

Some combinatorial properties of Ultimate L and V

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Abstract

This paper establishes a number of constraints on the structure of large cardinals under strong compactness assumptions. These constraints coincide with those imposed by the Ultrapower Axiom [1], a principle that is expected to hold in Woodin's hypothesized Ultimate L , providing some evidence for the Ultimate L Conjecture [2].

We show that every regular cardinal above the first strongly compact that carries an indecomposable ultrafilter is measurable, answering a question of Silver [3] for large enough cardinals. We show that any successor almost strongly compact cardinal of uncountable cofinality is strongly compact, making progress on a question of Boney, Unger, and Brooke-Taylor [4]. We show that if there is a proper class of strongly compact cardinals then there is no nontrivial cardinal preserving elementary embedding from the universe of sets into an inner model, answering a question of Caicedo [5] granting large cardinals. Finally, we show that if κ is strongly compact, then V is a set forcing extension of the inner model κ -HOD consisting of sets that are hereditarily ordinal definable from a κ -complete ultrafilter over an ordinal; κ -HOD seems to be the first nontrivial example of a ground of V whose definition does not involve forcing.

1 Introduction

1.1 The Ultimate L Conjecture

Since Cohen's proof of the independence of the Continuum Hypothesis [6], it has become clear that many of the fundamental features of the universe of sets will never be decided on the basis of the currently accepted axioms of set theory. Woodin's Ultimate L Conjecture [2], however, raises the possibility that the fundamental objects of set theory can be transferred into a substructure of the set theoretic universe (namely, Ultimate L) that is as tractable as the conventional structures of

mathematics.¹ The fundamental objects in question are *large cardinals*, strong closure points in Cantor’s hierarchy of infinities whose existence, taken axiomatically, suffices to interpret and compare the vast array of mutually incompatible formal systems studied in contemporary set theory.

If Woodin’s conjecture is true, the downward transference of large cardinal properties from the universe of sets into Ultimate L would necessitate an upward transference of combinatorial structure from Ultimate L back into the universe of sets. (For example, see [1, Theorem 8.4.40].) This motivates the prediction that assuming large cardinal axioms, the universe of sets resembles Ultimate L in certain ways. This paper presents a collection of theorems confirming this prediction by showing that various consequences of the Ultrapower Axiom, a principle expected to hold in Ultimate L , are actually provable from large cardinal axioms alone.

1.2 The Ultrapower Axiom

The Ultrapower Axiom (UA) asserts that the category of wellfounded ultrapowers of the universe of sets and internally definable ultrapower embeddings is directed.² In the author’s thesis [1, Theorem 2.3.10], it is shown that UA holds in any model whose countable elementary substructures satisfy a weak form of the *Comparison Lemma* of inner model theory.

The Comparison Lemma is really a series of results (for example, [7, 8, 9, 10, 11]) each roughly asserting the directedness of some subcategory of the category of countable canonical models of set theory and iterated ultrapower embeddings. These canonical models are known as mice. (We warn that the “category of canonical models” is not yet precisely defined; so far, only certain subcategories of this category have been identified, namely, those for which the Comparison Lemma has been proved. The term “iterated ultrapower” is used in a similarly open-ended sense.)

As it is currently conceived, the ongoing search for more powerful canonical models of set theory (including Ultimate L) amounts to an attempt to generalize the Comparison Lemma to larger subcategories of the category of canonical models. As a consequence, the current methodology of inner model theory simply cannot produce a canonical model in which the Ultrapower Axiom fails. For this reason, it seems likely that if Ultimate L exists, it will satisfy the Ultrapower Axiom.

1.3 Consequences of UA from large cardinal axioms alone

The Ultrapower Axiom can be used to develop a structure theory in the context of very large cardinals, proving, for example, that the Generalized Continuum Hy-

¹*The axiom $V = \text{Ultimate } L$:* (1) There is a proper class of Woodin cardinals. (2) If some level of the von Neumann hierarchy satisfies a sentence φ in the language of set theory, then there is a universally Baire set $A \subseteq \mathbb{R}$ such that some level of the von Neumann hierarchy of $\text{HOD}^{L(A, \mathbb{R})}$ satisfies φ .

The Ultimate L Conjecture: If κ is extendible, then there is an inner model M that satisfies ZFC plus the axiom $V = \text{Ultimate } L$ and has the property that for all cardinals $\lambda \geq \kappa$, there is a κ -complete normal fine ultrafilter \mathcal{U} over $P_\kappa(\lambda)$ with $P_\kappa(\lambda) \cap M \in \mathcal{U}$ and $\mathcal{U} \cap M \in M$.

²A category theorist would say *filtered*.

pothesis holds above the least strongly compact cardinal and that the universe is a set generic extension of HOD. One can also develop the theory of large cardinals, obtaining equivalences between a number of large cardinal axioms that are widely believed to have the same strength (e.g., strong compactness and supercompactness).

All of these results are impossible to prove in ZFC alone, but it turns out that each has an analog that is provable from large cardinal axioms. For example, the analog of the UA theorem that the GCH holds above a strongly compact cardinal is Solovay’s result that the Singular Cardinals Hypothesis holds above a strongly compact cardinal. This paper establishes analogs of the other theorems using techniques that are quite different from those used under the Ultrapower Axiom. The main methods of this paper actually derive from a lemma used by Woodin in his analysis of the downward transference of large cardinal axioms to Ultimate L , namely, that assuming large cardinal axioms, any ultrapower of the universe absorbs all sufficiently complete ultrafilters (Theorem 3.7). This fact enables us to simulate the Ultrapower Axiom in certain restricted situations.

We now summarize the results of this paper.

1.4 Indecomposable ultrafilters and Silver’s question

Our first theorem, the subject of Section 4, concerns a question posed by Silver [3] in the 1970s. If $\delta \leq \lambda$ are cardinals, X is a set, and U is an ultrafilter over X , U is said to be (δ, λ) -*indecomposable* if any $< \lambda$ -sized family of disjoint subsets of X whose union belongs to U has a $< \delta$ -sized subfamily whose union belongs to U . Indecomposability refines the concept of λ -completeness: an ultrafilter U over X is λ -*complete* if U is $(2, \lambda)$ - or equivalently, (ω, λ) -indecomposable, or in other words, U meets every $< \lambda$ -sized family of disjoint subsets of X whose union belongs to U .

The precise relationship between indecomposability and completeness, however, is not at all clear. A uniform ultrafilter on a cardinal λ is said to be *indecomposable* if it is (ω_1, λ) -indecomposable, the maximum degree of indecomposability short of λ -completeness. Silver asked whether an inaccessible cardinal λ that carries an indecomposable ultrafilter is necessarily measurable, that is, whether λ carries a λ -complete uniform ultrafilter. If λ is measurable, then λ carries an indecomposable ultrafilter that is not itself ω_1 -complete, but the hope is that one can extract a λ -complete ultrafilter from any indecomposable ultrafilter over λ (in the same way, perhaps, that one extracts a normal ultrafilter from an arbitrary λ -complete ultrafilter).

Jensen showed that in the canonical inner models, the answer to Silver’s question is yes. On the other hand, by forcing, Sheard [12] produced a model in which the answer is no. Thus the question appears to be “settled” in the usual way: no answer can be derived from the standard axioms.

The Ultrapower Axiom does not help with Silver’s question itself, but it does answer the natural generalization of Silver’s question to countably complete ultrafilters: assuming UA, for any cardinal δ , if $\lambda > \delta$ is inaccessible and carries a uniform *countably complete* (δ, λ) -indecomposable ultrafilter, then λ is measurable.

Despite Jensen and Sheard’s independence results, we will show that for sufficiently large cardinals λ , the answer to Silver’s question is yes:

Theorem 4.5. *Suppose $\delta < \kappa \leq \lambda$ are cardinals, κ is strongly compact, and λ carries a uniform (δ, λ) -indecomposable ultrafilter. Then either λ is a measurable cardinal or λ has cofinality less than δ and is a limit of measurable cardinals.*

As a special case, if a cardinal λ above the least strongly compact cardinal carries an indecomposable ultrafilter, then λ is either a measurable cardinal or the limit of countably many measurable cardinals.

1.5 Almost strong compactness

Our second result, proved in Section 5, concerns a generalization of strong compactness defined by Bagaria-Magidor [13]. A cardinal κ is *strongly compact* if every κ -complete filter extends to a κ -complete ultrafilter. Many applications of strong compactness only seem to require that κ be *almost strongly compact*: for any cardinal $\nu < \kappa$, every κ -complete filter extends to a ν -complete ultrafilter.

The Ultrapower Axiom’s most interesting consequences relate to the structure of strong compactness. Most notably, UA implies that the least strongly compact cardinal is supercompact. In fact, UA also implies that the least *almost* strongly compact cardinal is supercompact; in particular, the least almost strongly compact cardinal is strongly compact. Whether this is provable outright is an open question, posed by Boney and Brooke-Taylor. We will obtain the following partial answer:

Theorem 5.7 (SCH). *If the least almost strongly compact cardinal has uncountable cofinality, it is strongly compact.*

It is not true in general that every almost strongly compact cardinal is strongly compact, since any limit of strongly compact cardinals is almost strongly compact, while every strongly compact cardinal is regular. UA does imply that every *successor* almost strongly compact cardinal is strongly compact. Here we will show that this is almost a theorem of ZFC:

Theorem 5.8. *For any ordinal α , if the $(\alpha + 1)$ -st almost strongly compact limit cardinal has uncountable cofinality, it is strongly compact.*

(We must say “limit cardinal” in Theorem 5.8 because technically the successor of any strongly compact cardinal is almost strongly compact.)

1.6 Cardinal preserving embeddings

Next, in Section 6, we take up the problem of cardinal preserving embeddings, posed by Caicedo [5]. If M is an inner model, an elementary embedding $j : V \rightarrow M$ is said to be *cardinal preserving* (up to λ) if every cardinal of M (less than λ) is a cardinal in V .

Caicedo asked whether cardinal preserving embeddings exist. The Ultrapower Axiom implies that they do not. In fact, under UA, if λ is an aleph fixed point and

$j : V \rightarrow M$ is an elementary embedding that fixes λ and is cardinal preserving up to λ , then $V_\lambda \subseteq M$. Since every elementary embedding has an ω -closed unbounded class of fixed points, it follows that under UA, no elementary embedding $j : V \rightarrow M$ can be fully cardinal preserving: otherwise $V_\lambda \subseteq M$ for a proper class of λ , violating the Kunen inconsistency [14], which states that there is no elementary embedding from V to V .

We show that one can refute the existence of cardinal preserving embeddings from large cardinal axioms alone:

Theorem 6.6. *Suppose there is a proper class of strongly compact cardinals. Then there are no cardinal preserving embeddings.*

This theorem can be viewed as a version of the Kunen inconsistency, but the proof is completely different from all of the usual proofs of Kunen's theorem.

1.7 Definability from ultrafilters

Finally, Section 7 studies the structure of ordinal definability under large cardinal assumptions. The most prominent question here is Woodin's HOD Conjecture [2]. It turns out that UA implies the HOD Conjecture,³ and more:

Theorem (UA). *If there is a supercompact cardinal, then V is a generic extension of HOD.* \square

The proof appears in the author's thesis [1, Theorem 6.2.8].

It is impossible to prove that V is a generic extension of HOD from any of the standard large cardinal hypotheses. We will instead consider a generalization of HOD.

Definition. Let κ -OD denote the class of sets definable from a κ -complete ultrafilter over an ordinal, and let κ -HOD denote the class of hereditarily κ -OD sets.⁴

An ultrafilter over a set X can be thought of as a generalized element of X . From this perspective, an ultrafilter over an ordinal is a generalized ordinal. For this reason, definability from an ultrafilter over an ordinal seems to be a natural extension of ordinal definability.

Arguably, the more complete an ultrafilter over an ordinal is, the more it should resemble an ordinal. Thus as κ increases, κ -HOD should become more like HOD; for example, the ∞ -complete ultrafilters (i.e., those that are κ -complete for all cardinals κ) are just the principal ultrafilters over ordinals, which are essentially just ordinals. Therefore ∞ -HOD is just the usual HOD. On the other hand, ω -HOD, the class of sets definable from an arbitrary ultrafilter over an ordinal, turns out to be equal to V (Proposition 7.5). The remaining models $\langle \kappa\text{-HOD} \rangle_{\kappa \in \text{Card}}$ form a decreasing sequence of structures between V and HOD.

³Technically UA implies the HOD Hypothesis. The HOD Conjecture is that the HOD Hypothesis is provable in ZFC.

⁴Note that x is κ -OD if and only if x is in OD_U for some κ -complete ultrafilter U over an ordinal, so κ -OD and κ -HOD are first-order definable.

Standard arguments show that for any cardinal κ , κ -HOD is an inner model of ZF. A little bit more surprisingly, if κ is strongly compact, then κ -HOD satisfies the Axiom of Choice.

It is consistent with all known large cardinal axioms that $V \neq \kappa$ -HOD for any uncountable cardinal κ , since this holds after adding a Cohen real (Proposition 7.9). We will show, however, that V is almost equal to κ -HOD. If M is an inner model of ZFC, M is said to be a *ground* of V if there is a partial order $\mathbb{P} \in M$ and an M -generic filter G on \mathbb{P} such that $M[G] = V$.

Theorem 7.8. *Suppose κ is strongly compact. Then κ -HOD is a ground of V .*

It follows that for all sufficiently large cardinals λ , $(\lambda^+)^{\kappa\text{-HOD}} = \lambda^+$, $(2^\lambda)^{\kappa\text{-HOD}} = 2^\lambda$, κ -HOD correctly computes stationary subsets of λ , large cardinals are transferred into and out of κ -HOD, etc. In fact, this is true for all $\lambda \geq (2^\kappa)^+$. Therefore unlike HOD, κ -HOD is provably very similar to V . The model κ -HOD is, as far as we know, the first nontrivial example of a ground of V that is not defined in terms of set theoretic geology.

