The potentialist principle

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Abstract

We settle a question of Woodin motivated by the philosophy of potentialism in set theory. A Σ_2-sentence φ is *possible* if for any ordinal α, φ holds in a forcing extension W of the universe of sets V such that V and W contain the same sets of rank α. We show in Theorem 6.1 that it is consistent relative to a supercompact cardinal that each possible sentence is true; this is Woodin's Σ_2-potentialist principle. We accomplish this by generalizing Gitik's method of iterating distributive forcings by embedding them into Príkry-type forcings [5, Section 6.4]; our generalization, Theorem 5.2, works for forcings that add no bounded subsets to a strongly compact cardinal, which requires a completely different proof. Finally, using the concept of mutual stationarity, we show in Theorem 7.5 that the Σ_2-potentialist principle implies the consistency of a Woodin cardinal.[1]

1 Introduction

The introduction of the forcing technique in the 1960s revolutionized set theory by demonstrating the unsolvability of many classical questions in the framework of Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). Forcing seems to offer us a glimpse of alternate universes of set theory, universes where all the axioms of ZFC hold but various classical questions, for example the continuum hypothesis, can go either way.

One response to this situation is to say that these questions are unanswerable or even meaningless; another is to declare instead that the ZFC axioms are simply inadequate to the task. Perhaps one should seek out new axioms, inspired presumably by new intuitions about the nature of sets. In this direction, a starting point has often been forcing itself. Broadly construed, a *forcing axiom* asserts that the universe of sets resembles the alternate universes revealed by the method of forcing. The issue, however, is that these universes by design have conflicting properties: for example, in some, the continuum hypothesis holds, and in others, it fails.

A proponent of forcing axioms must therefore explain which of the alternate universes the true universe resembles and in what respect it resembles them. This has been done

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[1]We would like to thank Menachem Magidor and Martin Zeman for suggestions that ultimately led to the proof of this consistency strength lower bound.

in several ways, the most fruitful of which is the hierarchy of forcing axioms leading from Martin's Axiom to Martin's Maximum. Here we investigate another approach, suggested by Woodin, based on a potentialist account of the cumulative hierarchy of sets.[2]

The idea behind such a potentialist stance is that the hierarchy of sets does not exist as a completed totality, since if it did there seems to be no reason why its union should not itself form a set. Rather, the hierarchy is a "potential" object, extending as far as *possible* or *imaginable* or *consistent*. We direct the reader to [8, Section 2] for a more compelling discussion of these issues.

The forcing axiom studied here, which was proposed by Woodin, roughly asserts that if it is possible for a level of the hierarchy of sets to have some property, then some level of the hierarchy actually does have this property. The notion of "possibility" intended here combines both possibility in the sense of set-theoretic potentialism and possibility in the sense of forcing. Let us say that ψ is *possible* if for every ordinal α, there is a forcing extension W of the universe of sets V such that the α-th level W_α of the hierarchy of sets in W is equal to V_α and for some ordinal β, $W_\beta \vDash \psi$.

The Σ_2*-potentialist principle* states that if ψ is possible, then there is some ordinal α such that $V_\alpha \vDash \psi$.[3]

For a certain kind of set theoretic potentialist, the unfinished nature of the universe of sets renders many sentences in the language of set theory meaningless. The only meaningful set theoretic questions, on this view, ask whether there is some level of the hierarchy of sets with some first-order property. For this sort of set theoretic potentialist, if ψ is possible, then there can be no meaningful evidence against the actual existence of some V_α satisfying ψ.

Woodin proposed the Σ_2-potentialist principle with the hope that it might be refutable, the idea being that this would amount to a mathematical refutation of the form of potentialism described in the previous paragraph. Others, however, have argued in favor of such principles, at least implicitly. For example, Shelah writes:

> Maybe the following analogy will explain my attitude; we use the standard American ethnic prejudice system, as it is generally familiar. So a typical universe of set theory is the parallel of Mr. John Smith, the typical American; my typical universe is quite interesting (even pluralistic), it has long intervals where GCH holds, but others in which it is violated badly, many λ's such that λ^+-Suslin trees exist and many λ's for which every λ^+-Aronszajn is special, and it may have lots of measurables, with a huge cardinal being a marginal case but certainly no supercompact.

We will see in Section 3 that all of the properties of Shelah's "typical universe" are consequences of the Σ_2-potentialist principle.[4]

The first result of this paper (Theorem 6.1) is that the Σ_2-potentialist principle is consistent. Instead of a knock-down argument against a certain form of set-theoretic potentialism, the result provides some evidence in favor of this perspective, or at least some reason to believe the perspective is coherent.

[2]This is the sequence $\langle V_\alpha : \alpha \in \mathrm{Ord}\rangle$ defined by transfinite recursion by setting $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$, and $V_\gamma = \bigcup_{\beta<\gamma} V_\beta$ for limit ordinals γ.

[3]The reason the principle is named the Σ_2-potentialist principle is that a sentence φ is Σ_2 in Lévy hierarchy if and only if it is equivalent to a sentence of the form "there is a level of the hierarchy of sets satisfying ψ" for some sentence ψ.

[4]More accurately, they are consequences of the version of this principle that allows the possible sentences to have parameters.

To be precise, we prove that the Σ_2-potentialist principle is consistent relative to a supercompact cardinal. The second result of this paper (Theorem 7.5) shows that one cannot prove the consistency of the Σ_2-potentialist principle without resorting to fairly large cardinals: the Σ_2-potentialist principle implies the existence of an inner model with a Woodin cardinal.

To conclude the introduction, let us spell out the main technical difficulty that this paper overcomes. It is not hard to show that for any finite set of sentences $\{\psi_n\}_{n<m}$ in the language of set theory, one can force the Σ_2-potentialist principle restricted to these sentences. Suppose this is the case for m, and let us prove it for $m+1$. First, using the induction hypothesis, let W be a forcing extension in which the Σ_2-potentialist principle holds for $\{\psi_n\}_{n<m}$. If ψ_m is not possible in W, then the Σ_2-potentialist principle holds in W for $\{\psi_n\}_{n<m+1}$, and we are done. Otherwise, we would like to pass to a forcing extension W' such that ψ_m holds in some level of the hierarchy of sets of W'.

There are two possible issues with this. First, if $n<m$ and ψ_n is true in the α_n-th level of the hierarchy of sets of W for some ordinal α_n, then we would like this to remain the case in W'. But this can be achieved by making sure $W'_{\alpha_n} = W_{\alpha_n}$. Second, if $n<m$ and ψ_n is not true in any level of the hierarchy of sets of W, it follows that ψ_n is not possible in W. This means that there is some ordinal α_n such that for any forcing extension W'' of W that preserves W_{α_n}, ψ_n does not hold in any level of the hierarchy of sets of W''. If we make sure $W'_{\alpha_n} = W_{\alpha_n}$, then ψ_n will remain impossible in W'.

Thus we can simply take W' to be a forcing extension such that ψ_m hold in some level of W' and $W'_\alpha = W_\alpha$ where $\alpha = \max_{n<m} \alpha_n$. But we can do this since ψ_m is possible in W!

If one tries to extend this method to infinitely many sentences, one faces the seemingly insurmountable problem of showing that if all the finite stages of the iteration preserve V_α, then the whole iteration preserves V_α. The issue is that no matter what support one chooses — full support seems like a natural choice here —- it is not clear how to show, for example, that no new subsets of ω are added in the limit. And it seems essential to be able to preserve levels of the cumulative hierarchy to make the argument work.

Our solution to this problem is to extend a technique of Gitik for iterating distributive forcings. The idea is to realize these forcings as Príkry-type forcings and perform a Magidor iteration rather than the full-support iteration. Our main contribution to Gitik's technique is to show that if κ is strongly compact, then any forcing that does not change V_κ is equivalent to a Príkry-type forcing (Theorem 5.2). Combining this with Gitik's technique allows us to answer Woodin's question.

