

Measures in Nairian models, I

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Abstract

Assuming $\text{AD}^+ + V = L(P(\mathbb{R}))$, we show that the Ultrapower Axiom holds in HOD_s for every ω -sequence of ordinals s . We use this result to analyze the structure of Nairian models [2, 3], showing that they contain many measurable cardinals. As an application, we establish the consistency of the existence of infinitely many consecutive measurable cardinals in ZF set theory, relative to a Woodin limit of Woodin cardinals, solving a question of Henle, Mathias, and Woodin [16]. From the same large cardinal hypothesis, we produce a model of ZF in which every uncountable cardinal is singular; Gitik's original construction requires a proper class of strongly compact cardinals.

1 Introduction

The Axiom of Determinacy (AD) produces an intricate structure theory for sets of reals and cardinals below Θ , the supremum of all ordinals that $\mathbb{R}$ surjects onto. A natural question is whether this theory can be generalized in some way to a global structure theory for all sets and all cardinals. Recent work of Blue–Larson–Sargsyan [2, 3] offers a precise way to approach this question: under strong determinacy hypotheses, a candidate global theory emerges in *Nairian models*, a family of inner models of ZF built using definability from countable sequences of ordinals.

Blue–Larson–Sargsyan's results on Nairian models rely on tools from descriptive inner model theory, particularly the fine-structural analysis of the inner model HOD in models of determinacy. An obstacle to our current understanding of Nairian models is that no general fine-structural analysis is known for models of the form HOD_s , where s is a countable sequence of ordinals. Building on work of Woodin, we develop an abstract comparison process for such models, which shows that they satisfy the Ultrapower Axiom (UA), a set-theoretic principle introduced by the first author [7] that captures part of the combinatorial content of fine-structural comparison without reference to fine structure.

Theorem 2.8. *Assume $\text{AD}^+ + V = L(P(\mathbb{R}))$. Then for all $s \in \text{Ord}^\omega$, HOD_s satisfies UA.*

A central conjecture of inner model theory is that under $\text{AD}^+ + V = L(P(\mathbb{R}))$, HOD is a fine-structural inner model. Woodin confirmed a precise consequence of this conjecture by showing that under $\text{AD}^+ + V = L(P(\mathbb{R}))$, HOD satisfies UA. We reorganize the proof

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of Woodin's result and extend it to HOD_s for every countable sequence of ordinals s . The theorem hints that HOD_s might admit a fine-structural analysis, but this remains unclear.

We use Theorem 2.8 to study measures in Nairian models. The basic idea is to analyze the restrictions of these measures to HOD_s . Instead of using fine structure in HOD_s , we use the theory of supercompact cardinals under UA [7]. An interesting feature of our approach is that this theory, which was originally developed to predict the structure of inner models with supercompact cardinals, finds concrete applications to models that do not contain an inner model with a supercompact.

We state our theorems in the context of

$$\text{AD}^+ + V = L(P(\mathbb{R}))$$

Fix an ordinal $\xi \leq \Theta$. The *extended Nairian model at ξ* is the class $\mathcal{N}_\xi$ of all sets hereditarily ordinal definable from a bounded countable subset of ξ . Thus,

$$\mathcal{N}_\xi = \text{HOD}_{I_\xi}, \quad I_\xi = \{s \in \xi^\omega : \sup(s) < \xi\}$$

Blue–Larson–Sargsyan denote this model by $\mathcal{N}_\xi^+$.

We are interested in the situation where ξ is strongly inaccessible in $\mathcal{N}_\xi$. Blue–Larson–Sargsyan show that this is consistent relative to a Woodin limit of Woodin cardinals. In this case, the *Nairian model* $N = \mathcal{N}_\xi \cap V_\xi$ is a model of ZF exhibiting global determinacy phenomena. *From now on we assume ξ is strongly inaccessible in $\mathcal{N}_\xi$.*

As usual, $|X| \leq |Y|$ means there is an injection from X to Y and $|X| \leq^* |Y|$ means there is a surjection from Y to X . Let

$$\aleph(X) = \sup\{\alpha \in \text{Ord} : |\alpha| \leq |X|\}, \quad \Theta(X) = \sup\{\alpha \in \text{Ord} : |\alpha| \leq^* |X|\}$$

Theorem 3.23. *Let $\mathcal{H} = \text{HOD}$. Inside $\mathcal{N}_\xi$, suppose $\eta < \xi$ is a regular cardinal satisfying:*

(1) *For all $\alpha < \eta$, $P(\alpha) \subseteq \mathcal{H}(\eta^\omega)$.*

(2) *For all $\alpha < \eta$, $\Theta(\alpha^\omega) \leq \eta$.*

Then either η is measurable in $\mathcal{N}_\xi$ or $\mathcal{H}$ correctly computes all finite successors of η^{++} .