Last of all, we prove the following theorem:

Theorem 7.11. *Suppose κ is supercompact. Then κ is supercompact in κ -HOD.*

Since supercompactness is defined in terms of κ -complete normal fine ultrafilters, which are necessarily κ -OD, Theorem 7.11 may not seem very surprising. The issue one must overcome, however, is that these ultrafilters might not concentrate on κ -HOD and therefore might not witness that κ is supercompact in κ -HOD. This corresponding question for strongly compact cardinals remains open.

2 Preliminaries

We put down some definitions and notational conventions, most of which are completely standard.

2.1 Ultrafilters

Definition 2.1. If (\mathbb{P}, \leq) is a partial order, a proper subset $F \subseteq \mathbb{P}$ is a *filter* on \mathbb{P} if it is closed upwards under \leq and for any $p, q \in F$, there is some $r \in F$ with $r \leq p$ and $r \leq q$. A filter U on \mathbb{P} is an *ultrafilter* on \mathbb{P} if it is \subseteq -maximal among all filters on \mathbb{P} .

We are really only interested in the following special case:

Definition 2.2. Suppose M is a model of set theory and $X \in M$. We say $U \subseteq P^M(X)$ is an *M -filter* (resp. *M -ultrafilter*) over X if U is a filter (resp. ultrafilter) on the partial order $(P^M(X), \subseteq^M)$.

A fundamental concept in the theory of large cardinals is the completeness of an ultrafilter. We will need the generalization of this concept to M -ultrafilters.

Definition 2.3. Suppose M is a model of set theory, U is an M -ultrafilter, ρ is an M -cardinal, and κ is a cardinal.

- U is M - ρ -complete if for any $\sigma \subseteq U$ with $\sigma \in M$ and $|\sigma|^M < \rho$, $\bigcap \sigma \in U$.
- U is M - κ -complete if for any $\sigma \subseteq U$ with $\sigma \in M$ and $|\sigma| < \kappa$, $\bigcap \sigma \in U$.⁵

If U is a V -ultrafilter over X , we say that U is an *ultrafilter over X* , and if U is V - κ -complete, we say U is κ -complete.

We denote the ultrapower of a model P by an P -ultrafilter U by

$$j_U : P \rightarrow M_U^P$$

The ultrapower of V by an ultrafilter U is denoted $j_U : V \rightarrow M_U$.

The following terminology is probably self-explanatory:

Definition 2.4. Suppose M is a model of set theory and W is an M -ultrafilter. Then W is an *ultrafilter of M* if $W \in M$.

We now turn to some basic combinatorial definitions.

Definition 2.5. Suppose M is a model of set theory and $X \in M$. An M -ultrafilter U over X is *uniform* if every set in U has M -cardinality $|X|^M$.

Note that if M is a wellfounded model of ZFC, then for any M -ultrafilter U , there is some $A \in U$ such that $U \cap P^M(A)$ is uniform. In particular, this holds for any V -ultrafilter, so in the theory of ultrafilters, one can usually work with uniform ultrafilters with no loss of generality.

A notion similar to uniformity, but distinct from it, is fineness:

Definition 2.6. An ultrafilter U over a family of sets F is *fine* if for all $x \in \bigcup F$, the set $\{A \in F : x \in A\}$ belongs to U .

This is a slight generalization of the standard definition of fineness. Note that an ultrafilter U over an ordinal α is fine if and only if every set in U is cofinal in α .

Definition 2.7. Suppose f is a function, U is an ultrafilter over a set X , and Y is a set such that $f^{-1}[Y] \cap X \in U$. The *pushforward of U under f over Y* is the ultrafilter defined by $f_*(U) = \{A \subseteq Y : f^{-1}[A] \cap X \in U\}$.

Our notation for pushforwards ignores the choice of Y , which we ask the reader to infer from context. For notational convenience, we allow that $\text{dom}(f) \neq X$ and $\text{ran}(f) \neq Y$, and instead require just that f is defined U -almost everywhere and sends U -almost every element of X to an element of Y . This is not really an important point.

What *is* important is the relationship between pushforwards and derived ultrafilters.

⁵We often identify a point $\sigma \in M$ with its extension $\text{ext}_M(\sigma) = \{x \in M : M \models x \in \sigma\}$, which is a subset of M , *even when M is illfounded*. For each $\sigma \in M$, let $P^\infty(\sigma)$ be denote the maximum ZFA model with atom set σ as computed in M (see [?, (15.33)]). There is a unique isomorphism i from the wellfounded part $W(\sigma)$ of $P^\infty(\sigma)$ to a transitive model $N(\sigma)$ of ZFA with atom set $\text{ext}_M(\sigma)$ such that i is the identity on $\text{ext}_M(\sigma)$. Our abuse of notation amounts to identifying $W(\sigma)$ with $N(\sigma)$ via this isomorphism.

Definition 2.8. If $j : M \rightarrow N$ is an elementary embedding, $X \in M$, and $a \in j(X)$, then the M -ultrafilter over X derived from j using a is the set $\{A \in P^M(X) : a \in j(A)\}$.

Proposition 2.9. Suppose U and W are ultrafilters over sets X and Y and f is a function such that $f^{-1}[Y] \cap X \in U$. Then the following are equivalent:

- (1) $f_*(U) = W$.
- (2) W is the ultrafilter on Y derived from j_U using $[f]_U$.
- (3) There exists an elementary embedding $k : M_W \rightarrow M_U$ such that $k \circ j_W = j_U$ and $k([\text{id}]_W) = [f]_U$. □

Notice that there is at most one embedding witnessing (3).

2.2 The approximation and cover properties

For our results, it is important to define covering properties for models that are not necessarily wellfounded.

Definition 2.10. Suppose M is a model of set theory, $X \in M$ is a set, ρ is an M -cardinal, and κ is a cardinal.

- M has the (κ, ρ) -cover property if for all $\sigma \subseteq M$ with $|\sigma| < \kappa$, there is some $\tau \in M$ with $|\tau|^M < \rho$ such that $\sigma \subseteq \tau$.
- κ -cover property if for all $\sigma \subseteq M$ with $|\sigma| < \kappa$, there is some $\tau \in M$ with $|\tau| < \kappa$ such that $\sigma \subseteq \tau$.

We will also discuss the Hamkins approximation property [15], but we pass over the illfounded case:

Definition 2.11. Suppose M is a model of set theory and κ is a cardinal.

- A set $A \subseteq M$ is κ -approximated by M if for all $\sigma \in M$ with $|\sigma| < \kappa$, $A \cap \sigma \in M$.
- M has the κ -approximation property if every set that is κ -approximated by M belongs to M .

These two properties combined define the notion of a pseudoground:

Definition 2.12. Suppose $M \subseteq N$ are transitive models of ZFC and κ is an N -cardinal. We say M is a κ -pseudoground of N if N satisfies that M has the κ -approximation and cover properties.⁶ We say M is a pseudoground of N if there is some N -cardinal κ such that M is a κ -pseudoground of N .

We will refer to pseudogrounds of V simply as pseudogrounds.

Note that if M is a pseudoground of N then $\text{Ord} \cap N \subseteq M$, or in other words, M is an inner model of N . In particular, M is not an element of N , but it turns out that M must be definable over N :

⁶Formally this is expressed in the structure (N, M) .

Theorem 2.13 (Laver-Hamkins). *Suppose M is a κ -pseudoground of N . Then M is the unique κ -pseudoground P of N such that $P \cap H(\kappa^{+N}) = M \cap H(\kappa^{+N})$ and M is Δ_2 -definable over N from the parameter $M \cap H(\kappa^+)$. \square*

The following is Woodin's Universality Theorem for pseudogrounds:

Theorem 2.14 (Woodin). *Suppose M is a κ -pseudoground and E is an M -extender of length ν whose critical point is at least κ . If $j_E(A) \cap [\nu]^{<\omega} \in M$ for all $A \in M$, then $E \in M$. \square*

The Hamkins Universality Theorem shows that for nice embeddings, one does not even have to assume closure under the extender:

Theorem 2.15 (Hamkins). *Suppose M is a κ -pseudoground.*

- *Every κ -complete M -ultrafilter belongs to M .*
- *If E is an extender with critical point greater than κ such that M_E is closed under κ -sequences, then $E \cap M \in M$. \square*

Theorem 2.16 (Hamkins-Reitz). *Suppose κ is a cardinal and M is a κ -pseudoground. Then M is a λ -pseudoground for all $\lambda \geq \kappa$. \square*

2.3 Compactness principles

In this section, we define various notions of strong compactness, the most famous of which is of course due to Tarski [16], and the rest of which were introduced by Bagaria-Magidor [13].

Definition 2.17. Suppose $\delta \leq \kappa \leq \lambda$ are cardinals. Then κ is (δ, λ) -strongly compact if there is a δ -complete fine ultrafilter over $P_\kappa(\lambda)$.

This principle is degenerate in the sense that if κ is (δ, λ) -strongly compact, then all ordinals above κ are (δ, λ) -strongly compact.

Definition 2.18. Suppose $\delta \leq \kappa \leq \lambda$ are cardinals.

- κ is (δ, ∞) -strongly compact if it is (δ, γ) -strongly compact for all cardinals $\gamma \geq \kappa$.
- κ is λ -strongly compact if it is (κ, λ) -strongly compact.
- κ is strongly compact if it is (κ, ∞) -strongly compact.
- κ is almost λ -strongly compact if it is (γ, λ) -strongly compact for all cardinals $\gamma < \kappa$.
- κ is almost strongly compact if it is almost η -strongly compact for all cardinals $\eta \geq \kappa$.

These principles can be reformulated in terms of either the filter extension property, elementary embeddings, or uniform ultrafilters on cardinals. We will actually use all four characterizations below without much comment.

Theorem 2.19 (Solovay, Ketonen). *Suppose $\delta \leq \kappa \leq \lambda$ are cardinals. Then the following are equivalent:*

- κ is (δ, λ) -strongly compact.
- There is an elementary embedding $j : V \rightarrow M$ with $\text{crit}(j) \geq \delta$ such that M has the $(\lambda^+, j(\kappa))$ -cover property.
- Every κ -complete filter that is generated by at most λ sets extends to a δ -complete ultrafilter.

If $\text{cf}(\lambda) \geq \kappa$, one can add to the list:

- Every regular cardinal in the interval $[\kappa, \lambda]$ carries a δ -complete uniform ultrafilter. □

We also use the following theorem, which is essentially due to Solovay:

Theorem 2.20 (Solovay). *The Singular Cardinals Hypothesis holds above the least almost strongly compact cardinal κ : if $\lambda \geq \kappa$ is a singular cardinal, then*

$$\lambda^{\text{cf}(\lambda)} = \max(2^{\text{cf}(\lambda)}, \lambda^+) \quad \square$$

3 Ultrafilters in ultrapowers

Suppose D is an ultrafilter and W is an M_D -ultrafilter. It is often useful to know whether W belongs to M_D . The Ultrapower Axiom yields many instances in which $W \in M_D$ must occur for D and W countably complete ultrafilters; this fact is leveraged to prove most of the consequences of UA in [1]. But it turns out that in certain situations, one can prove that $W \in M_D$ from large cardinal axioms alone.

The idea is that if one can W extend to a sufficiently complete V -ultrafilter W^* , then using a result known as Kunen's *commuting ultrapowers lemma* one obtains that $j_{W^*} \upharpoonright M_D$ is definable over M_D , and hence W belongs to M_D . In Section 3.1, we give a proof of Kunen's result. (The reason we include this is to verify that the proof goes through in the case that D is countably incomplete, which we need in order to answer to Silver's question above a strongly compact cardinal.)

In Section 3.2, we prove that if κ is a strong limit cardinal and W is κ -complete with respect to sets in M_D , then W generates a κ -complete filter in V . Thus if κ is strongly compact, W extends to a κ -complete ultrafilter W^* , and so by the observation in the previous paragraph, one can conclude that $W \in M_D$.

Finally, Section 3.3 is devoted to applications of the results of Section 3.2 to the theory of *pseudogrounds*, a generalization due to Hamkins [17] of the concept of a set forcing ground of V that appears to have a deep relationship with the theory of inner models for supercompact cardinals. These applications digress from the main

thread of this paper, and are not strictly speaking necessary to prove our main results. What we show is that if κ is strongly compact, then κ -pseudogrounds are characterized by their most basic properties:

Theorem 3.22. *Suppose κ is strongly compact and M is an inner model. Then the following are equivalent:*

- (1) M is a κ -pseudoground.
- (2) κ is strongly compact in M and the following hold:
 - Every κ -complete ultrafilter over a set in M extends an ultrafilter of M .
 - Every κ -complete ultrafilter of M extends to a κ -complete ultrafilter of V .
- (3) Every regular cardinal of M above κ has cofinality at least κ and every κ -complete ultrafilter over a set in M extends an ultrafilter of M .

3.1 Commuting ultrafilters and ultrapowers

The following definition explains how elementary embeddings act on amenable classes.

Definition 3.1. Suppose M is a model of set theory. A class A is *amenable to M* if $A \subseteq M$ and $A \cap x \in M$ for all $x \in M$.

An elementary embedding $j : M \rightarrow N$ is *cofinal* if for every $a \in N$, there is some $X \in M$ such that $a \in j(X)$.

If $j : M \rightarrow N$ is a cofinal elementary embedding, and A is an amenable class of M , then $j(A) = \bigcup_{x \in P} j(A \cap x)$.

If $j' : M' \rightarrow N'$ extends $j : M \rightarrow N$, and $j' \upharpoonright M : M \rightarrow j'(M)$ is a cofinal embedding, then $j'(A) = j(A)$ for all A in M' amenable to M .

If $j_0 : M \rightarrow N_0$ is a cofinal elementary embedding and $j_1 : M \rightarrow N_1$ is an amenable elementary embedding, then $j_0(j_1) : N_0 \rightarrow j_0(N_1)$ is an elementary embedding.