2 Preliminaries

In this section we lay out the well-known connection between Σ_2-sentences and the hierarchy of sets and introduce the various principles that will be studied in this paper.

Proposition 2.1. *For each Σ_2-formula $\varphi(x)$ in the language of set theory, there is a formula $\varphi'(x)$, obtained uniformly from $\varphi(x)$, such that* ZFC *proves that $\varphi(x)$ is equivalent to the existence of a level of the hierarchy of sets containing x and satisfying $\varphi'(x)$.*

Proof. Let ψ be the conjunction of the Axiom of Infinity with the statement that every set is in bijection with an ordinal number. Let $\varphi'(x) = \psi \wedge \varphi(x)$. Then ZFC proves that $\varphi(x)$ is equivalent to the existence of a level of the hierarchy of sets containing x and satisfying $\varphi'(x)$. This is because if κ is an ordinal, then $V_\kappa \models \psi$ if and only if V_κ is a Σ_1-elementary substructure of V. □

We say a sentence φ in the language of set theory is *possible* if for all ordinals α, φ holds in some forcing extension W of the universe of sets such that $W_\alpha = V_\alpha$. More generally, if $\varphi(x)$ is a formula in the language of set theory and a is a set, then $\varphi(a)$ is *possible* if for all ordinals α, $\varphi(a)$ holds in some forcing extension W of the universe of sets such that $W_\alpha = V_\alpha$.

The Σ_2-potentialist principle, defined in the introduction, is equivalent to the statement that every possible Σ_2-sentence is true. But we can now also define, for each $n < \omega$, the Σ_n-potentialist principle, asserting that every possible Σ_n-sentence is true. Similarly, we define the Π_n-potentialist principle for $n < \omega$.

The *boldface* Σ_n*-potentialist principle* asserts that if a is a set and $\varphi(a)$ is a possible Σ_n-formula, then $\varphi(a)$ holds. The boldface Σ_2-potentialist principle seems equal in intuitive appeal to the Σ_2-potentialist principle, but it is a more powerful global principle; for example, see Proposition 3.3 and Proposition 3.6.

For $n \geq 3$, the Σ_n-potentialist principle does not seem well-motivated at all, since the formulas involved are not pinned down by a single level of the hierarchy of sets. To see the difference, note that if ZFC proves two Σ_2-sentences φ_0 and φ_1 are possible, then ZFC proves that $\varphi_0 \wedge \varphi_1$ is possible. But this is not at all clear for more complicated sentences.

We will show below that the Π_3-potentialist principle is consistent. In an earlier preprint of this paper, we showed that the boldface Σ_3-potentialist principle is inconsistent and posed the problem of proving the Σ_3-potentialist principle inconsistent as well. This was quickly accomplished by Taranovsky, and we are grateful to include his argument below (Theorem 3.5).

3 The inconsistency of the Σ_3-Potentialist Principle

Proposition 3.1. *The boldface* Π_2*-potentialist principle is a consequence of* ZFC*.*

Proof. Suppose $\varphi(x)$ is a Π_2-formula and a is a set such that for all α, there is a forcing extension W of V such that $W_\alpha = V_\alpha$ and W satisfies $\varphi(a)$.

Fix a first-order formula $\varphi'(x)$ in the language of set theory such that ZFC proves that $\varphi(x)$ is equivalent to the statement that $\varphi'(x)$ holds in every level of the hierarchy of sets that contains x.

Fix $\alpha > \text{rank}(a)$, and let us show that $V_\alpha \vDash \varphi'(a)$. Let W be a forcing extension of V such that $W_\alpha = V_\alpha$ and W satisfies $\varphi(a)$. Since $\varphi(x)$ is equivalent to $\forall\alpha \in \text{Ord}(\text{rank}(x) < \alpha \to V_\alpha \vDash \varphi'(x)$, $W_\alpha \vDash \theta(a)$, and so $V_\alpha \vDash \varphi'(a)$.

This proves that for all $\alpha > \text{rank}(a)$, $V_\alpha \vDash \theta(a)$, or equivalently, $\varphi(a)$ holds. □

Proposition 3.2. *The boldface* Σ_2*-potentialist principle implies the boldface* Π_3*-potentialist principle.*

Proof. Assume the boldface Σ_2-potentialist principle, and suppose $\varphi(a)$ is a possible Π_3 formula. Then $\varphi(a)$ has the form $\forall y\, \psi(a, y)$. For each $b \in V$, the formula $\psi(a, b)$ is a possible Σ_2-formula, and so by the Σ_2-potentialist principle, it is true. Therefore $\varphi(a)$ is true. □

Proposition 3.3. *The boldface* Σ_2*-potentialist principle implies* $V = \text{HOD}$*.*

Proof. For each set of ordinals A, the statement that A is coded into the continuum function is Σ_2 and possible, and so the boldface Σ_2-potentialist principle implies it is true. Therefore the principle implies that every set of ordinals belongs to HOD, and so $V = \text{HOD}$. □

Proposition 3.4. *The boldface Σ_3-potentialist principle is inconsistent.*

Proof. The Σ_3-potentialist principle implies that there is a set that is not in HOD, since this is a possible Σ_3-sentence. Therefore the Σ_3-potentialist principle contradicts the boldface Σ_2-potentialist principle. It follows that the boldface Σ_3-potentialist principle is inconsistent. □

Taranovsky answered one of the questions raised in an earlier preprint of this paper by showing that the Σ_3-potentialist principle itself is already inconsistent.

Theorem 3.5 (Taranovsky)**.** *The Σ_3-potentialist principle is inconsistent.*

Proof. Let α_0 denote the least ordinal α such that there is a subset of $\aleph_\alpha$ that is not ordinal definable. Let φ_{even} be the sentence stating that α_0 is even and let φ_{odd} state that α_0 is odd. Then φ_{even} and φ_{odd} are Σ_3 sentences and both are possible. The lightface Σ_3-potentialist principle implies that both are true, which is a contradiction. □

Proposition 3.6. *The boldface Σ_2-potentialist principle implies there are no strongly compact cardinals.*

Proof. The boldface Σ_2-potentialist principle implies that there are arbitrarily large cardinals λ for which $\square_\lambda$ holds, and therefore there are no strongly compact cardinals. □

4 Iterating Príkry-type forcing

Suppose $\mathbb{P} = (P, \leq)$ is a partial order and $\mathbb{P}^* = (P, \leq^*)$ is a weak suborder of $\mathbb{P}$; that is, for all $p, q \in P$, if $p \leq^* q$, then $p \leq q$. Then $(P, \leq, \leq^*)$ is a *Príkry-type forcing* if it has the *Príkry property*: for all conditions $p \in P$ and all statements φ in the forcing language associated to P, there is some $q \leq^* p$ such that either $q \Vdash_{\mathbb{P}} \varphi$ or $q \Vdash_{\mathbb{P}} \neg\varphi$. The triple $(P, \leq, \leq^*)$ is a *κ-complete Príkry-type forcing* if in addition, for all $p \in P$, every subset of $\mathbb{P}^*_p$ of cardinality less than κ has a $\leq^*$-lower bound. This is equivalent to the assumption that $\mathbb{P}^*$ is κ-closed and for all $p \in P$, for any $q, r \leq^* p$, there is some $s \leq^* q, r$. Given a partial order $\mathbb{P}$ for which some direct extension order $\mathbb{P}^*$ exists that makes $(P, \leq, \leq^*)$ a κ-complete Príkry-type forcing, we say that $\mathbb{P}$ is a *κ-complete Príkry-type poset.*

For the definition of a Magidor iteration of Príkry-type forcings, see Gitik's article in the Handbook of Set Theory [5, Section 6.1].

