$, and so $\pi^{-1}[S]$ is *aa*-definable over $\mathcal{M}$. □

A theorem of Enayat [2, Theorem 2.2] states that if $\mathcal{M}$ is a model of ZFC, then an elementary embedding $\pi : \mathcal{M} \to \mathcal{N}$ is close to $\mathcal{M}$ if and only if for all $A \subseteq \mathcal{N}$ definable over $\mathcal{N}$ from parameters, $\pi^{-1}[A]$ is definable over $\mathcal{M}$ from parameters.

It seems vital in applications of Weak Comparison, however, that the embeddings be close in the usual sense, not in the sense of Lemma 3.5; in particular, we need them to be cofinal. (For example, see the proof of Theorem 3.12.) Our solution to this issue is to note that certain types of *aa*-mice do have cofinal *aa*-ultrapowers. An *aa*-premouse $\mathcal{M}$ is *ω-cofinal* if

$$\mathcal{M} \vDash aa\sigma\, \forall x\, \exists y \in \sigma\, x \in y$$

Note that $\mathcal{C}_\nu$ is ω-cofinal if and only if $\mathrm{cf}(\nu) = \omega$, while a Skolem argument shows that there are countable *aa*-mice that are not ω-cofinal.

Lemma 3.6. *If $\mathcal{M}$ is an ω-cofinal aa-premouse, then all aa-iterations of $\mathcal{M}$ give rise to cofinal embeddings.* □

To obtain close embeddings, we impose one further property. We say an *aa*-premouse $\mathcal{M}$ *satisfies* ZFC if it satisfies the separation and replacement schema for *aa*-formulas.

Lemma 3.7. *If $\mathcal{M}$ is an ω-cofinal aa-premouse that satisfies* ZFC*, then all aa-iterations of $\mathcal{M}$ give rise to close embeddings.*

Proof. This follows immediately from Lemma 3.5 and Lemma 3.6. □

Theorem 3.8. *Suppose λ is an ordinal of countable cofinality such that $\mathcal{C}_\lambda \vDash$* ZFC*. Then $\mathcal{C}_\lambda$ satisfies Weak Comparison.*

Proof. This is obtained by combining Proposition 3.3 and Lemma 3.7. □

For Theorem 3.8 to be useful, we must show that there exist some levels of $C(aa)$ to which it applies. Say an ordinal λ is *good* if $\mathrm{cf}(\lambda) = \omega$, $C_\lambda = V_\lambda \cap C(aa)$, and λ is strongly inaccessible in $C(aa)$. The following lemma implies the existence of a proper class of good ordinals:

Lemma 3.9. *Suppose κ is a countably closed regular cardinal. Then the set of good ordinals less than κ contains an ω-club subset of κ.*

Recall that a cardinal κ is *countably closed* if for all $\gamma < \kappa$, $\gamma^{\aleph_0} < \kappa$. There is a proper class of countably closed regular cardinals: namely, the cardinals of the form $\beth_{\alpha+1}$ where α is either 0, a successor ordinal, or an ordinal of uncountable cofinality.

The following argument is due to Magidor:

Proposition 3.10. *If κ is a countably closed cardinal, then $C_\kappa = C(aa) \cap V_\kappa$.*

Proof. It suffices to show that every $x \in C(aa) \cap V_\kappa$ belongs to C_κ. Fix $\gamma > \kappa$ such that $x \in C(aa)^{V_\gamma}$. Let H be a countably closed elementary substructure of V_γ such that $\mathrm{tc}(x \cup \{x\}) \subseteq H$ and $|H| < \kappa$. Let M be the transitive collapse of H. By elementarity, $x \in C(aa)^M$, since x collapses to itself and $x \in C(aa)^{V_\gamma}$. Since M is countably closed, club determinacy implies $C(aa)^M = C_{o(M)}$. Thus $x \in C_\kappa$, as desired. □

Proof of Lemma 3.9. Let F denote the ω-club filter on κ. By club determinacy, $U = F \cap C(aa)$ is a normal ultrafilter in $C(aa)$. It follows that U concentrates on the set I of all ordinals $\lambda < \kappa$ such that λ is strongly inaccessible in $C(aa)$. On the other hand, since $C_\kappa = V_\kappa \cap C(aa)$, $S = \{\lambda < \kappa : C_\lambda = V_\lambda \cap C(aa)\}$ is closed unbounded. Since $S \in C(aa)$, it follows that $S \in U$. Thus the set $S \cap I$ of good ordinals less than κ belongs to U. Since $U \subseteq F$, this means that the set of good ordinals less than κ contains an ω-club. □