Definition 3.2. Suppose $j_0 : V \rightarrow M_0$ and $j_1 : V \rightarrow M_1$ are cofinal elementary embeddings. We say j_0 and j_1 *commute* if there is an isomorphism $k : j_0(M_1) \rightarrow j_1(M_0)$ such that

$$j_1 \upharpoonright M_0 = k \circ j_0(j_1) \text{ and } j_0 \upharpoonright M_1 = k^{-1} \circ j_1(j_0)$$

Note that we do not assume that M_0 and M_1 are wellfounded. If M_0 and M_1 are *transitive*, then k must be the identity, and hence j_0 and j_1 commute if and only if

$$j_0(j_1) = j_1 \upharpoonright M_0 \text{ and } j_1(j_0) = j_0 \upharpoonright M_1$$

For ultrafilters U and W , whether j_U and j_W commute is influenced by the relationship between the filter product $U \times W$ and the ultrafilter product $U \otimes W$ of U and W .

Definition 3.3. Suppose U and W are ultrafilters over sets X and Y .

- W is U -complete if for any sequence $\langle B_x : x \in X \rangle \subseteq W$, there is some $A \in U$ such that $\bigcap_{x \in A} B_x \in W$.
- The *filter product of U and W* is the filter $U \times W$ generated by sets of the form $A \times B$ for $A \in U$ and $B \in W$.
- The *ultrafilter product of U and W* is the ultrafilter

$$U \otimes W = \{A \subseteq X \times Y : \forall^U x \forall^W y (x, y) \in A\}$$

The filter product is commutative up to canonical isomorphism, but in general the ultrafilter product is not.

Theorem 3.4 (Blass). *Suppose U and W are ultrafilters. Then the following are equivalent:*

- (1) W is U -complete.
- (2) $j_U[W]$ generates $j_U(W)$.
- (3) $U \times W$ is an ultrafilter.
- (4) $U \times W = U \otimes W$.

Proof. Let X and Y be the underlying sets of U and W .

(1) *implies* (2): Fix $[B_x]_U \in U$. The U -completeness of W yields that for some $A \in U$, $\bigcap_{x \in A} B_x \in W$. Note that $j_U(\bigcap_{x \in A} B_x) \subseteq [B_x]_U$ since $\bigcap_{x \in A} B_x \subseteq B_x$ for U -almost all $x \in X$. Thus $[B_x]_U$ contains an element of $j_U[W]$, as desired.

(2) *implies* (3): Fix $R \subseteq X \times Y$. We will prove that either R or its complement contains a set in U . Let $R_x = \{y \in Y : (x, y) \in R\}$. Assume without loss of generality that $R_x \in W$ for U -almost all $x \in X$. Then $[R_x]_U \in j_U(W)$, so by (2), there is some $B \in W$ such that $j_U(B) \subseteq [R_x]_U$. Fix $A \in U$ such that for all $x \in A$, B is contained in R_x . Then $A \times B \subseteq R$ and $A \times B \in U \times W$.

(3) *implies* (4): This is trivial since by definition $U \times W \subseteq U \otimes W$, so if $U \times W$ is an ultrafilter, then $U \times W = U \otimes W$ by maximality.

(4) *implies* (1): Fix $\langle B_x \rangle_{x \in X} \subseteq W$. Let $R = \{(x, y) \in X \times Y : y \in B_x\}$. Then $R \in U \otimes W$ by definition. Therefore $R \in U \times W$ by (4), so fix $A \in U$ and $B \in W$ such that $A \times B \subseteq R$. Then for all $x \in A$, $B \subseteq B_x$. In other words, $B \subseteq \bigcap_{x \in A} B_x$, so since $B \in W$, $\bigcap_{x \in A} B_x \in W$. This shows that W is U -complete. \square

The equivalence of (1) and (3) in Theorem 3.4 implies that an ultrafilter W is U -complete if and only if U is W -complete, which is a bit surprising given the original definition.

Lemma 3.5. *Suppose U and W are ultrafilters over X and Y . The following are equivalent:*

- (1) j_U and j_W commute.

(2) Let $\text{flip}(x, y) = (y, x)$. Then $\text{flip}_*(U \otimes W) = W \otimes U$.⁷

(3) The quantifiers associated to U and W commute. That is, for any predicate R on $X \times Y$,

$$\forall^U x \forall^W y R(x, y) \iff \forall^W y \forall^U x R(x, y)$$

Proof. (1) if and only if (2): There is a natural isomorphism between $M_{U \otimes W}$ and $j_U(j_W)(M_U)$ sending a point $[f]_{U \otimes W}$, where $f : X \times Y \rightarrow V$, to the point $[[\lambda f]_U]_{j_U(j_W)}$ where $\lambda f : X \rightarrow V^Y$ is the function defined by $\lambda f(x)(y) = f(x, y)$. For notational convenience, we will identify the two models via this isomorphism.

This identification results in the following equalities:

$$\begin{aligned} j_{U \otimes W} &= j_U(j_W) \circ j_U \\ [\text{id}]_{U \otimes W} &= (j_U(j_W)([\text{id}]_U), j_U([\text{id}]_W)) \end{aligned}$$

Under the corresponding identification of $M_{W \otimes U}$ with $j_W(j_U)(M_W)$,

$$\begin{aligned} j_{W \otimes U} &= j_W(j_U) \circ j_W \\ [\text{id}]_{W \otimes U} &= (j_W(j_U)([\text{id}]_W), j_W([\text{id}]_U)) \end{aligned}$$

Given these equalities and Proposition 2.9, the function $\text{flip}(x, y) = (y, x)$ satisfies $\text{flip}_*(U \otimes W) = W \otimes U$ if and only if there is an elementary embedding $k : M_{W \otimes U} \rightarrow M_{U \otimes W}$ satisfying

$$k \circ j_W(j_U) \circ j_W = j_U(j_W) \circ j_U \tag{1}$$

$$k(j_W(j_U)([\text{id}]_W), j_W([\text{id}]_U)) = \text{flip}(j_U(j_W)([\text{id}]_U), j_U([\text{id}]_W)) \tag{2}$$

We claim that an embedding satisfies (1) and (2) if and only if it is an isomorphism such that $k \circ j_W(j_U) = j_U \upharpoonright M_W$, and $k^{-1} \circ j_U(j_W) = j_W \upharpoonright M_U$, or in other words, j_U and j_W commute.

For the forwards direction, assume k satisfies (1) and (2). We claim k is surjective. Let

$$S = j_U(j_W) \circ j_U[V] \cup \{j_U(j_W)([\text{id}]_U), j_U([\text{id}]_W)\}$$

Then Loś's Theorem implies every element of $j_U(j_W)(M_U)$ is definable in $j_U(j_W)(M_U)$ from parameters in S . But (1) and (2) imply that $S \subseteq \text{ran}(k)$. Since $\text{ran}(k)$ is closed under definability in $j_U(j_W)(M_U)$, every point in $j_U(j_W)(M_U)$ is in $\text{ran}(k)$, so k is surjective. It follows that k is an isomorphism.

To see that $k \circ j_W(j_U) = j_U \upharpoonright M_W$, notice that $k \circ j_W(j_U)$ agrees on $j_W[V]$ with j_U by (1), and $k(j_W(j_U)([\text{id}]_W)) = j_U([\text{id}]_W)$. Hence $k \circ j_W(j_U)$ and j_U agree on $j_W[V] \cup \{[\text{id}]_W\}$. Since every point in M_W is definable in M_W from parameters in $j_W[V] \cup \{[\text{id}]_W\}$, $k \circ j_W(j_U) = j_U \upharpoonright M_W$ by elementarity. The fact that $k^{-1} \circ j_U(j_W) = j_W \upharpoonright M_U$ is proved by a similar argument.

The reverse direction of the claim is very similar, so we omit the proof. We also omit the proof of the equivalence of (2) and (3), since there are no ideas there, and anyway (3) was included only for aesthetic reasons. \square

⁷Recall that $f_*(U) = \{A : f^{-1}[A] \in U\}$; see Definition 2.7.

We now prove the Commuting Ultrapowers Lemma using Blass's result.

Theorem 3.6 (Kunen). *Suppose U and W are ultrafilters such that W is U -complete. Then j_U and j_W commute.*

Proof. By Theorem 3.4, $U \otimes W = U \times W$. Therefore $\text{flip}_*(U \otimes W) = \text{flip}_*(U \times W) = W \times U$. Since $W \times U$ is an ultrafilter, by Theorem 3.4 (with the roles of U and W exchanged), $W \times U = W \otimes U$. Putting these equations together, $\text{flip}_*(U \otimes W) = W \otimes U$. Applying Lemma 3.5, j_U and j_W commute. \square

Whether the converse of Theorem 3.6 is provable in ZFC an open question. The converse restricted to countably complete ultrafilters is an easy consequence of the Ultrapower Axiom. The author has also proved that the converse follows from the Generalized Continuum Hypothesis. Thus another consequence of UA can be verified by a classical axiom.

3.2 Sufficiently complete M_D -ultrafilters are in M_D

In this subsection we prove our main theorem on the amenability of ultrafilters:

Theorem 3.7. *Suppose δ is a cardinal and $\kappa \geq \delta$ is a strong limit cardinal. Suppose D is an ultrafilter over a set of size less than δ and X is a set in M_D . Suppose W is an M_D - κ -complete M_D -ultrafilter over $X \in M_D$. Assume that every κ -complete filter over X extends to a δ -complete ultrafilter. Then $W \in M_D$.*

The main point is that in the situation of Theorem 3.7, the M_D -ultrafilter W can be extended to a δ -complete ultrafilter:

Proposition 3.8. *Suppose κ is a strong limit cardinal and M is a model of set theory with the κ -cover property. Suppose W is an M - κ -complete M -ultrafilter. Then W generates a κ -complete filter.*

This in turn follows from Kunen's analysis of weakly amenable ultrafilters, which we state in a very general form:

Theorem 3.9 (Kunen). *Suppose M is a model of set theory, U is an M -ultrafilter over $X \in M$, and ι is an M -cardinal. Let $j : M \rightarrow N$ be the ultrapower of M by U . Then the following are equivalent:*

- (1) *For all $\sigma \subseteq P^M(X)$ with $\sigma \in M$ and $|\sigma|^M = \iota$, $U \cap \sigma \in M$.*
- (2) *For all $B \in P^N(j(\iota))$, $j^{-1}[B] \in M$.*

Proof. (1) implies (2): Fix $B \in P^N(j(\iota))$. Let $f : X \rightarrow P^M(\iota)$ be a function in M such that $B = [f]_U^M$. For $\xi < \iota$, let $A_\xi = \{x \in X : \xi \in f(x)\}$. Note that the sequence $\langle A_\xi \rangle_{\xi < \iota}$ belongs to M . Let $\sigma = \{A_\xi : \xi < \iota\}$. Now $j^{-1}[B] = \{\xi < \iota : A_\xi \in U\} = \{\xi < \iota : A_\xi \in U \cap \sigma\}$. But by (1), $U \cap \sigma \in M$. Hence $j^{-1}[B] \in M$.

(2) implies (1): Fix $\sigma \subseteq P^M(X)$ and a surjection $f : \iota \rightarrow \sigma$ that belongs to M . Let $a = [\text{id}]_U$. Let $B = \{\xi < j(\iota) : a \in j(f)(\xi)\}$. Clearly $B \in P^N(j(\iota))$. Note that $j(\xi) \in B$ if and only if $a \in j(f)(\xi)$, which happens if and only if $f(\xi) \in U$. In other words, $U \cap \sigma = \{f(\xi) : \xi \in j^{-1}[B]\}$. By (2), $j^{-1}[B] \in M$, and so $U \cap \sigma \in M$. \square

Proof of Proposition 3.8. To show that W generates a κ -complete filter, it suffices to show that for all $\sigma \subseteq W$ with $|\sigma| < \kappa$, $\bigcap \sigma$ is nonempty. By the κ -cover property, there is some $\tau \in M$ containing σ of cardinality less than κ . Let $\iota = |\tau|^M$.

Let $j : M \rightarrow N$ be the ultrapower of M by W . Since κ is a strong limit cardinal, W is M - $((2^\iota)^+)^M$ -complete. This implies that $j[P^M(\iota)] = j(P^M(\iota)) = P^N(j(\iota))$. (Recall that an M -ultrafilter U is M - δ -complete if and only if $j(X) = j[X]$ for every $X \in M$ with $|X|^M < \delta$.) In particular, for any $B \in P^N(j(\iota))$, there is some $\bar{B} \in M$ with $j(\bar{B}) = B$, so $j^{-1}[B] \in M$ since $j^{-1}[B] = \bar{B}$.

Applying Theorem 3.9, it follows that $W \cap \tau \in M$. Since W is M - ρ -complete, $\bigcap(W \cap \tau)$ is nonempty. But $\sigma \subseteq W \cap \tau$, so $\bigcap \sigma$ is nonempty, as desired. \square

To obtain the cover hypothesis in Proposition 3.8, we establish a general fact about the covering properties of ultrapowers.

Proposition 3.10. *Suppose κ is a strong limit cardinal and D is an ultrafilter over a set X of size less than κ . Then M_D has the κ -cover property.*

Proof. Fix $\sigma \subseteq M_D$ with $|\sigma| < \kappa$. Let $\delta = |\sigma|$ and choose functions $\langle f_\alpha \rangle_{\alpha < \delta}$ such that

$$\sigma = \{[f_\alpha]_U : \alpha < \delta\} = \{j_U(f_\alpha)([id]_U) : \alpha < \delta\}$$

Let $\langle g_\beta \rangle_{\beta < j_D(\delta)} = j_D(\langle f_\alpha \rangle_{\alpha < \delta})$. Then $\{j_U(f_\alpha) : \alpha < \delta\} \subseteq \{g_\beta : \beta < j_D(\delta)\}$, so $\sigma \subseteq \{g_\beta([id]_U) : \beta < j_D(\delta)\}$. Clearly $\{g_\beta([id]_U) : \beta < j_D(\delta)\} \in M_D$ and has cardinality at most $|j_D(\delta)| \leq \delta^{|X|} < \kappa$ since κ is a strong limit cardinal. \square

Theorem 3.7 is now a matter of citing the preceding results in the right order.

