THE JOURNAL OF SYMBOLIC LOGIC
Volume 00, Number 0, XXX 0000

RANK-TO-RANK EMBEDDINGS AND STEEL'S CONJECTURE

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Abstract. This paper establishes a conjecture of Steel [6] regarding the structure of elementary embeddings from a level of the cumulative hierarchy into itself. Steel's question is related to the Mitchell order on these embeddings, studied in [5] and [6]. Although this order is known to be illfounded, Steel conjectured that it has certain large wellfounded suborders, which is what we establish. The proof relies on a simple and general analysis of the much broader class of extender embeddings and a variant of the Mitchell order called the internal relation.

§1. Introduction. A basic result in the theory of large cardinals states that a countably complete ultrafilter U is never an element of the inner model $\text{Ult}(V, U)$, the ultrapower of the universe of sets V by U. This suggests a more general question, instances of which arise all over large cardinal theory: which countably complete ultrafilters belong to which ultrapowers? In [4], Mitchell defined the partial order on countably complete ultrafilters now known as the *Mitchell order*, by setting $U \lhd W$ whenever U belongs to $\text{Ult}(V, W)$. Intuitively, this order arranges the countably complete ultrafilters according to the degree to which their ultrapowers resemble V. The fact that an ultrafilter does not belong to its own ultrapower is the most basic structural property of the Mitchell order: $\lhd$ is irreflexive. But having defined the order, one can prove a stronger structural property: the Mitchell order is *wellfounded*. (Mitchell proved that his order is wellfounded on normal ultrafilters, but in fact this result generalizes quite easily to arbitrary countably complete ultrafilters; a proof can be found in in [1, Theorem 4.2.47].)

It makes sense to generalize the Mitchell order to a broader class of objects than ultrafilters, namely, certain directed systems of ultrafilters called *extenders,* defined in Section 2. If E is an extender, or more generally a directed system of ultrafilters, then $\text{Ult}(V, E)$ denotes the direct limit of the ultrapowers of V associated to the ultrafilters of the system E; one requirement in the definition of an extender is that the limit $\text{Ult}(V, E)$ be wellfounded, so that it can be identified with the proper class inner model to which it is isomorphic. For extenders E_0

2000 *Mathematics Subject Classification.* 03E55.
Key words and phrases. Extender, Mitchell order, internal relation, rank-to-rank.
This research was partially supported by NSF Grant MSPRF 1902884.

and E_1, set $E_0 \lhd E_1$ if E_0 belongs to the ultrapower $\text{Ult}(V, E_1)$. The wellfoundedness of this generalized Mitchell order turns out to be a very subtle question. Applying the theory of iteration trees, Steel [6] showed that the Mitchell order is wellfounded on certain restricted classes of extenders (for example, short extenders, and more generally, amenable extenders). It turns out, however, that the Mitchell order is *not* wellfounded on arbitrary extenders.

The infinite descending sequences of extenders that thwart the wellfoundedness of the Mitchell order were first discovered by Martin as he investigated one of the strongest known large cardinal hypotheses, dangerously close to the inconsistent hypotheses proposed by Reinhardt and refuted by Kunen [3]. This is the axiom I_2, which states that there is a cardinal λ and a non-trivial elementary embedding $j : V_\lambda \to V_\lambda$ that extends to an elementary embedding from the universe of sets into an inner model. If E is the extender of length λ derived from the embedding j (Definition 2.1), then the ultrapower embedding $j_E : V \to \text{Ult}(M, E)$ itself furnishes a canonical extension of j to an elementary embedding from the universe of sets into an inner model.

Such an extender is called a λ-*extender*. One can produce a $\lhd$-descending sequence of λ-extenders as follows. Let E_0 be a λ-extender. For $n > 0$, let $E_n = j_{E_{n-1}}(E_0)$. Then one can show that for all $n < \omega$, E_n is a λ-extender, and obviously $E_n \in \text{Ult}(V, E_{n-1})$, so $E_n \lhd E_{n-1}$. So $E_0 \rhd E_1 \rhd E_2 \rhd \cdots$, and hence the Mitchell order is illfounded on λ-extenders.

Are such large cardinals really necessary to prove the illfoundedness of the Mitchell order? This question was taken up by Neeman [5], who showed that the Mitchell order is wellfounded on all downward closed extenders that are not λ-extenders. (An extender E is *downward closed* if for all ordinals δ with $j_E(\delta) < \text{length}(E)$, $j_E[\delta] \in \text{Ult}(V, E)$.) The wellfoundedness of the Mitchell order on arbitrary extenders that are not λ-extenders remains an open question.

Steel conjectured that the wellfoundedness of the Mitchell order generalizes to certain sets of λ-extenders as well:

CONJECTURE (Steel [6]). For any $\delta < \lambda$, the Mitchell order is wellfounded on the set of λ-extenders whose critical points lie below δ.