In the known Nairian models, $\mathcal{H}$ does not correctly compute a single successor cardinal below ξ , so the measurability conclusion holds. Letting $\lambda = (\eta^{+\omega})^{\mathcal{N}_\xi}$, correct computation of successors in particular implies that some ordinal $\kappa \leq \eta^{++}$ is $<\lambda$ -supercompact in HOD , whereas Blue–Larson–Sargsyan always assume Steel's *HOD pair capturing* hypothesis (HPC) [23].

Indeed, for our applications of Theorem 3.23 we adopt Blue–Larson–Sargsyan's hypotheses:

$$\text{AD}_{\mathbb{R}} + \Theta \text{ is regular} + V = L(P(\mathbb{R})) + \text{HPC} \tag{1}$$

We also assume that

$$N = \mathcal{N}_\xi \cap V_\xi$$

is a minimal Nairian model in the sense of [3, Definition 1.1].

We apply Theorem 3.23 in N to the sequence of cardinals

$$\langle \Theta(V_\alpha) : \alpha < \xi \rangle$$

These are the choiceless analogs of the Beth numbers. These cardinals are closely related to the strong cardinals of $\text{HOD} \cap V_\xi$. Note that $\Theta(V_\omega) = \omega_1$ and $\Theta(V_{\omega+1}) = \Theta$. It is a theorem of ZF that if α is a limit ordinal, then $\Theta(V_\alpha) = (\sup_{\beta < \alpha} \Theta(V_\beta))^+$, and a similar argument shows that under AD, $\Theta(V_{\omega+2}) = \Theta^+$.

Blue–Larson–Sargsyan compute that in N , for $n \leq \omega$,

$$\Theta(V_{\omega+n+1}) = \Theta^{+n}$$

More generally, although the value of $(\Theta(V_{\alpha+1}))^N$ for limit ordinals $\alpha > \omega$ of countable cofinality remains open, for all other $\alpha > \omega$, Blue–Larson–Sargsyan show

$$\Theta(V_{\alpha+1}) = \Theta(V_\alpha)^+$$

This can be seen as a version of the generalized continuum hypothesis: in ZFC, GCH is equivalent to $\Theta(V_{\alpha+1}) = \Theta(V_\alpha)^+$ for all $\alpha \geq \omega$.

Our main measurability result for minimal Nairian models is as follows:

Theorem 3.27. *In N , for all infinite ordinals α , $\Theta(V_\alpha)$ is measurable.*

Applying Blue–Larson–Sargsyan’s calculation above, we have our first application:

Corollary 3.28. *The theory “ZF + there are infinitely many consecutive measurable cardinals” is consistent relative to a Woodin limit of Woodin cardinals.*

Theorem 3.27 also allows us to compute $\aleph(V_{\alpha+1})$.

Corollary 3.32. *In N , for all infinite ordinals α , $\aleph(V_{\alpha+1}) = \Theta(V_\alpha)$.*

In other words, there is no $\Theta(V_\alpha)$ -sequence of distinct subsets of V_α . This is further evidence that the theory of Nairian models yields a global generalization of determinacy. The equation $\aleph(V_{\omega+1}) = \Theta(V_\omega)$ asserts that there is no ω_1 -sequence of distinct reals, while $\aleph(V_{\omega+2}) = \Theta(V_{\omega+1})$ states that there is no Θ -sequence of distinct subsets of $\mathbb{R}$. These are well-known consequences of AD and $\text{AD}_{\mathbb{R}}$ respectively.

Our second application is a reduction of the large cardinal hypothesis used in Gitik’s construction of a model of ZF in which every uncountable cardinal is singular [5]. This uses our results on the global structure of measures in arbitrary Nairian models. We return to the setting of $\text{AD}_{\mathbb{R}} + \text{DC} + V = L(P(\mathbb{R}))$, with $\xi < \Theta$ inaccessible in $\mathcal{N}_\xi$.