Proof of Theorem 3.7. By Proposition 3.10, M_D has the κ -cover property, so by Proposition 3.8, the M_D - κ -complete M_D -ultrafilter W is κ -complete, or in other words, it generates a κ -complete filter.

Let F be the κ -complete filter generated by W . The filter F extends to a δ -complete ultrafilter U by our large cardinal hypothesis. Now we apply the Commuting Ultrapowers Lemma (Theorem 3.6) to conclude that $U \cap M_D$ belongs to M_D . More precisely, the Commuting Ultrapowers Lemma yields an isomorphism $k : j_U(M_D) \rightarrow j_D(M_U)$ such that $k \circ j_U \upharpoonright M_D = j_D(j_U)$. We therefore have $A \in U \cap M$ if and only if $A \in P^M(X)$ and $[id]_U \in j_U(A)$, and this holds if and only if $k([id]_U) \in j_D(j_U)(A)$. Clearly the set

$$\{A \in P^{M_D}(X) : k([id]_U) \in j_D(j_U)(A)\}$$

belongs to M_D , since it is definable from parameters over M_D . Therefore $U \cap M_D \in M_D$.

But $U \cap M_D = W$. This completes the proof. \square

3.3 The approximation property

This section proves some basic structural results about pseudogrounds under large cardinal assumptions. We will show that if there is a proper class of strongly compact cardinals, then the pseudogrounds are closed under the fundamental model constructions of set theory: generic extensions and extender ultrapowers.

For the sake of exposition, let us recall a theorem of Woodin and Usuba that motivates the results of this section. This requires some definitions.

Definition 3.11. An ultrafilter \mathcal{U} over a family F of subsets of X is *normal* if for any $\langle A_x : x \in X \rangle$ with $A_x \in \mathcal{U}$ for all $x \in X$, the *diagonal intersection*

$$\Delta_{x \in X} A_x = \left\{ \sigma \in F : \sigma \in \bigcap_{x \in \sigma} A_x \right\}$$

belongs to \mathcal{U} .

A cardinal κ is *supercompact* if for all $\lambda \geq \kappa$, there is a κ -complete normal fine ultrafilter over $P_\kappa(\lambda)$, or equivalently, there is an elementary embedding $j : V \rightarrow M$ where M is an inner model closed under λ -sequences.

An inner model M is a *weak extender model for the supercompactness of κ* if for all $\lambda \geq \kappa$, there is a κ -complete normal fine ultrafilter \mathcal{U} over $P_\kappa(\lambda)$ such that $P_\kappa(\lambda) \cap M \in \mathcal{U}$ and $\mathcal{U} \cap M \in M$.

Woodin and Usuba independently proved the following theorem:

Theorem 3.12. *If M is a weak extender model for the supercompactness of κ , then M is a κ -pseudoground.*

Woodin asked whether the converse holds: if κ is supercompact, must every κ -pseudoground be a weak extender model for the supercompactness of κ ? The author found the following counterexample, based on Magidor's identity crisis [18] as treated by Mitchell [19]:

Theorem 3.13. *Suppose κ is strongly compact. Then there is a κ -pseudoground in which κ is the least measurable cardinal.*

We sketch the proof after Theorem 3.17.

This raises a natural question in the context of strongly compact cardinals. Although Theorem 3.13 shows that a supercompact cardinal need not be supercompact in a κ -pseudoground, a theorem of Hamkins [15] shows that there is no corresponding counterexample for strong compactness. Therefore one might hope that by considering weak extender models for strong compactness, one might obtain a theorem like Theorem 3.12 that is an equivalence.

Definition 3.14. An inner model M is a *weak extender model for the strong compactness of κ* if for every $\lambda \geq \kappa$, there is a κ -complete fine ultrafilter \mathcal{U} over $P_\kappa(\lambda)$ such that $P_\kappa(\lambda) \cap M \in \mathcal{U}$ and $\mathcal{U} \cap M \in M$.

The question becomes whether every weak extender model for the strong compactness of κ is a κ -pseudoground. The answer is again no:

Proposition 3.15. *Suppose κ is strongly compact and M is an inner model. Then M is a weak extender model for the strong compactness of κ if and only if κ is strongly compact in M and M has the κ -cover property.*

Proof. We begin with the forwards direction. To see that κ is strongly compact in M , just note that if \mathcal{U} is a fine ultrafilter on $P_\kappa(\lambda)$ such that $P_\kappa(\lambda) \cap M \in \mathcal{U}$ and $\mathcal{U} \cap M \in M$, then $\mathcal{U} \cap M$ is a fine ultrafilter in M . To see that M has the κ -cover property, fix a cardinal λ and a κ -complete fine ultrafilter \mathcal{U} on λ such that $P_\kappa(\lambda) \cap M \in \mathcal{U}$ and $\mathcal{U} \cap M \in M$. We will show that every $\sigma \in P_\kappa(\lambda)$ is contained in some $\tau \in P_\kappa(\lambda) \cap M$. For any $\sigma \in P_\kappa(\lambda)$ $\{\tau \in P_\kappa(\lambda) : \tau \subseteq \sigma\} \in \mathcal{U}$ by fineness and κ -completeness. Therefore $\{\tau \in P_\kappa(\lambda) : \tau \subseteq \sigma\} \cap M \in M$ since $P_\kappa(\lambda) \cap M \in \mathcal{U}$. In other words, there is some $\tau \in P_\kappa(\lambda) \cap M$ such that $\sigma \subseteq \tau$.

For the converse, fix $\lambda \geq \kappa$. Let \mathcal{W} be a κ -complete fine ultrafilter on $P_\kappa(\lambda)$ that belongs to M . Then by the κ -cover property, \mathcal{W} generates a κ -complete filter F in V . Since κ is strongly compact, F extends to a κ -complete ultrafilter \mathcal{U} . Now \mathcal{U} is a fine ultrafilter, $P_\kappa(\lambda) \cap M \in \mathcal{W} \subseteq \mathcal{U}$, and $\mathcal{U} \cap M = \mathcal{W} \in M$. Since $\lambda \geq \kappa$ was arbitrary, this shows that M is a weak extender model for the strong compactness of κ . \square

Corollary 3.16. *Suppose κ is strongly compact and U is a κ^+ -complete ultrafilter. Then M_U is a weak extender model for the strong compactness of κ , but M_U does not have the κ -approximation property.*

Proof. It is clear that M_U is a weak extender model for the strong compactness of κ by Proposition 3.15 since M_U is closed under κ -sequences. On the other hand, M_U does not have the κ -approximation property by the Laver-Hamkins uniqueness theorem Theorem 2.13, since $H(\kappa^+) \cap M_U = H(\kappa^+)$ yet $M_U \neq V$. \square

Despite Proposition 3.15, we will show that there is a variant of the notion of a weak extender model for strong compactness that coincides with the property of being a pseudoground.

Theorem 3.17. *Suppose κ is strongly compact and M is a model of set theory with the κ -cover property. Then the following are equivalent:*

- (1) *M has the κ -approximation property.*
- (2) *Every κ -complete ultrafilter over a set in M extends an ultrafilter of M .*

For the proof, we need the concept of a *close embedding*. This is a special case of a fine-structural notion introduced by Mitchell-Steel [9]. Its utility in the coarse large cardinal setting was first realized by Woodin [2].

Definition 3.18. Suppose M and N are models of set theory. An elementary embedding $j : M \rightarrow N$ is *close to M* if j is cofinal and every M -ultrafilter derived from j belongs to M .

The author noticed that closeness has a very simple model theoretic characterization that simplifies a number of proofs.

Theorem 3.19. *Suppose M and N are models of set theory and $j : M \rightarrow N$ is an elementary embedding. Then the following are equivalent:*

(1) j is close to M .

(2) For any $A \in N$, $j^{-1}[A] \in M$.

Proof. (1) implies (2): Fix $A \in N$. Since j is cofinal, there is some $X \in M$ such that N satisfies $A \in j(X)$, and let U be the M -ultrafilter over X derived from j using A . Let $i : M \rightarrow P$ be the ultrapower embedding associated with U and let $k : P \rightarrow N$ be the unique factor embedding such that $k \circ i = j$ and $k([\text{id}]_U) = A$. Let $\bar{A} = [\text{id}]_U$. Then since i is definable over M , $i^{-1}[\bar{A}] \in M$. But $i^{-1}[\bar{A}] = (k \circ i)^{-1}[i(\bar{A})] = j^{-1}[A]$.

(2) implies (1): Fix $X \in M$ and $a \in N$ such that N satisfies that $a \in j(X)$. We will show that the M -ultrafilter U over X derived from j using a belongs to N . Indeed, let P be the principal ultrafilter over $j(X)$ concentrated at a , as computed in N . Then $P \in N$ and $U = j^{-1}[P]$.

To see that j is cofinal, fix $a \in N$. Let $\alpha \in \text{Ord}^N$ be least such that N satisfies $a \in V_\alpha$. Let $\bar{\alpha} = j^{-1}[\alpha]$. By (2), $\bar{\alpha} \in M$, and so $\bar{\alpha}$ is an ordinal of M . But $\alpha < j(\bar{\alpha})$: otherwise N satisfies that $\bar{\alpha} \in j^{-1}[\alpha] = \bar{\alpha}$, which is impossible. Therefore N satisfies that $a \in j(X)$ where $X = (V_{\bar{\alpha}})^M$. \square

Notice that Woodin's lemma that close embeddings are closed under composition is completely transparent given this characterization.

Proof of Theorem 3.17. (1) implies (2): This implication is due to Hamkins [15] and does not require that κ is strongly compact. If κ is a strong limit cardinal, we can use Proposition 3.8 to obtain the stronger theorem that every M - κ -complete M -ultrafilter belongs to M . (This seems to be a new result.)

Assume M has the κ -approximation property. Let U be a κ -complete ultrafilter over $X \in M$. We must show that $U \cap M \in M$. It suffices to show that $U \cap M$ is κ -approximated by M . Suppose $\sigma \subseteq P^M(X)$ such that $|\sigma| < \kappa$. We want to show that $U \cap \sigma \in M$. Clearly it suffices to prove this in the case that σ is closed under relative complements in X . By the κ -completeness of U , $\bigcap(U \cap \sigma) \neq \emptyset$, so fix $x \in \bigcap(U \cap \sigma)$. Then since σ is closed under complements, $U \cap \sigma = \{A \in \sigma : x \in A\}$. It follows that $U \cap \sigma \in M$.

(2) implies (1): Suppose $X \in M$ and $A \subseteq X$ is κ -approximated by M . Let $j : V \rightarrow N$ be an elementary embedding with critical point κ such that $j[X]$ is contained in a set $B \in N$ with $|B|^N < j(\kappa)$. By replacing B with $B \cap j(M)$, we may assume without loss of generality that $B \subseteq j(M)$. Since $j(M)$ has the $j(\kappa)$ -cover property in N , there is some $C \in j(M)$ with $|C|^N < j(\kappa)$ such that $B \subseteq C$. Since $j(A)$ is $j(\kappa)$ -approximated by $j(M)$, $j(A) \cap C \in j(M)$. We have assumed that $U \cap M \in M$ for every κ -complete ultrafilter U over a set in M ; therefore for every M -ultrafilter W derived from j , $W \in M$. In other words, j is close to M . Therefore by Theorem 3.19, $j^{-1}[j(A) \cap C] \in M$. But $j^{-1}[j(A) \cap C] = A$, so $A \in M$, as desired. \square

Using Theorem 3.17, we prove Theorem 3.13.

Sketch of Theorem 3.13. Let Meas denote the class of measurable cardinals. Choose $\vec{U} = \langle U_\delta : \delta \in \text{Meas} \cap \kappa \rangle$ such that U_δ is a normal ultrafilter on δ with $\text{Meas} \cap \delta \notin U_\delta$. We define a sequence of ordinals $\langle \delta_\alpha : \alpha < \kappa \rangle$ and an iterated ultrapower

$$\langle M_\alpha, W_\beta, j_{\beta, \alpha} : \beta < \alpha \leq \kappa \rangle$$

by simultaneous recursion, letting δ_α be the least measurable cardinal δ of M_α such that the set of preimages $\{\beta < \alpha : j_{\beta, \alpha}(\delta_\beta) = \delta\}$ of δ is finite. Let $W_\alpha = j_{0\alpha}(\vec{U})_{\delta_\alpha}$. The rest of the data of the iterated ultrapower is uniquely determined by the sequence $\langle W_\alpha : \alpha < \kappa \rangle$ in the usual way.

For $\gamma \leq \kappa$, let $G_\gamma = \{\delta_\alpha : \alpha < \gamma\}$. One can show that κ is the least measurable cardinal of $M_\kappa[G_\kappa]$. The proof appears in [19, Theorem 1.2].

It is easy to see that M_κ has the κ -cover property: this follows from the fact that $j_{0\kappa} = \kappa$ and the proof of Proposition 3.10. Since $M_\kappa \subseteq M_\kappa[G]$, $M_\kappa[G]$ has the κ -cover property as well.

To finish we must show that $M_\kappa[G]$ has the κ -approximation property. By Theorem 3.17, it suffices to show that for any elementary embedding $i : V \rightarrow N$ such that $N^\kappa \subseteq N$ and $\text{crit}(i) \geq \kappa$, $i \upharpoonright M_\kappa$ is amenable to M_κ . The proof is due to Mitchell and appears in [19, Theorem 1.2]. We have

$$i(j_{0\kappa}) = (j_{0, i(\kappa)})^N = j_{\kappa, i(\kappa)}^N \circ j_{0\kappa} \upharpoonright N$$

The final equality uses the κ -closure of N to deduce that $(j_{0\kappa})^N = j_{0\kappa} \upharpoonright N$. We claim

$$i \upharpoonright M_\kappa = j_{\kappa, i(\kappa)}^N \circ j_{0\kappa}(i)$$

We have

$$i \circ j_{0\kappa} = i(j_{0\kappa}) \circ i = j_{\kappa, i(\kappa)}^N \circ j_{0\kappa} \circ i = j_{\kappa, i(\kappa)}^N \circ j_{0\kappa}(i) \circ j_{0\kappa}$$

In other words, i and $j_{\kappa, i(\kappa)}^N \circ j_{0\kappa}(i)$ agree on $j_{0\kappa}[V]$. They also agree on κ , since both embeddings are the identity on κ . Since $M_\kappa = H^{M_\kappa}(j_{0\kappa}[V] \cup \kappa)$, it follows that $i = j_{\kappa, i(\kappa)}^N \circ j_{0\kappa}(i)$, as claimed. As a consequence, i is amenable to M_κ .