If $\mathcal{M}$ is an aa-premouse that satisfies ZFC and $\mathcal{M}$ satisfies that U is a countably complete ultrafilter, the ultrapower of $\mathcal{M}$ by U, denoted $\mathrm{Ult}(\mathcal{M}, U)$, is the aa-premouse defined by the usual ultrapower construction. (One has to shift the aa-satisfaction predicate of $\mathcal{M}$ to a predicate on $\mathrm{Ult}(\mathcal{M}, U)$. This is possible since $\mathcal{M}$ satisfies ZFC.) If $\mathcal{M}$ is finitely generated, then so is $\mathrm{Ult}(\mathcal{M}, U)$.

The following "realizability lemma" is proved by the standard argument, which is originally due to Jensen; for example, see [8, Theorem 10.3]:

Lemma 3.11. *Suppose that in $C(aa)$, $\mathcal{M}$ is a countable aa-premouse satisfying* ZFC, *and $P^{\mathcal{M}} = \emptyset$. Suppose $\pi : \mathcal{M} \to \mathcal{C}_\lambda$ is an aa-elementary embedding that belongs to $C(aa)$. If $\mathcal{M}$ satisfies that U is a countably complete ultrafilter, then there is an aa-elementary embedding $\pi' : \mathrm{Ult}(\mathcal{M}, U) \to \mathcal{C}_\lambda$ such that $\pi' \circ j_U = \pi$.* □

Theorem 3.12. *$C(aa)$ satisfies the Ultrapower Axiom.*

Proof. The proof is very similar to the proof of the proof of [4, Theorem 2.3.10], which shows that assuming $V = \mathrm{HOD}$, a slightly different formulation of Weak Comparison implies the Ultrapower Axiom. The difference here is that we cannot assume $V = \mathrm{HOD}$, since this is an open question. This does not cause any real difficulties, since each level of $C(aa)$ has definable Skolem functions, which is all that is really needed for [4, Theorem 2.3.10]. For the reader's convenience, we repeat this argument with the necessary adjustments.

Fix a good ordinal λ. By Theorem 3.8, $\mathcal{C}_\lambda$ satisfies weak comparision. Since $\mathcal{C}_\lambda$ has definable Skolem functions, there is a pointwise definable aa-mouse $\mathcal{M}$ that is aa-elementarily embeddable into $\mathcal{C}_\lambda$.

Suppose $U \in \mathcal{M}$ is a countably complete ultrafilter. By Lemma 3.11, there is an aa-elementary embedding from $\mathrm{Ult}(\mathcal{M}, U)$ to $\mathcal{C}_\lambda$.

Suppose now that $\mathcal{M}$ satisfies that U_0 and U_1 are countably complete ultrafilters. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be the ultrapowers of $\mathcal{M}$ associated to U_0 and U_1 respectively. Then $\mathcal{M}_0$ and $\mathcal{M}_1$ are finitely generated and aa-embeddable in $\mathcal{C}_\lambda$.

Since $o(\mathcal{M}_0) = o(\mathcal{M}_1)$, neither model belongs to the other, and therefore Weak Comparison (Theorem 3.8) yields close embeddings $j_0 : \mathcal{M}_0 \to \mathcal{N}$ and $j_1 : \mathcal{M}_1 \to \mathcal{N}$. By replacing $\mathcal{N}$ with the elementary hull of $j_0[\mathcal{M}_0] \cup j_1[\mathcal{M}_1]$ inside of $\mathcal{N}$, one can assume j_0 and j_1 are internal ultrapower embeddings, which shows that the Ultrapower Axiom holds for U_0 and U_1. (Here it seems essential that j_0 and j_1 are cofinal embeddings.) The details appear in the proof of [3, Theorem 2.3.10].

Since UA is equivalent to a Π_2-sentence and $C(aa) \cap V_\lambda$ satisfies UA for a proper class of λ, $C(aa)$ satisfies UA. □

Theorem 3.13. *If $C(aa) \vDash U$ is countably complete, then in V, U extends to a countably complete filter.*

To prove this, we need the following lemma, which says that any element of an iterated aa-ultrapower has a preimage in a finite aa-iteration. This is a common pattern in the theory of iterated ultrapowers, and the proof is essentially the standard one.