Theorem 4.1 (Gitik [5, Lemma 6.4])**.** *If κ is strongly compact and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle$ is a Magidor iteration such that $\dot{\mathbb{Q}}_\alpha \in V_\kappa^{\mathbb{P}_\alpha}$ is forced by $\mathbb{P}_\alpha$ to be an α-complete Príkry-type forcing, then κ remains strongly compact in $V^{\mathbb{P}}$ where $\mathbb{P}$ is the Magidor support limit of the iteration.*

5 Distributivity, compactness, and Príkry-type forcings

This section is motivated by the following result [5, Lemma 6.22]:

Theorem 5.1 (Gitik). *If κ is supercompact, then every κ-distributive partial order is forcing-equivalent to a κ-complete Príkry-type poset.* □

We will improve this result by reducing the large cardinal hypothesis to strong compactness and, more significantly, relaxing the assumption of κ-distributivity to the optimal one. Note that a κ-complete Príkry-type forcing $\mathbb{P}$ adds no bounded subsets of κ.[5] Indeed, this was Príkry's original motivation for establishing that the Príkry forcing associated to a normal ultrafilter has the Príkry property [9].

A partial order $\mathbb{P}$ that adds no bounded subsets of κ is called $(\kappa, 2)$-*distributive*, since this property is equivalent to the statement that the Boolean completion of $\mathbb{P}$ satisfies the following distributive law for all $\gamma < \kappa$:

$$\bigwedge_{i<\gamma} (a_i^0 \vee a_i^1) = \bigvee_{s\in 2^\gamma} \bigwedge_{i<\gamma} a_i^{s_i}$$

Theorem 5.2. *If κ is strongly compact, then a partial order is $(\kappa, 2)$-distributive if and only if it is forcing-equivalent to a κ-complete Príkry-type poset.*

The idea that such a theorem might be possible comes from a striking observation of Hamkins-Seabold [6, Theorem 83]:

Theorem 5.3 (Hamkins-Seabold). *If κ is strongly compact and $\mathbb{B}$ is a $(\kappa, 2)$-distributive complete Boolean algebra, then $\mathbb{B}$ carries a κ-complete ultrafilter.* □

Proposition 5.4 is a corollary of this result along with a proposition that requires no large cardinal hypothesis whatsoever. A partial order $\mathbb{P}$ is *centered by κ-complete ultrafilters* if it is the disjoint union of a family of sets each of which generates a κ-complete ultrafilter on the Boolean completion of $\mathbb{P}$.

Proposition 5.4. *For any partial order $\mathbb{P}$, the following are equivalent:*

(1) $\mathbb{P}$ is a κ-complete Príkry-type poset.

(2) Every $p \in \mathbb{P}$ belongs to a set $A \subseteq \mathbb{P}$ that generates a κ-complete ultrafilter on the Boolean completion of $\mathbb{P}$.

(3) A dense suborder of $\mathbb{P}$ is centered by κ-complete ultrafilters.

(4) $\mathbb{P}$ is centered by κ-complete ultrafilters.

Proof. (1) implies (2): Fix a partial order $\leq^*$ such that $(\mathbb{P}, \leq^*)$ is a κ-complete Príkry-type forcing. Then for any $p \in \mathbb{P}$, $\{q \in \mathbb{P} : q \leq^* p\}$ generates a κ-complete ultrafilter on the Boolean completion of $\mathbb{P}$.

(2) implies (3): Assume (2). By transfinite recursion, we construct a sequence $\langle A_\alpha \rangle_{\alpha<\nu}$ such that $\bigcup_{\alpha<\nu} A_\alpha$ is dense in $\mathbb{P}$ and centered by κ-complete ultrafilters. Suppose A_β has been defined from all $\beta < \alpha$. If $\bigcup_{\beta<\alpha} A_\beta$ is dense in $\mathbb{P}$, set $\nu = \alpha$ and terminate the

[5] If $\gamma < \kappa$ and $p \Vdash \dot{A} \subseteq \gamma$, then using the Príkry property and κ-completeness, one can find a condition $q \leq^* p$ such that for all $\alpha < \gamma$, either $q \Vdash_{\mathbb{P}} \alpha \in \dot{A}$ or $q \Vdash_{\mathbb{P}} \alpha \notin \dot{A}$. Letting $A = \{\alpha < \gamma : q \Vdash_{\mathbb{P}} \alpha \in \dot{A}\}$, we have $q \Vdash \dot{A} = A$, and hence $q \Vdash \dot{A} \in \check{V}$.

construction. Otherwise, choose some $p \in \mathbb{P}$ that has no extension in $\bigcup_{\beta<\alpha} A_\beta$. Choose a set A containing p that generates a κ-complete ultrafilter on the Boolean completion of $\mathbb{P}$. Let $A_\alpha = \{q \leq p : q \in A\}$.

(3) implies (4): Let $\langle A_\alpha\rangle_{\alpha<\gamma}$ witness that the dense suborder $\mathbb{Q}$ of $\mathbb{P}$ is centered by κ-complete ultrafilters. We will define $\langle B_\alpha\rangle_{\alpha<\gamma}$ by recursion witnessing that $\mathbb{P}$ is centered by κ-complete ultrafilters. Having defined B_β for all $\beta < \alpha$, let F be the filter on $\mathbb{P}$ generated by A_α, and let $B_\alpha = F \setminus \bigcup_{\beta<\alpha} B_\beta$. Then the sets B_α form a partition of $\mathbb{P}$: for any $p \in \mathbb{P}$, let α be least such that there is some $q \in A_\alpha$ extending p, and note that $p \in B_\alpha$. Moreover for all $\alpha < \gamma$, $A_\alpha \subseteq B_\alpha$, and so B_α generates a κ-complete ultrafilter on the Boolean completion of $\mathbb{P}$. (Since $\mathbb{Q}$ is dense in $\mathbb{P}$, the Boolean completion of $\mathbb{P}$ is the Boolean completion of $\mathbb{Q}$.)

(4) implies (1): Let $\langle A_i\rangle_{i\in I}$ witness that $\mathbb{P}$ is centered by κ-complete ultrafilters. Set $p \leq^* q$ if $p \leq q$ and both p and q belong to the same piece A_i of this partition. Then $\leq^*$ witnesses that $\mathbb{P}$ is a κ-complete Príkry-type forcing. □

It is unclear to us whether every partial order that is forcing-equivalent to a κ-complete Príkry-type forcing is itself a Príkry type forcing, although this statement is true for complete Boolean algebras:

Corollary 5.5. *A partial order $\mathbb{P}$ is forcing-equivalent to a κ-complete Príkry-type poset if and only if its Boolean completion is a κ-complete Príkry-type complete Boolean algebra.*

Proof. By Proposition 5.4, if a κ-complete Príkry-type forcing $\mathbb{Q}$ is dense in a forcing $\mathbb{Q}'$, then $\mathbb{Q}'$ is a κ-complete Príkry-type forcing. Therefore to prove the corollary, it suffices to show that if a complete Boolean algebra $\mathbb{B}$ is forcing equivalent to a κ-complete Príkry-type complete Boolean algebra $\mathbb{C}$, then $\mathbb{B}$ is κ-complete Príkry-type.

Fix a maximal antichain $A \subseteq \mathbb{B}$ such that for each $a \in A$, $\mathbb{B}_a$ is isomorphic to $\mathbb{C}_w$ for some $w \in \mathbb{C}$; since $\mathbb{C}$ is κ-complete Príkry type, so is $\mathbb{C}_w$, and hence so is $\mathbb{B}_a$.

