REFLECTING MEASURES

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Abstract. We give new, purely combinatorial characterizations of several kinds of large cardinals, such as strongly $C^{(n)}$-compact and $C^{(n)}$-extendible, in terms of *reflecting measures*. We then study the key property of *tightness* of elementary embeddings that witness strong $C^{(n)}$-compactness, which prompts the introduction of the new large-cardinal notion of *tightly $C^{(n)}$-compact* cardinal. Then we prove, assuming the Ultrapower Axiom, that a cardinal is tightly $C^{(n)}$-compact if and only if it is either $C^{(n-1)}$-extendible or a measurable limit of $C^{(n-1)}$-extendible cardinals. In the last section we also give new characterizations of Σ_n-strong cardinals in terms of *reflecting extenders*.

1. Introduction

Some of the most prominent large cardinals are given by two-valued measures, i.e., ultrafilters, satisfying certain conditions. Thus, a cardinal κ is *measurable* if there exists a κ-complete uniform measure over κ; and a cardinal κ is *λ-supercompact* if there exists a κ-complete fine and normal measure over $\mathcal{P}_\kappa\lambda$, etc. Nevertheless, the most common presentation, and usage, of large cardinals is in the form of *reflection principles*. For instance, κ is measurable iff it is the critical point (i.e., the least ordinal moved) of an elementary embedding $j : V \to M$, with M transitive; and κ is λ-supercompact iff it is the critical point of such an embedding with M closed under λ-sequences. That these are reflection principles is easily seen by noticing that any elementary property of κ that holds in M is reflected, via j, to some cardinal smaller than κ. Further, higher-order reflection properties analogous to the Skolem-Löwenheim theorem have been shown to characterize many large cardinals. Thus, κ is the least *supercompact* cardinal (i.e., λ-supercompact for every λ) iff it is the least cardinal such that every second-order statement (in a language of size less than κ) true in any structure $\mathcal{A}$ is true in some substructure of $\mathcal{A}$ of size less than κ ([Mag71]). In fact, more general forms of Skolem-Löwenheim-type reflection, namely *Structural Reflection*, have been shown to characterize most of the best known large-cardinal notions (see [Bag23]).

In this paper we show how to incorporate the reflection properties of large cardinals into their combinatorial definitions in terms of measures. The main notion is that of an *n-reflecting measure* (Definition 2.1),

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namely a measure that concentrates on the subset of $\mathcal{P}_\kappa\lambda$ consisting of those sets whose order-type, α, is Σ_n-correct in V, i.e., V_α is a Σ_n-elementary substructure of V.

We begin with the observation that, for λ a Σ_1-correct cardinal, a cardinal $\kappa < \lambda$ is λ-supercompact iff there exists a fine and normal 1-reflecting measure over $\mathcal{P}_\kappa\lambda$ (Theorem 2.3). This yields a characterization of supercompactness in terms of 1-reflecting measures that is then generalized to give analogous characterizations of extendible and $C^{(n)}$-extendible cardinals in terms of 2-reflecting and n-reflecting measures, respectively (Theorems 2.5 and 2.7). Let us emphasize that these are the first known characterizations of extendible and $C^{(n)}$-extendible cardinals in terms of measures.

In [Tsa12, Tsa14], Tsaprounis defines strongly $C^{(n)}$-compact cardinals in accordance with the general framework for the study of $C^{(n)}$-cardinals initiated in [Bag12]. Namely, a strongly $C^{(n)}$-compact cardinal is given by the usual characterization of a strongly compact cardinal via elementary embeddings, but with the additional requirement that the image of the critical point belongs to the class $C^{(n)}$, namely the class of Σ_n-correct cardinals (see [Tsa14, Definition 1.6]). However, he shows that a cardinal is strongly compact if and only if it is strongly $C^{(n)}$-compact, for all n ([Tsa14, Theorem 3.6 and Corollary 3.7]), which demonstrates that this notion of strong $C^{(n)}$-compactness does not yield new large cardinals.

In section 3, starting with the (trivial) observation that a cardinal κ is strongly compact iff there exists a 0-reflecting measure over $\mathcal{P}_\kappa\lambda$, for a proper class of λ, we re-define the notion of a cardinal being *strongly $C^{(n)}$-compact* as carrying a fine n-reflecting measure over $\mathcal{P}_\kappa\lambda$, for a proper class of λ (Definition 3.1). But our analysis of strongly $C^{(n)}$-compact cardinals turns out to yield rather disappointing results, for we can show that a cardinal is strongly $C^{(n)}$-compact iff it is strongly compact and a limit of Σ_n-correct cardinals (Corollary 3.4). Because of this, we shift our attention to a key property of elementary embeddings that indicates their degree of compactness, namely the property of being *tight* (Definition 4.1).

In section 4, we begin by proving several lemmas about the tightness function for elementary embeddings, which prompt the definition of the notion of *tightly $C^{(n)}$-compact cardinal* (Definition 4.6). We show that tightly $C^{(n)}$-compact cardinals κ can be characterized in terms of fine reflecting measures on $\mathcal{P}_\kappa\lambda$, for all $\lambda \in C^{(n)}$ greater than κ (Proposition 4.8). We also show that they are given by κ-complete measures on some cardinals $\delta \geq \kappa$ (Corollary 4.9). Further, we show that for every regular cardinal in $C^{(n)}$, a cardinal $\kappa < \lambda$ is tightly λ-$C^{(n)}$-compact iff there is a κ-complete weakly normal measure over λ concentrating on the set of ordinals whose cofinality is Σ_n-correct and less than κ (Proposition 4.10). These results may give the impression that the notion of tightly $C^{(n)}$-compact cardinal, while naturally motivated, is somewhat technical. But this is only apparent. The main result of section 4 shows that assuming the Ultrapower Axiom, a cardinal is tightly $C^{(n)}$-compact ($n > 1$) iff it is either $C^{(n-1)}$-extendible or a measurable limit of $C^{(n-1)}$-extendible cardinals (Theorem 4.15). The proof uses several results from [Gol22] as well as the characterization of $C^{(n)}$-extendible cardinals in terms of reflecting measures from section 2.1. This characterization of extendibility under the Ultrapower Axiom extends the theorem from [Gol22, 8.3.10] which shows, assuming the Ultrapower Axiom, that a cardinal is strongly compact iff it is either supercompact or a measurable limit of supercompact cardinals.

The characterization of large cardinals in terms of reflecting measures, given in the previous sections, is not exclusive of the region in the large-cardinal hierarchy that lies between supercompactness (or strong compactness) and Vopěnka's Principle (i.e., the existence of a $C^{(n)}$-extendible cardinal, for every n [Bag12]). We show this in section 5, the last one, where we consider the, much lower, region that spans between the first strong cardinal and the large-cardinal principle "OR is Woodin" (i.e., the existence of a Σ_n-strong cardinal, for every n [BW22, 5.14]). As with strong cardinals, which cannot be given by single measures, requiring the use of coherent systems of measures known as *extenders*, we define the corresponding notion of *(κ, λ)-n-reflecting extender* (Definition 5.1) which is then used to characterize Σ_n-strong cardinals and "OR is Woodin" in terms of the existence of *n-reflecting extenders* (Theorem 5.6 and Corollary 5.7).

2. Fine and normal reflecting measures

For each natural number n, $C^{(n)}$ is the Π_n-definable club proper class of ordinals κ that are *Σ_n-correct*, i.e., V_κ is a Σ_n-elementary substructure of V (written as $V_\kappa \preceq_{\Sigma_n} V$).

Note that $\lambda \in C^{(1)}$ iff λ is an uncountable cardinal and $V_\lambda = H_\lambda$. Also, $\lambda \in C^{(1)}$ iff λ is uncountable and $|V_\lambda| = \lambda$, hence iff λ is a fixed point of the Beth function. (See [Bag12] for further properties of the $C^{(n)}$ classes.)

As customary, for any set S and cardinal κ, we write $\mathcal{P}_\kappa S$ for the set of all subsets of S of cardinality less than κ.

Definition 2.1. *Let κ be an infinite cardinal, and let λ be an ordinal greater than or equal to κ. An n-*reflecting measure *on $\mathcal{P}_\kappa\lambda$ is a κ-complete ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa\lambda$ which contains the set $T^n_{\kappa,\lambda}$ of all $\sigma \in \mathcal{P}_\kappa\lambda$ such that* $\mathrm{ot}(\sigma) \in C^{(n)}$.

Notice that $T^{n+1}_{\kappa,\lambda} \subseteq T^n_{\kappa,\lambda}$, and therefore every $(n+1)$-reflecting measure on $\mathcal{P}_\kappa\lambda$ is also n-reflecting. Note also that if $\mathrm{ot}(\sigma) \in C^{(n)}$, then $|\sigma| \in C^{(n)}$.

Recall that an ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa S$ is *fine* if for every $\mu \in S$ the set $\{\sigma \in \mathcal{P}_\kappa S : \mu \in \sigma\}$ belongs to $\mathcal{U}$. And $\mathcal{U}$ is *normal* if it is closed under diagonal intersections, i.e., if $\mathcal{X} = \langle X_i : i \in S\rangle$ is a sequence of elements of $\mathcal{U}$, then $\Delta\mathcal{X} = \{\sigma : \sigma \in \bigcap_{i\in\sigma} X_i\} \in \mathcal{U}$.

Lemma 2.2. *If there exists a fine and normal* 1*-reflecting measure on* $\mathcal{P}_\kappa\lambda$*, then* $\lambda \in C^{(1)}$.

Proof. Let $\mathcal{U}$ be a 1-reflecting measure on $\mathcal{P}_\kappa\lambda$, and let $j : V \to M \cong \mathrm{Ult}(V,\mathcal{U})$ be the corresponding ultrapower embedding. Then standard arguments (see [Kan03, §22]) show that $\mathrm{crit}(j) = \kappa$, $[\mathrm{Id}]_\mathcal{U} = j[\lambda]$, and M is closed under λ-sequences. Since $T^1_{\kappa,\lambda} \in \mathcal{U}$, by Łos' Theorem, in M the order-type of $j[\lambda]$, i.e., λ, belongs to $C^{(1)}$. So, in M, λ is an uncountable cardinal and $|V_\lambda| = \lambda$. But since M is closed under λ-sequences, $V^M_\lambda = V_\lambda$, and thus, in V, λ is also an uncountable cardinal and $|V_\lambda| = \lambda$, hence $\lambda \in C^{(1)}$. □

Recall that for any ordinal λ, a cardinal $\kappa < \lambda$ is λ*-supercompact* if there exists a κ-complete fine normal ultrafilter on $\mathcal{P}_\kappa\lambda$. Equivalently, if there exists an elementary embedding $j : V \to M$, M transitive, with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and M closed under λ-sequences (see [Kan03, 22.7]). A cardinal κ is *supercompact* if it is λ-supercompact for all $\lambda > \kappa$ (equivalently, for a proper class of $\lambda > \kappa$).