The purpose of this paper is to prove Steel's conjecture. This is the content of Corollary 6.8. The proof involves a variant of the Mitchell order called the internal relation, which coincides with the Mitchell order on λ-extenders: an extender F is said to be internal to an extender E if $j_F \restriction \text{Ult}(V, E)$ is definable over $\text{Ult}(V, E)$. The internal relation is not wellfounded, but by Theorem 5.8, its restriction to extenders with a common discontinuity point is. This identifies an aspect of the wellfoundedness of the Mitchell order orthogonal to the notions underlying the wellfoundedness proofs of Neeman and Steel. Theorem 5.8 easily yields a proof of Steel's conjecture.

§2. Extenders. This section contains a terse account of the basic theory of extenders.

Suppose M is a model of set theory. If U is an M-ultrafilter on a set X in M and $f : X \to Y$ is a function in M then the *pushforward of U by f* is the M-ultrafilter $f_*(U) = \{A \in P^M(Y) : f^{-1}[A] \in U\}$. The function f

induces an embedding of ultrapowers in the opposite direction: one can define $k : \mathrm{Ult}(M, f_*(U)) \to \mathrm{Ult}(M, U)$ by $k([g]_W) = [g \circ f]_U$.

A *directed system of M-ultrafilters* is a sequence

$$D = \langle U_a, X_a, f_{b,a} : a \leq b \in \mathbb{D}\rangle$$

where $\mathbb{D}$ is a directed partial order, U_a is an M-ultrafilter over $X_a \in M$, and $f_{b,a} : X_b \to X_a$ is a partial function in M, defined on a U_b-measure one set, such that U_a is the pushforward of U_b under $f_{b,a}$.

Associated to a directed system of ultrafilters D is the directed system of ultrapowers

$$\langle M_a, j_{a,b} : a \leq b \in \mathbb{D}\rangle$$

where $M_a = \mathrm{Ult}(M, U_a)$ is the ultrapower of M by U_a and $j_{a,b} : M_a \to M_b$ is the elementary embedding induced by $f_{a,b}$:

$$j_{a,b}([g]_{U_a}) = [g \circ f_{a,b}]_{U_b}$$

Let $\mathrm{Ult}(M, D)$ denote the direct limit of this system of ultrapowers, and for each $a \in D$, let $j_{a,D} : M_a \to \mathrm{Ult}(M, D)$ denote the direct limit embedding. For each $a \in D$, let $j_a : M \to M_a$ denote the the ultrapower embedding associated to U_a, and let

$$j_D : M \to \mathrm{Ult}(M, D)$$

be the embedding $j_{a,D} \circ j_a$, which is independent of the choice of a. Finally, we denote $j_{a,D}([g]_{U_a})$ by $[g, a]_D$ so that

$$\mathrm{Ult}(M, D) = \{[g, a]_D : a \in \mathbb{D} \text{ and } g \in M^{X_a} \cap M\}$$

An arbitrary elementary embedding $i : M \to N$ between two transitive models of set theory can be approximated by a particularly simple kind of directed system of ultrafilters called an *extender derived from i*. More precisely, for each ordinal $\lambda \in N$, will show how to form a directed system E from i that approximates i up to λ in the sense that for any set $A \subseteq [\mathrm{Ord}]^{<\omega}$,

$$j_E(A) \cap [\lambda]^{<\omega} = i(A) \cap [\lambda]^{<\omega}$$

The ultrafilters that make up our derived extender will themselves come from the *derived ultrafilter construction*: given $X \in M$, and $a \in i(X)$, the *M-ultrafilter on X derived from i using a* is the ultrafilter $U = \{A \in P^M(X) : a \in i(A)\}$. The associated *factor embedding* is the elementary embedding $k : \mathrm{Ult}(M, U) \to N$ defined by

$$k([g]_U) = i(g)(a)$$

This is the unique elementary embedding from $\mathrm{Ult}(M, U)$ to N such that $k_{a,i} \circ j_U = i$ and $k_{a,i}([\mathrm{id}]_U) = a$.

Definition 2.1. Suppose $i : M \to N$ is elementary embedding between transitive models of set theory and λ is an ordinal in N. Then the *M-extender of length λ derived from i* is the directed system of ultrafilters

$$E = \langle U_a, X_a, f_{b,a} : a \subseteq b \in [\lambda]^{<\omega}\rangle$$

where

- $X_a = [\delta_a]^{|a|}$ for δ_a the least ordinal δ such that $a \in i(\delta)$.
- U_a is the M-ultrafilter on X_a derived from i using a.
- $f_{b,a} : X_b \to X_a$ is the partial function sending each $w \in X_b$ to the unique $u \in X_a$ such that $(w, u, \in) \cong (b, a, \in)$.