Again let $N = \mathcal{N}_\xi \cap V_\xi$. Corollary 3.15 states that in N , each regular cardinal δ carries a unique weakly normal measure concentrating on ordinals of countable cofinality, which we denote by W_δ . Theorem 3.22 shows that in N , the map $\Phi : \text{Reg} \rightarrow \text{Reg}$ that sends δ to the completeness of W_δ is monotone. Proposition 3.34 and Proposition 3.35 show that if either HPC holds or ξ is a cardinal, then Φ is unbounded. In Theorem 3.38, assuming Φ is unbounded, we adapt a forcing technique of Adkisson–Ben-Neria [1], originally developed in the context of choiceless large cardinals, to work over N . This produces a model of ZF in which every uncountable cardinal is singular, leading to our second application:

Corollary 3.40. *The theory “ZF + every uncountable cardinal is singular” is consistent relative to a Woodin limit of Woodin cardinals.* $\square$

Finally, we develop a comparison process for Nairian models themselves. This allows us to prove the following results for $N = \mathcal{N}_\xi \cap V_\xi$:

Theorem 4.2. *In N , for every set X , there is a unique normal fine measure on $P_{\omega_1}(X)$.*

Theorem 4.3. *UA holds in N .*

In this choiceless context, UA states that any two ultrapowers of V by measures on ordinals can be compared by further internal ultrapowers. Note that these ultrapowers will not in general be fully elementary.

As an application of Theorem 4.3, we prove the following result. We return to the Blue–Larson–Sargsyan hypotheses 1 and assume $N = \mathcal{N}_\xi \cap V_\xi$ is a minimal Nairian model.

Theorem 4.4. *In N , if $0 < \alpha < \xi$ is a limit ordinal and $\lambda = \sup_{\beta < \alpha} \Theta(V_\beta)$, then every λ -complete filter on an ordinal extends to a λ -complete ultrafilter.*

We can also prove the corresponding statement for $\lambda = \Theta(V_{\alpha+1})$ unless α is a limit ordinal of countable cofinality; we defer the proof to a later draft.

2 Definability from a countable sequence of ordinals

Definition 2.1. Suppose M is a transitive class. The M -ultrafilter hypothesis states that if $U \in M$ is a measure in M , then U extends to a countably complete filter in V .

The main result of this section is an abstract comparison theorem for models of the form HOD_s , where s is a countable sequence of ordinals. Assume $\text{AD}^+ + V = L(P(\mathbb{R}))$. We will show that for all $s \in \text{Ord}^\omega$, HOD_s satisfies the Ultrapower Axiom, and the HOD_s -ultrafilter hypothesis holds.

The proof is obtained by relativizing Woodin’s theorem that UA holds in the HOD of any model of $\text{AD}^+ + V = L(P(\mathbb{R}))$. The relativization uses the following ZF fact, essentially due to Paris [4, 20]:

Definition 2.2. Let $\mathcal{T}_X$ denote the Σ_2 -theory of V with parameters in $\text{Ord} \cup \{X\}$. Let $\mathcal{H}_X$ denote the structure $(\text{HOD}_X, \mathcal{T}_X, \in)$. Also let $\mathcal{T} = \mathcal{T}_\emptyset$ and $\mathcal{H} = \mathcal{H}_\emptyset$.

Proposition 2.3. *Suppose X is a set, H is a countable transitive structure, and*

$$\rho : H \rightarrow \mathcal{H}_X$$

is an elementary embedding. Then there is a countable transitive model $\bar{V}$ of ZF and a set $\bar{X} \in \bar{V}$ such that $H = (\mathcal{H}_{\bar{X}})^{\bar{V}}$. Consequently, ρ is partial elementary from $(\bar{V}, \bar{X})$ to (V, X) .

We begin by providing a new proof of Woodin’s original theorem on HOD. Our proof reorganizes, simplifies, and generalizes a proof that Woodin gave in the case $\text{AD}_{\mathbb{R}} + \Theta$ is regular; see [13]. We use one of the central AD^+ theorems on derived models:

Theorem 2.4 (Woodin). *Assume $\text{AD}^+ + V = L(P(\mathbb{R}))$. There is a definable inner model $M \subseteq \text{HOD}$ with a limit of Woodin cardinals η and a generic elementary embedding*

$$j : V \rightarrow D$$

where D is a derived model of M at η .

Here we mean that M is definable without parameters over V .

We can make a number of further conclusions abstractly from the statement of Theorem 2.4 (which are also immediate consequences of Woodin’s unpublished construction). By homogeneity, $\mathcal{H}^D$ is definable over M . Since $\text{HOD} = L[S]$ for a set of ordinals S , $j \restriction \mathcal{H}$

is an extender embedding. By the uniqueness of generic extender embeddings, $j \restriction \mathcal{H}$ is definable without parameters over V , or equivalently over $\mathcal{H}$. A basic fact about derived models is that $\Theta^D \leq (\eta^+)^M$. It follows easily that $\eta \geq \Theta$ and that η is the supremum of the measurable cardinals of M ; it follows that η is the supremum of the Woodin cardinals of M .

Though we do not really need it below, an immediate corollary of Theorem 2.4 is Woodin's theorem that HOD relativizes by adjunction.