Notice that $T^0_{\kappa,\lambda} = \mathcal{P}_\kappa\lambda$. Thus, a cardinal κ is λ-supercompact iff there exists a fine and normal 0-reflecting measure on $\mathcal{P}_\kappa\lambda$.

Theorem 2.3. *For* $\lambda \in C^{(1)}$*, a cardinal* $\kappa \leq \lambda$ *is* λ*-supercompact iff there exists a fine and normal* 1*-reflecting measure on* $\mathcal{P}_\kappa\lambda$.

Proof. Assume κ is supercompact. Let $\lambda \in C^{(1)}$ be greater than κ, and let $j : V \to M$, M transitive, be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and M closed under λ-sequences. Define

$$X \in \mathcal{U} \quad \text{iff} \quad X \subseteq \mathcal{P}_\kappa\lambda \text{ and } j[\lambda] \in j(X).$$

We claim that $\mathcal{U}$ is a 1-reflecting measure. First, it readily follows that $\mathcal{U}$ is a κ-complete ultrafilter on $\mathcal{P}_\kappa\lambda$. Moreover, standard arguments (see [Kan03, §22]) show that $\mathcal{U}$ is fine and normal. Now, by elementarity, $j(T^1_{\kappa,\lambda})$ is, in M, the set of all $\sigma \subseteq j(\lambda)$ of cardinality less than $j(\kappa)$ such that $\mathrm{ot}(\sigma) \in C^{(1)}$. Hence, $j[\lambda] \in j(T^1_{\kappa,\lambda})$, which yields $T^1_{\kappa,\lambda} \in \mathcal{U}$.

The converse is well-known. In fact, the existence of a fine normal ultrafilter on $\mathcal{P}_\kappa\lambda$ yields that κ is λ-supercompact (see [Kan03, §22]). □

Corollary 2.4. *A cardinal* κ *is supercompact iff there exists a fine and normal* 1*-reflecting measure on* $\mathcal{P}_\kappa\lambda$ *for every* λ *in* $C^{(1)}$ *greater than or equal to* κ.

2.1. **2-reflecting measures and extendible cardinals.** Recall that, for any ordinal λ, a cardinal $\kappa < \lambda$ is λ-*extendible* if there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. A cardinal κ is *extendible* if it is λ-extendible for all $\lambda > \kappa$ (equivalently, for a proper class of $\lambda > \kappa$).

Let us note that the assertion "κ is λ-extendible" is Σ_2 (with κ and λ as parameters), hence extendibility is a Π_3 property.

The following theorem gives a characterization of extendibility in terms of 2-reflecting measures.

Theorem 2.5. *A cardinal κ is extendible iff there exists a fine and normal 2-reflecting measure on $\mathcal{P}_\kappa\lambda$ for every (equivalently, for a proper class of) λ in $C^{(2)}$ greater than or equal to κ.*

Proof. Assume κ is extendible. Let $\lambda \in C^{(2)}$ be greater than or equal to κ, and let $j : V_{\lambda+2} \to V_{\lambda'+2}$, some λ', be an elementary embedding with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. Define

$$X \in \mathcal{U} \quad \text{iff} \quad X \subseteq \mathcal{P}_\kappa\lambda \text{ and } j[\lambda] \in j(X).$$

We claim that $\mathcal{U}$ is a 2-reflecting measure. It readily follows that $\mathcal{U}$ is a κ-complete ultrafilter on $\mathcal{P}_\kappa\lambda$. Also, standard arguments show that $\mathcal{U}$ is fine and normal (see [Kan03, §22]). Now notice that since $\lambda \in C^{(2)}$, and therefore by elementarity of j, $\lambda' \in C^{(1)}$, $V_{\lambda'}$ satisfies that $\lambda \in C^{(2)}$. Also note that $T^2_{\kappa,\lambda} \in V_{\lambda+2}$. By elementarity, $j(T^2_{\kappa,\lambda})$ is the subset of $V_{\lambda'}$ consisting of all $\sigma \subseteq \lambda'$ of cardinality less than $j(\kappa)$ and such that $V_{\lambda'}$ satisfies that $\mathrm{ot}(\sigma) \in C^{(2)}$. Hence, $j[\lambda] \in j(T^2_{\kappa,\lambda})$, which yields $T^2_{\kappa,\lambda} \in \mathcal{U}$.

Conversely, suppose $\mathcal{U}$ is a fine and normal 2-reflecting measure, with $\lambda \in C^{(2)}$. Let $j : V \to M \cong \mathrm{Ult}(V, \mathcal{U})$ be the corresponding ultrapower embedding, with M transitive. We have that $\mathrm{crit}(j) = \kappa$, $[\mathrm{Id}]_\mathcal{U} = j[\lambda]$, and M is closed under λ-sequences (see [Kan03, §22]).

Since $\lambda \in C^{(1)}$, the restriction of j to V_λ belongs to M (because $|V_\lambda| = \lambda$ and M is closed under λ-sequences). So, M satisfies that κ is γ-extendible for all $\gamma < \lambda$.

As $j[\lambda]$ is represented in M by the identity function, and since $T^2_{\kappa,\lambda} \in \mathcal{U}$, Łos' theorem yields that the order-type of $j[\lambda]$ (namely, λ) is Σ_2-correct in M. Hence, $V_\lambda = V^M_\lambda$ satisfies that κ is γ-extendible, for all $\gamma < \lambda$, and since λ is Σ_2-correct, this is true in V. □

Remark 2.6. *Note that if we use the definition of $T^2_{\kappa,\lambda}$ given in* [Gol21], *namely as the set of all subsets of λ of cardinality less than κ whose order-type is Σ_2-correct in V_κ, then the proof above shows that if $\mathcal{U}$ is a fine and normal 2-reflecting measure on $\mathcal{P}_\kappa\lambda$, with $\lambda \in C^{(2)}$, then κ is an extendible cardinal in V_λ.*

Recall ([Bag12]) that for cardinals $\kappa < \lambda$, κ is λ-$C^{(n)}$-*extendible* if there is an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with critical point κ and such that $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. We say that κ is $C^{(n)}$-*extendible* if it is λ-$C^{(n)}$-extendible for all $\lambda > \kappa$.

Note that a cardinal κ is extendible iff it is $C^{(1)}$-extendible. Also note that the assertion "κ is λ-$C^{(n)}$-extendible" is Σ_{n+1} (with κ and λ as parameters), hence $C^{(n)}$-extendibility is a Π_{n+2} property.

Arguing similarly as in the proof of Theorem 2.3, we can now give the following characterization of $C^{(n)}$-extendible cardinals.

Theorem 2.7. *A cardinal κ is $C^{(n)}$-extendible iff there exists a fine and normal $(n+1)$-reflecting measure on $\mathcal{P}_\kappa\lambda$, for all (equivalently, a proper class of) cardinals $\lambda \in C^{(n+1)}$ greater than or equal to κ.*

Proof. Assume κ is $C^{(n)}$-extendible. Hence, $\kappa \in C^{(n+2)}$ ([Bag12, 3.4]). Let $\lambda \in C^{(n+1)}$ be greater than or equal to κ, and let $j : V_{\lambda+2} \to V_{\lambda'+2}$, some λ', be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $j(\kappa) \in C^{(n)}$. Define

$$X \in \mathcal{U} \quad \text{iff} \quad X \subseteq \mathcal{P}_\kappa\lambda \text{ and } j[\lambda] \in j(X).$$

We claim that $\mathcal{U}$ is an $(n+1)$-reflecting measure. First, notice that since $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$, $V_{j(\kappa)}$ satisfies that $\lambda \in C^{(n+1)}$. Also, by elementarity of j, $V_{\lambda'}$ satisfies that $j(\kappa) \in C^{(n+2)}$, and therefore $V_{\lambda'} \models$ "$\lambda \in C^{(n+1)}$". We have that $T^{n+1}_{\kappa,\lambda} \in V_{\lambda+2}$, and since $\lambda \in C^{(n+1)}$, $V_{\lambda+1}$ satisfies that $T^{n+1}_{\kappa,\lambda}$ is the set of all $\sigma \subseteq \lambda$ of cardinality less than κ and such that V_λ satisfies that $\mathrm{ot}(\sigma) \in C^{(n+1)}$. So, by elementarity, $V_{\lambda'+1}$ satisfies that $j(T^{n+1}_{\kappa,\lambda})$ is the subset of $V_{\lambda'}$ consisting of all $\sigma \subseteq \lambda'$ of cardinality less than $j(\kappa)$ and such that $V_{\lambda'}$ satisfies that $ot(\sigma) \in C^{(n+1)}$. Hence, $j[\lambda] \in j(T^{n+1}_{\kappa,\lambda})$. Now as in 2.3, it follows that $\mathcal{U}$ is as required.

Conversely, suppose there exists a fine and normal $(n+1)$-reflecting measure $\mathcal{U}$ on $\mathcal{P}_\kappa\lambda$, where $\lambda \in C^{(n+1)}$. By induction on n, κ is $C^{(n-1)}$-extendible, and therefore $\kappa \in C^{(n+1)}$. Let $j : V \to M \cong \mathrm{Ult}(V, \mathcal{U})$ be the corresponding ultrapower embedding, with M transitive. We have that $\mathrm{crit}(j) = \kappa$, $[\mathrm{Id}]_\mathcal{U} = j[\lambda]$, and M is closed under λ-sequences.

Since $\lambda \in C^{(1)}$, the restriction of j to V_λ belongs to M. By elementarity, in M, $j(\kappa)$ belong to $C^{(n+1)}$. Thus, M satisfies that for every $\gamma < \lambda$ greater than κ, there exists an elementary embedding $j : V_\gamma \to V_{j(\gamma)}$ with critical point κ, $j(\kappa) > \gamma$, and $j(\kappa) \in C^{(n+1)}$, i.e., M satisfies that κ is γ-$C^{(n+1)}$-extendible.

Since $j[\lambda]$ is represented in M by the identity function, and $T^{n+1}_{\kappa,\lambda} \in \mathcal{U}$, by Łos' theorem M satisfies that the order-type of $j[\lambda]$ (namely, λ) is Σ_{n+1}-correct. Hence, $V_\lambda = V^M_\lambda$ satisfies that κ is γ-$C^{(n+1)}$-extendible, for all $\gamma < \lambda$, and since $\lambda \in C^{(n+1)}$ this is true in V. □

Corollary 2.8. *A cardinal κ is $C^{(n)}$-extendible iff for all (equivalently, a proper class of) cardinals $\lambda \in C^{(n+1)}$ greater than or equal to κ, there exists a fine and normal measure $\mathcal{U}$ on $\mathcal{P}_\kappa\lambda$ such that $M_\mathcal{U} \models$ "$\lambda \in C^{(n+1)}$", where $M_\mathcal{U}$ is the transitive collapse of* $\mathrm{Ult}(V, \mathcal{U})$.