In the context of Definition 2.1, letting $\langle M_a, j_{a,b} : a \subseteq b \in [\lambda]^{<\omega}\rangle$ be the associated directed system of ultrapowers associated to E, one can check that $k_{b,E} \circ j_{a,b} = k_{a,E}$. Therefore the universal property of the direct limit supplies a canonical embedding

$$k_{E,i} : \mathrm{Ult}(M, E) \to N$$

called *factor embedding* associated to the derived extender E, given by the formula $k_{E,i}([g, a]_E) = i(g)(a)$. By setting g equal to the identity function, one sees that a lies in the range of $k_{E,i}$ for all $a \in [\lambda]^{<\omega}$. In other words, the critical point of $k_{E,i}$, if it exists, is greater than or equal to λ.[1] As a consequence, for any $B \in \mathrm{Ult}(M, E)$ contained in $[\mathrm{Ord}]^{<\omega}$, $k_{E,i}(B) \cap [\lambda]^{<\omega} = B \cap [\lambda]^{<\omega}$, and therefore for any $A \in M$ with $A \subseteq [\mathrm{Ord}]^{<\omega}$,

$$j_E(A) \cap [\lambda]^{<\omega} = k_{E,i}(j_E(A)) \cap [\lambda]^{<\omega} = i(A) \cap [\lambda]^{<\omega}$$

as promised. The derived extender ultrapower can be characterized as the minimum embedding with this property:

Lemma 2.2. *Suppose $i : M \to N$ is an elementary embedding and E is the M-extender of length λ derived from i. Suppose $M \xrightarrow{j} P \xrightarrow{k} N$ is a sequence of embeddings with $k \circ j = i$ and $\mathrm{crit}(k) \geq \lambda$. Then there is a unique elementary embedding $\ell : \mathrm{Ult}(M, E) \to P$ with $\mathrm{crit}(\ell) \geq \lambda$ such that $\ell \circ j_E = j$.* ⊣

Note that if E is the M-extender of length λ derived from some embedding $i : M \to N$, then in fact E is the M-extender of length λ derived from its own ultrapower embedding $j_E : M \to \mathrm{Ult}(M, E)$. Therefore, one can kick away the initial embedding to obtain the notion of an extender simpliciter:

Definition 2.3. Suppose M is a transitive model of set theory. A directed system of M-ultrafilters E is an *M-extender of length λ* if E is the M-extender of length λ derived from its own ultrapower embedding $j_E : M \to \mathrm{Ult}(M, E)$. An *extender* is a V-extender.

We will need some basic facts about iterated of extender ultrapowers.

Lemma 2.4. *Suppose E is an M-extender and F is an $\mathrm{Ult}(M, E)$-extender. Then there is an M-extender G such that $\mathrm{Ult}(M, G) = \mathrm{Ult}(\mathrm{Ult}(M, E), F)$ and $j_G = j_F \circ j_E$.*

Sketch. Let $\lambda = \mathrm{length}(E)$ and $\mu = \mathrm{length}(F)$. Let G be the extender of length $\max\{j_F(\lambda), \mu\}$ derived from j_G. ⊣

Definition 2.5. Suppose M is a transitive model of set theory, E is an M-extender, and F is an $\mathrm{Ult}(M, E)$-extender. Then $E * F$ denotes the M-extender G of minimum length such that $\mathrm{Ult}(M, G) = \mathrm{Ult}(\mathrm{Ult}(M, E), F)$ and $j_G = j_F \circ j_E$.

[1] Recall that if $k : P \to Q$ is an elementary embedding of transitive models of set theory, its *critical point* of k is the least ordinal κ such that $k(\kappa) > \kappa$.

$$
\begin{tikzcd}
 & \mathrm{Ult}(V, E * F^*) = \mathrm{Ult}(V, F * E^*) & \\
\mathrm{Ult}(V, E) \arrow[ur, "j_{F^*}"] & & \mathrm{Ult}(V, F) \arrow[ul, "j_{E^*}"'] \\
 & V \arrow[ul, "j_E"] \arrow[ur, "j_F"'] &
\end{tikzcd}
$$

FIGURE 1. The diagram of a comparison

There is a sort of dual to Lemma 2.4:

LEMMA 2.6. *Suppose E and G are M-extenders. Then for any elementary embedding $k : \mathrm{Ult}(M, E) \to \mathrm{Ult}(M, G)$ such that $k \circ j_E = j_G$, there is an* $\mathrm{Ult}(M, E)$*-extender F such that* $\mathrm{Ult}(M, E * F) = \mathrm{Ult}(M, G)$ *and* $j_F = k$.

SKETCH. Let $\nu = \mathrm{length}(G)$ and let F be the extender of length ν derived from k. $\dashv$

§3. Comparison. The fundamental *Comparison Lemma* of inner model theory roughly states that any pair of canonical models of large cardinal axioms can be amalgamated into a common model. The notion of a comparison turns out to be a convenient construct outside the context of fine structure theory.

DEFINITION 3.1. Suppose M is a transitive model of set theory and E and F are M-extenders. A *comparison of* (E, F) is a pair (F^*, E^*) with the following properties:

- F^* is an $\mathrm{Ult}(M, E)$-extender.
- E^* is an $\mathrm{Ult}(M, F)$-extender.
- $j_{F^*} \circ j_E = j_{E^*} \circ j_F$.

The comparison (F^*, E^*) is *right-internal* if $E^* \in \mathrm{Ult}(M, F)$ and *left-internal* if $F^* \in \mathrm{Ult}(M, E)$. A comparison that is simultaneously left-internal and right-internal is said to be *internal*.