Corollary 2.5 (Woodin). *In the context of Theorem 2.4, for all $\sigma \in [\text{Ord}]^{<\omega_1}$, $\mathcal{T}_\sigma$ is uniformly definable over $(\text{HOD}[\sigma], \mathcal{T}, \sigma)$, and hence $\text{HOD}_\sigma = \text{HOD}[\sigma]$.*

Proof. Let M, η, j be as in Theorem 2.4, and set $i = j \restriction \mathcal{H}$. Since σ is countable, $j(\sigma) = j[\sigma] = i[\sigma]$. Let $\mathcal{X}_\sigma = (\text{HOD}[\sigma], \mathcal{T}, \sigma)$. Note that $M[i[\sigma]] \subseteq \text{HOD}[\sigma]$ and is definable over $\mathcal{X}_\sigma$, and so by homogeneity, $\mathcal{H}_{i[\sigma]}^D$ is definable over $\mathcal{X}_\sigma$. Finally, $\mathcal{T}_\sigma$ is definable over $\mathcal{X}_\sigma$ because

$$\mathcal{T}_\sigma = i^{-1} \left[\mathcal{T}_{i[\sigma]}^D \right]$$

and therefore $\text{HOD}_\sigma = L[\mathcal{T}_\sigma] \subseteq \text{HOD}[\sigma]$. $\square$

Theorem 2.6 (Woodin). *Assume $\text{AD}^+ + V = L(\mathbb{R}, \text{Hom})$. Then $\mathcal{H}$ satisfies weak comparison: if H_0 and H_1 are finitely generated structures elementarily embeddable into $\mathcal{H}$, then either $H_0 \in H_1$, $H_1 \in H_0$, or there are close embeddings $k_0 : H_0 \rightarrow H^*$ and $k_1 : H_1 \rightarrow H^*$.*

Proof. Let M and η be as in Theorem 2.4, and let $\mathbb{R}^* = \mathbb{R} \cap M$.

Suppose that $H \in \mathcal{H}$ is a countable transitive structure that is elementarily embeddable into $\mathcal{H}$.¹ We begin by constructing a close elementary embedding $k : H \rightarrow H^*$ where H^* is $\mathcal{H}^{D^*}$ for a model D^* of the form $L_\alpha(\mathbb{R}^*, \Gamma)$ with Γ an initial segment of $\text{Hom}_{<\eta}^M$.

By Proposition 2.3, choose a countable transitive model $\bar{V} \equiv V$ such that $\mathcal{H}^{\bar{V}} = H$. Applying Theorem 2.4 inside $\bar{V}$, let $\bar{M} \subseteq H$, $\bar{\eta}$, and the generic elementary embedding $\bar{j} : \bar{V} \rightarrow D$ be as in that theorem, where D is a derived model of $\bar{M}$ at $\bar{\eta}$. Define

$$i : H \rightarrow \mathcal{H}^D$$

by $i = \bar{j} \restriction H$. By the observations following Theorem 2.4, i is definable over H .

Note that $\bar{M} \in M$ and by absoluteness, M sees that $\bar{M}$ is elementarily embeddable into M . Therefore using Neeman-style $\mathbb{R}$ -genericity iterations,² we can iterate $(\bar{M}, \bar{\eta})$ to a model (M^*, η^*) , again elementarily embeddable into M , such that $\mathbb{R}^* = \mathbb{R} \cap M$ are the reals of a symmetric collapse of M^* at η^* , and if D^* is the derived model of M^* at η^* in $M^*(\mathbb{R}^*)$, then $\Gamma = \text{Hom}^{D^*}$ is an initial segment of $\text{Hom}_{<\eta}^M$.

Let

$$\pi : \bar{M} \rightarrow M^*$$

be the iteration map. Then $\pi \restriction \mathcal{H}^D$ is elementary from $\mathcal{H}^D$ to $\mathcal{H}^{D^*}$. Set $H^* = \mathcal{H}^{D^*}$. Define an elementary embedding $k : H \rightarrow H^*$ by

$$k = \pi \circ i$$

¹Throughout this section, we work in choiceless Morse–Kelley set theory, which allows us to refer to the first-order satisfaction predicate of V . By standard methods, the proof can be carried out in ZF.

²Woodin's original construction used Woodin-style genericity iterations, which requires establishing certain instances of the unique branch hypothesis. Steel suggested replacing them with Neeman-style iterations using [22].