3. Reflecting measures and compact cardinals

By dropping the normality condition we may obtain similar characterizations for strongly compact and strongly $C^{(n)}$-compact cardinals (defined below).

Recall that, for any ordinal λ, a cardinal $\kappa < \lambda$ is *λ-compact* iff there exists a κ-complete fine ultrafilter on $\mathcal{P}_\kappa\lambda$ (see [Kan03, §22]). And κ is *strongly compact* if it is λ-compact for all (equivalently, a proper class of) ordinals $\lambda > \kappa$.

Thus, a cardinal κ is λ-compact iff there exists a fine 0-reflecting measure on $\mathcal{P}_\kappa\lambda$. And κ is strongly compact iff there exists a fine 0-reflecting measure on $\mathcal{P}_\kappa\lambda$ for a proper class of λ. This suggests the following definition.

Definition 3.1. *A cardinal κ is λ-$C^{(n)}$-*compact *if there exists a fine n-reflecting measure on $\mathcal{P}_\kappa\lambda$.*

A cardinal κ is strongly $C^{(n)}$-compact *if there exists a fine n-reflecting measure on $\mathcal{P}_\kappa\lambda$ for a proper class of λ.*

Proposition 3.2. *For cardinals $\kappa \leq \lambda$, the following are equivalent:*

(1) *κ is λ-$C^{(n)}$-compact.*
(2) *There is an elementary embedding $j : V \to M$, M transitive, with $\mathrm{crit}(j) = \kappa$ and such that for some $D \in M$, $j[\lambda] \subseteq D$ and $M \models$ "$\mathrm{ot}(D) \in C^{(n)} \cap j(\kappa)$".*

Proof. Assume κ is λ-$C^{(n)}$-compact, some λ greater than or equal to κ. Let $\mathcal{U}$ be a fine n-reflecting measure. Let $j : V \to M \cong \mathrm{Ult}(V,\mathcal{U})$ be the corresponding ultrapower embedding, with M transitive. We have that $\mathrm{crit}(j) = \kappa$, and by fineness $j[\lambda] \subseteq \pi([\mathrm{Id}]_{\mathcal{U}})$, where $\pi : \mathrm{Ult}(V,\mathcal{U}) \cong M$ is the transitive collapse. Now note that, in M,

$$\lambda = \mathrm{ot}(j[\lambda]) \leq \mathrm{ot}(\pi([\mathrm{Id}]_{\mathcal{U}})) < j(\kappa).$$

Let $D = \pi([\mathrm{Id}]_{\mathcal{U}})$. Then $D \in M$, $j[\lambda] \subseteq D$, and, since $T^n_{\kappa,\lambda} \in \mathcal{U}$, by Łós' Theorem $M \models$ "$\mathrm{ot}(D) \in C^{(n)} \cap j(\kappa)$".

Now assume (2) and let $j : V \to M$, M transitive, be an elementary embedding with $\mathrm{crit}(j) = \kappa$ and such that for some $D \in M$ we have that $j[\lambda] \subseteq D$ and $M \models$ "$\mathrm{ot}(D) \in C^{(n)} \cap j(\kappa)$". Define

$$X \in \mathcal{V} \quad \text{iff} \quad X \subseteq \mathcal{P}_\kappa\lambda \text{ and } D \cap j(\lambda) \in j(X).$$

We claim that $\mathcal{V}$ is an n-reflecting measure. By elementarity, $j(T^n_{\kappa,\lambda})$ is, in M, the set of all $\sigma \subseteq j(\lambda)$ of cardinality less than $j(\kappa)$ such that $ot(\sigma) \in C^{(n)}$. Hence, $D \cap j(\lambda) \in j(T^n_{\kappa,\lambda})$. It readily follows that $\mathcal{V}$ is a κ-complete ultrafilter on $T^n_{\kappa,\lambda}$. To check that $\mathcal{V}$ is fine note that if $\mu < \lambda$, then $j(\mu) \in D \cap j(\lambda)$, and therefore $D \cap j(\lambda)$ belongs to the image under j of the set $X_\mu := \{\sigma \in T^n_{\kappa,\lambda} : \mu \in \sigma\}$, and so $X_\mu \in \mathcal{V}$. □

Proposition 3.3. *A cardinal κ is λ-$C^{(n)}$-compact iff it is λ-compact and the class $C^{(n)}$ is unbounded below κ.*

Proof. One direction follows easily from the definition (3.1) of λ-$C^{(n)}$-compactness. For the other direction, suppose κ is λ-compact and the class $C^{(n)}$ is unbounded below κ. Let $j : V \to M$, M transitive, be an elementary embedding such that $\mathrm{crit}(j) = \kappa$, and there is some $D \in M$ such that $j[\lambda] \subseteq D$ and $M \models$ "$|D| < j(\kappa)$".

In M, the class $C^{(n)}$ is unbounded below $j(\kappa)$. So, we can find some D' of cardinality less than $j(\kappa)$ such that $D \subseteq D'$ and $\mathrm{ot}(D') \in C^{(n)}$. By Proposition 3.2, κ is λ-$C^{(n)}$-compact. □

It follows from the proposition above that every λ-$C^{(n)}$-compact cardinal belongs to $C^{(n)}$.

A reader familiar with [Bag12] may think that the first $C^{(n)}$-supercompact cardinal (see [Bag12, Definition 5.1]) is greater than or equal to the first strongly $C^{(n)}$-compact cardinal. But the proposition above shows that this is not the case. For instance, it is consistent for the first $C^{(3)}$-supercompact cardinal to be the first strongly compact cardinal (see [HMP22, Theorem 1.3]), but we have showed that the first strongly $C^{(3)}$-compact cardinal is necessarily a limit of cardinals in $C^{(3)}$.

Corollary 3.4.
(1) *A cardinal κ is λ-compact iff it is λ-$C^{(1)}$-compact.*
(2) *A cardinal κ is strongly compact iff it is strongly $C^{(1)}$-compact.*
(3) *A cardinal κ is strongly $C^{(n)}$-compact iff it is strongly compact and the class $C^{(n)}$ is unbounded below κ.*

Thus, the definitions of λ-$C^{(n)}$-compact and strongly $C^{(n)}$-compact cardinal given above (3.1), while natural, turned out to be, perhaps, too simple. An alternative stronger definition, which takes into account the *tightness* of the λ-$C^{(n)}$-compact embeddings, will be given in the next section.

4. The tightness of an elementary embedding

An important function that determines the degree of compactness of an elementary embedding is the following:

Definition 4.1. *For any elementary embedding $j : M \to N$, with M and N transitive and any cardinal λ in M, let*

$$t_j(\lambda) = \min\{|A|^N : j[\lambda] \subseteq A \in N\}.$$

Thus $t_j : CARD^M \to CARD^N$, and we call it the tightness map *for j.*

Clearly, $\lambda \leq t_j(\lambda) \leq j(\lambda)$, for every cardinal λ. Also, it is easily seen that t_j is strictly increasing. Note that if $j : V \to M$ is an elementary embedding, with M transitive and closed under λ-sequences, then $t_j(\mu) = \mu$ for every cardinal $\mu \leq \lambda$.

Let us say that, for M a model of ZFC, an elementary embedding $k : M \to N$ is *close to M* if for all $A \in N$, $k^{-1}[A] \in M$. (Note that if $M = V$, then j is close to M.)

Lemma 4.2. *If $i : V \to M$ and $k : M \to N$ are elementary embeddings, k is close to M, and $j = k \circ i$, then $t_j = t_k \circ t_i$.*

Proof. Fix a cardinal λ. We first show that $t_j(\lambda) \leq t_k(t_i(\lambda))$. For this, fix $A \in M$ with $i[\lambda] \subseteq A$ and $|A|^M = t_i(\lambda)$. Fix $B \in N$ with $|B|^N = t_k(t_i(\lambda))$ and $k[A] \subseteq B$. Then $j[\lambda] = k \circ i[\lambda] \subseteq B$, which shows $t_j(\lambda) \leq t_k(t_i(\lambda))$.

Next we show that $t_k(t_i(\lambda)) \leq t_j(\lambda)$. Suppose $B \in N$ and $j[\lambda] \subseteq B$. We will show that $|B|^N \geq t_k(t_i(\lambda))$. Let $A = k^{-1}[B]$. Then $A \in M$ since k is close to M, and so since $i[\lambda] = k^{-1}[j[\lambda]] \subseteq A$, $|A|^M \geq t_i(\lambda)$. But $k[A] \subseteq B$, and it follows that $|B|^N \geq t_k(|A|^M) \geq t_k(t_i(\lambda))$. □

In general, the value of $t_j(\lambda)$ may not be easy to compute. Nevertheless, for particular j and λ, some information about its value may be easily obtained, as shown in the next three lemmas.

Lemma 4.3. *If $j : V \to M$ is an ultrapower embedding, via some κ-complete ultrafilter, and λ is a cardinal such that $\mathrm{cf}(\lambda) \leq \kappa$, then $t_j(\lambda) = \sup_{\delta<\lambda} t_j(\delta)$.*

Proof. We may assume $\lambda > \kappa$. Let $\eta = \mathrm{cf}(\lambda)$, and let $f : \eta \to \lambda \cap CARD$ be cofinal. Then $j[\lambda] = \bigcup_{\delta<\eta} j[f(\delta)]$. For each $\delta < \eta$, let $A_\delta \in M$ be such that $j[f(\delta)] \subseteq A_\delta$ and $|A_\delta|^M = t_j(f(\delta))$. Since M is closed under η-sequences, the union A of all A_δ, $\delta < \eta$, belongs to M and witnesses that $t_j(\lambda) = \sup_{\delta<\eta} t_j(f(\delta)) = \sup_{\delta<\lambda} t_j(\delta)$. □

Lemma 4.4. *If $j : V \to M$ is an elementary embedding and δ is a regular cardinal such that $j(\delta) = \sup j[\delta]$, then $t_j(\delta) = j(\delta)$.*

Proof. Since $j[\delta] \subseteq j(\delta)$, $t_j(\delta) \leq j(\delta)$. For the other inequality, if $j[\delta] \subseteq A \in M$, then $A \cap j(\delta)$ is a cofinal subset of $j(\delta)$. Since $j(\delta)$ is regular in M, $|A|^M \geq j(\delta)$. □

Lemma 4.5. *Suppose $\kappa < \lambda$ are cardinals with $\mathrm{cf}(\lambda) < \kappa$ and $j : V \to M$ is an ultrapower embedding via a κ-complete ultrafilter on a set of cardinality $\gamma < \lambda$. Then $t_j(\lambda) = j(\lambda)$.*

Proof. First, note that if δ is a regular cardinal greater than γ, then $j(\delta) = \sup j[\delta]$. For if $\alpha < j(\delta)$, then $\alpha = [f]_{\mathcal{U}}$ for some $f : \gamma \to \delta$. Take β such that $f[\gamma] \subseteq \beta < \delta$. Then $\alpha < j(\beta)$.