Lemma 3.14. *Suppose $\mathcal{M}$ is an aa-mouse, $j : \mathcal{M} \to \mathcal{Q}$ is an iterated aa-ultrapower. For any finite sets $\overline{a} \subseteq \mathcal{Q}$ and $\overline{\sigma} \subseteq P^{\mathcal{Q}}$, Q in a finite signature, there is a finite iterated aa-ultrapower $i : \mathcal{M} \to \mathcal{N}$ and an aa-elementary embedding $k : \mathcal{N} \to \mathcal{Q}$ such that $k \circ i = j$ and $\overline{a}, \overline{\sigma} \subseteq \mathrm{ran}(k)$.* □

We will also use the following lemma, which is due to Woodin [12, Lemma 5.32] in a slightly different context:

Lemma 3.15. *Suppose $\mathcal{M}$ is a finitely generated aa-premouse and $j_0, j_1 : \mathcal{M} \to \mathcal{N}$ are close aa-elementary embeddings. Then $j_0 = j_1$.* □

The argument in the proof of Theorem 3.13 below is based on an unpublished idea of Woodin [12] for showing that the HOD-ultrafilter conjecture holds granting the HOD analysis.

Proof of Theorem 3.13. Fix a good ordinal λ. We claim that for any countably complete ultrafilter $U \in \mathcal{C}_\lambda$, there is a finite aa-iteration $i : \mathcal{C}_\lambda \to \mathcal{N}$ and some $a \in \mathcal{N}$ such that $U = \{A \in P^{\mathcal{C}_\lambda}(X) : a \in i(A)\}$. Then the fact that U extends to a countably complete filter in V follows from countable completeness of the closed unbounded filter and its iterated products.

To prove the claim, fix such a countably complete ultrafilter $U \in \mathcal{C}_\lambda$, and let $\mathcal{M}$ be a finitely generated aa-mouse admitting an aa-elementary embedding $\pi : \mathcal{M} \to \mathcal{C}_\lambda$ with $U \in \operatorname{ran}(\pi)$. It suffices to show that $W = \pi^{-1}[U]$ is of the form $\{A \in P^{\mathcal{M}}(X) : a \in i(A)\}$ for some finite aa-iteration $i : \mathcal{M} \to \mathcal{N}$ and some $a \in \mathcal{N}$: then the same holds of U in $\mathcal{C}_\lambda$ by aa-elementarity.

First note that as in Theorem 3.12, for some ordinal α there are close embeddings $j : \mathcal{M} \to \mathcal{C}_\alpha$ and $\ell : \operatorname{Ult}(\mathcal{M}, W) \to \mathcal{C}_\alpha$. We have $\ell \circ j_W = j$ by the uniqueness of close embeddings, and so $U = \{A \in P^{\mathcal{M}}(X) : \ell([\mathrm{id}]_W) \in j(A)\}$. Now by Lemma 3.14, there is a finite iterated aa-ultrapower $i : \mathcal{M} \to \mathcal{N}$ and an aa-elementary embedding $k : \mathcal{N} \to \mathcal{C}_\alpha$ such that $k \circ i = j$, and $\ell([\mathrm{id}]_W) \in \operatorname{ran}(k)$. Thus $U = \{A \in P^M(X) : a \in i(A)\}$ where $a = k^{-1}(\ell([\mathrm{id}]_W))$. $\square$

Corollary 3.16. *The least measurable cardinal of $C(aa)$ is ω_1.* $\square$

If S is a set of ordinals, a sequence of ultrafilters $\langle U_\alpha \rangle_{\alpha \in S}$ is *regressive* if for all $\alpha \in S$, U_α is an ultrafilter on α. If δ is an ordinal and U and W are countably complete ultrafilters on δ, then U lies below W in the *Ketonen order*, denoted $U <_{\Bbbk} W$, if for some $S \in W$, there is a regressive sequence $\langle U_\alpha \rangle_{\alpha \in S}$ of countably complete ultrafilters such that $U = W\text{-}\lim_{\alpha \in S} U_\alpha$; that is, $A \in U$ if $A \cap \alpha \in U_\alpha$ for W-almost all $\alpha \in S$. (One can always assume $S = \kappa \setminus \{0\}$.)