Note that $i(G) = G \cup H$ where H is the sequence of indiscernibles generated by $j_{\kappa, i(\kappa)}^N$ in much the same way that G is generated from $j_{0\kappa}$. Therefore $i(G) \in M_\kappa[G]$. It follows that i is amenable to $M_\kappa[G]$: every element of $M_\kappa[G]$ is Σ_2 -definable from parameters in $M_\kappa \cup \{G\}$, and $i \upharpoonright M_\kappa \cup \{G\}$ is amenable to $M_\kappa[G]$.

In particular, it follows that for every κ -complete ultrafilter U over a set X in M , $j_U \upharpoonright M$ is amenable to M , and therefore $U \cap M \in M$, since $U \cap M$ is the M -ultrafilter over X derived from $j_U \upharpoonright M$ using $[\text{id}]_U$. Applying Theorem 3.17, it follows that M has the κ -approximation property. \square

Combining Theorem 3.7, Proposition 3.10, and Theorem 3.17 immediately yields that small ultrapowers are pseudogrounds:

Corollary 3.20. *Suppose κ is a strongly compact cardinal and $D \in V_\kappa$ is an ultrafilter. Then M_D has the κ -approximation property. \square*

In fact, by generalizing the Commuting Ultrapowers Lemma (Theorem 3.6) to work in the case where one embedding is an external extender, one can prove a much stronger result:

Corollary 3.21. *Suppose κ is a strongly compact cardinal. Suppose N is κ -pseudoground and E is an N -extender in V_κ . Then M_E^N is a κ -pseudoground. \square*

It is natural to ask whether one can drop the cover assumption in Theorem 3.17. Suppose there is a proper class of extendible cardinals and M is an inner model such that every sufficiently complete ultrafilter extends an ultrafilter of M . Must M be a pseudoground? The answer is probably no, but our next theorem reaches in this direction:

Theorem 3.22. *Suppose κ is strongly compact and M is an inner model. Then the following are equivalent:*

- (1) M is a κ -pseudoground.
- (2) κ is strongly compact in M and the following hold:
 - Every κ -complete ultrafilter over a set in M extends an ultrafilter of M .
 - Every κ -complete ultrafilter of M extends to a κ -complete ultrafilter of V .
- (3) Every regular cardinal of M above κ has cofinality at least κ and every κ -complete ultrafilter over a set in M extends an ultrafilter of M .

Proof. (1) implies (2). The first bullet is just the theorem of Hamkins proved in Theorem 3.17. The second bullet uses Proposition 3.8 to conclude that M - κ -complete ultrafilters generate κ -complete filters.

The fact that κ is strongly compact in M is also due to Hamkins. We give a different proof. By the κ -cover property, the M -filter over $P_\kappa(\lambda) \cap M$ generated by sets of the form $\{\sigma \in P_\kappa(\lambda) \cap M : \alpha \in \sigma\}$ generates a κ -complete filter, and therefore extends to a κ -complete ultrafilter of V , which in turn extends a κ -complete ultrafilter \mathcal{W} of M ; \mathcal{W} is fine and therefore witnesses that κ is λ -strongly compact in M .

(2) implies (3): Fix an M -regular cardinal δ above κ . Since κ is strongly compact in M , M satisfies that there is a κ -complete uniform ultrafilter U on δ . Since U extends to a κ -complete ultrafilter, δ must have cofinality at least κ : indeed, any ordinal that carries a κ -complete fine ultrafilter is necessarily of cofinality at least κ .

(3) implies (1): By Theorem 3.17, it suffices to prove that M has the κ -cover property. Fix a regular uncountable cardinal δ . We will find a set $X \in M$ with $|X|^M = \delta$ and a κ -complete fine ultrafilter over $P_\kappa(X)$ such that $P_\kappa(X) \cap M \in \mathcal{U}$. It then follows immediately that M has the κ -cover property at X : indeed, if $\sigma \in P_\kappa(X)$, then $\{\tau \in P_\kappa(X) : \sigma \subseteq \tau\} \in \mathcal{U}$ since \mathcal{U} is κ -complete and fine, so $\{\tau \in P_\kappa(X) : \sigma \subseteq \tau\} \cap M$ is nonempty since $P_\kappa(X) \cap M \in \mathcal{U}$. In other words, there is some $\tau \in P_\kappa(X) \cap M$ with $\sigma \subseteq \tau$. This proves that M has the κ -cover property

at X . Since we can make X arbitrarily large, it follows that M has the κ -cover property.

Let $j : V \rightarrow N$ be an elementary embedding with critical point κ such that for some $B \in N$ with $|B|^N < j(\kappa)$, $j[\delta] \subseteq B$. In particular, N satisfies that $\text{cf}(\sup j[\delta]) < j(\kappa)$. It follows that $j(M)$ satisfies that $\text{cf}(\sup j[\delta]) < j(\kappa)$: if $\text{cf}^{j(M)}(\sup j[\delta]) \geq j(\kappa)$, then by our assumption shifted to $j(M)$, its cofinality in N is at least $j(\kappa)$, a contradiction. Therefore we may fix a closed unbounded set $C \subseteq \sup j[\delta]$ such that $C \in j(M)$ and $|C|^{j(M)} < j(\kappa)$.

Since every κ -complete ultrafilter over a set in M extends an ultrafilter of M , the embedding j is close to M . Therefore by Theorem 3.19, since $C \in j(M)$, $j^{-1}[C] \in M$.

Let $X = j^{-1}[C]$. Since $j[\delta]$ and C are ω -closed unbounded subsets of the ordinal $\sup j[\delta]$ (which has uncountable cofinality), $j[\delta] \cap C$ is unbounded in $\sup j[\delta]$. But $j[X] = j[\delta] \cap C$, so X must also be unbounded in δ . Therefore since δ is regular, $|X| = \delta$.

Let $D = C \cap j(X)$. Then $D \in j(M)$, $j[X] \subseteq D$, $D \subseteq j(X)$, and $|D|^{j(M)} < j(\kappa)$. In particular, $D \in j(P_\kappa(X))$, so it makes sense to derive an ultrafilter \mathcal{U} over $P_\kappa(X)$ from j using D . Since j has critical point κ , \mathcal{U} is κ -complete. Since $j[X] \subseteq D$, \mathcal{U} is fine. Since $D \in j(M)$, $P_\kappa(X) \cap M \in \mathcal{U}$ by the definition of a derived ultrafilter.

Thus we have found a set X of cardinality δ and a κ -complete fine ultrafilter \mathcal{U} over $P_\kappa(X)$ such that $P_\kappa(X) \cap M \in \mathcal{U}$. This completes the proof. \square

Of course, Theorem 3.22 (3) is equivalent to the statement that ultrafilters extend ultrafilters of M and M correctly computes the class of cardinals of cofinality less than κ .

Corollary 3.23. *Suppose κ is strongly compact and M is an inner model. Assume M is cardinal correct and every κ -complete ultrafilter over a set in M extends an ultrafilter of M . Then M is a κ -pseudoground.*

Proof. We will show that any M -regular cardinal δ greater than or equal to κ has cofinality at least κ , which by Theorem 3.22 implies the corollary.

A theorem of Viale [20, Theorem 27] states that if κ is strongly compact, M is an inner model, and $\lambda \geq \kappa$ is an M -regular cardinal such that $\lambda^{+M} = \lambda^+$, then $\text{cf}(\lambda) \geq \kappa$.

Now fix an M -regular cardinal $\delta \geq \kappa$. Since $\delta^{+M} = \delta^+$, $\text{cf}(\delta) \geq \kappa$ by Viale's theorem. \square

An immediate consequence of Theorem 3.17 is the transitivity of the pseudoground order:

Corollary 3.24. *If κ is a strongly compact cardinal, then a κ -pseudoground of a κ -pseudoground is a κ -pseudoground.* \square

Corollary 3.25. *Suppose κ is a strongly compact cardinal and M is a κ -pseudoground of V . Assume $\mathbb{P} \in V_\kappa \cap M$ is a partial order and $G \subseteq \mathbb{P}$ is an M -generic filter that belongs to V . Then $M[G]$ is a κ -pseudoground of V .*

Proof. Clearly $M[G]$ has the κ -cover property. We will show that $M[G]$ inherits κ -complete ultrafilters. It easily suffices to show that $M[G]$ inherits κ -complete ultrafilters over sets in M . Suppose U is a κ -complete ultrafilter over a set in M . We must show that $U \cap M[G] \in M[G]$. Note that $U \cap M \in M$ by Theorem 3.17. By the Lévy-Solovay theorem, the filter W generated by $U \cap M$ in $M[G]$ is an $M[G]$ -ultrafilter. Clearly $W \in M[G]$ since $U \cap M \in M[G]$. But since $W \subseteq U \cap M[G]$, in fact $W = U \cap M[G]$ (by the maximality of ultrafilters). \square

It is tempting to doubt that this corollary really requires a strongly compact cardinal, but the following fact comes close to showing that this hypothesis is optimal:

Proposition 3.26. *Suppose $\kappa \leq \lambda$ are regular cardinals and κ is $<\lambda$ -strongly compact. Then there is a cofinality preserving forcing extension N of V with the following properties:*

- κ remains $<\lambda$ -strongly compact in N .
- V is an ω_1 -pseudoground of N .
- For some V -generic Cohen real $g \in N$, $V[g]$ does not have the λ -approximation property in N .

Proof. Let g be a V -generic Cohen real and $G \subseteq (\text{Add}(\lambda, 1))^{V[g]}$ be $V[g]$ -generic. Let $N = V[g][G]$. Then V has the ω_1 -approximation and cover properties in N by a theorem of Hamkins [17, Lemma 13], and obviously $V[g]$ does not have the λ -approximation property in N , since by the λ -closure of $(\text{Add}(\lambda, 1))^{V[g]}$, N satisfies that G is λ -approximated by $V[g]$, and yet $G \notin V[g]$ by genericity.

The Lévy-Solovay theorem implies that κ remains $<\lambda$ -strongly compact in $V[g]$. The λ -closure of $(\text{Add}(\lambda, 1))^{V[g]}$ therefore implies that κ remains $<\lambda$ -strongly compact in N : every N -regular cardinal δ with $\kappa \leq \delta < \lambda$ carries a uniform κ -complete ultrafilter in $V[g]$, which is in fact an N -ultrafilter since $P^N(\delta) = P^{V[g]}(\delta)$. \square

4 Silver's question

4.1 Indecomposable ultrafilters

Let us reintroduce some concepts defined in the introduction.

Definition 4.1. Suppose $\delta < \lambda$ are cardinals. An ultrafilter U over a set X is (δ, λ) -*indecomposable* if any partition $\langle A_\nu \rangle_{\nu < \alpha}$ of X with $\alpha < \lambda$ has a subsequence $\langle A_{\nu_\xi} \rangle_{\xi < \beta}$ with $\beta < \delta$ whose union belongs to U .

The following combinatorial characterization of indecomposability is sometimes convenient.

Definition 4.2. Suppose U is an ultrafilter over a set X and γ is a cardinal. A γ -*decomposition* of U is a function $f : X \rightarrow \gamma$ such that for any $A \in U$, $f[A]$ has cardinality γ ; U is γ -*decomposable* if it has a γ -decomposition, and U is γ -*indecomposable* otherwise.

Thus U is γ -indecomposable if and only if U is (γ, γ^+) -indecomposable and U is (δ, λ) -indecomposable if and only if U is not γ -decomposable for any cardinal γ such that $\delta \leq \gamma < \lambda$.

The concept of a γ -decomposition is best understood in terms of pushforwards (Definition 2.7). Notice that $f : X \rightarrow \gamma$ is a γ -decomposition of U if and only if the pushforward $f_*(U)$ is a uniform ultrafilter over γ . Thus U is γ -decomposable if and only if U pushes forward to a uniform ultrafilter over γ .

We remark that if γ is regular, then an ultrafilter U is γ -indecomposable if and only if U is closed under intersections of descending sequences of sets of length γ .

The following theorem, due to Silver, is a key element in all of our applications:

Theorem 4.3 (Silver). *Suppose δ and κ are cardinals with δ regular and $2^\delta < \kappa$. Suppose U is a (δ, κ) -indecomposable ultrafilter over a set X . Then $j_U = j_W^{M_D} \circ j_D$ where D is an ultrafilter over a cardinal less than δ and W is an M_D - κ -complete M_D -ultrafilter over $j_U(X)$. \square*

Silver [3, Lemma 2] sketches a proof in the case that $\delta = \omega_1$. The author's thesis [1, Theorem 7.5.24] contains a more detailed proof, assuming for superficial reasons that U is countably complete.

4.2 Silver's question above a strongly compact

As an immediate consequence of Theorem 3.7 and Silver's factorization theorem (Theorem 4.3), we obtain a factorization theorem for indecomposable ultrafilters:

Theorem 4.4. *Suppose $\delta < \kappa$ are cardinals and X is a set. Assume δ is regular, κ is a strong limit cardinal, and every κ -complete filter over X extends to a δ -complete ultrafilter. Suppose U is a (δ, κ) -indecomposable ultrafilter over X . Then*

$$j_U = (j_W)^{M_D} \circ j_D$$

where D is an ultrafilter over a cardinal less than δ and W is a κ -complete ultrafilter of M_D over $j_D(X)$.