Thus, by Lemma 4.4, for all regular cardinals $\delta < \lambda$ greater than γ, we have that $t_j(\delta) = j(\delta)$. Now by Lemma 4.3, $t_j(\lambda) = \sup_{\delta<\lambda} t_j(\delta) = \sup_{\delta<\lambda} j(\delta) = j(\lambda)$, the last equality holding because $\mathrm{cf}(\lambda) < \kappa$. □

Definition 4.6. *A cardinal κ is* tightly λ-$C^{(n)}$-compact *if $\lambda \in C^{(n)}$ and there is an elementary embedding $j : V \to M$, M transitive, with critical point κ such that $t_j(\lambda) \in (C^{(n)})^M \cap j(\kappa)$.*

A cardinal κ is tightly $C^{(n)}$-compact *if it is tightly λ-$C^{(n)}$-compact for all $\lambda \in C^{(n)}$ greater than κ.*

By Proposition 3.2, if κ is tightly λ-$C^{(n)}$-compact, then it is λ-$C^{(n)}$-compact. Hence, by Proposition 3.3, every tightly λ-$C^{(n)}$-compact cardinal belongs to $C^{(n)}$.

Question 4.7. *Does κ being tightly $C^{(n)}$-compact imply $\kappa \in C^{(n+1)}$?*

We shall prove below (Corollary 4.17) that if the Ultrapower Axiom holds, then the answer is yes. In contrast, Toshimichi Usuba has just sent us a note in which he shows that, modulo a supercompact cardinal, it is consistent that the first measurable cardinal is also the first tightly $C^{(1)}$-compact cardinal, and therefore it does not belong to $C^{(2)}$.

Let us give next an exact characterization of a cardinal κ being tightly λ-$C^{(n)}$-compact, for any $\lambda \in C^{(n)}$, in terms of fine n-reflecting measures over $\mathcal{P}_\kappa\lambda$.

Proposition 4.8. *If $\lambda \in C^{(n)}$, then a cardinal $\kappa < \lambda$ is tightly λ-$C^{(n)}$-compact iff there is a fine n-reflecting measure $\mathcal{U}$ over $\mathcal{P}_\kappa\lambda$, with the property that for every function $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ such that $|f(\sigma)| < |\sigma|$ for almost all σ, there exists $\alpha < \lambda$ such that $\alpha \in \sigma \setminus f(\sigma)$ for almost all σ.*

Proof. Fix $\lambda \in C^{(n)}$ and assume $\kappa < \lambda$ is tightly λ-$C^{(n)}$-compact. Let $j : V \to M$ witness this, so that κ is the critical point of j, and $t_j(\lambda) \in (C^{(n)})^M \cap j(\kappa)$.

Fix $A \in M$ such that $j[\lambda] \subset A \subseteq j(\lambda)$ and $|A|^M = t_j(\lambda)$. Let $\mathcal{U}$ be the κ-complete fine ultrafilter on $\mathcal{P}_\kappa\lambda$ given by

$$X \in \mathcal{U} \quad \text{iff} \quad A \in j(X)$$

and let $j_{\mathcal{U}} : V \to \mathrm{Ult}(V,\mathcal{U})$ be the ultrapower embedding. The map $k : \mathrm{Ult}(V,\mathcal{U}) \to M$ given by $k([f]_{\mathcal{U}}) = j(f)(A)$ is such that $k \circ j_{\mathcal{U}} = j$. Let $\bar{A} = [\mathrm{Id}]_{\mathcal{U}}$. Thus, $k(\bar{A}) = A$. We claim that $t_{j_{\mathcal{U}}}(\lambda) = |\bar{A}|^M$. First notice that $j_{\mathcal{U}}[\lambda] \subseteq \bar{A}$, and so $t_{j_{\mathcal{U}}}(\lambda) \leq |\bar{A}|^M$. But if $t_{j_{\mathcal{U}}}(\lambda) < |\bar{A}|^M$, then

$$t_j(\lambda) = k(t_{j_{\mathcal{U}}}(\lambda)) < k(|\bar{A}|^M) = |A|^M = t_j(\lambda)$$

which yields the claim. So,

$$k(t_{j_{\mathcal{U}}}(\lambda)) = k(|\bar{A}|^M) = t_j(\lambda)$$

and therefore, by the elementarity of k,

$$t_{j_{\mathcal{U}}}(\lambda) \in (C^{(n)})^M \cap j_{\mathcal{U}}(\kappa).$$

Thus, as $t_{j_{\mathcal{U}}}(\lambda) = |[\mathrm{Id}]_{\mathcal{U}}|^M \in (C^{(n)})^M$, by Łós' Theorem we have that $T^n_{\kappa,\lambda} \in \mathcal{U}$, and so $\mathcal{U}$ is an n-reflecting measure.

Now fix a function $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ such that $\{\sigma : |f(\sigma)| < |\sigma|\} \in \mathcal{U}$. So, $|j(f)(A)|^M < |A|^M$. Since $|A|^M = t_j(\lambda)$, we must then have that $j[\lambda] \not\subseteq j(f)(A)$. Pick $\alpha < \lambda$ such that $j(\alpha) \notin j(f)(A)$. Then, as $j[\lambda] \subseteq A$, we have that $\{\sigma : \alpha \in \sigma \setminus f(\sigma)\} \in \mathcal{U}$, as wanted.

For the converse, let $\mathcal{U}$ be a fine n-reflecting measure over $\mathcal{P}_\kappa\lambda$, such that for every function $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ with $|f(\sigma)| < |\sigma|$ for almost all σ, there exists $\alpha < \lambda$ such that $\alpha \in \sigma \setminus f(\sigma)$ for almost all σ.

Let $j : V \to M \cong \mathrm{Ult}(V,\mathcal{U})$. By κ-completeness, $\mathrm{crit}(j) = \kappa$. Also, by fineness, $j[\lambda] \subseteq [\mathrm{Id}]_\mathcal{U}$. Moreover, since $T^n_{\kappa,\lambda} \in \mathcal{U}$, by Łós' Theorem we have that $|[\mathrm{Id}]_\mathcal{U}|^M \in (C^{(n)})^M$. So it only remains to check that $|[\mathrm{Id}]_\mathcal{U}|^M = t_j(\lambda)$. Let $A \in M$ be such that $j[\lambda] \subseteq A \subsetneq j(\lambda)$ and $|A|^M = t_j(\lambda)$. Aiming for a contradiction, assume that $|A|^M < |[\mathrm{Id}]_\mathcal{U}|^M$. Let $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ be such that $[f]_\mathcal{U}$ represents A in the ultrapower. Then $|f(\sigma)| < |\sigma|$ for almost all σ. So let $\alpha < \lambda$ be such that $\alpha \in \sigma \setminus f(\sigma)$ for almost all σ. Then $j(\alpha) \notin A$, yielding a contradiction. □

It follows easily from the proposition above that, for any $\lambda \in C^{(n)}$, the property of a cardinal $\kappa < \lambda$ being tightly λ-$C^{(n)}$-compact is Δ_2, with parameter λ. Moreover, the property of κ being tightly $C^{(n)}$-compact is Π_{n+1}.

The following direct consequence of Proposition 4.8 shows that an elementary embedding witnessing the tight λ-$C^{(n)}$-compactness of κ may be assumed to be an ultrapower embedding by some κ-complete ultrafilter on some ordinal greater than or equal to λ.

Corollary 4.9. *If κ is a tightly λ-$C^{(n)}$-compact cardinal, then there exists a κ-complete ultrafilter U on some cardinal $\delta \geq \lambda$ such that letting $j_U : V \to M_U \cong \mathrm{Ult}(V,U)$ be the ultrapower embedding, we have that $t_{j_U}(\lambda) \in (C^{(n)})^{M_U} \cap j_U(\kappa)$.*

Proof. Let $j : V \to M$ be an elementary embedding, M transitive, with critical point κ, and such that $t_j(\lambda) \in (C^{(n)})^M \cap j(\kappa)$. As in the proof of Proposition 4.8, fix $A \in M$ such that $j[\lambda] \subset A \subseteq j(\lambda)$ and $|A|^M = t_j(\lambda)$. Let $\mathcal{U}$ be the κ-complete fine ultrafilter on $\mathcal{P}_\kappa\lambda$ given by

$$X \in \mathcal{U} \quad \text{iff} \quad A \in j(X)$$

and let $j_\mathcal{U} : V \to \mathrm{Ult}(V,\mathcal{U})$ be the ultrapower embedding. Then $\kappa = \mathrm{crit}(j_\mathcal{U})$, and as in 4.8 we can show that

$$t_{j_\mathcal{U}}(\lambda) \in (C^{(n)})^M \cap j_\mathcal{U}(\kappa).$$

The claim now follows by taking a bijection between $\mathcal{P}_\kappa\lambda$ and $\delta = |\mathcal{P}_\kappa\lambda|$, and letting U be the induced ultrafilter on δ given by $\mathcal{U}$. □

Recall that an ultrafilter $\mathcal{U}$ over a cardinal λ is *weakly normal* if it is uniform (i.e., all elements of $\mathcal{U}$ have the same cardinality), and for any set $A \in \mathcal{U}$ and every regressive function $f : A \to \lambda$ there exists some $B \subseteq A$ such that $B \in \mathcal{U}$ and $f[B]$ has cardinality less than λ.

Proposition 4.10. *If $\lambda \in C^{(n)}$ is regular, then a cardinal $\kappa < \lambda$ is tightly λ-$C^{(n)}$-compact iff there is a κ-complete weakly normal ultrafilter $\mathcal{U}$ over λ such that $S := \{\alpha < \lambda : \mathrm{cf}(\alpha) \in C^{(n)} \cap \kappa\} \in \mathcal{U}$.*

Proof. Assume $\lambda \in C^{(n)}$ is regular and $\kappa < \lambda$ is tightly λ-$C^{(n)}$-compact. Let $j : V \to M$ be an elementary embedding with M transitive and with critical point κ such that $t_j(\lambda) \in (C^{(n)})^M \cap j(\kappa)$. By Ketonen's Theorem (see [Gol22, 7.2.12]), $t_j(\lambda) = \mathrm{cf}^M(\sup j[\lambda])$. Thus, $\sup j[\lambda] < j(\lambda)$, and so the ultrafilter $\mathcal{U}$ over λ derived from j using $\sup j[\lambda]$ is κ-complete and weakly normal (see [Gol22, 4.4.18]). Now note that

$$\sup j[\lambda] \in j(S) = \{\alpha < j(\lambda) : \mathrm{cf}^M(\alpha) \in (C^{(n)})^M \cap j(\kappa)\}$$

and so $S \in \mathcal{U}$.