If U is a subset of $P(\delta)$ and σ is a countable subset of $P(\delta)$, then $A_U(\sigma) = \bigcap_{A \in \sigma \cap U} A$ and, if $A_U(\sigma) \neq \emptyset$, $\chi_U(\sigma) = \min(A_U)$. By Theorem 3.13, if U is a countably complete ultrafilter in $C(aa)$, then $\chi_U(\sigma)$ is defined for almost all $\sigma \subseteq P(\delta)$.

Theorem 3.17. *Suppose δ is an ordinal and in $C(aa)$, U and W are countably complete ultrafilters on δ. Then $C(aa)$ satisfies $U <_{\Bbbk} W$ if and only if $aa\sigma \subseteq P(\delta)\, \chi_U(\sigma) < \chi_W(\sigma)$.*

Proof. Suppose $C(aa)$ satisfies $U <_{\Bbbk} W$ and let $\langle U_\alpha \rangle_{\alpha \in S}$ witness this. The key observation is that

$$aa\sigma \subseteq P(\delta)\, \forall A \in \sigma\, (A \in U \iff A \cap \chi_W(\sigma) \in U_{\chi_W(\sigma)})$$

To see this, let $h : P(\delta) \to P(\delta)$ be the function $h(A) = \{\alpha \in S : A \cap \alpha \in U_\alpha\}$. If $S \in \sigma$ and σ is closed under h, then for all $A \in \sigma$,

$$A \in U \iff h(A) \in W \iff \chi_W(\sigma) \in h(A) \iff A \cap \chi_W(\sigma) \in U_{\chi_W(\sigma)}$$

If σ is such that for all $A \in \sigma$, $A \in U \iff A \cap \chi_W(\sigma) \in U_{\chi_W(\sigma)}$, then $\chi_U(\sigma) < \chi_W(\sigma)$. This implies the theorem. $\square$

4 The Generalized Continuum Hypothesis

For any cardinal μ, let $u_2(\mu) = \sup\{\mu^{+L[A]} : A \in H_\mu\}$. We will need to apply the following theorem in $C(aa)$:

Theorem 4.1. *Assume there is no inner model with a Woodin cardinal and for every set A, there is an iterable model M containing A with two measurable cardinals above* $\sup(A)$. *Then for each regular cardinal $\mu \geq \omega_2$, $u_2(\mu) < \mu^+$.*

Proof. Fix μ, and let

$$\mathcal{M} = \{N \in H_\mu \mid N \text{ is a properly 1-small mouse}\}.$$

Let $<^*$ be the mouse order on $\mathcal{M}$.

For any particular $N \in \mathcal{M}$, $|N|_{<^*} < \mu^+$, because $<^*$ is contained in the mouse order on mice that are countable in $V[g]$, for g generic on $\text{Col}(\omega, <\mu)$, and in $V[g]$, the mouse order on countable $P <^* N$ is a $\Sigma^1_2(N)$ wellfounded relation.

Since $\mu \geq \omega_2$, $K|\mu$ is universal for mice of height $\leq \mu$, by [7, Theorem 3.4]. It follows that if $N \in \mathcal{M}$, then $N <^* K|\xi$ for some $\xi < \mu$. Thus $<^*$ has cofinality μ, and hence

$$|<^*| < \mu^+.$$

Thus it suffices to show that $u_2(\mu) \leq |<^*|$. For that, fix some bounded subset A of μ. We need to see that $\mu^{+L[A]} < |<^*|$. By hypothesis, there is a transitive $P_0 \models \text{ZFC}$ such that $A \in P_0$, $|P_0| < \mu$, and P_0 has two measurable cardinals above $\sup(A)$. Let P come from iterating P_0 at its bottom measurable μ times, so μ is measurable in P and $A \in H^P_\mu$.