For each $a \in A$, let $\leq^*_a$ be a direct extension order witnessing that $\mathbb{B}_a$ is κ-complete Príkry type. For $u, v \in \mathbb{B}^+$, set $u \leq^* v$ if for all $a \in A$, either $u \wedge a = v \wedge a = 0$ or $(u \wedge a) \leq^*_a (v \wedge a)$. Then it is easy to check that $\leq^*$ witnesses that $\mathbb{B}$ is a κ-complete Príkry-type complete Boolean algebra. □

Proof of Theorem 5.2. Let $\mathbb{B}$ be the Boolean completion of $\mathbb{P}$. Since $\mathbb{B}$ is a $(\kappa, 2)$-distributive complete Boolean algebra, for every $p \in \mathbb{B}$, $\mathbb{B}_p$ is a $(\kappa, 2)$-distributive complete Boolean algebra. By Theorem 5.3, $\mathbb{B}_p$ carries a κ-complete ultrafilter. Applying Proposition 5.4, it follows that $\mathbb{B}$ is a Príkry-type poset. □

6 Forcing the Σ_2-potentialist principle

We now turn to the main theorem of this paper.

Theorem 6.1. *The Σ_2-potentialist principle is consistent relative to a supercompact cardinal.*

Proof. In fact, we will only need to assume the existence of a strongly compact cardinal κ such that for some $\alpha \leq \kappa$, $V_\alpha \preceq_{\Sigma_2} V$.

We define an iteration $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n : n < \omega\rangle$ of Príkry-type forcings as follows. Let us fix in advance an ordering $\preceq$ of all formulas in the language of set theory of order-type ω. Suppose $\mathbb{P}_n$ has been defined, along a sequence of formulas $\langle \psi_k : k < n\rangle$. Fix $G_n \subseteq \mathbb{P}_n$ is V-generic. In $V[G_n]$, let κ_n be the least Beth fixed-point λ such that $V[G_n]_\lambda \vDash \bigwedge_{k<n} \psi_k$. Let ψ_n be

the first Σ_2-sentence (according to the order $\preceq$) that is false in $V[G_n]$ but can be forced by a κ_n-complete Príkry-type forcing. Let $\dot{\mathbb{Q}}_n$ be a $\mathbb{P}_n$-name that is forced by the empty condition to be such a forcing.

Let $\mathbb{P}$ be the Magidor iteration of the forcings $\mathbb{P}_n$. Since there is some $\alpha \leq \kappa$ such that $V_\alpha \preceq_{\Sigma_2} V$, $\mathbb{P} \in V_\kappa$. Let $G \subseteq \mathbb{P}$ be V-generic. We will show that the Σ_2-potentialist principle holds in $V[G]$.

Assume towards a contradiction that in $V[G]$, φ is the first possible Σ_2-sentence that is not true. Since κ remains strongly compact in $V[G]$, there is a κ-complete Príkry-type forcing $\mathbb{Q}$ that forces φ over $V[G]$.

For each n, let G_n be the projection of G to $\mathbb{P}_n$, and let κ_n and ψ_n be defined in $V[G_n]$ as above. Then for all n, φ can be forced over $V[G_n]$ by a κ_n-complete Príkry-type forcing, namely the two-step iteration $\mathbb{P}_{n,\omega} * \dot{\mathbb{Q}}$ where $\mathbb{P}_{n,\omega}$ denotes the factor forcing from $V[G_n]$ to $V[G]$ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}_{n,\omega}$-name for $\mathbb{Q}$ in $V[G_n]$.

It follows that there is some $n < \omega$ such that $\psi_n = \varphi$: in fact, if $n < \omega$ is least such that $\varphi \preceq \psi_n$, then $\psi_n = \varphi$. Therefore φ holds in $V[G_{n+1}]$, and in fact, φ holds in $V[G_{n+1}]_{\kappa_{n+1}}$. Since $V[G_{n+1}]_{\kappa_{n+1}} = V[G]_{\kappa_{n+1}}$, φ is true in $V[G]$. □

Recall that the *boldface Σ_2-potentialist principle* states that every possible Σ_2-formula (using an arbitrary set as a parameter) is true. Let us generalize the proof of Theorem 6.1 to show that this stronger principle is consistent.

Theorem 6.2. *The boldface Σ_2-potentialist principle is consistent relative to a proper class of strongly compact cardinals κ such that $V_\kappa \preceq_{\Sigma_2} V$.*

Proof. It simplifies matters to assume instead that there is a strongly inaccessible cardinal Ω such that for arbitrarily large $\kappa < \Omega$, κ is strongly compact in V_Ω and $V_\kappa \preceq_{\Sigma_2} V_\Omega$. In Lemma 6.3, we explain how to dispense with Ω.

Fix in advance a well-ordering W of V_Ω and an ordering $\preceq$ of all formulas in the language of set theory of order-type ω. We define a Magidor iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \Omega \rangle$ of Príkry-type forcings. Simultaneously, we define sequences $\langle \dot\kappa_\alpha : \alpha < \Omega \rangle$, $\langle \dot y_\alpha : \alpha < \Omega \rangle$, $\langle \psi_\alpha : \alpha < \Omega \rangle$ such that:

- Each $\dot y_\alpha$ is a $\mathbb{P}_\alpha$-name for a set in $V[\dot G_\alpha]_\Omega$.
- Each ψ_α is a Σ_2-formula.
- For every $\xi < \alpha$, $V[\dot G_\alpha]_{\kappa_\alpha} \vDash \psi_\xi(\dot y_\xi)$.
- If, in $V[\dot G_\alpha]$, there exists a κ_α-complete Príkry-type extension which forces $\psi_\alpha(\dot y_\alpha)$, then the weakest condition of $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is such a forcing.

Suppose that $\mathbb{P}_\alpha$, $\langle \dot\kappa_\xi : \xi < \alpha \rangle$, $\langle \dot y_\xi : \xi < \alpha \rangle$ and $\langle \psi_\xi : \xi < \alpha \rangle$ have been defined, and let us define $\dot{\mathbb{Q}}_\alpha$, $\dot\kappa_\alpha$, $\dot y_\alpha$ and ψ_α. Let $G_\alpha \subseteq \mathbb{P}_\alpha$ be a V-generic filter.

Let κ_α be the least Beth-fixed point in $V[G_\alpha]_\Omega$, such that $\kappa_\alpha \geq \sup\{\kappa_\xi : \xi < \alpha\}$, $\kappa_\alpha \geq \sup\{\mathrm{rank}(\dot{\mathbb{Q}}_\alpha) : \xi < \alpha\}$ and $V[G_\alpha]_{\kappa_\alpha} \vDash \psi(\dot y_\xi)$ for every $\xi < \alpha$. We argue that such κ_α exists. Assume that $\xi < \alpha$. If $\xi + 1 < \alpha$, then by our construction

$$V[G_\alpha]_{\kappa_{\xi+1}} = V[G_{\xi+1}]_{\kappa_{\xi+1}} \vDash \psi_\xi(y_\xi)$$

and since $\psi_\xi(y_\xi)$ is a Σ_2-formula, for every cardinal κ^* of $V[G_\alpha]_\Omega$ above $\kappa_{\xi+1}$, $V[G_\alpha]_{\kappa^*} \vDash \psi_\xi(y_\xi)$. If $\xi + 1 = \alpha$, then $V[G_\alpha] \vDash \psi_\xi(y_\xi)$, and again since ψ_ξ is a Σ_2-formula, there exists a

Beth fixed-point $\kappa_\alpha < \Omega$ such that $V[G_\alpha]_{\kappa_\alpha} \models \psi_\xi(y_\xi)$. Overall, by picking κ_α high enough, $V[G_\alpha]_{\kappa_\alpha} \models \psi_\xi(y_\xi)$ for all $\xi < \alpha$.