For the converse, suppose $\mathcal{U}$ is a κ-complete weakly normal ultrafilter on λ such that $S \in \mathcal{U}$. Let $j : V \to M$ be the corresponding ultrapower embedding. By κ-completeness, $\mathrm{crit}(j) = \kappa$, and by weak normality $[\mathrm{Id}]_{\mathcal{U}} = \sup j[\lambda]$ (see [Gol22, 4.4.17]). Since $S \in \mathcal{U}$, by Łoś's Theorem in M we have that

$$\mathrm{cf}([\mathrm{Id}]_{\mathcal{U}}) \in C^{(n)} \cap j(\kappa)$$

and so, since λ is regular, by Ketonen's Theorem, in M,

$$t_j(\lambda) = \mathrm{cf}(\sup j[\lambda]) \in C^{(n)} \cap j(\kappa)$$

hence κ is tightly λ-$C^{(n)}$-compact. □

For γ a successor cardinal, let γ^- denote its predecessor.

Proposition 4.11. *If $\lambda \in C^{(n)}$ and $\mathrm{cf}(\lambda) < \kappa < \lambda$, then the following are equivalent:*

(1) *κ is tightly λ-$C^{(n)}$-compact, witnessed by an embedding $j : V \to M$ such that $t_j(\lambda^+) = (t_j(\lambda)^+)^M$.*
(2) *There is a κ-complete weakly normal ultrafilter $\mathcal{U}$ on λ^+ such that $S := \{\alpha < \lambda^+ : \mathrm{cf}(\alpha)^- \in C^{(n)} \cap \kappa\} \in \mathcal{U}$ and, moreover, if $j_{\mathcal{U}} : V \to M_{\mathcal{U}}$ is the corresponding ultrapower embedding, then $t_{j_{\mathcal{U}}}(\lambda^+) = (t_{j_{\mathcal{U}}}(\lambda)^+))^{M_{\mathcal{U}}}$.*

Proof. Assume $\mathrm{cf}(\lambda) < \kappa < \lambda$ and κ is tightly λ-$C^{(n)}$-compact. Let $j : V \to M$ be an elementary embedding with critical point κ such that $t_j(\lambda) \in (C^{(n)})^M \cap j(\kappa)$ and $t_j(\lambda^+) = (t_j(\lambda)^+)^M$. As in Corollary 4.9, let U be a κ-complete ultrafilter on $\delta = |\mathcal{P}_\kappa \lambda|$ such that letting $j_U : V \to M_U$ be the ultrapower embedding, $t_{j_U}(\lambda) \in (C^{(n)})^{M_U} \cap j_U(\kappa)$. Since we are assuming $t_j(\lambda^+) = (t_j(\lambda)^+)^M$, arguing as in the proof of Proposition 4.8 we also have that $t_{j_U}(\lambda^+) = (t_{j_U}(\lambda)^+)^{M_U}$. Note that since $\mathrm{cf}(\lambda) < \kappa$, $\delta \geq \lambda^+$. By Ketonen's Theorem (see [Gol22, 7.2.12]), $t_{j_U}(\lambda^+) = \mathrm{cf}^{M_U}(\sup j_U[\lambda^+])$.

Claim 4.12. $\sup j_U[\lambda^+] < j_U(\lambda^+)$

Proof of Claim. Since $t_{j_U}(\lambda) < j_U(\kappa)$, and $t_{j_U}(\lambda^+) = (t_{j_U}(\lambda)^+)^{M_U}$, it follows that

$$\mathrm{cf}^{M_U}(\sup j_U[\lambda^+]) = t_{j_U}(\lambda^+) = (t_{j_U}(\lambda)^+)^{M_U} < j_U(\kappa) < j_U(\lambda) < j_U(\lambda^+)$$

hence, since $j_U[\lambda^+] \subseteq j_U(\lambda^+)$ and $j_U(\lambda^+)$ is a regular cardinal in M_U, the Claim follows. □

So the ultrafilter $\mathcal{U}$ over λ^+ derived from j_U using $\sup j_U[\lambda^+]$ is weakly normal (see [Gol22, 4.4.18]).

Now using the fact that $t_{j_U}(\lambda) \in (C^{(n)})^{M_U} \cap j_U(\kappa)$, we have

$$\sup j_U[\lambda^+] \in j_U(S) = \{\alpha < j_U(\lambda^+) : \mathrm{cf}^{M_U}(\alpha)^- \in (C^{(n)})^{M_U} \cap j_U(\kappa)\}$$

and so $S \in \mathcal{U}$.

For the converse, suppose $\mathcal{U}$ is a κ-complete weakly normal ultrafilter on λ^+ such that $S \in \mathcal{U}$ and $t_{j_{\mathcal{U}}}(\lambda^+) = (t_{j_{\mathcal{U}}}(\lambda)^+))^{M_{\mathcal{U}}}$. By κ-completeness, $\mathrm{crit}(j) \geq \kappa$, and by weak normality $[\mathrm{Id}]_{\mathcal{U}} = \sup j_{\mathcal{U}}[\lambda^+]$ (see [Gol22, 4.4.17]). Since $S \in \mathcal{U}$, by Łoś's Theorem, in $M_{\mathcal{U}}$ we have that

$$\mathrm{cf}([\mathrm{Id}]_{\mathcal{U}})^- \in C^{(n)} \cap j_{\mathcal{U}}(\kappa).$$

Since λ^+ is regular, by Ketonen's Theorem, in $M_{\mathcal{U}}$,

$$t_{j_{\mathcal{U}}}(\lambda^+) = \mathrm{cf}(\sup j_{\mathcal{U}}[\lambda^+]).$$

Thus, in $M_{\mathcal{U}}$,

$$t_{j_{\mathcal{U}}}(\lambda) = t_{j_{\mathcal{U}}}(\lambda^+)^- = \mathrm{cf}(\sup j_{\mathcal{U}}[\lambda^+])^- = \mathrm{cf}([\mathrm{Id}]_{\mathcal{U}})^- \in C^{(n)} \cap j_{\mathcal{U}}(\kappa)$$

hence κ is tightly λ-$C^{(n)}$-compact. □

The concept of tightly $C^{(n)}$-compact cardinal leads to a characterization of extendibility under the Ultrapower Axiom analogous to the theorem from [Gol22, 8.3.10], which shows that a cardinal is strongly compact iff it is either supercompact or a measurable limit of supercompact cardinals. To show this, let us first prove the following lemmas.

Lemma 4.13. *If κ is $C^{(n-1)}$-extendible, then it is tightly $C^{(n)}$-compact.*

Proof. If κ is $C^{(n-1)}$-extendible, then by Corollary 2.8, for all cardinals $\lambda \in C^{(n)}$ greater than or equal to κ, there exists a fine and normal measure $\mathcal{U}$ on $\mathcal{P}_\kappa\lambda$ such that $\mathrm{Ult}(V,\mathcal{U}) \models$ "$\lambda \in C^{(n)}$". The corresponding ultrapower embedding $j_{\mathcal{U}} : V \to M_{\mathcal{U}}$ then witnesses that κ is tightly λ-$C^{(n)}$-compact, because $t_{j_{\mathcal{U}}}(\lambda) = \lambda$. □

Lemma 4.14. *If κ is a measurable limit of tightly $C^{(n)}$-compact cardinals, then κ is tightly $C^{(n)}$-compact.*

Proof. Fix a cardinal $\lambda > \kappa$ in $C^{(n)}$, and we will show that κ is tightly λ-$C^{(n)}$-compact. Let D be a κ-complete nonprincipal ultrafilter on κ. Let $i : V \to N$ be the elementary embedding given by the ultrapower of V by D. In N, let δ be a tightly $i(\lambda)$-$C^{(n)}$-compact cardinal such that $\kappa \leq \delta \leq i(\kappa)$. Let $k : N \to M$ witness that δ is tightly $i(\lambda)$-$C^{(n)}$-compact. Therefore

$$t_k(i(\lambda)) \in (C^{(n)})^M \cap k(\delta).$$

We claim that $j = k \circ i$ witnesses that κ is tightly λ-$C^{(n)}$-compact. For this it suffices to show that $t_j(\lambda) \in (C^{(n)})^M \cap j(\kappa)$, which will follow by showing that $t_j(\lambda) = t_k(i(\lambda))$. By Proposition 4.8, we may assume that k is an ultrapower embedding defined in N via a δ-complete ultrafilter $\mathcal{U}$ over $\mathcal{P}_\delta i(\lambda)$, hence k is close to N, and therefore by Lemma 4.2 we have that $t_j = t_k \circ t_i$. Moreover, by Lemma 4.5 we have that $t_i(\lambda) = i(\lambda)$. □

It now follows from Proposition 4.13 and the lemma above that, for any $n \geq 2$, if a cardinal κ is either $C^{(n-1)}$-extendible or a measurable limit of $C^{(n-1)}$-extendible cardinals, then κ is tightly $C^{(n)}$-compact. We will show next that if we assume the Ultrapower Axiom, then the converse is also true.

Theorem 4.15. *Assuming the Ultrapower Axiom, for any $n \geq 2$, a cardinal κ is tightly $C^{(n)}$-compact if and only if it is either $C^{(n-1)}$-extendible or a measurable limit of $C^{(n-1)}$-extendible cardinals.*

To keep this paper as self-contained as possible, let us define here some notions associated with the Ultrapower Axiom that are treated more thoroughly in [Gol22].

The *Ketonen order* partially orders the countably complete ultrafilters on an ordinal δ by $U <_{\Bbbk} W$ if for a W-large set of $\alpha < \delta$, there is an ultrafilter U_α on α such that for $A \subseteq \delta$

$$A \in U \iff \{\alpha < \delta : A \cap \alpha \in U_\alpha\} \in W.$$

For a countably complete ultrafilter U, let M_U denote the transitive collapse of the ultrapower $\mathrm{Ult}(V, U)$, with $j_U : V \to M_U$ being the corresponding ultrapower embedding.