REMARK 3.2. The first bullet-point in the definition of a comparison can be weakened slightly. Suppose E and F are M-extenders, E^* is an $\mathrm{Ult}(M, F)$-extender, and

$$k : \mathrm{Ult}(M, E) \to \mathrm{Ult}(M, F * E^*)$$

is an elementary embedding such that $k \circ j_E = j_{F^*} \circ j_E$. Then there is an $\mathrm{Ult}(M, E)$-extender F^* such that $k = j_{F^*}$. To see this, let $G = E * F^*$, so that $j_G = j_{F^*} \circ j_E = k \circ j_E$. One can then apply Lemma 2.6 to obtain an $\text{Ult}(M, E)$-extender F^* such that $j_G = j_{F^*} \circ j_E$.

One cannot fail to notice the somewhat strange mirroring convention in the definition of a comparison, in which (E, F) is compared by (F^*, E^*). Heuristically, we treat E^* as a copy of E into $\text{Ult}(M, F)$, and j_{F^*} as an embedding of E into E^*.

The most important example of a comparison is the shift comparison.

Example 3.3. Suppose E and F are M-extenders and E belongs to M. Then $\text{Ult}(M, E)$ is a definable inner model of M. Let $E_* = j_F(E)$, so E_* is an $\text{Ult}(M, F)$-extender that belongs to $\text{Ult}(M, F)$. Then by elementarity, for any $x \in M$,

$$j_F(j_E(x)) = j_{E^*}(j_F(x))$$

In other words, $j_F \circ j_E = j_{E^*} \circ j_F$. As a consequence, j_F restricts to an elementary embedding $k : \text{Ult}(M, E) \to \text{Ult}(M, F * E^*)$. By Remark 3.2, there is an $\text{Ult}(M, E)$-extender F^* such that $j_{F^*} = k$. Therefore $j_{F^*} \circ j_E = j_{E^*} \circ j_F$. This shows that (E, F) admits a right-internal comparison.

Definition 3.4. In the context of Example 3.3, the comparison (F^*, E^*) is called the *shift comparison* of (E, F).

§4. Pointed extenders. In [2], Ketonen introduces a combinatorial generalization of the Mitchell order to countably complete weakly normal ultrafilters which he proved to be wellfounded. In [1], the author (independently) introduced a generalization of this order to arbitrary countably complete ultrafilters on ordinals, now called the *Ketonen order.* This order is again wellfounded, and the author showed it can be linear.

Here it turns out to be useful to generalize Ketonen order to extenders:

Definition 4.1. A *pointed extender* is a pair (E, α) such that E is an extender and α is an ordinal. The *Ketonen order on pointed extenders* is defined as follows:

- $(E, \alpha) <_{\Bbbk} (F, \beta)$ if there is a right-internal comparison (F^*, E^*) of (E, F) such that $j_{F^*}(\alpha) < j_{E^*}(\beta)$.
- $(E, \alpha) \leq_{\Bbbk} (F, \beta)$ if there is a right-internal comparison (F^*, E^*) of (E, F) such that $j_{F^*}(\alpha) \leq j_{E^*}(\beta)$.
- $(E, \alpha) =_{\Bbbk} (F, \beta)$ if $E \leq_{\Bbbk} F$ and $F \leq_{\Bbbk} E$.

We do not know whether in general $(E, \alpha) \leq_{\Bbbk} (F, \beta)$ if and only if either $(E, \alpha) <_{\Bbbk} (F, \beta)$ or $(E, \alpha) =_{\Bbbk} (F, \beta)$. (This is true, however, if E and F have an internal comparison.) The Ketonen order on ultrafilters embeds into the Ketonen order on extenders by sending U to $(E_U, [\text{id}]_U)$, where E_U is the extender E of least length such that $\text{Ult}(V, E) = \text{Ult}(V, U)$ and $j_E = j_U$.

The main result of this section is that the Ketonen order is wellfounded. First we show it is transitive, which involves a stacking operation on comparisons.

Lemma 4.2. *Suppose E, F, and G are M-extenders. Suppose*

- *$(F^\bullet, E^\bullet)$ is a comparison of (E, F).*
- *(G°, F°) is a comparison of (F, G).*

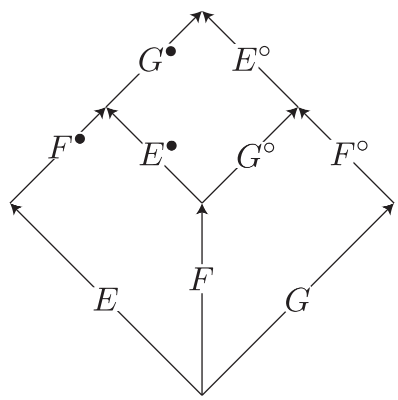

FIGURE 2. Stacking comparisons

- *$(G^\bullet, E^\circ)$ is a comparison of $(E^\bullet, G^\circ)$.*

Then $(F^\bullet * G^\bullet, F^\circ * E^\circ)$ is a comparison of (E, G).