We claim that k is close to H . To prove this, fix $A \in H^*$, and let us show that $k^{-1}[A] \in H$. We have $\pi^{-1}[A] \in \bar{M}$ since π is close to $\bar{M}$ and $A \in M^*$. Since $\bar{M} \subseteq H$, it follows that $\pi^{-1}[A] \in H$ and hence $i^{-1}[\pi^{-1}[A]] \in H$ since i is definable over H . But $k^{-1}[A] = i^{-1}[\pi^{-1}[A]]$.

We will use that

$$D^* = \text{Hull}^{D^*}(k[H] \cup \Lambda)$$

where $\Lambda = P(\mathbb{R}) \cap D^*$. This follows from the fact that $H^* = \text{Hull}^{H^*}(k[H] \cup k(\Theta^H))$, which holds because k is close to H and every uniform measure of H lies on a cardinal less than or equal to Θ^H .

We now show that Weak Comparison holds for $\mathcal{H}$. Fix any two finitely generated models H_0 and H_1 that are elementarily embeddable into $\mathcal{H}$. We will construct close embeddings $k_0 : H_0 \rightarrow H_0^*$ and $k_1 : H_1 \rightarrow H_1^*$ such that H_0^* and H_1^* are *comparable*: either they are equal or one belongs to the other.

For $\ell = 0, 1$, the construction above yields a close embedding $k_\ell : H_\ell \rightarrow H_\ell^*$ where $H_\ell^* = \mathcal{H}^{D_\ell^*}$ for a model D_ℓ^* satisfying $\mathbb{R} \cap D_\ell^* = \mathbb{R} \cap M$, $\Gamma_\ell = \text{Hom}^{D_\ell^*} \triangleleft \text{Hom}_{<\eta}^M$, and $D_\ell^* = \text{Hull}^{D_\ell^*}(k_\ell[H_\ell] \cup \Lambda_\ell)$ where $\Lambda_\ell = P(\mathbb{R}^*) \cap D_\ell^*$.

We claim that D_0^* and D_1^* are comparable. By Wadge determinacy in $\text{Hom}_{<\eta}^M$, Γ_0 and Γ_1 are comparable pointclasses. It follows from our assumption that $V = L(\mathbb{R}, \text{Hom})$ that Λ_0 and Λ_1 are comparable as well. (If $\Gamma_0 = \Gamma_1$, this is trivial, and if $\Gamma_0 \triangleleft \Gamma_1$, then $\Gamma_0^\# \in D_1^*$, so $\Lambda_0 \triangleleft \Lambda_1$.) By symmetry, it suffices to show that if $\Lambda_0 \triangleleft \Lambda_1$, then $D_0^* \in D_1^*$.

Suppose first that every set of reals has a sharp. We will use that D_1^* is correct about sharps, which means that if $A \in D_1^*$ and $D_1^* \models B = A^\#$, then in fact, $B = A^\#$. To see this, note that M_1^* is correct about sharps because M_1^* is elementarily embeddable into M . It follows that D_1^* is correct about sharps, since D_1^* is an inner model of a forcing extension of M_1^* .

Since $D_0^* = \text{Hull}^{D_0^*}(k_0[H_0] \cup \Lambda_0)$ and H_0 is finitely generated, D_0^* is coded by a subset of Λ_0 in $L(\mathbb{R}^*, \Lambda_0)$. Since $\Lambda_0^\# \in D_1^*$, $D_0^* \in D_1^*$.

Suppose instead that some set of reals has no sharp. We then have that $\theta = \Theta^{H_0}$ is regular in H_0 . It follows that $k_0(\theta) = \sup k_0[\theta]$ since θ carries no uniform measures in H_0 and k_0 is close to H_0 . In particular, $\text{cf}^M(k_0(\theta)) = \omega$. (We use that $k_0 \in M$, which follows from the fact that $H_0^* \in M$ and H_0 is finitely generated.) Therefore $D_0^* = \text{Hull}^{D_0^*}(\mathbb{R}^* \cup \mathcal{A})$ where $\mathcal{A} = \{a_n : n < \omega\}$ is a countable set in M . Letting $T_n = \text{Th}_{\Sigma_n}^{D_0^*}(\mathbb{R}^* \cup \{a_0, \dots, a_{n-1}\})$, we have $\{T_n\}_{n < \omega} \subseteq D_0^*$ and $\langle T_n \rangle_{n < \omega} \in M$, and hence $\langle T_n \rangle_{n < \omega} \in D_1^*$. Since D_0^* can be recovered from $\langle T_n \rangle_{n < \omega}$, it follows that $D_0^* \in D_1^*$.