The *Rudin-Frolík order* preorders the class of countably complete ultrafilters by $D \leq_{\mathrm{RF}} U$ if in M_D, there is a countably complete ultrafilter U^* such that $\mathrm{Ult}(M_D, U^*) = M_U$ and $j_{U^*} \circ j_D = j_U$ where $j_{U^*} : M_D \to M_U$ denotes the ultrapower embedding.[1]

The *Ultrapower Axiom* states that the Rudin-Frolík order on countably complete ultrafilters is upwards directed. In other words, for any countably complete ultrafilters D_0 and D_1, there is a countably complete ultrafilter U above both of them in the Rudin-Frolík order; equivalently, there are countably complete ultrafilters U_0 and U_1 in M_{D_0} and M_{D_1}, respectively, such that $\mathrm{Ult}(M_{D_0}, U_0) = \mathrm{Ult}(M_{D_1}, U_1)$ and $j_{U_0} \circ j_{D_0} = j_{U_1} \circ j_{D_1}$. This statement holds in all the known canonical inner models of set theory (for example, $L[U]$ and $\mathrm{HOD}^{L(\mathbb{R})}$ assuming $\mathrm{AD}^{L(\mathbb{R})}$) as an immediate consequence of the Comparison Lemma. UA is expected to hold in all canonical inner models of set theory, and

[1]Note that U^* may not be an ultrafilter in V. Instead, it is assumed to be an ultrafilter *in the model M_D*. For this reason, the ultrapower $\mathrm{Ult}(M_D, U^*)$ is constructed using only functions in M_D. Equivalently, we relativize the definition of the ultrapower to the inner model M_D, forming $(M_{U^*})^{M_D}$.

it is hoped that there will be canonical models for all large cardinal hypotheses. Thus it makes sense to study UA in the presence of extendible cardinals.

By [Gol22, Theorem 3.5.1], the Ultrapower Axiom is equivalent to the statement that for all ordinals δ, the Ketonen order wellorders the countably complete ultrafilters on δ.

It is quite surprising that there could be such a simply definable wellorder of all countably complete ultrafilters in the presence of strong assumptions like supercompactness (granting the consistency of UA with very large cardinals). Still, the proof of the linearity of the Ketonen order from the Ultrapower Axiom is straightforward compared to the converse, which uses a concept that will also be needed for the proof of Theorem 4.15. This is the notion of the *translation of an ultrafilter*, whose relationship to the tightness function is close enough to justify the following very similar notation.

In the context of the Ultrapower Axiom, if D and U are countably complete ultrafilters and the underlying set of U is an ordinal δ, then $t_D(U)$ denotes the Ketonen least ultrafilter U^* of M_D on $j_D(\delta)$ such that $j_D[U] \subseteq U^*$. The ultrafilter $t_D(U)$ clarifies the connection between the Rudin-Frolík order and the Ketonen order: if $D \leq_{\mathrm{RF}} U$, then $U^* = t_D(U)$ witnesses this, in the sense that $\mathrm{Ult}(M_D, U^*) \cong M_U$ and $j_{U^*} \circ j_D = j_U$. Moreover, given countably complete ultrafilters D_0 and D_1 on an ordinal δ, $U_0 = t_{D_0}(D_1)$ and $U_1 = t_{D_1}(D_0)$ witness the Ultrapower Axiom for D_0 and D_1.

Proof. We will show that under the Ultrapower Axiom, if κ is the least tightly $C^{(n)}$-compact cardinal greater than some ordinal γ, then κ is $C^{(n-1)}$-extendible. From this, it follows from Lemma 4.14 that every tightly $C^{(n)}$-compact cardinal is either $C^{(n-1)}$-extendible or a limit of $C^{(n-1)}$-extendible cardinals, which implies the result (since tightly $C^{(n)}$-compact cardinals are measurable).

Fix a cardinal $\lambda > \kappa$ in $C^{(n)}$ such that $\mathrm{cf}(\lambda) < \kappa$. (It is important that λ is singular in order to apply [Gol22, Lemma 5.4.4] below. The fact that $\mathrm{cf}(\lambda) < \kappa$ will be used to apply Lemma 4.5.) By taking λ sufficiently large, we may assume that κ is the least tightly λ-$C^{(n)}$-compact cardinal above γ.

By Corollary 4.9 there exists a γ^+-complete ultrafilter U on some ordinal $\delta \geq \lambda$ such that letting $j_U : V \to M_U$ be the associated ultrapower embedding, $t_{j_U}(\lambda) \in (C^{(n)})^{M_U} \cap j_U(\kappa)$. Let U be the Ketonen minimum ultrafilter with this property. We will show that U is λ-*irreducible*, namely, if D is a countably complete ultrafilter on a cardinal less than λ, and D is a Rudin-Frolík predecessor of U, then D is principal.

So let D be such an ultrafilter. There is an internal ultrapower embedding $k : M_D \to M_U$ such that $k \circ j_D = j_U$. In M_D, let U' be the ultrafilter on $j_D(\delta)$ derived from k using $[\mathrm{id}]_U$. Namely,

$$X \in U' \quad \text{iff} \quad [\mathrm{id}]_U \in k(X).$$

Claim 4.16. $U' = t_D(U)$

Proof. Let $j_{U'} : M_D \to M_{U'}$ be the ultrapower embedding defined in M_D. Then the embedding $i : M_{U'} \to M_U$ given by

$$i([f]_{U'}) = j_D(f)([\mathrm{Id}]_U)$$

is such that $i \circ j_{U'} = k$, and we have the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{j_U} & M_U \\ {\scriptstyle j_D}\downarrow & \overset{k}{\nearrow} & \uparrow{\scriptstyle i} \\ M_D & \xrightarrow[j_{U'}]{} & M_{U'} \end{array}$$

We claim that i is surjective. Note that, as in [Kan03, 5.13],

$$M_U = \{j_U(f)([\mathrm{Id}]_U) : f : \delta \to V\}$$

and so M_U is the Skolem hull in M_U of the set $j_U[V] \cup \{[\mathrm{Id}]_U\}$. Thus it suffices to show that $j_U[V] \cup \{[\mathrm{Id}]_U\} \subseteq i[M_{U'}]$. One can easily check that $i([\mathrm{Id}]_{U'}) = [\mathrm{Id}]_U$. And for every $a \in V$,

$$j_U(a) = k(j_D(a)) = i(j_{U'}(j_D(a))).$$

Hence, M_U and $M_{U'}$ are isomorphic, and therefore identical. It then follows from [Gol22, Lemma 5.2.6] that $U' = t_D(U)$. □

To show D is principal, by [Gol22, Lemma 5.4.4], it suffices to show that $U' = j_D(U)$. By [Gol22, Lemma 5.4.3], we have that $U' \leq_{\Bbbk} j_D(U)$ in M_D, so it remains to prove the reverse inequality. For this, invoking the minimality of $j_D(U)$ in M_D under the Ketonen ordering, it suffices to see that in M_D, U' is a $j_D(\gamma^+)$-complete ultrafilter such that $t_{j_{U'}}(j_D(\lambda)) \in (C^{(n)})^{M_{U'}} \cap j_{U'}(j_D(\kappa))$. On the one hand, the $j_D(\gamma^+)$-completeness of U' follows easily from the fact that $j_D(\gamma^+) = \gamma^+$. On the other hand, to see that $t_{j_{U'}}(j_D(\lambda)) \in (C^{(n)})^{M_{U'}} \cap j_{U'}(j_D(\kappa))$, since $t_{j_U}(\lambda) \in (C^{(n)})^{M_U} \cap j_U(\kappa)$ and $M_{U'} = M_U$, it suffices to show that $t_{j_{U'}}(j_D(\lambda)) = t_{j_U}(\lambda)$. But since $j_D(\lambda) = t_{j_D}(\lambda)$ by Lemma 4.5, and since $j_{U'}$ is close to M_D, the fact that $t_{j_{U'}}(j_D(\lambda)) = t_{j_U}(\lambda)$ follows from Lemma 4.2. This completes the proof that U is λ-irreducible.

We now apply one of the main theorems of [Gol22, Corollary 8.2.21]: if λ is a singular strong limit cardinal and U is a countably complete λ-irreducible ultrafilter, then j_U is a λ-supercompact embedding, which means that M_U is closed under λ-sequences. Note that this implies $t_{j_U}(\lambda) = \lambda$.

The rest of the proof makes no further use of UA. Thus the assumption is used in two places: first, in the proof that a $\leq_{\Bbbk}$-minimal ultrafilter U witnessing our large cardinal property is λ-irreducible, and second, in concluding that this irreducibility leads to supercompactness.

Next, we claim that κ is the critical point of j_U. Let $\eta = \mathrm{crit}(j_U)$. So $\gamma < \eta \leq \kappa$, since $j_U(\kappa) > t_{j_U}(\lambda) = \lambda$. Notice that for some $m < \omega$, $j_U^m(\eta) > \lambda$: otherwise, letting $\eta_\omega = \sup_{\ell<\omega} j_U^\ell(\eta)$, we contradict the Kunen inconsistency theorem since $j_U(\eta_\omega) = \eta_\omega$ and M_U is closed under η_ω-sequences (see [Kan03, 23.12 and 23.14]). We claim that j_U^m witnesses that η is strongly λ-$C^{(n)}$-compact, where j_U^m is defined as in [Kan03, 23.15]. For this, one must check that $\lambda \in (C^{(n)})^N$ where $N = j_U^m(V)$. This is true for $m = 1$, i.e., for $N = M_U$. So let us assume it is true for m and show it holds for $m + 1$ as well. Let $j = j_U(j_U^m) : j_U^m(V) \to N$ be so that $j \circ j_U^m = j_U^{m+1}$. Since λ belongs to $C^{(n)}$ and also to $(C^{(n)})^{M_U}$, and since M_U is closed under λ-sequences, and therefore $V_\lambda = (V_\lambda)^{M_U}$, we may assume, by induction, that

(1) $(V_{j_U^m(\lambda)})^{j_U^m(V)} = (V_{j_U^m(\lambda)})^N$
(2) $j_U^m(\lambda)$ belongs to both $(C^{(n)})^{j_U^m(V)}$ and $(C^{(n)})^N$
(3) $\lambda \in (C^{(n)})^{j_U^m(V)}$.

Hence, since $\lambda < j_U^m(\lambda)$, it easily follows that $\lambda \in (C^{(n)})^N$, as wanted.

Since κ was taken to be the least tightly λ-$C^{(n)}$-compact cardinal above γ, we have that $\kappa \leq \eta$, and therefore $\kappa = \eta$.

We have shown that $j_U : V \to M_U$ is an elementary embedding with critical point κ such that M_U is closed under λ-sequences, $j_U(\kappa) > \lambda$, and $\lambda \in (C^{(n)})^{M_U}$. Since λ was chosen to be in $C^{(n)}$, by defining $\mathcal{U}$ as

$$X \in \mathcal{U} \quad \text{iff} \quad X \subseteq \mathcal{P}_\kappa\lambda \text{ and } j[\lambda] \in j(X)$$

and arguing as in the proof of Theorem 2.3, we have that $\mathcal{U}$ is an n-reflecting measure over $\mathcal{P}_\kappa\lambda$. Therefore, since λ was chosen arbitrarily large, by Theorem 2.7, κ is a $C^{(n-1)}$-extendible cardinal. □

Since $C^{(n)}$-extendible cardinals belong to $C^{(n+2)}$ ([Bag12]), we have the following corollary:

Corollary 4.17. *Assuming the Ultrapower Axiom, if κ is tightly $C^{(n)}$-compact, then $\kappa \in C^{(n+1)}$.*

Recall the following principle of Structure Reflection from [Bag23]:

$\mathbf{\Pi_n}$-SR(κ): ($\mathbf{\Pi_n}$-*Structural Reflection at κ*) For every Π_n-definable, with parameters in V_κ, class $\mathcal{C}$ of relational structures of the same type, and for every $A \in \mathcal{C}$, there exists $B \in \mathcal{C} \cap V_\kappa$ together with an elementary embedding $j : B \to A$.