Proof. Applying Silver's Theorem (Theorem 4.3) to U yields that $j_U = j_W^{M_D} \circ j_D$ where D is an ultrafilter over a cardinal less than δ and W is an M_D -ultrafilter over $j_D(\lambda)$ that is M_D - κ -complete. By Theorem 3.7, $W \in M_D$. Moreover by Proposition 3.8 and Proposition 3.10, W is κ -complete. \square

Theorem 4.5. *Suppose $\delta < \kappa \leq \lambda$ are cardinals. Assume δ is regular, κ is a strong limit cardinal and every κ -complete filter over λ extends to a δ -complete ultrafilter. Suppose there is a (δ, λ) -indecomposable ultrafilter over λ . Then λ either is a measurable cardinal or λ has cofinality less than δ and λ is a limit of measurable cardinals.*

Proof. Applying Theorem 4.4,

$$j_U = (j_W)^{M_D} \circ j_D$$

where D is an ultrafilter over a cardinal less than δ and W is a κ -complete ultrafilter of M_D over $j_D(\lambda)$.

Since U is (δ, λ) -indecomposable, W must be $(j_D(\delta), j_D(\gamma))$ -indecomposable in M_D for all cardinals $\gamma < \lambda$. To see this, fix an M_D -cardinal η and an η -decomposition $f : j_D(\lambda) \rightarrow \eta$ of W in M_D . Let γ be the least cardinal such that $j_D(\gamma) \geq \eta$, and assume $\gamma < \lambda$. We must show that $\eta < j_D(\delta)$. Let $g : \lambda \rightarrow \gamma$ be such that $[g]_U = [f]_W^{M_D}$. Since W is γ -indecomposable, there is a set $A \in U$ such that $|g[A]| < \gamma$.

Since γ is least such that $j_D(\gamma) \geq \eta$, $j_D(|g[A]|) < \eta$. Since

$$[f]_W^{M_D} = j_U(g)([id]_U) \in j_U(g[A]) = j_W^{M_D}(j_D(g[A]))$$

Loś's Theorem implies that $f(\alpha) \in j_D(g[A])$ for all α in a W -large set B . In other words, $f[B] \subseteq j_D(g[A])$, and hence $|f[B]|^{M_D} \leq j_D(|g[A]|) < \eta$. This contradicts that f is an η -decomposition in M_D .

Let ρ be the completeness of W as computed in M_D , the least M_D -cardinal such that W is not ρ -complete in M_D . The completeness of a countably complete ultrafilter is always a measurable cardinal, and so since W is κ -complete, $\rho \geq \kappa$ and ρ is measurable. Moreover, W is ρ -decomposable, so since W is $(j_D(\delta), j_D(\gamma))$ -indecomposable for all $\gamma < \lambda$, ρ must be greater than $j_D(\gamma)$ for all $\gamma < \lambda$. On the other hand, since W is an M_D -ultrafilter over λ , W is γ -indecomposable for all M_D -cardinals greater than λ , and hence $\rho \leq j_D(\lambda)$.

Assume first that λ has cofinality at least δ . Then $j_D(\lambda) = \sup j_D[\lambda]$. This is a general fact; to see it let ι be the underlying set of D , and note that any $\alpha < j_D(\lambda)$ is equal to $[f]_U$ for some function $f : \iota \rightarrow \lambda$, and $\text{ran}(f) \subseteq \beta$ for some $\beta < \lambda$ since $\iota < \delta \leq \text{cf}(\lambda)$, so $\alpha < j_D(\beta)$. Since $j_D(\lambda) = \sup j_D[\lambda]$ and $j_D(\gamma) < \rho \leq j_D(\lambda)$ for all $\gamma < \lambda$, $\rho = j_D(\lambda)$. Therefore $j_D(\lambda)$ is measurable in M_D , and so λ is measurable by elementarity. This proves the theorem in the case that λ has cofinality at least δ .

Otherwise, λ has cofinality less than δ . We finish by proving that in this case λ is a limit of measurable cardinals. Since ρ is regular in M_D , $\rho \neq j_D(\lambda)$. Therefore $\rho < j_D(\lambda)$. This is a standard reflection argument. Suppose $\gamma < \lambda$. We will show that there is a measurable cardinal between γ and λ . Of course, M_D satisfies that there is a measurable cardinal between $j_D(\gamma)$ and $j_D(\lambda)$, namely ρ . Therefore by elementarity, there is a measurable cardinal between γ and λ , as desired. \square

We state a special case which answers Silver's question above the least strongly compact cardinal:

Theorem 4.6. *Suppose λ is greater than or equal to the least strongly compact cardinal and carries an indecomposable ultrafilter. Either λ is measurable or else $\text{cf}(\lambda) = \omega$ and λ is a limit of measurable cardinals.* \square

From Theorem 4.4, one can extract a topological characterization of indecomposable ultrafilters above the least strongly compact cardinal. Recall that $\beta(\lambda)$ denotes the space of ultrafilters on λ with the Stone-Ćech topology.

Theorem 4.7. *Suppose λ is greater than or equal to the least strongly compact cardinal and U is an ultrafilter over λ . Then the following are equivalent:*

- U is indecomposable.
- Either U is λ -complete or U lies in the closure of a countable discrete set of ultrafilters $S \subseteq \beta(\lambda)$ such that for any $\gamma < \lambda$, all but finitely many ultrafilters in S are γ^+ -complete. \square

5 Almost strong compactness

The principles of Bagaria-Magidor laid out in Section 2.3 offer a spectrum of strong compactness properties. In a perfect world (for example, assuming UA), these would be characterized in terms of the classical notion of strong compactness. But given Bagaria-Magidor's theorem [13] that it is consistent with ZFC that the first ω_1 -strongly compact cardinal is singular, it is natural to wonder whether there are any nontrivial relationships between these principles at all. The results of this section show that there are subtle implications between classical strong compactness and Bagaria-Magidor's notion of almost strong compactness.

5.1 Decomposability spectra

Our results in this section make use of the observation that assuming the Singular Cardinals Hypothesis, countably complete ultrafilters have very simple *decomposability spectra*, a concept first studied by Lipparini.

Definition 5.1. If U is an ultrafilter, the *decomposability spectrum* of U , denoted K_U , is the set of all cardinals λ such that U is λ -decomposable.

We use the following theorem of Lipparini:

Theorem 5.2 (Lipparini). *Suppose U is an ultrafilter and $(\lambda_\alpha)_{\alpha < \eta}$ is an increasing sequence of infinite cardinals in K_D . Then there is a cardinal $\delta \in K_D$ with $\sup_{\alpha < \eta} \lambda_\alpha \leq \delta \leq \prod_{\alpha < \eta} \lambda_\alpha$.*

Proof. For each $\alpha < \eta$, choose a λ_α -decomposition f_α of U . Thus f_α is a function from the underlying set X of U to λ_α . Define $f : X \rightarrow \prod_{\alpha < \eta} \lambda_\alpha$ by $f(x) = (f_\alpha(x))_{\alpha < \eta}$.

Fix $A \in U$ such that $|f[A]| = \delta$ is as small as possible. Note that $\delta \leq \prod_{\alpha < \eta} \lambda_\alpha$. Moreover, for all $\alpha < \eta$, $\delta \geq \lambda_\alpha$: for $A \in U$, $|f[A]| \geq |f_\alpha[A]| \geq \lambda_\alpha$ since f_α is a λ_α -decomposition.

Let $p : \prod_{\alpha < \eta} \lambda_\alpha \rightarrow \delta$ be injective on A and 0 on the complement of A . Then $p \circ f$ is a δ -decomposition of U . \square

Lemma 5.3 (SCH). *Suppose U is a countably complete ultrafilter. Suppose K_U is unbounded below a limit cardinal λ . Then all sufficiently large regular cardinals less than λ belong to K_U .*

For this, we need a well-known fact, a special case of a more general theorem of Ketonen [21]. For one approach, see [1, Theorem 7.2.12].

Lemma 5.4. *Suppose U is a θ^+ -complete ultrafilter, λ is a singular cardinal of cofinality θ , and $\lambda^+ \in K_U$. Then all sufficiently large regular cardinals below λ are in K_U . \square*

Proof of Lemma 5.3. We first handle the case in which λ has countable cofinality. Assume towards a contradiction that the lemma fails. Let $(\lambda_n)_{n < \omega}$ be a sequence of cardinals unbounded in λ such that $\lambda_n \notin K_U$ for all $n < \omega$. By Theorem 5.2 there is some $\delta \in K_U$ with $\sup_{n < \omega} \lambda_n \leq \delta \leq \prod_{n < \omega} \lambda_n$. Since U is countably complete, $\delta \neq \sup_{n < \omega} \lambda_n$. Note however that $\prod_{n < \omega} \lambda_n \leq \lambda^\omega = \lambda^+$ by SCH. Therefore $\delta = \lambda^+$. Since U is countably complete, λ has countable cofinality, and $\lambda^+ \in K_U$, Lemma 5.4 implies that all sufficiently large regular cardinals less than λ belong to K_U .

Now we take on the case that λ has uncountable cofinality. Let S be the set of limit points of K_U of countable cofinality. Then S is ω -closed unbounded in λ . Define $f : S \rightarrow \lambda$ by setting $f(\alpha)$ equal to the least $\gamma < \alpha$ such that every regular cardinal between γ and α belongs to K_U . Note that there is such a cardinal γ by the previous case. The function f is nondecreasing and regressive, so there is some $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for all sufficiently large $\alpha < S$. In other words, every regular cardinal between γ and κ belongs to K_U , as desired. \square

5.2 On the next almost strongly compact cardinal

To discover the nontrivial relationships between compactness principles, one must first dispense with the trivial ones. For example, any limit of strongly compact cardinals is almost strongly compact. One is therefore led to ask whether every almost strongly compact cardinal is either strongly compact or a limit of strongly compacts. This is provable under the Ultrapower Axiom ([1, Proposition 8.3.7]), but it is conceivable that this hypothesis is unnecessary.

There is an easy characterization of precisely those almost strongly compact cardinals that are strongly compact, essentially due to Menas, although he proved it before the concept of an almost strongly compact cardinal had been formulated:

Theorem 5.5 (Menas). *An almost strongly compact cardinal is strongly compact if and only if it is measurable. \square*

This theorem would seem to be optimal (since after all it is an equivalence). We will show, however, that there are a priori weaker notions than measurability that suffice to conclude that an almost strongly compact cardinal is strongly compact.

Proposition 5.6 (SCH). *Suppose ν is a cardinal such that the least (ν, ∞) -strongly compact cardinal κ is almost strongly compact. Then κ is strongly compact.*

Proof. By Theorem 5.5, it suffices to show that κ is measurable. Let U be a ν -complete uniform ultrafilter over κ^+ . We claim that K_U is bounded below κ . Assume K_U is unbounded, towards a contradiction. By Lemma 5.3 (using our SCH

assumption), there is some cardinal $\eta < \kappa$ such that every regular cardinal δ with $\eta \leq \delta < \kappa$ is in K_U . In other words, U can be pushed forward to a uniform ultrafilter on δ for every regular cardinal δ with $\eta \leq \delta < \kappa$. A pushforward of U is necessarily ν -complete. It follows that every regular cardinal greater than or equal to η carries a ν -complete uniform ultrafilter. Therefore by Ketonen's Theorem (Theorem 2.19), η is (ν, ∞) -strongly compact, and this contradicts the fact that κ is the least (ν, ∞) -strongly compact cardinal.

Since K_U is bounded below κ , we can now apply Theorem 4.4 to factor U : thus $j_U = (j_W)^{M_D} \circ j_D$ where D is an ultrafilter on a cardinal $\gamma < \kappa$ and $W \in M_D$ is a uniform κ -complete M_D -ultrafilter over $j_D(\kappa^+)$.

We will show that κ is regular. Given this, we can conclude the proposition by the following argument. Since D lies on a cardinal less than κ and κ is regular, $j_D(\kappa) = \sup j_D[\kappa]$, and so every M_D -cardinal $\delta < j_D(\kappa)$ has true cardinality strictly less than κ . Therefore since W is κ -complete, W is $j_D(\kappa)$ -complete in M_D . By elementarity, κ^+ carries a κ -complete uniform ultrafilter, as desired.

To finish, we show κ is regular. Suppose towards a contradiction that κ is singular, and let $\theta = \text{cf}(\kappa)$. Since M_D satisfies that W is M_D - $j_D(\theta^+)$ -complete and $j_D(\kappa^+)$ -decomposable, $(K_W)^{M_D}$ contains all sufficiently large M_D -regular cardinals below $j_D(\kappa)$ by Lemma 5.4. Therefore W witnesses that there is a $(j_D(\nu), \infty)$ -strongly compact cardinal below $j_D(\kappa)$ in M_D , contradicting the definition of κ (since j_D is elementary). \square

At first glance, the proof appears to show that the least (ν, ∞) -strongly compact cardinal is always strongly compact, but by a theorem of [13], one cannot prove (assuming ZFC + GCH) that the least (ω_1, ∞) -strongly compact cardinal is *regular*. Where does Proposition 5.6 use the almost strong compactness of κ ? The answer is that this hypothesis is required to apply Theorem 4.4.

Theorem 5.7 (SCH). *Suppose κ is an almost strongly compact cardinal of uncountable cofinality. Then one of the following holds:*

- κ is a strongly compact cardinal.
- κ is the successor of a strongly compact cardinal.
- κ is a limit of almost strongly compact cardinals.

Proof. We may assume that κ is not a limit of almost strongly compact cardinals. We may also assume that κ is a limit cardinal.

Let $\delta < \kappa$ be the supremum of the almost strongly compact cardinals below κ . For each $\alpha < \kappa$ with $\alpha > \delta$, let $f(\alpha)$ be the least cardinal ν such that α is not (ν, ∞) -strongly compact. The function $f : \kappa \setminus \delta \rightarrow \kappa$ is regressive and nondecreasing, so since κ has uncountable cofinality, f assumes a constant value at all sufficiently large ordinals below κ . In other words, there is a cardinal $\nu < \kappa$ and an ordinal α_0 such that for all $\alpha > \alpha_0$, $f(\alpha) = \nu$. Thus κ is the least (ν, ∞) -strongly compact cardinal. By Proposition 5.6, κ is strongly compact. \square

Theorem 5.8. *For any ordinal α , if the $(\alpha + 1)$ -st almost strongly compact limit cardinal has uncountable cofinality, it is strongly compact.*

Proof. Let κ be the $(\alpha + 1)$ -st almost strongly compact limit cardinal and assume that κ has uncountable cofinality. Note that κ is not the least almost strongly compact cardinal. We work in the collapse forcing extension N of V in which the least strongly compact is countable. Notice that SCH holds in N since SCH holds above the least almost strongly compact cardinal in V . Therefore we can apply Theorem 5.7 to conclude that κ is strongly compact in N . By Lévy-Solovay, it follows that κ is strongly compact in V .