Thus we have shown that D_0^* and D_1^* are comparable. If $D_0^* \in D_1^*$, for example, then D_0^* is OD in D_1^* and hence $H_0^* \in H_1^*$. It follows that H_0^* and H_1^* are comparable, and this establishes weak comparison. $\square$

The conclusion of Theorem 2.6 implies HOD satisfies UA by [7, Theorem 2.3.10], and a further reflection argument allows us to make this conclusion assuming $V = L(P(\mathbb{R}))$ rather than $V = L(\mathbb{R}, \text{Hom})$. We omit this argument since it is similar to the one from Theorem 2.8 below. It is open whether $\mathcal{H}$ satisfies weak comparison assuming only $\text{AD}^+ + V = L(P(\mathbb{R}))$.

We now prove our main relativizations of Woodin's result. The proof will again require a reduction to the case that $V = L(\mathbb{R}, \text{Hom})$. This uses a bit more of Woodin's analysis of the largest Suslin cardinal:

Theorem 2.7 (Woodin). *Assume $\text{AD}^+ + V = L(P(\mathbb{R}))$ and δ is the largest Suslin cardinal. Then for any $\lambda < \Theta$, there is a normal measure U on δ such that $j_U(\delta) > \lambda$, $P(\lambda) \cap V \subseteq M_U$, and $P(\lambda) \cap j_U(\text{HOD}) = P(\lambda) \cap \text{HOD}$. $\square$*

The proof of this theorem is unpublished in the key case $V \neq L(\mathbb{R}, \text{Hom})$.

Theorem 2.8. *Under $\text{AD}^+ + V = L(P(\mathbb{R}))$, HOD_s satisfies UA for every $s \in \text{Ord}^\omega$.*

Proof. We start by assuming $V = L(\mathbb{R}, \text{Hom})$. Let H be a pointwise definable transitive structure that is elementarily embeddable into $\mathcal{H}$.

Fix a countable transitive model $\bar{V} \equiv V$ in HOD , constructed by type omission, so that $H = \mathcal{H}^{\bar{V}}$. Fix $s \in \text{Ord}^\omega \cap \bar{V}$. We will show UA holds in $P = \mathcal{H}_s^{\bar{V}}$.

Fix internal ultrapowers $e_0 : P \rightarrow P_0$ and $e_1 : P \rightarrow P_1$, and let $s_n = e_n(s)$. For $n = 0, 1$, we imitate the construction from Theorem 2.6. That is, construct $\bar{V}_n \equiv V$ in HOD by relativizing Proposition 2.3 so that

$$P_n = \mathcal{H}_{s_n}^{\bar{V}_n}$$

Applying Theorem 2.4 in $\bar{V}_n$, fix $\bar{M}_n \subseteq \mathcal{H}^{\bar{V}_n}$, $\bar{\eta}_n$, and an elementary $\bar{j}_n : \bar{V}_n \rightarrow D_n$ where D_n is a derived model of $\bar{M}_n$ at $\bar{\eta}_n$. Let $t_n = \bar{j}_n(s_n)$ and define

$$i_n : P_n \rightarrow \mathcal{H}_{t_n}^{D_n}$$

by $i_n = \bar{j}_n \upharpoonright P_n$.

By Theorem 2.4, fix $\gamma_n < \bar{\eta}_n$ such that t_n is $< \gamma_n$ -generic over $\bar{M}_n$. Let $\pi_n : \bar{M}_n \rightarrow M_n^*$ be a Neeman-style $\mathbb{R}^*$ -genericity iteration above γ_n , where $\mathbb{R}^* = \mathbb{R} \cap \text{HOD}$, such that if D_n^* is the derived model of M_n^* as computed in $M_n^*(\mathbb{R}^*)$, then D_0^* and D_1^* are comparable. Let $s_n^* = \pi_n[t_n]$, and let

$$\pi_n^+ : \bar{M}_n[t_n] \rightarrow M_n^*[s_n^*]$$

be the unique extension coming from Lévy–Solovay. Finally let $k_n = \pi_n^+ \circ i_n$, so

$$k_n : P_n \rightarrow \mathcal{H}_{s_n^*}^{D_n^*}$$

is a close embedding.

We first show that $D_0^* = D_1^*$. We have elementary embeddings

$$\begin{aligned} (k_0 \circ e_0) \upharpoonright H : H &\rightarrow \mathcal{H}^{D_0^*} \\ (k_1 \circ e_1) \upharpoonright H : H &\rightarrow \mathcal{H}^{D_1^*} \end{aligned}$$

Assuming towards a contradiction that $D_0^* \in D_1^*$, for example, we would have that H is coded by a real in $\mathcal{H}^{D_1^*}$. Since $\text{crit}(k_1 \circ e_1) > (2^{\aleph_0})^H$, H and $\mathcal{H}^{D_1^*}$ have the same reals, and so H is coded by a real in H , which is a contradiction. This proves that $D_0^* = D_1^*$, so let us denote this model by D^* .