The following is a corollary of Theorem 4.15 above and the results from section 3 of [Bag23]. The case $n = 1$ uses [Gol22, 8.3.10].

Corollary 4.18. *Assuming the Ultrapower Axiom, for any $n \geq 1$, a cardinal is tightly $C^{(n)}$-compact if and only if it is measurable and* $\mathbf{\Pi_n}$-SR(κ) *holds.*

5. Reflecting extenders

The characterization in terms of reflecting measures, given in the previous sections, of large cardinals lying in the region of the large-cardinal hierarchy between supercompactness (or strong compactness) and Vopěnka's Principle (i.e., the existence of a $C^{(n)}$-extendible cardinal, for every n [Bag12]) can similarly be given for large cardinals lying in the, much lower, region that spans between the first strong cardinal and the principle "OR is Woodin" (i.e., the existence of a Σ_n-strong cardinal, for every n [BW22, 5.14]). However, as with strong cardinals, which cannot be given by single measures and thus require the use of *extenders*, we need to define the corresponding notion of a *(κ,λ)-n-reflecting extender* (Definition 5.1) which is then applied to characterize Σ_n-strong cardinals and "OR is Woodin" in terms of the existence of *n-reflecting extenders* (Theorem 5.6 and Corollary 5.7).

Definition 5.1. *Given a cardinal κ and an ordinal λ greater than or equal to κ, a (κ,λ)-n-*reflecting extender *is a set $\mathcal{E} := \{E_a : a \in [\lambda]^{<\omega}\}$ such that*

(1) (a) *Each E_a is a κ-complete ultrafilter over $[\kappa]^{|a|}$.*
(b) *If $a \subseteq C^{(n)}$, then E_a is n-*reflecting*, i.e., $[C^{(n)} \cap \kappa]^{|a|} \in E_a$.*
(c) *E_a is not κ^+-complete for some a.*
(d) *For each $\xi \in \kappa$, there is some $a \in [\lambda]^{<\omega}$ with*

$$\{s \in [\kappa]^{|a|} : \xi \in s\} \in E_a.$$

(2) Coherence: *If $a \subseteq b$ are in $[\lambda]^{<\omega}$ and such that $b = \{\alpha_1,\ldots,\alpha_n\}$ and $a = \{\alpha_{i_1},\ldots,\alpha_{i_n}\}$, and $\pi_{ba} : [\kappa]^{|b|} \to [\kappa]^{|a|}$ is the projection map given by $\pi_{ba}(\{\xi_1,\ldots,\xi_n\}) = \{\xi_{i_1},\ldots,\xi_{i_n}\}$, then*

$$X \in E_a \quad \textit{if and only if} \quad \{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\} \in E_b\,.$$

(3) Normality: *Whenever $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to V$ are such that $\{s \in [\kappa]^{|a|} : f(s) \in \max(s)\} \in E_a$, there is $b \in [\lambda]^{<\omega}$ with $a \subseteq b$ such that*

$$\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b\,.$$

(4) Well-foundedness: *Whenever $a_m \in [\lambda]^{<\omega}$ and $X_m \in E_{a_m}$ for $m \in \omega$, there is a function $d : \bigcup_m a_m \to \kappa$ such that $d``a_m \in X_m$ for every m.*

Note that the definition above is just the usual definition of a (κ,λ)-extender (as in [Kan03, §26]), plus the requirement, given in (1)(b), of E_a being n-reflecting in the case $a \subseteq C^{(n)}$.

Recall the following standard direct limit ultrapower construction, given by a (κ,λ)-extender $\mathcal{E} := \{E_a : a \in [\lambda]^{<\omega}\}$ (for details see [Kan03, §26]). Namely, for each $a \in [\lambda]^{<\omega}$, let $\mathrm{Ult}(V, E_a)$ be the ultrapower of

V by E_a. Since E_a is κ-complete, the ultrapower is well-founded, so we let

$$j_a : V \to M_a \cong \mathrm{Ult}(V, E_a)$$

with M_a transitive, be the corresponding elementary embedding. As usual, we denote the elements of M_a by their corresponding elements in $\mathrm{Ult}(V, E_a)$.

For each $a \subseteq b$ in $[\lambda]^{<\omega}$, let π_{ba} be the projection map given as in (2) above, and let $i_{ab} : M_a \to M_b$ be the map given by

$$i_{ab}([f]_{E_a}) = [f \circ \pi_{ba}]_{E_b}$$

for all $f : [\kappa]^{|a|} \to V$. By coherence, the maps i_{ab} are well-defined and commute with the ultrapower embeddings j_a. Thus we can form the direct limit $M_{\mathcal{E}}$ of the directed system

$$\langle\langle M_a : a \in [\lambda]^{<\omega}\rangle, \langle i_{ab} : a \subseteq b\rangle\rangle$$

and let $j_{\mathcal{E}} : V \to M_{\mathcal{E}}$ be the corresponding limit elementary embedding, given by

$$j_{\mathcal{E}}(x) = [a, [c^a_x]_{E_a}]$$

for some (any) $a \in [\lambda]^{<\omega}$, and where $c^a_x : [\kappa]^{|a|} \to \{x\}$.

For each $a \in [\lambda]^{<\omega}$, let $k_{a\mathcal{E}} : M_a \to M_{\mathcal{E}}$ be given by

$$k_{a\mathcal{E}}([f]_{E_a}) = [a, [f]_{E_a}]\,.$$

It is easily checked that $j_{\mathcal{E}} = k_{a\mathcal{E}} \circ j_a$ and $k_{b\mathcal{E}} \circ i_{ab} = k_{a\mathcal{E}}$, for all $a \subseteq b$ in $[\lambda]^{<\omega}$.

Let us recall the following definitions from [BW22]:

Definition 5.2. [BW22] *For $n \geq 1$, a cardinal κ is* λ-Σ_n-strong *if for every Σ_n-definable (without parameters) class A there is an elementary embedding $j : V \to M$ with M transitive,* $\mathrm{crit}(j) = \kappa$*, $V_\lambda \subseteq M$, and $A \cap V_\lambda \subseteq j(A)$.*

κ is Σ_n-strong *if it is λ-Σ_n-strong for every ordinal λ.*

Every strong cardinal is Σ_2-strong ([BW22, 5.2]), and every Σ_n-strong cardinal belongs to $C^{(n)}$ ([BW22, 5.6]). The following gives a characterization of Σ_n-strong cardinals in terms of Σ_n-strong extenders.

Definition 5.3. [BW22, 5.7] *Given $n \geq 1$ and given cardinals $\kappa < \lambda$, a* Σ_n-strong (κ, λ)-extender *is a $(\kappa, |V_\lambda|^+)$-extender $\mathcal{E}$ such that $j(\kappa) > \lambda$, $V_\lambda \subseteq \overline{M}_{\mathcal{E}}$, and $\overline{M}_{\mathcal{E}} \models$ "$\lambda \in C^{(n-1)}$", where $\overline{M}_{\mathcal{E}}$ is the transitive collapse of the direct limit ultrapower of V by $\mathcal{E}$, and $j : V \to \overline{M}_{\mathcal{E}}$ is the corresponding elementary embedding.*

Proposition 5.4. [BW22, 5.8] *If $n \geq 2$ and $\lambda \in C^{(n)}$, then a cardinal $\kappa < \lambda$ is λ-Σ_n-strong if and only if there exists a Σ_n-strong (κ, λ)-extender.*

We will give a simpler, purely combinatorial characterization of λ-Σ_n-strong cardinals in terms of reflecting extenders (Theorem 5.6 below). First, let us prove the following lemma.

Lemma 5.5. *For every Σ_n-definable class A, every cardinal $\lambda \in C^{(n)}$, and every m, if κ is λ-A-strong (i.e., there exists an elementary embedding $j : V \to M$ with M transitive,* $\mathrm{crit}(j) = \kappa$*, $V_\lambda \subseteq M$, and $A \cap V_\lambda \subseteq j(A)$), then there is such an embedding j which in addition satisfies $j(\kappa) \in C^{(m)}$.*

Proof. Let $j : V \to M$ be an elementary embedding with M transitive, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $A \cap V_\lambda \subseteq j(A)$. Since in M, $j(\kappa)$ is a measurable cardinal, we can obtain, via an iterated ultrapower construction of length some $\alpha \in C^{(m)}$, an elementary embedding $j_\alpha : M \to M'$, some transitive M', with $\mathrm{crit}(j_\alpha) = j(\kappa)$ and with $j_\alpha(j(\kappa)) = \alpha$. Letting $k = j_\alpha \circ j$, we have that k is a λ-strong embedding with $\mathrm{crit}(k) = \kappa$ and with $k(\kappa) \in C^{(m)}$. Moreover, since $A \cap V_\lambda \subseteq j(A)$, for every $\gamma \in A \cap V_\lambda$ we have that $M \models$ "$\gamma \in j(A)$", hence by elementarity, and since $\mathrm{crit}(j_\alpha) = j(\kappa) > \lambda$, $M' \models$ "$\gamma \in j_\alpha(j(A))$". Hence, $\gamma \in k(A)$. Thus, $A \cap V_\alpha \subseteq k(A)$. □

Theorem 5.6. *If $n \geq 1$ and $\lambda \in C^{(1)}$, then a cardinal $\kappa < \lambda$ is λ-Σ_{n+1}-strong if and only if there exists a $(\kappa, \lambda+1)$-n-reflecting extender.*

Proof. Suppose first that κ is λ-Σ_{n+1}-strong. Since $C^{(n)}$ is a Π_n-definable class, hence also Σ_{n+1}-definable, there is an elementary embedding $j : V \to M$ with M transitive, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $C^{(n)} \cap \lambda \subseteq j(C^{(n)})$. By Lemma 5.5, we may assume $j(\kappa)$ is a cardinal.

We shall obtain a $(\kappa, \lambda+1)$-n-reflecting extender $\mathcal{E}$ from j, as follows: for every $a \in [\lambda + 1]^{<\omega}$ let E_a be given by:

$$X \in E_a \text{ if and only if } X \subseteq [\kappa]^{|a|} \text{ and } a \in j(X).$$

We need to check that $\mathcal{E} := \{E_a : a \in [\lambda + 1]^{<\omega}\}$ satisfies conditions $(1) - (4)$ of Definition 5.1.