PROOF. The lemma is clear from Figure 2. ⊣

LEMMA 4.3. *Suppose $(E,\alpha) \leq_{\Bbbk} (F,\beta) \leq_{\Bbbk} (G,\gamma)$. Then $(E,\alpha) \leq_{\Bbbk} (G,\gamma)$, with strict inequality if $(E,\alpha) <_{\Bbbk} (F,\beta)$ or $(F,\beta) <_{\Bbbk} (G,\gamma)$.*

PROOF. We start by choosing some comparisons:

- Let $(F^\bullet, E^\bullet)$ witness $(E,\alpha) \leq_{\Bbbk} (F,\beta)$.
- Let (G°, F°) witness $(F,\beta) \leq_{\Bbbk} (G,\gamma)$.
- Let $(G^\bullet, E^\circ)$ be the shift comparison of $(E^\bullet, G^\circ)$ (Example 3.3).

By Lemma 4.2, $(F^\bullet * G^\bullet, F^\circ * E^\circ)$ is a comparison of (E, G), and clearly it is right-internal since E° and F° belong to their domain models. Finally,

$$\begin{aligned} j_{G^\bullet}(j_{F^\bullet}(\alpha)) &\leq j_{G^\bullet}(j_{E^\bullet}(\beta)) && (1)\\ &= j_{E^\circ}(j_{G^\circ}(\beta)) \\ &\leq j_{E^\circ}(j_{F^\circ}(\gamma)) && (2) \end{aligned}$$

so $j_{G^*}(\alpha) \leq j_{E^*}(\alpha)$. One has strict inequality if and only if either (1) or (2) is strict, which one can secure by choosing $(F^\bullet, E^\bullet)$ to witness $(E,\alpha) <_{\Bbbk} (F,\beta)$ or (G°, F°) to witness $(F,\beta) <_{\Bbbk} (G,\gamma)$. ⊣

The proof of the wellfoundedness of the Ketonen order is a matter of iterating the stacking construction. The proof is reminiscent of both the proof of the Dodd-Jensen Lemma from fine structure theory, exposited in [7] and [8], and also the proof of the wellfoundedness of the Mitchell order on normal ultrafilters.

THEOREM 4.4. *The Ketonen order on pointed extenders is wellfounded.*

PROOF. We isolate the main construction of the proof in a claim:

CLAIM 1. *Suppose $(E_0,\alpha_0) >_{\Bbbk} (E_1,\alpha_1) >_{\Bbbk} (E_2,\alpha_2) >_{\Bbbk} \cdots$ is a descending sequence of pointed extenders. Then there is a descending sequence of pointed extenders $(F_1,\beta_1) >_{\Bbbk} (F_2,\beta_2) >_{\Bbbk} (F_3,\beta_3) >_{\Bbbk} \cdots$ and an $\mathrm{Ult}(V,E_0)$ extender $G \in \mathrm{Ult}(V,E_0)$ such that $F_1 = E_0 * G$ and $j_G(\alpha_0) > \beta_1$.*

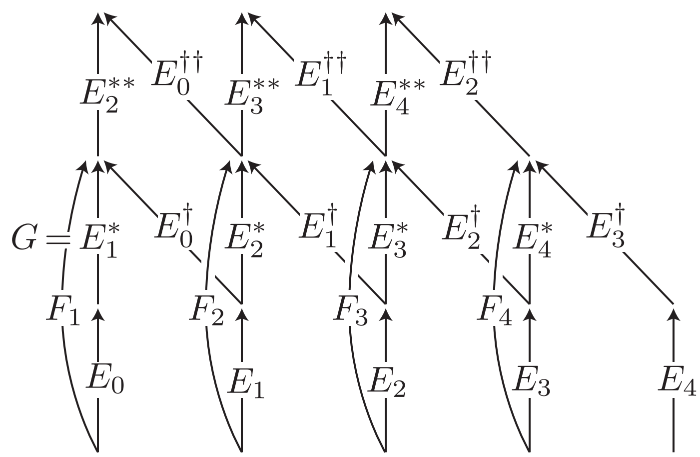

FIGURE 3. A descending sequence self-replicates

PROOF. The proof is illustrated by Figure 3.

For each $n < \omega$, let $(E^*_{n+1}, E^\dagger_n)$ be a left-internal comparison of (E_n, E_{n+1}) witnessing $(E_n, \alpha_n) >_{\Bbbk} (E_{n+1}, \alpha_{n+1})$. For $n < \omega$, let $F_{n+1} = E_n * E^*_{n+1}$ and let $\beta_{n+1} = j_{E^\dagger_n}(\alpha_{n+1})$. Let $G = E^*_1$. To prove the claim, it remains to show that $(F_1, \beta_1) >_{\Bbbk} (F_2, \beta_2) >_{\Bbbk} (F_3, \beta_3) >_{\Bbbk} \cdots$. For this, let $(E^{**}_{n+2}, E^{\dagger\dagger}_n)$ be a comparison of $(E^\dagger_n, E^*_{n+2})$, which is guaranteed to exist by Example 3.3. Then