Since H is pointwise definable, it follows that $k_0 \circ e_0 \upharpoonright H = k_1 \circ e_1 \upharpoonright H$. Therefore $k_0 \circ e_0[s] = k_1 \circ e_1[s]$, and hence $k_0 \circ e_0(s) = k_1 \circ e_1(s)$. In other words, $k_0(s_0) = k_1(s_1)$, and so $\mathcal{H}_{k_0(s_0)}^{D^*} = \mathcal{H}_{k_1(s_1)}^{D^*}$. Now $k_0 \circ e_0$ and $k_1 \circ e_1$ have the same target model, and since every element of $P = \mathcal{H}_s^{\bar{V}}$ is definable in P from s , and $k_0 \circ e_0(s) = k_1 \circ e_1(s)$, we obtain $k_0 \circ e_0 = k_1 \circ e_1$. This yields UA for e_0 and e_1 , as desired.

Now assume $V \neq L(\mathbb{R}, \text{Hom})$. Let δ be the largest Suslin cardinal. We will show that for all $s \in \Theta^\omega$, HOD_s satisfies UA. This suffices for the result: if $t \in \text{Ord}^\omega$, then since HOD_t is

a Θ -cc extension of HOD , there is a set $S \in \text{HOD}$ such that $\text{ot}(S) < \Theta$ and $\text{ran}(t) \subseteq S$, and therefore $\text{HOD}_t = \text{HOD}_s$ for some $s \in \Theta^\omega$.

Let δ be the largest Suslin cardinal, which exists since $\text{Hom} \neq P(\mathbb{R})$. By Suslin-co-Suslin reflection, $\text{Hom} = \text{Hom}^{L(\mathbb{R}, \text{Hom})}$ and for all $s \in \delta^\omega$, $\text{HOD}_s|\delta = \text{HOD}_s^{L(\mathbb{R}, \text{Hom})}|\delta$. Thus, $L(\mathbb{R}, \text{Hom})$ satisfies $\text{AD}^+ + V = L(\mathbb{R}, \text{Hom})$, so by the previous case, UA holds in $\text{HOD}_s^{L(\mathbb{R}, \text{Hom})}$ for all $s \in \delta^\omega$. In the context of ZFC, since UA is a Π_2 -sentence, UA is downwards absolute to any V_ρ with $|V_\rho| = \rho$. Thus for all $s \in \delta^\omega$, UA holds in $\text{HOD}_s^{L(\mathbb{R}, \text{Hom})}|\delta = \text{HOD}_s|\delta$.

Assume towards a contradiction that UA fails in HOD_s for some $s \in \Theta^\omega$. Fix a limit ordinal $\gamma < \Theta$ such that $\sup(s) < \gamma$ and UA fails in $\text{HOD}_s \cap V_\gamma$. Find $\lambda < \Theta$ large enough that $\text{HOD}_s|\gamma \subseteq (\text{HOD}|\lambda)[s]$: to see this is possible, let $T \in \text{HOD}_s$ be a bounded subset of Θ , and note that T has a nice name in $\text{HOD} \cap V_\Theta$ by the Θ -cc of the Vopěnka algebra. Let $j : V \rightarrow M$ be a Spector ultrapower by a δ -complete measure on δ such that $j(\delta) > \lambda$, $\text{HOD}_s^M|\lambda = \text{HOD}|\lambda$, and $s \in M$.

Since all measures are OD, $\text{HOD}^M \subseteq \text{HOD}$, and hence $\text{HOD}_s^M \subseteq \text{HOD}_s$. On the other hand, $\text{HOD}|\lambda \subseteq \text{HOD}_s^M$, so $\text{HOD}_s|\gamma \subseteq (\text{HOD}|\lambda)[s] \subseteq \text{HOD}_s^M$. It follows that $\text{HOD}_s|\gamma = \text{HOD}_s^M|\gamma$. Therefore UA fails in $\text{HOD}_s^M|\gamma$, and hence it fails in $\text{HOD}_s^M|j(\delta)$, contradicting that in V , UA holds in $\text{HOD}_s|\delta$ for any $s \in \delta^\omega$. $\square$

Using Theorem 2.4, we can generalize Woodin's theorem on the HOD -ultrafilter hypothesis to the context of $\text{AD}^+ + V = L(P(\mathbb{R}))$. As in the case of Theorem 2.6, Woodin proved the slightly weaker $(\text{HOD} \cap V_\Theta)$ -ultrafilter hypothesis, again using more AD^+ -theory.