(1): Since $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$, each E_a is easily seen to be a κ-complete ultrafilter over $[\kappa]^{|a|}$. And since $C^{(n)} \cap \lambda \subseteq j(C^{(n)})$, if $a \subseteq C^{(n)}$, then $[C^{(n)} \cap \kappa]^{|a|} \in E_a$. Moreover, $E_{\{\kappa\}}$ is not κ^+-complete, as the set $[\kappa \setminus \alpha]^1$ is in $E_{\{\kappa\}}$, for every $\alpha < \kappa$. As for (d), for each $\xi \in \kappa$, $\{\xi\} \in j(\{s \in [\kappa]^1 : \xi \in s\})$. Hence, $\{s \in [\kappa]^1 : \xi \in s\} \in E_{\{\xi\}}$.

(2) Coherence: Assume $a \subseteq b$ and are in $[\lambda+1]^{<\omega}$. Suppose $X \in E_a$. Thus, $X \subseteq [\kappa]^{|a|}$ and $a \in j(X)$. We need to show that

$$b \in j(\{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\}).$$

Now notice that

$$j(\{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\}) = \{s \in [j(\kappa)]^{|b|} : \pi_{ba}(s) \in j(X)\}.$$

But since $\pi_{ba}(b) = a$, and $a \in j(X)$, we have that

$$b \in \{s \in [\lambda+1]^{|b|} : \pi_{ba}(s) \in j(X)\} \subseteq \{s \in [j(\kappa)]^{|b|} : \pi_{ba}(s) \in j(X)\}$$

as wanted.

Conversely, if $\{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\} \in E_b$, we have that

$$b \in j(\{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in X\}) = \{s \in [j(\kappa)]^{|b|} : \pi_{ba}(s) \in j(X)\}.$$

Hence, $\pi_{ba}(b) = a \in j(X)$, and therefore $X \in E_a$.

(3) Normality: Assume $a \in [\lambda+1]^{<\omega}$ and $f : [\kappa]^{|a|} \to V$ are such that

$$X := \{s \in [\kappa]^{|a|} : f(s) \in \max(s)\} \in E_a.$$

We need to find some $b \in [\lambda+1]^{<\omega}$ with $a \subseteq b$ such that

$$\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b\,.$$

We have

$$j(X) = \{s \in [j(\kappa)]^{|a|} : j(f)(s) \in \max(s)\}\,.$$

Also, since $X \in E_a$, we have that $a \in j(X)$, and therefore $j(f)(a) \in \max(a)$.

Let $\delta = j(f)(a)$, and let $b = a \cup \{\delta\}$. Thus,

$$b \in \{s \in [\lambda+1]^{|b|} : j(f)(\pi_{ba}(s)) \in s\}$$

where $\pi_{ba} : [\lambda+1]^{|b|} \to [\lambda+1]^{|a|}$ is the projection function. So, since

$$\{s \in [\lambda+1]^{|b|} : j(f)(\pi_{ba}(s)) \in s\} \subseteq j(\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\})$$

we have

$$\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b$$

as wanted.

(4) Well-foundedness: Assume $a_m \in [\lambda+1]^{<\omega}$ and $X_m \in E_{a_m}$, for every $m \in \omega$. We need to find a function $d : \bigcup_m a_m \to \kappa$ such that $d``a_m \in X_m$ for every m. Assuming such function does not exist, one can show similarly as in [Kan03, 15.7 (a)] that the direct limit ultrapower $M_{\mathcal{E}}$ is not well-founded. Now letting $k_{\mathcal{E}} : M_{\mathcal{E}} \to M$ be given by $k_{\mathcal{E}}([a,[f]_{E_a}]) = j(f)(a)$, we have that $k_{\mathcal{E}} \circ j_{\mathcal{E}} = j$. But this implies $M_{\mathcal{E}}$ is well-founded, as any infinite $\in_{\mathcal{E}}$-descending sequence in $M_{\mathcal{E}}$ would yield an infinite $\in$-descending sequence in V, thus yielding a contradiction.

We have thus shown that $\mathcal{E}$ is a $(\kappa, \lambda+1)$-n-reflecting extender.

For the converse, assume $\mathcal{E} = \{E_a : a \in [\lambda+1]^{<\omega}\}$ is a $(\kappa, \lambda+1)$-n-reflecting extender and we will show that κ is λ-Σ_{n+1}-strong.

Let $j_{\mathcal{E}} : V \to M_{\mathcal{E}}$ be the elementary embedding given by $\mathcal{E}$. Let us write M for the transitive collapse of $M_{\mathcal{E}}$, and let $j : V \to M$ be the corresponding elementary embedding, i.e., $j = \pi \circ j_{\mathcal{E}}$, where $\pi : M_{\mathcal{E}} \to \overline{M}_{\mathcal{E}}$ is the transitive collapse.

Condition (1) ensures that κ is the critical point of j (see [Kan03, 26.2].

By [BW22, 5.5], we only need to show that $j(\kappa) > \lambda$, $M \models$ "$\lambda \in C^{(n)}$", and $V_\lambda \subseteq M$.

Let us first show that $j(\kappa) > \lambda$.

For each $a \in [\lambda+1]^{<\omega}$, let $k_a : M_a \to M$ be the composition of $k_{a\mathcal{E}}$ with the transitive collapse. Thus, $j = k_a \circ j_a$.

Let Id_1 be the identity function on $[\kappa]^1$, let $c_{[\kappa]^1} : [\kappa]^1 \to \{[\kappa]^1\}$, and let $c_\kappa : [\kappa]^1 \to \{\kappa\}$. Now, for every $\alpha < \lambda+1$, in $M_{\{\alpha\}}$ we have

$$[\mathrm{Id}_1]_{E_{\{\alpha\}}} \in [c_{[\kappa]^1}]_{E_{\{\alpha\}}} = [[c_\kappa]_{E_{\{\alpha\}}}]^1 = [j_{\{\alpha\}}(\kappa)]^1.$$

Hence in M,

$$k_{\{\alpha\}}([\mathrm{Id}_1]_{E_{\{\alpha\}}}) \in k_{\{\alpha\}}([j_{\{\alpha\}}(\kappa)]^1) = [j(\kappa)]^1.$$

Similarly as in [Jec02, 26.2 (a)], one can show that for every $\alpha < \lambda+1$, $k_{\{\alpha\}}([\mathrm{Id}_1]_{E_{\{\alpha\}}}) = \{\alpha\}$. Therefore,

$$\{\alpha\} \in [j(\kappa)]^1.$$

In particular, $\{\lambda\} \in [j(\kappa)]^1$, which yields $\lambda < j(\kappa)$.

We next show that $M \models$ "$\lambda \in C^{(n)}$".

Since $\lambda \in C^{(n)}$, $E_{\{\lambda\}}$ is n-reflecting, hence $[C^{(n)} \cap \kappa]^1 \in E_{\{\lambda\}}$, which implies $M_{\{\lambda\}} \models$ "$[\mathrm{Id}_1]_{E_{\{\lambda\}}} \in [C^{(n)}]^1$". Now, since we observed above, $k_{\{\lambda\}}([\mathrm{Id}_1]_{E_{\{\lambda\}}}) = \{\lambda\}$, by the elementarity of $k_{\{\lambda\}}$ we have that $M \models$ "$\{\lambda\} \in [C^{(n)}]^1$", and so $M \models$ "$\lambda \in C^{(n)}$", as wanted.

It only remains to show that $V_\lambda \subseteq M$.

Let $f \in V$ be a bijection between $[\kappa]^1$ and V_κ with the property that, if $\mu < \kappa$ belongs to $C^{(1)}$, then the restriction of f to $[\mu]^1$ is a bijection between $[\mu]^1$ and V_μ.

Then $j(f) : [j(\kappa)]^1 \to V^M_{j(\kappa)}$ is a bijection in M with the same property. Thus, since $\lambda \in (C^{(1)})^M$, for every $x \in V^M_\lambda$ there exists $\gamma < \lambda$ such that $j(f)(\{\gamma\}) = x$.

Letting $D := \{[\{\gamma\},[f]] : \gamma < \lambda\}$, we have just shown that the map $i : \langle D, \in_{\mathcal{E}} \restriction D\rangle \to \langle V_\lambda, \in\rangle$ given by

$$i([\{\gamma\},[f]]) = j(f)(\{\gamma\})$$

is onto. Moreover, if $[\{\gamma\},[f]] \in_{\mathcal{E}} [\{\delta\},[f]]$, then for some $X \in E_{\{\gamma,\delta\}}$, we have

$$(f \circ \pi_{\{\gamma,\delta\}\{\gamma\}})(s) \in (f \circ \pi_{\{\gamma,\delta\}\{\delta\}})(s)$$

for every $s \in X$. Hence, for every $s \in j(X)$,

$$(j(f) \circ \pi_{\{\gamma,\delta\}\{\gamma\}})(s) \in (j(f) \circ \pi_{\{\gamma,\delta\}\{\delta\}})(s)\,.$$

In particular, since $\{\gamma,\delta\} \in j(X)$,

$$j(f)(\{\gamma\}) \in j(f)(\{\delta\})\,.$$

A similar argument shows that i is one-to-one. Hence, i is an isomorphism, and so i is just the transitive collapsing map. Since $D \subseteq M_{\mathcal{E}}$, to conclude that $V_\lambda \subseteq M$ it will be sufficient to show that the transitive collapse of D is the same as the restriction to D of the transitive collapse of $M_\mathcal{E}$. For this, it suffices to see that every $\in_\mathcal{E}$-element of an element of D is $=_\mathcal{E}$-equal to an element of D. So, suppose $[\{\gamma\},[f]] \in D$ and $[a,[g]] \in_\mathcal{E} [\{\gamma\},[f]]$, with $[a,[g]] \in M_\mathcal{E}$. Then $j(g)(a) \in j(f)(\gamma)$. Since the restriction of $j(f)$ to $[\lambda]^1$ is surjective on V_λ, and V_λ is transitive, there is some $\delta < \lambda$ such that $j(f)(\{\delta\}) = j(g)(a)$. Hence, $[\{\delta\},[f]] =_\mathcal{E} [a,[g]]$. □

Recall from [BW22, 5.14] that the statement"OR *is Woodin*" is the schema asserting that for every (definable, with parameters) proper class A, there exists some cardinal κ which is A-strong, i.e., for every λ, there is an elementary embedding $j : V \to M$, with M transitive, such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \gamma$, $V_\gamma \subseteq M$ and $A \cap V_\gamma = j(A) \cap V_\gamma$.

In [BW22, 5.13] it is shown that "OR is Woodin" is equivalent to the schema asserting that, for every $n \geq 1$, there exist a proper class of Σ_n-strong cardinals. Thus the following is a corollary to Theorem 5.6:

Corollary 5.7. "OR *is Woodin" iff for every $n \geq 1$ there is a proper class of cardinals κ such that for every $\lambda \in C^{(1)}$ greater that κ there exists a $(\kappa, \lambda+1)$-n-reflecting extender.*

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