If one wants to avoid forcing, one can just check that all the previous theorems go through under the assumption that SCH holds at all sufficiently large cardinals below κ . \square

6 Cardinal preserving embeddings

6.1 Strong compactness and the Kunen inconsistency

Kunen famously proved the inconsistency of Reinhardt’s “ultimate large cardinal axiom” asserting the existence of an elementary embedding from the universe of sets to itself. From a technical perspective, the Kunen inconsistency places a bound on the degree of supercompactness an elementary embedding $j : V \rightarrow M$ can exhibit: there is always some $\lambda < \kappa_\omega(j)$ such that $M^\lambda \not\subseteq M$. Here $\kappa_\omega(j)$ is the supremum of the critical sequence of j :

Definition 6.1. Suppose $j : M \rightarrow N$ is an elementary embedding between two transitive models of set theory. The *critical sequence of j* is the sequence $\langle \kappa_n(j) \rangle_{n < \omega}$ defined by setting $\kappa_n(j) = j^{(n)}(\text{crit}(j))$. The ordinal $\kappa_\omega(j)$ is the supremum of the critical sequence of j .

A natural (vague) question is whether there is a similar inconsistency theorem for strong compactness, or in other words, a limitation on the covering properties of inner models M such that there is an elementary embedding $j : V \rightarrow M$.

For example, one might ask whether there can be an elementary embedding $j : V \rightarrow M$ where M is an inner model that has the λ -cover property for all cardinals λ ; in other words, every $A \subseteq M$ is contained in some $B \in M$ with $|B| = |A|$. The answer to this question, perhaps surprisingly, is yes. Suppose U is a κ -complete ultrafilter over κ . If $2^\kappa > \kappa^+$, then M_U does not have the κ^+ -cover property, but assume instead that the Generalized Continuum Hypothesis holds. Then M_U has the λ -cover property for every cardinal λ . To see this, it suffices to see that for any cardinal λ , $j_U[\lambda]$ is covered by a set $A \in M_U$ with $|A| = \lambda$. In fact, we can just take $A = \sup j_U[\lambda]$.

A second question is whether there can be an elementary embedding $j : V \rightarrow M$ where M is an inner model with the *tight cover property* at every cardinal: every $A \subseteq M$ is contained in some $B \in M$ with $|B|^M = |A|$. (This is easily equivalent to the question of whether there can be an embedding $j : V \rightarrow M$ where M is an inner

model with the λ -cover property for all cardinals λ as in the previous paragraph, with the additional requirement that M and V have the same cardinals.) The answer here is an easy *no*.

First note that M must be closed under ω -sequences. To see this, fix a countable set $\sigma \subseteq M$. Let $\tau \in M$ be an M -countable set containing σ , and let $f : \omega \rightarrow \tau$ be a surjection. Let $x = f^{-1}[\sigma]$. Since $j : V \rightarrow M$ is elementary and $j(\omega) = \omega$, $x = j(x) \in M$. Since $\sigma = f[x]$, $\sigma \in M$.

We now reach a contradiction following Zapletal's proof of the Kunen inconsistency. Now let $\lambda = \kappa_\omega(j)$. Applying Shelah's Representation Theorem [22], there exist regular cardinals $\langle \delta_n \rangle_{n < \omega}$ cofinal in λ for which there is a scale, or in other words an increasing cofinal sequence $\langle f_\alpha \rangle_{\alpha < \lambda^+}$ in the preorder $(\prod_{n < \omega} \delta_n, <^*)$. Here $f <^* g$ if $f(n) < g(n)$ for all but finitely many $n < \omega$. Let $\langle g_\alpha \rangle_{\alpha < \lambda^+} = j(\langle f_\alpha \rangle_{\alpha < \lambda^+})$. By elementarity $\langle g_\alpha \rangle_{\alpha < \lambda^+}$ is a scale for $\langle j(\delta_n) \rangle_{n < \omega}$ in M , but since M is closed under countable sequences, this is upwards absolute to V . Since $j[\lambda^+]$ is cofinal in λ^+ , $\langle g_\alpha \rangle_{\alpha \in j[\lambda^+]}$ is also a scale for $\langle j(\delta_n) \rangle_{n < \omega}$. Of course $g_{j(\alpha)} = j(f_\alpha)$, so $\langle j(f_\alpha) \rangle_{\alpha < \lambda^+}$ is a scale for $\langle j(\delta_n) \rangle_{n < \omega}$.

Finally let $h = \langle \sup j[\delta_n] \rangle_{n < \omega}$. We have $\sup j[\delta_n] < j(\delta_n)$ since $j(\delta_n)$ is a regular cardinal larger than δ_n . Therefore $h \in \prod_{n < \omega} j(\delta_n)$. But $j(f_\alpha) < h$ for all $\alpha < \lambda^+$, $g_\alpha = j \circ f_{j^{-1}(\alpha)}$ for any $\alpha \in j[\lambda^+]$. This contradicts that $\langle g_\alpha : \alpha \in j[\lambda^+] \rangle$ is cofinal in $(\prod_{n < \omega} j(\delta_n), <^*)$.

6.2 Strongly discontinuous embeddings

The preceding proof shows that the tight cover property is not really the right notion in this context. A much more difficult question seems to be whether there can be an elementary embedding $j : V \rightarrow M$ such that for all cardinals λ , $j[\lambda]$ is contained in a set in M of M -cardinality λ . We call this the *local tight cover property* because for any $a \in M$ and any cardinal λ , j factors as $V \xrightarrow{i} N \xrightarrow{k} M$ where N has the tight cover property at λ and $a \in \text{ran}(k)$.

We will show that if there is a proper class of strongly compact cardinals, then no such embedding can exist. In fact, our proof rules out the weaker concept of a strongly discontinuous embedding:

Definition 6.2. Suppose $j : V \rightarrow M$ is an elementary embedding. Then j is *strongly discontinuous* if for all cardinals λ , if $j(\lambda^+) \neq \lambda^+$ then $j[\lambda^+]$ is bounded below $j(\lambda^+)$.

Obviously if δ is regular, $j(\delta) > \delta$, and $j[\delta]$ is covered by a set C in M of M -cardinality δ , then C , and hence $j[\delta]$, must be bounded below the M -regular cardinal $j(\delta)$. Thus every elementary embedding with the local tight cover property is strongly discontinuous.

Strongly discontinuous embeddings also generalize the concept of a cardinal preserving embedding:

Definition 6.3. Suppose M is an inner model. A nontrivial elementary embedding $j : V \rightarrow M$ is said to be *cardinal preserving* if $\text{Card}^M = \text{Card}$.

If $j(\lambda^+)$ is a cardinal, then $j(\lambda^+) = j(\lambda)^+$. In particular, $j(\lambda^+)$ is regular, so either $j[\lambda^+]$ is bounded below $j(\lambda^+)$ or $j(\lambda^+) = \lambda^+$. It follows that cardinal preserving embeddings are strongly discontinuous.

Proposition 6.4. *Suppose $j : V \rightarrow M$ is a strongly discontinuous elementary embedding with critical point κ and $\delta \geq \kappa$ is an almost strongly compact cardinal. Then $j(\delta) > \delta$.*

Proof. It suffices to prove this in the case that δ is an almost strongly compact limit cardinal. Suppose towards a contradiction that $j(\delta) = \delta$. Note that for all $\alpha < \kappa$, $j(\delta^\alpha) = (\delta^{+\alpha})^M \leq \delta^{+\alpha}$, and so $j(\delta^{+\alpha}) = \delta^{+\alpha}$. It follows that $(\delta^{+\kappa})^M = \delta^{+\kappa}$. In fact, $(\delta^{+\kappa+1})^M = \delta^{+\kappa+1}$ by a standard argument. (For any wellorder \preceq of $\delta^{+\kappa}$, $j(\preceq) \cap \delta^{+\kappa}$ belongs to M and has length at least $\text{ot}(\preceq)$. Thus $(\delta^{+\kappa+1})^M > \text{ot}(\preceq)$.)

On the other hand, $j(\delta^{+\kappa}) = (\delta^{+j(\kappa)})^M > (\delta^{+\kappa+1})^M = \delta^{+\kappa+1}$. Since j is strongly discontinuous, we must therefore have $j(\delta^{+\kappa+1}) > \sup j[\delta^{+\kappa+1}]$.

Let U be the ultrafilter over $\delta^{+\kappa+1}$ derived from j using $\sup j[\delta^{+\kappa+1}]$. Then for all $\alpha < \kappa$, $j_U(\delta^{+\alpha}) = \delta^{+\alpha}$, so U is $\delta^{+\alpha}$ -indecomposable.

Since δ is almost strongly compact, $j_U = (j_W)^{M_D} \circ j_D$ where D is an ultrafilter over a cardinal $\eta < \delta$ and $W \in M_D$ is an M_D -ultrafilter over $j_D(\delta^{+\kappa+1})$ that is M_D - $j_D(\gamma)$ -complete in M_D for all $\gamma < \delta^{+\kappa}$. Working in M_D , let $\zeta = \text{crit}(W)$. Then $\delta < \zeta \leq (\delta^{+j_D(\kappa)+1})^{M_D}$. This contradicts that ζ is measurable and therefore inaccessible. \square

Theorem 6.5. *If there is a proper class of almost strongly compact cardinals, then there are no strongly discontinuous embeddings.*

Proof. If $j : V \rightarrow M$ is an elementary embedding, then j must fix an almost strongly compact cardinal above its critical point κ since j is continuous at ordinals of cofinality ω and the class of almost strongly compact cardinals is closed. (Let δ_0 be the least almost strongly compact cardinal, and for $n < \omega$, let δ_{n+1} be the least almost strongly compact cardinal above $j(\delta_n)$. Then $\sup_{n < \omega} \delta_n$ is an almost strongly compact cardinal that is fixed by j .) Therefore j is not strongly discontinuous by Proposition 6.4. \square

Given our observations above, the following is an immediate corollary:

Theorem 6.6. *If there is a proper class of almost strongly compact cardinals, then there are no cardinal preserving embeddings.* \square

We note the following fact, which improves on an observation due to Caicedo:

Proposition 6.7. *Suppose $j : V \rightarrow M$ is a strongly discontinuous embedding with critical point κ . Then κ is λ -strongly compact for every $\lambda < \kappa_\omega(j)$.*

Proof. Suppose δ is a regular cardinal such that $\kappa \leq \delta < \kappa_\omega(j)$. Then δ^+ carries a uniform κ -complete ultrafilter U , namely the ultrafilter derived from j using

$\sup j[\delta^+]$. Since δ is regular, then U is necessarily δ -decomposable by a theorem of Kunen-Prikry [23]. In particular, δ carries a uniform κ -complete ultrafilter. Applying Ketonen's Theorem (Theorem 2.19), κ is λ -strongly compact for all $\lambda < \kappa_\omega(j)$. \square

On the other hand, $\kappa_\omega(j)$ cannot be a limit of $\kappa_\omega(j)^{+\kappa}$ -strongly compact cardinals by the proof of Theorem 6.6.

7 Definability and ultrafilters

The results of this section are a ZFC analog of the following theorem:

Theorem 7.1. *Assume the Ground Axiom,⁸ the Ultrapower Axiom, and the existence of a strongly compact cardinal. Then every set is definable from an ordinal.* \square

We will prove the following generalization:

Theorem 7.2. *Assume the Ground Axiom. Then for any strongly compact cardinal κ , every set is definable from a κ -complete ultrafilter over an ordinal.* \square

Under UA, every countable complete ultrafilter over an ordinal is ordinal definable, so Theorem 7.2 implies Theorem 7.1.

The proof (which appears below Theorem 7.8) involves the following collection of structures:

Definition 7.3. Let κ -OD denote the class of sets that are definable from a κ -complete ultrafilter over an ordinal, and let κ -HOD denote the class of hereditarily κ -OD sets.

Note that x is κ -OD if and only if x is in OD_U for some κ -complete ultrafilter U over an ordinal, so κ -OD is first-order definable, and therefore so is κ -HOD.

The following basic observation sets things in motion:

Theorem 7.4. *For any cardinal κ , the class κ -HOD is an inner model of ZF. If κ is strongly compact, then κ -HOD satisfies the Axiom of Choice.*

Proof. The proof that κ -HOD satisfies ZF is just like the usual proof that HOD satisfies ZF, so we omit it. The issue in showing that κ -HOD satisfies the Axiom of Choice is that the class of κ -complete ultrafilters over ordinals is not naturally wellordered.

Assume that κ is strongly compact. The key idea is that in this case, for any ordinal δ , there is actually a κ -OD wellorder of the κ -complete ultrafilters over δ . Let \mathcal{W} be a κ -complete fine ultrafilter over $P_\kappa(P(\delta))$. Let W be an ultrafilter over an ordinal such that \mathcal{W} and W are Rudin-Keisler equivalent, or in other words,

⁸The *Ground Axiom* asserts that V is not a set generic extension of any inner model $M \subsetneq V$.

$j_{\mathcal{W}} = j_W$. For each κ -complete ultrafilter U over δ , let α_U be the least ordinal such that U is the ultrafilter on δ derived from j_W using α_U .⁹

The function $U \mapsto \alpha_U$ is injective and κ -OD. It follows that there is a κ -OD wellorder of the set of κ -complete ultrafilters over δ .