We proceed and define $y_\alpha, \psi_\alpha, \mathbb{Q}_\alpha$. If there is no $y \in V[G_\alpha]_{\kappa_\alpha}$ such that some Σ_2 formula $\varphi(y)$ is false in $V[G_\alpha]_\Omega$, but can be forced over $V[G_\alpha]_\Omega$ by a κ_α-complete Príkry-type forcing, let $\mathbb{Q}_\alpha$ be the trivial forcing, take ψ_α to be a tautology and $y_\alpha = \emptyset$. Else, pick $y_\alpha \in V[G_\alpha]_\gamma$ with a W-least $\mathbb{P}_\alpha$-name among sets y as above. Let $\psi(y_\alpha)$ be the $\preceq$-minimal among Σ_2-formulas with parameter y_α which can be forced over $V[G_\alpha]_\Omega$ by a κ_α-complete Príkry-type forcing. Finally, let $\mathbb{Q}_\alpha$ be a κ_α-complete Príkry-type forcing notion which forces $\psi_\alpha(y_\alpha)$ over $V[G_\alpha]$.

Back in V, pick the W-minimal $\mathbb{P}_\alpha$-names for $\dot{\kappa}_\alpha, \dot{y}_\alpha, \dot{\mathbb{Q}}_\alpha$ which are forced by the weakest condition of $\mathbb{P}_\alpha$ to have the above properties.

This concludes the inductive definition. Let $\mathbb{P}$ be the Magidor support limit of the forcings $\langle \mathbb{P}_\alpha : \alpha < \Omega \rangle$. The reason it is convenient to use an inaccessible cardinal is that $\mathbb{P}$ is a subset of $V_{\Omega+1}$ rather than V_Ω.

We argue that $V[G]_\Omega$ satisfies the boldface Σ_2-potentialist principle.

Assume by contradiction that there are $y \in V[G]_\Omega$ and a Σ_2-formula $\varphi(y)$ which is possible in $V[G]_\Omega$, but doesn't hold in $V[G]_\Omega$. Let $\alpha < \Omega$ be the least for which y as above exists in $V[G_\alpha]_\Omega$. Choose such $y \in V[G_\alpha]_\Omega$ with the W-least $\mathbb{P}_\alpha$-name $\dot{y}$. Let φ be the $\preceq$-least Σ_2-formula such that $\varphi(y)$ is possible but false in $V[G]_\Omega$.

Fix $\kappa > \max\{\operatorname{rank}(y), \alpha\}$ such that κ is strongly compact in V_Ω and $V_\kappa \preceq_{\Sigma_2} V_\Omega$. We argue that for every $\xi < \kappa$, $\dot{\mathbb{Q}}_\xi \in V_\kappa$ and $1_{\mathbb{P}_\xi} \Vdash \dot{\kappa}_\xi < \kappa$.

Indeed, fix $\xi < \kappa$, and assume by induction that $\mathbb{P}_\xi \in V_\kappa$. Then there exists a $\mathbb{P}_\xi$-name $\dot{\zeta}$ such that the weakest condition in $\mathbb{P}_\xi$ forces that $V[\dot{G}_\xi]_{\dot{\zeta}} \models \psi_\xi(\dot{y}_\xi)$. This property is by itself a Σ_2-sentence, so it holds in V_κ, since $V_\kappa \preceq_{\Sigma_2} V_\Omega$. So the $\mathbb{P}_\xi$-name $\dot{\kappa}_\xi$ belongs to V_κ, and again from the fact that $V_\kappa \preceq_{\Sigma_2} V_\Omega$, it follows that $\dot{\mathbb{Q}}_\xi \in V_\kappa$.

By Theorem 4.1, κ remains strongly compact in $V[G_\kappa]_\Omega$.

Since $\varphi(y)$ is possible in $V[G]_\Omega$, there is some $\kappa \le \gamma < \Omega$ large enough that $V[G_\gamma]_\Omega$ satisfies that $\varphi(y)$ can be forced without adding bounded subsets to κ. Fix a forcing $\mathbb{Q} \in V[G_\gamma]_\Omega$ that achieves this, and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}_{\kappa,\gamma}$-name for $\mathbb{Q}$ where $\mathbb{P}_{\kappa,\gamma}$ is the factor forcing. So, in $V[G_\kappa]_\Omega$, the iteration $\mathbb{P}_{\kappa,\gamma} * \dot{\mathbb{Q}}$ witnesses that $\varphi(y)$ can be forced without adding bounded subsets to κ, and thus, by Proposition 5.4, $\mathbb{P}_{\kappa,\gamma} * \dot{\mathbb{Q}}$ can be realized as a κ-complete Príkry-type forcing.

Since $\kappa_\alpha < \kappa$, the forcing $\mathbb{P}_{\alpha,\gamma} * \mathbb{Q}$ can be realized as a κ_α-complete Príkry-type forcing. By the minimality of α, y, φ, it follows that $y = y_\alpha, \varphi = \varphi_\alpha$ and $\varphi(y)$ holds in $V[G_{\alpha+1}]_{\kappa_{\alpha+1}}$. In particular, it holds in $V[G]_{\kappa_{\alpha+1}}$ and thus also in $V[G]_\Omega$. □

In order to dispense with Ω, one has to adjust the proof of Theorem 6.2 and perform a class forcing. By forcing first Global Choice, we can assume that there exists a class well order W of the universe (the standard class forcing for achieving Global choice does not add new sets, and thus preserves the assumption that there exists a proper class of Σ_2-correct strongly compact cardinals). Then, perform an Ord-length iteration of Príkry type forcings as in Theorem 6.2. The technical justifications that such an iteration produces a model of ZFC appear in the following lemma.

Lemma 6.3. *Let $\mathbb{P}$ be the direct limit of the iterated forcing $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha \in \mathrm{Ord} \rangle$ from the proof of Theorem 6.2. Then:*

(1) $\mathbb{P}$ satisfies the Príkry property, with respect to the following direct extension order: given $p, q \in \mathbb{P}$, q direct extends p ($q \le^* p$) if and only if $q \le p$ and there exists a finite

set $x \subseteq supp(p)$ *such that for every* $\alpha \notin x$, $q \restriction \alpha \Vdash q(\alpha) \leq^*_{\dot{\mathbb{Q}}_\alpha} p(\alpha)$.

(2) For every $p \in \mathbb{P}$ *and a* $\mathbb{P}$*-name for an ordinal,* $\dot{\eta}$*, there exist a direct extension* $q \leq^* p$ *and a set* A *such that* $q \Vdash \dot{\eta} \in \check{A}$.

(3) Let $G \subseteq \mathbb{P}$ *be generic over* V*. Then* $V[G]$ *is a model of* ZFC.

Proof. (1) The same argument of [5, Lemma 6.2] works here. Let σ be a statement in the forcing language of $\mathbb{P}$, and let $p \in \mathbb{P}$. Assume for contradiction that no direct extension of p decides σ.

For every $\alpha \in \mathrm{supp}(p)$, consider in $V^{\mathbb{P}_\alpha}$ the statement σ_α in the forcing language of the quotient forcing $\mathbb{P} \setminus \alpha$,

$$\sigma_\alpha \equiv \exists r \leq^* p \setminus \alpha,\ r \parallel \sigma.$$

We will construct a direct extension $q \leq^* p$ with $\mathrm{supp}(q) = \mathrm{supp}(p)$, such that, for every $\alpha \in \mathrm{supp}(q)$, $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \neg\sigma_\alpha$.

Before constructing q, let us argue that this suffices. Take $s \leq q$ an extension which decides σ. Let $\alpha \in \mathrm{supp}(q)$ be the maximal coordinate such that $s \restriction \alpha+1 \Vdash_{\mathbb{P}_{\alpha+1}} s \setminus \alpha+1 \leq^* q \setminus \alpha+1$. Then $s \restriction \alpha+1 \Vdash_{\mathbb{P}_{\alpha+1}} \sigma_{\alpha+1}$, contradicting the fact that $s \restriction \alpha+1 \leq q \restriction \alpha+1 \Vdash_{\mathbb{P}_{\alpha+1}} \neg\sigma_{\alpha+1}$.