$$\begin{aligned} j_{E^{**}_{n+2}} \circ j_{F_{n+1}} &= j_{E^{**}_{n+2}} \circ j_{E^*_{n+1}} \circ j_{E_n} \\ &= j_{E^{**}_{n+2}} \circ j_{E^\dagger_n} \circ j_{E_{n+1}} \\ &= j_{E^{\dagger\dagger}_n} \circ j_{E^*_{n+2}} \circ j_{E_{n+1}} \\ &= j_{E^{\dagger\dagger}_n} \circ j_{F_{n+2}} \end{aligned}$$

so $(E^{**}_{n+2}, E^{\dagger\dagger}_n)$ is a comparison of (F_{n+1}, F_{n+2}). That this comparison witnesses $(F_{n+1}, \beta_{n+1}) >_{\Bbbk} (F_{n+2}, \beta_{n+2})$ is an easy computation, left to the reader. $\dashv$

Suppose now that the lemma fails. Let O denote the trivial extender. There is a minimum ordinal α such that the Ketonen order is illfounded below (O, α), and we denote this ordinal by α^0_0. Let $E^0_0 = O$, and let

$$(E^0_0, \alpha^0_0) >_{\Bbbk} (E^0_1, \alpha^0_1) >_{\Bbbk} (E^0_2, \alpha^0_2) >_{\Bbbk} \cdots$$

be a Ketonen descending sequence. Repeatedly applying Claim 1, one obtains, for each $m < \omega$, pointed extenders

$$(E^m_m, \alpha^m_m) >_{\Bbbk} (E^m_{m+1}, \alpha^m_{m+1}) >_{\Bbbk} (E^m_{m+2}, \alpha^m_{m+2}) >_{\Bbbk} \cdots$$

and an $\text{Ult}(V, E^m_m)$-extender G_m with $E^{m+1}_{m+1} = E^m_m * G_m$ and $j_{G_m}(\alpha^m_m) > \alpha^{m+1}_{m+1}$. Let H_1 be the trivial $\text{Ult}(V, G_0)$-extender, and for each integer $m \geq 1$, let

$$H_{m+1} = G_1 * \cdots * G_m$$

Then for all $m \geq 1$, H_m is an $\text{Ult}(V, G_0)$-extender, and in $\text{Ult}(V, G_0)$, the comparison (G_m, O) witnesses that $(H_m, \alpha^m_m) >_{\Bbbk} (H_{m+1}, \alpha^{m+1}_{m+1})$.

Thus the Ketonen order of $\mathrm{Ult}(V, G_0)$ is illfounded below $(H_1, \alpha^1_1) = (O, \alpha^1_1)$. By the absoluteness of wellfoundedness, $\mathrm{Ult}(V, G_0)$ satisfies that the Ketonen order is illfounded below (O, α^1_1). By the elementarity of $j_{G_0} : V \to \mathrm{Ult}(V, G_0)$, however, $\mathrm{Ult}(V, E^0_0)$ satisfies that $j_{G_0}(\alpha^0_0)$ is the least ordinal α such that for some extender E, the Ketonen order is illfounded below (E, α). But $\alpha^1_1 < j_{G_0}(\alpha^0_0)$, and this is a contradiction. $\dashv$

§5. The internal relation. This section introduces a variant of the Mitchell order which can be analyzed using the Ketonen order.

Definition 5.1 (Internal relation). Suppose E and F are extenders. Then F is *internal to* E, denoted $F \sqsubset E$, if $j_F \restriction \mathrm{Ult}(V, E)$ is definable over $\mathrm{Ult}(V, E)$.

By Remark 3.2, $F \sqsubset E$ if and only if every $\mathrm{Ult}(V, E)$-extender derived from $j_F \restriction \mathrm{Ult}(V, E)$ belongs to $\mathrm{Ult}(V, E)$. In fact, a closer look at this remark shows that it suffices that the $\mathrm{Ult}(V, E)$-extender of length $\sup j_E[\lambda]$ derived from $j_F \restriction \mathrm{Ult}(V, E)$ belongs to $\mathrm{Ult}(V, E)$, where λ is the length of F. In this sense, the internal relation looks a lot like the Mitchell order, except that instead of demanding $F \in \mathrm{Ult}(V, E)$, one demands that $F^* \in \mathrm{Ult}(V, E)$, where F^* is an $\mathrm{Ult}(V, F)$-extender that serves as a copy of F.

Even if $F \lhd E$, $\mathrm{Ult}(V, E)$ may not contain enough functions to correctly compute the ultrapower of F. In other words, $(j_F)^{\mathrm{Ult}(V,E)}$ need not be equal to $j_F \restriction \mathrm{Ult}(V, E)$. This raises a question that will become important later.

Definition 5.2. Suppose M is an inner model and F is an extender of length λ. Let $F \cap M$ denote the M-extender of length λ derived from $j_F \restriction M$.