Now by the proof of [10, Theorem 7.9], carried out inside P,

$$P \models u_2(\mu) \leq |<^*|.$$

(See the proof that "$u_2 \leq \delta$" starting on the bottom of p. 68.) But

$$\mu^{+L[A]} < u_2(\mu)^P \leq |<^*|^P \leq |<^*|.$$

Thus $\mu^{+L[A]} < |<^*|$, as desired. □

Lemma 4.2. *If $\mathcal{C}_\gamma$ is a model of* ZFC *and δ is a successor cardinal of M, then any close embedding $j : \mathcal{M} \to \mathcal{N}$ is continuous at δ.*

Proof. If j is discontinuous at δ, then the M-ultrafilter U on δ derived from j using $\sup j[\delta]$ belongs to M. Thus in $\mathcal{M}$ there is a countably complete uniform ultrafilter on the successor cardinal δ. (In the terminology of [3], δ is a *Fréchet cardinal.*) By the theory of supercompactness under UA [3, Corollary 7.4.10], it follows that in $\mathcal{M}$, there is a cardinal $\kappa < \delta$ that is δ-supercompact. This contradicts that $\mathcal{M}$ does not have an inner model with a Woodin cardinal. □

Theorem 4.3. *The inner model $C(aa)$ satisfies the Generalized Continuum Hypothesis.*

Proof. We will show that in $C(aa)$, if γ is regular or has countable cofinality, then $2^\gamma = \gamma^+$. By Silver's theorem that the minimal failure of GCH cannot occur at a singular cardinal of uncountable cofinality, it will then follow that $C(aa)$ satisfies GCH.

Fix a good ordinal $\lambda > \gamma$. For each $A \subseteq \gamma$, let $\mathcal{H}_A$ be the transitive collapse of the set of x definable in the structure $\mathcal{C}_\lambda$ from parameters from $(\gamma + 1) \cup \{A\}$, and let $\mathcal{N}_A$ be the iterated aa-ultrapower of $\mathcal{H}_A$ of length $\gamma^{+C(aa)}$.

We start by showing that for all $A_0, A_1 \subseteq \gamma$ and all aa-premice $\mathcal{M}$, any two close elementary embeddings $i_0 : \mathcal{H}_{A_0} \to \mathcal{M}$ and $i_1 : \mathcal{H}_{A_1} \to \mathcal{M}$ such that $\sup i_0[\gamma] = \sup i_1[\gamma]$ and $i_0(\gamma) = i_1(\gamma)$ must satisfy $i_0 \restriction \gamma = i_1 \restriction \gamma$.

To see this, first take the case that γ is regular, and let $\vec{S}$ be a partition of $(S^\gamma_\omega)^{C(aa)}$ into γ-many $C(aa)$-stationary sets that is definable from γ in $\mathcal{C}_\lambda$. Then $\vec{S}$ is definable from γ by the same formula in both $\mathcal{H}_{A_0}$ and $\mathcal{H}_{A_1}$, and so since $i_0(\gamma) = i_1(\gamma)$, $i_0(\vec{S}) = i_1(\vec{S})$. Letting $\vec{T} = i_0(\vec{S})$ and $\gamma^* = \sup i_0[\gamma]$, we have

$$i_0[\gamma] = \{\alpha < i_0(\gamma) : C(aa) \vDash T_\alpha \cap \gamma^* \text{ is stationary}\} = i_1[\gamma]$$

If instead γ has countable cofinality, fix a sequence $S \subseteq \gamma$ of successor cardinals of $C(aa)$ cofinal in γ that is definable in $\mathcal{C}_\lambda$ from γ. We have that $i_0(S) = i_1(S)$, and hence $\sup i_0[\delta] = i_0(\delta) = i_1(\delta) = \sup i_1[\delta]$ for all $\delta \in S$. This is because of Lemma 4.2. The argument from the case that γ is regular now implies that $i_0 \restriction \delta = i_1 \restriction \delta$ for all $\delta \in S$, or in other words, $i_0 \restriction \gamma = i_1 \restriction \gamma$, as desired.

Let $I = \{\mathcal{N}_B : B \subseteq \gamma\}$ and let $N = \bigcup I$. We will now show that in $C(aa)$, there is a surjection f from $I \times N$ onto $P(\gamma) \cap C(aa)$. Namely, for each $\mathcal{M} \in I$, each pair of ordinals $\gamma_0, \gamma_1 \in \mathcal{M}$, and each set $B \in P(\gamma_1) \cap \mathcal{M}$, we let $f(\mathcal{M}, (\gamma_0, \gamma_1, B))$ be the unique set $A \subseteq \gamma$ such that there is a close elementary embedding $i : \mathcal{H}_A \to \mathcal{M}$ such that $\sup i[\gamma] = \gamma_0$, $i(\gamma) = \gamma_1$, and $i(A) = B$, if such a set A exists. The uniqueness of A follows from the previous paragraph, since $A = (i \restriction \gamma)^{-1}[B]$, and $i \restriction \gamma$ is uniquely determined by M, γ_0, and γ_1.