We proceed to the construction of q. Assume that $\alpha \in \mathrm{supp}(p)$ and $q \restriction \alpha$ has been defined. Take $q(\alpha)$ to be a $\mathbb{P}_\alpha$-name for a direct extension of $p(\alpha)$ which decides the statement $\sigma_{\alpha+1}$.

We argue by induction that q is as desired, namely, for every $\alpha \in \mathrm{supp}(q)$, $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \neg\sigma_\alpha$. For $\alpha = 0$, this follows from the fact that no direct extension of p decides σ. Assume now that $\alpha \in \mathrm{supp}(q)$ and for every $\xi < \alpha$, $q \restriction \beta \Vdash_{\mathbb{P}_\beta} \neg\sigma_\beta$. We argue that $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \neg\sigma_\alpha$.

Case 1. *α is a limit ordinal.*

Assume, for contradiction, that there exists an extension $s \leq q \restriction \alpha$ such that $s \Vdash_{\mathbb{P}_\alpha} \sigma_\alpha$. Then there exists a $\mathbb{P}_\alpha$-name for a condition $r \leq^* p \setminus \alpha$ such that $s \Vdash_{\mathbb{P}_\alpha} r \parallel \sigma$. By extending s further, we can assume that it decides how r decides σ. Let $\beta < \alpha$ be the maximal coordinate such that $s \restriction \beta+1 \Vdash s \setminus \beta+1 \leq^* q \restriction [\beta+1, \alpha)$. Then $s \restriction \beta+1 \Vdash \sigma_{\beta+1}$, since $s \setminus \beta+1^\frown r$ is a direct extension of $p \setminus \beta+1$ which decides σ. This contradicts the fact that $s \restriction \beta+1 \leq q \restriction \beta+1 \Vdash_{\mathbb{P}_{\beta+1}} \neg\sigma_{\beta+1}$.

Case 2. *$\alpha = \beta+1$ is a successor ordinal.*

By our construction, $q \restriction \beta$ forces that $q(\beta) \parallel \sigma_\alpha$. It suffices to argue that $q \restriction \beta \Vdash_{\mathbb{P}_\beta} q(\beta) \Vdash_{\dot{\mathbb{Q}}_\beta} \neg\sigma_\alpha$. Else, there is $s \leq q \restriction \beta$ such that $s \Vdash_{\mathbb{P}_\beta} q(\beta) \Vdash_{\dot{\mathbb{Q}}_\beta} \sigma_\alpha$. Let r be a $\mathbb{P}_\alpha$-name for a direct extension of $p \setminus \alpha$ which decides σ, and direct extend $q^*(\beta) \leq^* q(\beta)$ (applying the Príkry property in the forcing $\dot{\mathbb{Q}}_\beta$) so that $q^*(\beta)$ decides how r decides σ. Assume without loss of generality that $q^*(\beta)^\frown r \Vdash_{\mathbb{P}\setminus\beta} \sigma$. Then $q^*(\beta)^\frown r$ is a $\mathbb{P}_\beta$-name for a direct extension of $p \setminus \beta$ which forces σ, namely $s \restriction \beta \Vdash_{\mathbb{P}_\beta} \sigma_\beta$, contradicting the fact that $s \restriction \beta \leq q \restriction \beta \Vdash_{\mathbb{P}_\beta} \neg\sigma_\beta$.

(2) Assume for contradiction that there is no direct extension $q \leq^* p$ and a set of ordinals A such that $q \Vdash \dot{\eta} \in \check{A}$. For every $\alpha \in \mathrm{supp}(p)$, consider in $V^{\mathbb{P}_\alpha}$ the statement σ_α in the forcing language of the quotient forcing $\mathbb{P} \setminus \alpha$,

$$\sigma_\alpha \equiv \exists r \leq^* p \setminus \alpha\ \exists A,\ r \Vdash_{\mathbb{P}\setminus\alpha} \dot{\eta} \in \check{A}.$$

We will construct a direct extension $q \leq^* p$ with $\mathrm{supp}(q) = \mathrm{supp}(p)$ such that, for every $\alpha \in \mathrm{supp}(q)$, $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \neg\sigma_\alpha$.

Before constructing q, let us argue that this suffices. Take any condition $r \leq q$ which decides the value of $\dot{\eta}$. Let $\alpha \in \mathrm{supp}(q)$ be the maximal coordinate such that $r \restriction \alpha + 1 \Vdash_{\mathbb{P}_{\alpha+1}} r \setminus \alpha + 1 \geq^* q \setminus \alpha + 1$. Then $r \restriction \alpha + 1 \Vdash_{\mathbb{P}_{\alpha+1}} \sigma_{\alpha+1}$, contradicting the fact that $r \restriction \alpha + 1 \leq q \restriction \alpha + 1 \Vdash_{\mathbb{P}_{\alpha+1}} \neg\sigma_{\alpha+1}$.

We proceed to the construction of q. Assume that $\alpha \in \mathrm{supp}(q)$ and $q \restriction \alpha$ has been defined. Take $q(\alpha)$ to be a $\mathbb{P}_\alpha$-name for a direct extension of $p(\alpha)$ which decides the statement $\sigma_{\alpha+1}$.

We argue by induction that q is as desired, namely, for every $\alpha \in \mathrm{supp}(q)$, $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \neg\sigma_\alpha$. For $\alpha = 0$, this follows from the fact that there is no direct extension of p which forces $\dot{\alpha}$ into a set. Assume now that $\alpha \in \mathrm{supp}(q)$ and for every $\beta < \alpha$, $q \restriction \beta \Vdash_{\mathbb{P}_\beta} \neg\sigma_\beta$. We argue that $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \neg\sigma_\alpha$.

Case 1. *α is a limit ordinal.*

Assume, for contradiction, that there exists an extension $s \leq q \restriction \alpha$ such that $s \Vdash_{\mathbb{P}_\alpha} \sigma_\alpha$. Then there exists a $\mathbb{P}_\alpha$-name for a condition $r \leq^* p \setminus \alpha$ such that $s \Vdash_{\mathbb{P}_\alpha} \exists A,\ r \Vdash \dot{\alpha} \in \check{A}$. Let $\dot{A}$ be a $\mathbb{P}_\alpha$-name for the set A whose existence in forced by s. Since $\mathbb{P}_\alpha$ is a set forcing, there exists a ground-model set A^* which is forced by s to cover $\dot{A}$. Therefore, $s^\frown r \Vdash \dot{\alpha} \in \check{A}^*$.

Let $\beta < \alpha$ be the maximal coordinate such that $s \restriction \beta + 1 \Vdash s \setminus \beta + 1 \leq^* q \restriction [\beta + 1, \alpha)$. Then $s \restriction \beta + 1 \Vdash_{\mathbb{P}_{\beta+1}} (s \setminus \beta + 1)^\frown r \Vdash \dot{\alpha} \in \check{A}^*$, namely $s \restriction \beta + 1 \Vdash_{\mathbb{P}_{\beta+1}} \sigma_{\beta+1}$. This contradicts the fact that $s \restriction \beta + 1 \leq q \restriction \beta + 1 \Vdash_{\mathbb{P}_{\beta+1}} \neg\sigma_{\beta+1}$.

Case 2. *$\alpha = \beta + 1$ is a successor ordinal.*

By our construction, $q \restriction \beta$ forces that $q(\beta) \parallel \sigma_\alpha$. It suffices to argue that $q \restriction \beta \Vdash_{\mathbb{P}_\beta} q(\beta) \Vdash_{\dot{\mathbb{Q}}_\beta} \neg\sigma_\alpha$. Else, there is $s \leq q \restriction \beta$ such that $s \Vdash_{\mathbb{P}_\beta} q(\beta) \Vdash_{\dot{\mathbb{Q}}_\beta} \sigma_\alpha$. Let r be a $\mathbb{P}_\alpha$-name for a direct extension of $p \setminus \alpha$ such that $s^\frown q(\beta) \Vdash \exists A, r \Vdash \dot{\alpha} \in \check{A}$. As in the limit case, there exists a set $A^* \in V$ such that $s^\frown q(\beta)^\frown r \Vdash \dot{\alpha} \in \check{A}^*$. Therefore, $s \Vdash_{\mathbb{P}_\beta} \sigma_\beta$, contradicting the fact that $s \restriction \beta \leq q \restriction \beta \Vdash_{\mathbb{P}_\beta} \neg\sigma_\beta$.