This notation is motivated by the fact that

$$U_{a,F\cap M} = U_{a,F} \cap \mathrm{Ult}(V, E)$$

Under what conditions is $j_{F\cap M}$ equal to $j_F \restriction M$? Obviously it suffices that $j_{U_a \cap M} = j_{U_a} \restriction M$ for cofinally many $a \in [\lambda]^{<\omega}$, so we first consider the case that F is a single ultrafilter. The answer then is fairly obvious:

Lemma 5.3. *Suppose M is a transitive model of set theory, $X \in M$, and U is an ultrafilter on X. Then $j_{U\cap M} = j_U \restriction M$ if and only if for every $f : X \to M$, there is some $g : X \to M$ such that $f = g$ mod U.* $\dashv$

When M is an extender ultrapower, this requirement can be relaxed slightly:

Lemma 5.4. *Suppose E is an extender of length λ. Let $M = \mathrm{Ult}(V, E)$ and suppose U is an ultrafilter on a set X in M. Then $j_{U\cap M} = j_U \restriction M$ if and only if the following hold:*

- *For all $f : X \to \lambda$, there is some $g : X \to [\lambda]^{<\omega}$ such that $f = g$ mod U.*
- *For some $h : X \to \lambda$ in M, $h = j_E \restriction X$ mod U.*

Proof. By Lemma 5.3, we need only show that the bullet points imply $j_{U\cap M} = j_U \restriction M$. Moreover it suffices to show that for every $f : X \to M$, there is a function $i \in M$ such that $f = i$ mod U.

First pick functions $\ell : X \to M$ and $a : X \to [\lambda]^{<\omega}$ such that for all $x \in X$, $f(x) = [\ell(x), a(x)]_E$. Then take $g : X \to [\lambda]^{<\omega}$ such that $g(x) = a(x)$ for

U-almost all x. Then let

$$i(x) = j_E(\ell)(h(x))(g(x))$$

Then for U-almost all x, $h(x) = j_E(x)$ and $g(x) = a(x)$, so

$$i(x) = j_E(\ell)(j_E(x))(a(x)) = j_E(\ell(x))(a(x)) = [\ell(x), a(x)]_E$$

as desired. ⊣

We now turn to the wellfoundedness properties of the internal relation. The internal relation is *not* wellfounded. Obviously the trivial extender is internal to itself, but more interestingly, there can be mutually internal extenders. For example, in the fairly common situation that E and F are extenders, κ is a cardinal, $E \in V_\kappa$, $\text{crit}(F) \geq \kappa$, and $\text{Ult}(V, F)$ is closed under κ-sequences, one has $E \sqsubset F$ (obviously) and $F \sqsubset E$ (nontrivially).

Definition 5.5. Suppose E is an extender. A limit ordinal δ is a *discontinuity point of* E if $j_E(\delta) > \sup j_E[\delta]$.

Proposition 5.6. *Suppose δ is an ordinal, E and F are extenders, F is δ-discontinuous, and $F \sqsubset E$. Then $(F, \sup j_F[\delta]) <_{\Bbbk} (E, \sup j_E[\delta])$.*

Proof. Let (F^*, E^*) be the shift comparison of (E, F), so $E^* = j_F(E)$ and $j_{F^*} = j_F \restriction \text{Ult}(V, E)$. Since $F \sqsubset E$, the shift comparison is internal. It therefore suffices to show that $j_{E^*}(\sup j_F[\delta]) < j_{F^*}(\sup j_E[\delta])$. Notice, however, that $j_{F^*}(\sup j_E[\delta]) = j_F(\sup j_E[\delta]) = \sup j_{E^*}[j_F(\delta)]$. Since $\sup j_F[\delta] < j_F(\delta)$, $j_{E^*}(\sup j_F[\delta])$ is strictly less that $\sup j_{E^*}[j_F(\delta)]$. ⊣

Proposition 5.6 has the following consequence:

Theorem 5.7. *If $E \sqsubset F$ and $F \sqsubset E$, then E and F have no common discontinuity points.*

Proof. Assume towards a contradiction that δ is a common discontinuity point of E and F. By Theorem 5.6,

$$(E, \sup j_E[\delta]) >_{\Bbbk} (F, \sup j_F[\delta]) >_{\Bbbk} (E, \sup j_E[\delta]) >_{\Bbbk} \cdots$$

contradicting the wellfoundedness of the Ketonen order (Theorem 4.4). ⊣

More generally:

Theorem 5.8. *The internal relation is wellfounded on any set of extenders with a common discontinuity point.* ⊣

§6. Steel's conjecture. In this section, we apply Theorem 5.8 to prove Steel's conjecture. First, recall the relevant definitions:

Definition 6.1 (Mitchell order). If E and F are extenders, then $E \lhd F$ if $E \in \text{Ult}(V, F)$.

Definition 6.2. Suppose λ is an ordinal. A nontrivial extender E of length λ is a *λ-extender* if $j_E(V_\lambda) = V_\lambda$.

In the context of ZFC, the structure of λ-extenders is highly constrained by the Kunen inconsistency theorem.