We claim that in $C(aa)$, $|I| \leq \gamma^+$. Note that for any $A_0, A_1 \subseteq \gamma$ in $C(aa)$, $\mathcal{N}_{A_0} = \mathcal{N}_{A_1}$ if and only if $o(\mathcal{N}_{A_0}) = o(\mathcal{N}_{A_1})$. To show that $C(aa)$ satisfies $|I| \leq \gamma^+$, it therefore suffices to prove that $C(aa)$ satisfies $|o(N)| \leq \gamma^+$.

Let $\mu = \gamma^{+C(aa)}$. Note that for each $B \subseteq \gamma$, $L[\mathcal{H}_B] \vDash |\mathcal{N}_B| \leq \mu$ since the iteration of $\mathcal{H}_B$ of length μ can be formed in this model. Therefore $o(\mathcal{N}_B) < \mu^{+L[\mathcal{H}_B]}$. Working in $C(aa)$ and applying Theorem 4.1,

$$o(N) = \sup_{B\subseteq\gamma} o(\mathcal{N}_B) \leq \sup_{B\subseteq\gamma} \mu^{+L[\mathcal{H}_B]} \leq (u_2(\mu))^{C(aa)} < \mu^{+C(aa)} = \gamma^{++C(aa)}$$

as desired.

Since $C(aa)$ satisfies that $|I| \leq \gamma^+$ and that $|\mathcal{N}_B| \leq \gamma^+$ for each $B \subseteq \gamma$, $C(aa)$ satisfies that $|N| \leq \gamma^+$. Since $C(aa)$ contains a partial surjection from $I \times N$ to $P(\gamma) \cap C(aa)$, it follows that $C(aa)$ satisfies $|P(\gamma)| \leq \gamma^+$, or in other words $2^\gamma = \gamma^+$. □

This also allows us to classify the measurable cardinals of $C(aa)$:

Theorem 4.4. *An ordinal δ is measurable in $C(aa)$ if and only if it has uncountable cofinality in V and is regular in $C(aa)$.*

Proof. Any such δ is a Fréchet cardinal in $C(aa)$, so by UA + GCH, some $\kappa \leq \delta$ is δ-strongly compact. (See [3, Theorem 7.4.9, Proposition 7.5.4].) Since $C(aa)$ has no inner model with a Woodin cardinal, we must have $\kappa = \delta$, and therefore δ is measurable. □

Theorem 4.5. *The inner model $C(aa)$ satisfies $V = \mathrm{HOD}$.*

Proof. Suppose γ is an ordinal and $A \subseteq \gamma$, and we will show that A is definable in $C(aa)$ from ordinal parameters. Let $\mathcal{H}_A$ be the aa-mouse defined in Theorem 4.3. Let $i : \mathcal{H}_A \to \mathcal{N}$ be the aa-iteration of length $\gamma^{+C(aa)}$. Let $\gamma_0 = \sup i[\gamma]$ and $\gamma_1 = i(\gamma)$. Finally let ξ be the rank of $i(A)$ in the canonical well-order of $\mathcal{N}$.

The proof of Theorem 4.3 shows that in $C(aa)$, if $\mathcal{H}'$ is an aa-mouse of cardinality γ whose aa-iteration $i' : \mathcal{H}' \to \mathcal{N}'$ of length $\gamma^{+C(aa)}$ satisfies $\sup i'[\gamma] = \gamma_0$ and $i'(\gamma) = \gamma_1$, then $i' \restriction \gamma = i \restriction \gamma$.

Therefore A is definable in $C(aa)$ from the ordinals γ, γ_0, γ_1, and ξ as the unique set of the form $(i')^{-1}[B]$ where $i' : \mathcal{H}' \to \mathcal{N}'$ is the $\gamma^{+C(aa)}$-length aa-iteration of some aa-mouse $\mathcal{H}'$ of cardinality γ such that $\sup i'[\gamma] = \gamma_0$, $i'(\gamma) = \gamma_1$, and B is the ξ-th element in the canonical well-order of $\mathcal{N}'$. □

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