(3) It is routine to verify that every axiom of ZFC other than Powerset and Replacement holds in $V[G]$.

Let us verify that the Powerset Axiom holds in $V[G]$. Let α be an ordinal, $p \in \mathbb{P}$ a condition and $\dot{A}$ a $\mathbb{P}$-name which is forced by p to be a subset of α. Factor $\mathbb{P} = \mathbb{P}_{\alpha+1} * \mathbb{P} \setminus \alpha + 1$. The forcing $\mathbb{P} \setminus \alpha + 1$ has a direct extension order which is more than α-closed. Thus, by applying the Príkry property over and over, we can find a $\mathbb{P}_{\alpha+1}$-name for a direct extension $r \leq^* p \setminus \alpha + 1$ such that $p \restriction \alpha + 1 \Vdash_{\mathbb{P}_{\alpha+1}} \exists B \subseteq \alpha, r \Vdash \dot{A} = B$. Namely, p has a direct extension which forces $\dot{A}$ to be a set in $V^{\mathbb{P}_{\alpha+1}}$. By density, the powerset of α exists in $V[G]$, and is equal to its powerset in $V[G \restriction \alpha + 1]$.

We proceed and verify that Replacement holds in $V[G]$. Since Comprehension holds in $V[G]$, it suffices to prove that for every ordinal α and $f\colon \alpha \to \mathrm{Ord}$ a class function definable over $V[G]$, there exists a set in $V[G]$ which contains the image of f. For every $\beta < \alpha$, consider the $\mathbb{P}$-name $\dot{\eta}_\beta = \mathrm{rank}(f(\beta))$. Then $\langle \dot{\eta}_\beta \colon \beta < \alpha \rangle$ is a definable class of V (by using the names of the parameters used to define f as a class over $V[G]$). In $V^{\mathbb{P}_{\alpha+1}}$, apply the Príkry property of $\mathbb{P} \setminus \alpha + 1$, together with the fact every $\mathbb{P} \setminus \alpha + 1$-name for an ordinal can be forced into a set by a direct extension, and the fact that $\langle \mathbb{P} \setminus \alpha + 1, \leq^* \rangle$ is more than α-closed, to find a $\mathbb{P}_{\alpha+1}$-name for an ordinal η and a direct extension $r \leq^* p \setminus \alpha + 1$ such that $r \Vdash_{\mathbb{P} \setminus \alpha+1} \sup_{\beta < \alpha} \dot{\eta}_\beta < \eta$. Thus, in V, there exists an ordinal η^* such that $p \restriction \alpha + 1^\frown r \Vdash \mathrm{im}(f) \subseteq V[G]_{\check{\eta}^*}$. By density, in $V[G]$, the image of f is contained in $V[G]_{\eta^*}$ for some ordinal η^*, as desired. $\square$

7 On the strength of the potentialist principle

In this section we prove that the Σ_2-potentialist principle has consistency strength beyond a Woodin cardinal, answering another question of Woodin. The main idea, which was independently suggested to the authors by Magidor and Zeman, is based on a connection between the potentialist principle and a failure of mutual stationarity. The exact implementation of this idea, as described below, is based on recent results by Adolf, the first author, Schindler and Zeman, regarding mutual stationarity and iterations of distributive forcings [2].

The notion of mutual stationarity was introduced by Foreman and Magidor in [4]. Assume that $\langle \kappa_n : n < \omega \rangle$ is a sequence of regular uncountable cardinals, and, for every $n < \omega$, $S_n \subseteq \kappa_n$ is stationary. The sequence $\langle S_n : n < \omega \rangle$ is called *mutually stationary* if every algebra $\mathfrak{A}$ on $\sup_{n<\omega} \kappa_n$ has a subalgebra $\mathfrak{A}' \prec \mathfrak{A}$ such that, for every $n < \omega$, $\sup(A' \cap \kappa_n) \in S_\kappa$ (here, A, A' denote the universes of $\mathfrak{A}, \mathfrak{A}'$, respectively).

As an example, we demonstrate the simple property that a sequence of clubs is mutually stationary.

Proposition 7.1. *Let $\vec{\kappa} = \langle \kappa_n : n < \omega \rangle$ be an increasing sequence of regular cardinals. Let $\vec{C} = \langle C_n : n < \omega \rangle$ be a sequence such that, for each $n < \omega$, C_n is a club subset of κ_n. Then for every sequence $\vec{S} = \langle S_n : n < \omega \rangle$ of sets such that for each $n < \omega$, $C_n \subseteq S_n \subseteq \kappa_n$, $\vec{S}$ is mutually stationary.*

Proof. By replacing algebras with their Skolemizations, it suffices to prove that every elementary substructure $\mathcal{M}$ of $(H_\theta, \vec{\kappa}, \vec{C})$ for θ large enough, has a substructure $\mathcal{N} \prec \mathcal{M}$ with $\sup(N \cap \kappa_n) \in S_n$ for every $n < \omega$.

Given such an $\mathcal{M}$, take any countable elementary substructure $\mathcal{N} \prec \mathcal{M}$ containing $\vec{\kappa}, \vec{S}$, and $\vec{C}$. For $n < \omega$, set $\eta_n = \sup(N \cap \kappa_n)$. Note that since N is countable, $\eta_n < \kappa_n$. Since $C_n \in N$ and $\mathcal{N} \vDash$ "C_n is a club in κ_n", C_n is unbounded in η_n and thus $\eta_n \in C_n$. In particular, $\eta_n \in S_n$. □

Very roughly, the argument that the Σ_2-potentialist principle has high consistency strength goes as follows: assuming the Σ_2-potentialist principle with suitable anti-large cardinals assumptions, one can prove that there are sequences $\langle \kappa_n : n < \omega \rangle$ and $\langle S_n : n < \omega \rangle$, such that for each $n < \omega$, κ_n is a regular uncountable cardinal, S_n contains a club subset of κ_n, but $\langle S_n : n < \omega \rangle$ is not mutually stationary. This contradicts Proposition 7.1.

A key feature of the stationary subsets we consider is that the standard forcing notion for shooting a club through them is highly distributive. Given a regular cardinal κ and a stationary subset $S \subseteq \kappa$, S is called *fat* if for every $\nu < \kappa$ and club $C \subseteq \kappa$, $S \cap C$ contains a closed set of order type ν. By a classical result of Abraham and Shelah, clubs can be shot through fat stationary subsets of κ without adding $<\kappa$-sequences.

Theorem 7.2 (Abraham–Shelah [1])**.** *Assume that κ is either inaccessible or the successor of a strong limit cardinal,*[6] *and $S \subseteq \kappa$ is a stationary set. Let $\mathbb{P}$ be the standard forcing notion for shooting a club through S.*[7] *Then $\mathbb{P}$ is κ-distributive if and only if S is a fat stationary subset of κ.*

[6] The version of the theorem for a successor of a strong limit singular cardinal, which will matter to us the most, is sketched in [1, Theorem 2].

[7] Namely, $\mathbb{P}$ consists of closed, bounded subsets of S ordered by reserve end-extension.