THEOREM 6.3 (Kunen [3]). *Suppose that $j : V \to M$ is an elementary embedding. Then for any ordinal $\delta \geq \operatorname{crit}(j)$, $j(V_{\delta+1}) \neq V_{\delta+1}$.* ⊣

The proof of Steel's conjecture requires two consequences of Kunen's theorem:

LEMMA 6.4. *Suppose λ is an ordinal and E is a λ-extender. Then for any ordinal δ such that $\operatorname{crit}(j_E) \leq \delta < \lambda$, $j_E(\delta) > \delta$. Therefore if δ is regular, then δ is a discontinuity point of j_E.*

PROOF. First, one cannot have $j_E(\delta) = \delta$, since otherwise $j_E(V_{\delta+1}) = V_{\delta+1}$ (because $\delta + 1 \leq \lambda$), contradicting Theorem 6.3. Second, if δ is regular, then $j_E(\delta)$ is regular and in $\operatorname{Ult}(V, E)$. Since $j_E(\delta) < j_E(\lambda) = \lambda$, $P(j_E(\delta)) \subseteq V_\lambda \subseteq \operatorname{Ult}(V, E)$, and therefore $j_E(\delta)$ is regular in V. Since $j_E(\delta) > \delta$, the set $j_E[\delta]$ cannot be cofinal in $j_E(\delta)$. Therefore $j_E(\delta) > \sup j_E[\delta]$, as desired. ⊣

LEMMA 6.5. *If there is a λ-extender, then λ is a limit cardinal of countable cofinality.*

PROOF. Let E be a λ-extender. Let $\kappa_0 = \operatorname{crit}(j_E)$ and for $n < \omega$, let $\kappa_{n+1} = j_E(\kappa_n)$. An obvious induction shows that for all $n < \omega$, $\kappa_n < \lambda$ and κ_n is a cardinal. Let $\delta = \sup_{n<\omega} \kappa_n$. By elementarity, $j_E(\delta) = \sup_{n<\omega} \kappa_{n+1} = \delta$. Therefore by Lemma 6.4, $\delta \geq \lambda$, so $\delta = \lambda$. In other words, $\lambda = \sup_{n<\omega} \kappa_n$, which implies the lemma. ⊣

In the context of λ-extenders, the Mitchell order and the internal relation coincide.

LEMMA 6.6. *Suppose E and F are λ-extenders and $F \lhd E$. Then $F \sqsubset E$, and in fact, $(j_F)^{\operatorname{Ult}(V,E)} = j_F \restriction \operatorname{Ult}(V, E)$.*

PROOF. Let $M = \operatorname{Ult}(V, E)$. It suffices to show that for all $a \in [\lambda]^{<\omega}$, $j_{U_{a,F} \cap M} = j_{U_{a,F}} \restriction M$. Note that $U_{a,F}$ is an ultrafilter on a set in V_λ, so it will be enough to show the following: for any countably complete ultrafilter U on a set $X \in V_\lambda$, $j_{U\cap M} = j_U \restriction M$.

For this, we verify the conditions of Lemma 5.4. First, we check that for all $f : X \to [\lambda]^{<\omega}$, there is some $g \in M$ such that $f = g \bmod U$. Since λ is singular, it is not necessarily the case that $f \in V_\lambda$. Recall, however, that λ has countable cofinality (Lemma 6.5), and fix a countable set $\{\kappa_n\}_{n<\omega}$ cofinal in λ. For every n, let $A_n = \{x \in X : f(x) \in [\kappa_n]^{<\omega}\}$. Since $\bigcup_{n<\omega} = X$ and U is countably complete, there is some $n < \omega$ such that $A_n \in U$. Therefore $f \restriction A_n \in V_\lambda$. Let $g = f \restriction A_n$. Then $g \in M$ and $g = f \bmod U$. Finally, we check that there is a function $h \in M$ such that $h = j_E \restriction X \bmod U$. In fact, since $j_E(V_\lambda) = V_\lambda$, $j_E \restriction X \in V_\lambda$, and therefore $j_E \restriction X \in M$. ⊣

This yields Steel's conjecture:

THEOREM 6.7. *Suppose λ is an ordinal and $E_0 \rhd E_1 \rhd E_2 \rhd \cdots$ is a descending sequence of λ-extenders. Then $\sup_{n<\omega} \operatorname{crit}(j_{E_n}) = \lambda$*

PROOF. Suppose not. Let $\delta = (\sup_{n<\omega} \operatorname{crit}(j_{E_n}))^+$. Since λ is a limit cardinal, $\delta < \lambda$. For each n, since $\operatorname{crit}(j_{E_n}) \leq \delta < \lambda$ and δ is regular, δ is a discontinuity point of E_n. By Lemma 6.6, $E_0 \sqsupset E_1 \sqsupset E_2 \sqsupset \cdots$. The existence of an internally descending sequence of extenders with a common discontinuity point contradicts Theorem 5.8. ⊣

COROLLARY 6.8. *Suppose $\delta < \lambda$ are cardinals. Then the Mitchell order is wellfounded on the set of λ-extenders with critical point less than δ.* $\dashv$

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