Now that one has a κ -OD wellorder of the κ -complete ultrafilters over δ for each δ , it is easy to construct a κ -OD wellorder of the sets that are ordinal definable from a κ -complete ultrafilter over δ . This suffices to show that κ -HOD satisfies the Axiom of Choice. The proof is the same as the proof that AC holds in HOD. \square

Note that the Axiom of Choice also holds ω -HOD, since in fact ω -HOD is V :

Proposition 7.5. $V = \omega$ -HOD.

Proof. Let $M = \omega$ -HOD. By the definition of M , ω -complete ultrafilters descend to M : in fact, if U is an M -ultrafilter, then $U \in M$, since U extends to an ultrafilter W which is isomorphic to an ultrafilter Z over an ordinal; j_Z is ω -OD, so $j_Z \upharpoonright M$ is close to M , and so since U is a derived M -ultrafilter of $j_Z \upharpoonright M$, $U \in M$.

Since M is closed under finite sequences, M has the ω -cover property. Although we have not shown that ω -HOD satisfies the Axiom of Choice, the proof of Theorem 3.17 still goes through with $\kappa = \omega$. It follows that M has the ω -approximation property, which of course implies that $V = M$. \square

We will need the analog of Vopenka's Theorem for κ -HOD. (The proof requires no real modification.)

Lemma 7.6. *For any strongly compact cardinal κ , for any set of ordinals A , κ -HOD $_A$ is a set-generic extension of κ -HOD.*

Proof. We first note that Theorem 7.4 relativizes to show that κ -HOD $_A$ is an inner model of ZFC. To show that κ -HOD $_A$ is a set-generic extension of κ -HOD, it therefore suffices to verify Bukovsky's criterion [24] by showing that κ -HOD has the $(2^\rho)^+$ -uniform cover property in κ -HOD $_A$ where $\rho = \sup A$.¹⁰

Let γ be an ordinal and let $f : \gamma \rightarrow \gamma$ be a function that is κ -OD from A . The function $g : \gamma \times P(\rho) \rightarrow \gamma$ defined by $f(\alpha) = g(\alpha, A)$ for all $\alpha < \gamma$ is then κ -OD. Let $F(\alpha) = \{g(\alpha, B) : B \subseteq \gamma\}$. Then F is κ -OD, $|F(\alpha)| < (2^\rho)^+$ for all $\alpha < \gamma$, and $f(\alpha) \in F(\alpha)$ for all $\alpha < \gamma$. This verifies that κ -HOD $_A$ has the $(2^\rho)^+$ -uniform covering property. \square

Proposition 7.7. *Suppose κ is strongly compact and A is a set of ordinals such that $V_\kappa \subseteq \kappa$ -HOD $_A$. Then $V = \kappa$ -HOD $_A$.*

⁹It is a standard fact that α_U exists, so we include the proof in fine print. Let $\sigma = [\text{id}]_{\mathcal{W}}$. Since \mathcal{W} is a fine ultrafilter over $P_\kappa(P(\delta))$, $|\sigma|^{M_{\mathcal{W}}} < j_{\mathcal{W}}(\kappa)$ and $j_{\mathcal{W}}[P(\delta)] \subseteq \sigma$. Therefore $j_{\mathcal{W}}[U] \subseteq \sigma \cap j_{\mathcal{W}}(U)$. Since $j_{\mathcal{W}}(U)$ is $j_{\mathcal{W}}(\kappa)$ -complete, $\bigcap(\sigma \cap j_{\mathcal{W}}(U))$ is nonempty. It follows that there is some ordinal $\alpha \in \bigcap j_{\mathcal{W}}[U]$. Clearly U is the ultrafilter over δ derived from j_W using α .

¹⁰To apply Bukovsky's Theorem, it is essential that κ -HOD is a model of AC; this is our only significant use of Theorem 7.4.

Proof. Let N denote $\kappa\text{-HOD}_A$. We first show that N is closed under κ -sequences. To show N is closed under κ -sequences, it therefore suffices to show that for all ordinals λ , ${}^\kappa\lambda \subseteq N$. Let U be a κ -complete ultrafilter over an ordinal ν such that $j_U(\kappa) > \lambda$. Then

$${}^\kappa\lambda \subseteq j_U(V_\kappa) \subseteq j_U(N) \subseteq N$$

We justify this last inclusion. It suffices to show that every set that M_U thinks is definable from a $j_U(\kappa)$ -complete ultrafilter over an ordinal is truly definable (in V) from a κ -complete ultrafilter over an ordinal. Since M_U is definable from the κ -complete ultrafilter U , it therefore suffices to show that every $j_U(\kappa)$ -complete ultrafilter W of M_U over an ordinal γ is definable from a κ -complete ultrafilter over an ordinal. But consider the following κ -complete ultrafilter:

$$U\text{-}\sum W = \{A \subseteq \nu \times \bar{\gamma} : [\alpha \mapsto A_\alpha]_U \in W\}$$

where $\bar{\gamma}$ is the least ordinal such that $j_U(\bar{\gamma}) \geq \gamma$. We have

$$W = \{[f]_U : \text{ran}(f) \subseteq P(\bar{\gamma}) \text{ and } \{(\alpha, \beta) : \beta \in f(\alpha)\} \in U\text{-}\sum W\}$$

so W is definable from $U\text{-}\sum W$.

We now show that N has the κ -approximation and cover properties. Since N is closed under $<\kappa$ -sequences, N certainly has the κ -cover property. Since N satisfies the Axiom of Choice and $U \cap N \in N$ for any κ -complete ultrafilter over an ordinal, it easily follows that $U \cap N \in N$ for any ultrafilter U over a set that belongs to N . Therefore by Theorem 3.17, N has the κ -approximation property.

It follows that $V = N$, since V is the unique inner model with the κ -approximation and cover properties that contains H_{κ^+} . (This last fact is a consequence of the proof of the definability of inner models with the κ -approximation and cover properties.) \square

Theorem 7.8. *For any strongly compact cardinal κ , V is a set generic extension of $\kappa\text{-HOD}$.*

Proof. Let A be a set of ordinals such that $V_\kappa \subseteq \kappa\text{-HOD}_A$; for example, A can be chosen to code a wellfounded extensional relation $R \subseteq \kappa \times \kappa$ whose transitive collapse is V_κ . By Proposition 7.7, $V = \kappa\text{-HOD}_A$, and by Lemma 7.6, $\kappa\text{-HOD}_A$ is a generic extension of $\kappa\text{-HOD}$, so V is a generic extension of $\kappa\text{-HOD}$. \square

Theorem 7.8 of course immediately implies Theorem 7.2.

Given Proposition 7.5, it is natural to speculate that Theorem 7.8 is just a precursor to a proof that $V = \kappa\text{-HOD}$ for all strongly compact cardinals κ . But of course this is not the case:

Proposition 7.9. *It is consistent with ZFC that there is a strongly compact cardinal κ such that $V \neq \kappa\text{-HOD}$.*

Proof. Assume there is a strongly compact cardinal. Let g be V -generic for Cohen forcing. Note that any κ -complete ultrafilter U of $V[g]$ over an ordinal is definable over $V[g]$ from a κ -complete ultrafilter in V : indeed, U is the unique ultrafilter extending $U \cap V$, and $U \cap V \in V$, by the Lévy-Solovay Theorem [25]. It follows that

$$\kappa\text{-HOD}^{V[g]} \subseteq \kappa\text{-HOD}^V$$

by the homogeneity of Cohen forcing. Therefore $g \notin \kappa\text{-HOD}^{V[g]}$. \square

Under large cardinal hypotheses, the Ground Axiom is equivalent to the statement that every set is ordinal definable from a countably complete ultrafilter on an ordinal in all generic extensions:

Theorem 7.10. *Assume there is a proper class of strongly compact cardinals. Then the following are equivalent:*

- (1) *The Ground Axiom.*
- (2) *V is the intersection of all models of the form $(\delta\text{-HOD})^{V^{\mathbb{B}}}$ where δ is a cardinal and \mathbb{B} is a complete Boolean algebra.*
- (3) *In any generic extension N , every set in V is ordinal definable using an internal ultrapower embedding of N as a predicate.*
- (4) *Every generic extension N satisfies that every set in V is ordinal definable using an elementary embedding from N into an inner model that is closed under $(\omega_1)^N$ -sequences.*

Proof. (1) *implies* (2): Fix a set x , a cardinal δ , and a complete Boolean algebra \mathbb{B} . We must show that x is δ -OD in $V^{\mathbb{B}}$. Let $\kappa > \delta \cdot |\mathbb{B}|$ be a strongly compact cardinal. Then x is κ -OD in V by Theorem 7.2. But since the Ground Axiom holds, V is definable over $V^{\mathbb{B}}$ without parameters (as the intersection of all grounds of $V^{\mathbb{B}}$). Moreover since $\kappa > |\mathbb{B}|$, every κ -complete ultrafilter of V is ordinal definable in $V^{\mathbb{B}}$ from its unique extension to $V^{\mathbb{B}}$. It follows that x is κ -OD in $V^{\mathbb{B}}$. Since $\kappa \geq \delta$, this implies (2).

(2) *implies* (3): This is easy given the observation that a countably complete ultrafilter over an ordinal is ordinal definable from its associated ultrapower embedding.

(3) *implies* (4): Trivial.

(4) *implies* (1): Fix a ground M of V . We must show $M = V$. Let \mathbb{B} be a complete Boolean algebra of M such that $V = M[G]$ for some M -generic ultrafilter G on \mathbb{B} . Let $\delta > |\mathbb{B}|$ and let $H \subseteq \text{Col}(\omega, \delta)$ be a V -generic filter. By the universality of collapse forcing, there is an M -generic filter $F \subseteq \text{Col}(\omega, \delta)$ in $M[G][H]$ such that $M[F] = M[G][H]$.

Fix a set of ordinals $x \in V$, and we will show that $x \in M$. Now let $N = M[G][H]$ and fix in N an elementary embedding $j : N \rightarrow P$ such that $P^{(\omega_1)^N} \cap N \subseteq P$ and

x is ordinal definable using j as a predicate. By the Reflection Theorem,¹¹ there is some ordinal α such that x is ordinal definable in N_α using $j \upharpoonright N_\alpha$ as a predicate. (Here $N_\alpha = N \cap V_\alpha$.)

Since M is an ω_1^N -pseudoground of N and $P^{(\omega_1)^N} \cap N \subseteq P$, the Hamkins Universality Theorem (Theorem 2.15) implies that $j \upharpoonright M_\alpha$ belongs to M . Note however that $j \upharpoonright N_\alpha$ is definable in N from $j \upharpoonright M_\alpha$ and M . It follows that x is definable in N from the predicate for M and parameters in M . Since $N = M[F]$ where $F \subseteq \text{Col}(\omega, \delta)$ is M -generic, the homogeneity of collapse forcing implies that $x \cap M$ is definable over M . Since x is a set of ordinals, $x \cap M = x$, so $x \in M$.

It follows that every set of ordinals belongs to M , which proves $V = M$. Since M was an arbitrary ground, the Ground Axiom holds. \square

Suppose κ is strongly compact. We do not know whether κ must be strongly compact in κ -HOD. We can, however, prove the following ‘‘cheap HOD Conjecture’’:

Theorem 7.11. *Suppose κ is supercompact. Then κ is supercompact in κ -HOD, and in fact κ -HOD is a weak extender model for the supercompactness of κ .*

Proof. Let N denote κ -HOD. By Theorem 7.8, N is a ground of V , so there is some cardinal λ such that for all regular cardinals $\delta \geq \lambda$, any set S that is stationary in N is stationary in V .

Fix $\delta \geq \lambda$. We will show that for any normal fine κ -complete ultrafilter \mathcal{U} over $P_\kappa(\delta)$, $\mathcal{U} \cap N \in N$ and $P_\kappa(\delta) \cap N \in \mathcal{U}$. This establishes that N is a weak extender model for the supercompactness of κ .

Of course, $\mathcal{U} \cap N \in N$ by the definition of N for any κ -complete normal fine ultrafilter \mathcal{U} over $P_\kappa(\delta)$; this is because \mathcal{U} is the unique normal fine ultrafilter on $P_\kappa(\delta)$ that is Rudin-Keisler equivalent to \mathcal{U} , and hence \mathcal{U} is ordinal definable from $j\mathcal{U}$.

We now show that $P_\kappa(\delta) \cap N \in \mathcal{U}$. Let $j : V \rightarrow M$ be the ultrapower embedding associated to \mathcal{U} . By Łoś’s Theorem, and since $[\text{id}]_{\mathcal{U}} = j[\delta]$, we just need to show that $j[\delta] \in j(N)$. Let T be the set of ordinals less than δ that have cofinality ω in N . Let $\langle S_\alpha \rangle_{\alpha < \delta} \in N$ be a partition of T into stationary sets. Let $\langle S_\alpha^* \rangle_{\alpha < j(\delta)} = j(\langle S_\alpha \rangle_{\alpha < \delta})$. Thus $\langle S_\alpha^* \rangle_{\alpha < j(\delta)} \in j(N)$. Then $j[\delta] = \{\alpha < j(\delta) : S_\alpha^* \cap \text{sup } j[\delta] \text{ is stationary}\}$ by the proof of Solovay’s Theorem; see [1, Corollary 4.4.31]. Thus $j[\delta]$ is ordinal definable in M from $\langle S_\alpha^* \rangle_{\alpha < j(\delta)}$, so $j[\delta] \in j(N)$. \square

8 Questions

Question 8.1 (Boney-Unger and Brooke-Taylor). If there is a proper class of almost strongly compact cardinals, is there a proper class of strongly compact cardinals?

¹¹Formally, we are working in von Neumann-Bernays-Gödel class theory (NBG). For any formula φ in the language of first-order set theory with an additional predicate symbol, NBG proves that for all classes A , there is an ordinal α such that for all $x \in V_\alpha$, $(V_\alpha, A \cap V_\alpha)$ satisfies $\varphi(x)$ if and only if (V, A) satisfies $\varphi(x)$.

Question 8.2 (Caicedo). Can there be cardinal preserving or cofinality preserving embeddings from the universe of sets into an inner model?

Question 8.3. Suppose κ is strongly compact. Is κ strongly compact in κ -HOD?

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