We sketch below a very partial list of fundamental results about mutual stationarity. Given an increasing sequence $\vec{\kappa} = \langle \kappa_n : n < \omega \rangle$ of regular uncountable cardinals and a regular cardinal $\lambda < \kappa_0$, the *Mutual Stationarity Property* $\mathrm{MS}(\vec{\kappa}, \lambda)$ is the assertion that every sequence of stationary sets $\vec{S} = \langle S_n : n < \omega \rangle$ with $S_n \subseteq \kappa_n \cap S^{\kappa_n}_\lambda$ is mutually stationary.

Foreman and Magidor proved in [4], that for every increasing sequence of regular uncountable cardinals $\vec{\kappa}$, $\mathrm{MS}(\vec{\kappa}, \omega)$ holds. In L, however, for every $1 \leq k < \omega$, $\mathrm{MS}(\langle \omega_n : k < n < \omega \rangle, \omega_k)$ is false.

The existence of a sequence of regular uncountable cardinals $\vec{\kappa} = \langle \kappa_n : n < \omega \rangle$ such that $\mathrm{MS}(\vec{\kappa}, \omega_1)$ holds, is equiconsistent with a single measurable cardinal: Cummings, Foreman and Magidor showed in [3] that, starting from a measurable cardinal and forcing a Príkry sequence into it, there exists a final segment $\langle \kappa_n : n < \omega \rangle$ of the generic Príkry sequence, such that every sequence $\langle S_n : n < \omega \rangle$ of sets, each S_n stationary in κ_n, is mutually stationary; for the other direction, Koepke and Welch proved in [7] that $\mathrm{MS}(\vec{\kappa}, \omega_1)$ for some sequence $\vec{\kappa}$ implies an inner model with a measurable cardinal.

In [7], Koepke and Welch also considered the principle $\mathrm{MS}(\langle \omega_n : 1 < n < \omega \rangle, \omega_1)$, showing that its consistency strength is higher than just a single measurable cardinal. Recently, this was improved in [2] to a lower bound of at least PD for the existence of a nonzero number k such that $\mathrm{MS}(\langle \omega_n : k < n < \omega \rangle, \omega_k)$. The same methods can be used to establish a similar lower bound on the consistency strength of the Σ_2-potentialist principle. (For simplicity, however, we only prove a lower bound of one Woodin cardinal.)

We quote below one of the main results of [2], and then apply it to prove the lower bound on the Σ_2-potentialist principle.

Theorem 7.3 (Adolf–Ben-Neria–Schindler–Zeman [2])**.** *Assume there is no inner model with a Woodin cardinal. Let $\vec{\kappa} = \langle \kappa_n : n < \omega \rangle$ be any increasing sequence of successors of strong limit singular cardinals.*[8] *Then for every regular $\lambda < \kappa_0$, $MS(\vec{\kappa}, \lambda)$ does not hold.*

For the consistency strength lower bound on the Σ_2-potentialist principle, we need some details from the proof of Theorem 7.3, which we summarize in the following proposition:

Proposition 7.4. *If there is no inner model with a Woodin cardinal, then there are partial functions $\pi_{\mu,\mu'} : \mu \to \mu'$ for every pair of successors of strong limit singular cardinals $\mu' < \mu$, which are lightface Σ_2-definable from parameters μ, μ', such that the following hold:*

(1) For every every increasing sequence $\mu_0 < \ldots < \mu_n$ of successors of singular strong limit cardinals, there is a sequence $\beta_0 < \mu_0, \ldots, \beta_{n-1} < \mu_{n-1}$ such that the set

$$S_{\vec{\beta}}(\mu_0, \ldots, \mu_n) = \{\xi < \mu_n : \forall i < n\, (\xi \in \mathrm{dom}(\pi_{\mu_n,\mu_i}) \text{ and } \pi_{\mu_n,\mu_i}(\xi) < \beta_i)\}$$

is a fat stationary subset of μ_n.

(2) Let $\vec{\beta}(\mu_0, \ldots, \mu_n) \in \prod_{i<n} \mu_i$ be the lex-least sequence such that $S_{\vec{\beta}}(\mu_0, \ldots, \mu_n)$ is fat, and let

$$S(\mu_0, \ldots, \mu_n) = S_{\vec{\beta}(\mu_0,\ldots,\mu_n)}(\mu_0, \ldots, \mu_n)$$

Then for each increasing sequence of successors of singular strong limit cardinals $\vec{\kappa} = \langle \kappa_i : i < \omega \rangle$, the stationary sets $\langle S(\kappa_0, \ldots, \kappa_{n-1}) : 1 \leq n < \omega \rangle$ are not mutually stationary.

[8] The original argument in [2] is more general and applies also in certain cases where the κ_is are not successor of singulars; in particular, it applies to the case where there exists some $n_0 < \omega$ such that for every $i < \omega$, $\kappa_i = \aleph_{i+n_0}$. The assumption that each κ_i is a successor of a singular has the advantage that by weak covering, each κ_i is a successor cardinal in the core model, which simplifies the fine structural analysis in the proof of the theorem.

It is not really relevant that $\vec{\beta}(\mu_0, \ldots, \mu_n)$ be chosen lexicographically least; any choice of $\vec{\beta} \in \prod_{i<n} \mu_i$ such that $S_{\vec{\beta}}(\mu_0, \ldots, \mu_n)$ is fat would do. We minimize $\vec{\beta}$ only to ensure that the function $(\mu_0, \ldots, \mu_n) \mapsto S(\mu_0, \ldots, \mu_n)$ is Σ_2-definable.

Theorem 7.5. *The Σ_2-potentialist principle implies that there is an inner model with a Woodin cardinal.*

Proof. Assume towards a contradiction that the Σ_2-potentialist principle holds but there is no inner model with a Woodin cardinal. For every $n < \omega$, consider the following Σ_2 statement:

$(*)^n$ There are $\kappa_0 < \ldots < \kappa_{n+1}$ such that κ_0 is the least successor of a strong limit singular cardinal, and for every $0 \leq i \leq n$, κ_{i+1} is the least successor of a strong limit singular cardinal above κ_i, such that $S(\kappa_0, \ldots, \kappa_{i+1})$ contains a club in κ_{i+1}.

We argue that all the statements $(*)^n$ hold. Furthermore, there exists a sequence $\langle \kappa_i : i < \omega \rangle$ of successors of strong limit singular cardinals, such that, for every $1 \leq n \leq \omega$, $\langle \kappa_0, \ldots, \kappa_{n+1} \rangle$ witnesses the truth of $(*)^n$.

For $n = 0$, let κ_0 be the least successor of a strong limit cardinal. By Proposition 7.4, for every cardinal κ above κ_0 which is a successor of a strong limit singular cardinal, $S(\kappa_0, \kappa)$ is a fat stationary subset of κ. By Theorem 7.2, the forcing which shoots a club through $S(\kappa_0, \kappa)$ is κ-distributive. Since κ can be picked arbitrarily high, $(*)^0$ is a possible statement. By the Σ_2-potentialist principle, $(*)^0$ holds.

Assuming that $(*)^n$ holds, let $\kappa_0 < \ldots < \kappa_n$ be the witnesses for its truth. Arguing as above, for every cardinal κ above κ_n which is a successor of a strong limit singular cardinal, the stationary set $S(\kappa_0, \ldots, \kappa_n, \kappa)$ is fat, and a club can be shot through it by a κ-distributive forcing. Since κ can be picked arbitrarily high, $(*)^{n+1}$ is a possible Σ_2-statement, and by the Σ_2-potentialist principle, $(*)^{n+1}$ is true, witnessed by a sequence $\kappa_0 < \ldots < \kappa_n < \kappa_{n+1}$, where κ_{n+1} is some cardinal above κ_n.

This concludes the inductive argument. By Proposition 7.4, the sets $\langle S(\kappa_0, \ldots, \kappa_n) : 1 \leq n < \omega \rangle$ are not mutually stationary. However, each such set contains a club, contradicting Proposition 7.1. ☐

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