The following theorem is due to Woodin when $s \in \omega^\omega$.

Theorem 2.9. *Assume $\text{AD}^+ + V = L(P(\mathbb{R}))$. Then for all $s \in \text{Ord}^\omega$, the HOD_s -ultrafilter hypothesis holds.*

Proof. We can reduce to the case that $V = L(\mathbb{R}, \text{Hom})$ by essentially the same argument from Theorem 2.8.

Suppose H is the pointwise definable hull of $\mathcal{H}$. By Proposition 2.3, we can form a model $\bar{V} \equiv V$ such that $H = \mathcal{H}^{\bar{V}}$. Fix $s \in \text{Ord}^\omega \cap \bar{V}$, and suppose $j : H_s \rightarrow H_1$ is the ultrapower of $H_s = (\mathcal{H}_s)^{\bar{V}}$ by a measure $U \in H_s$. It suffices to show that U extends to a countably complete filter in $\bar{V}$. Fix $\sigma \in (P_{\omega_1}(U))^{\bar{V}}$. We will show that $\bigcap \sigma \neq \emptyset$.

We use the construction from the proof of Theorem 2.8, with M and η as in Theorem 2.4 and $\mathbb{R}^* = \mathbb{R} \cap M$. Applying Theorem 2.4 inside $\bar{V}$, let $\bar{M} \subseteq H$, $\bar{\eta}$, and the generic elementary embedding $\bar{j} : \bar{V} \rightarrow D$ be as in that theorem, where D is a derived model of $\bar{M}$ at $\bar{\eta}$. Set $t = \bar{j}(s)$ and define

$$i : H_s \rightarrow \mathcal{H}_t^D.$$

by $i = \bar{j} \upharpoonright H_s$. Since σ is countable in $\bar{V}$ and $\sigma \subseteq H_s$,

$$\bar{j}(\sigma) = \bar{j}[\sigma] = i[\sigma] \in D.$$

In particular, $i[\sigma]$ and t belong to a common $< \bar{\eta}$ -generic extension of $\bar{M}$. Thus by taking the critical point of the genericity iteration $\pi : \bar{M} \rightarrow M^*$ sufficiently large, we can ensure that π lifts uniquely to an elementary embedding π^+ of $\bar{M}[i[\sigma], t]$. As in Theorem 2.8, we let $k = \pi^+ \circ i$. Note that $\pi^+(i[\sigma]) = k[\sigma]$.

Let Q be the target model of k . The construction from Theorem 2.8 yields $k_1 : H_1 \rightarrow Q$ such that $k = k_1 \circ j$.

Let $a = [\text{id}]_U$. Then $a \in \bigcap j[\sigma]$, so $k_1(a) \in \bigcap (k_1 \circ j)[\sigma] = \bigcap k[\sigma]$. In particular, $\bigcap k[\sigma] \neq \emptyset$, and since $k[\sigma] = \pi^+(i[\sigma])$, it follows that $\bigcap i[\sigma] \neq \emptyset$. Finally, $\bar{j}(\sigma) = i[\sigma]$, so the elementarity of $\bar{j} : \bar{V} \rightarrow D$ implies $\bigcap \sigma \neq \emptyset$, as desired. $\square$

3 Consequences for the Nairian model

In this section, we record some consequences of Section 2 for the structure of Nairian models.

Definition 3.1. Let $I_\xi = \bigcup_{\beta < \xi} \beta^\omega$. The *Nairian model* at ξ is defined by

$$\mathcal{N}_\xi = \text{HOD}_{I_\xi}$$

We only consider such models in the context of determinacy. We are interested in the case that $\xi < \Theta$ and $\mathcal{N}_\xi \cap V_\xi$ is a model of ZF. Blue–Larson–Sargsyan [2] prove that this is possible; in fact, ξ can be strongly inaccessible in $\mathcal{N}_\xi$. Here, a cardinal κ is *strongly inaccessible* if for all $\alpha < \kappa$, there is no cofinal map from V_α to κ .

3.1 UA preliminaries

Let $\text{meas}_\kappa(X)$ denote the set of κ -complete measures on X and $\text{meas}(X) = \text{meas}_{\omega_1}(X)$.

Definition 3.2. The *Ketonen order* is defined on $U, W \in \text{meas}(\delta)$ by setting $U <_{\mathbb{K}} W$ if for some $S \in W$, there are measures $(U_\alpha)_{\alpha \in S} \in \prod_{\alpha \in S} \text{meas}(\alpha)$ such that for all $A \subseteq \delta$,

$$A \in U \iff \{\alpha \in S : A \cap \alpha \in U_\alpha\} \in W$$

ZF + DC proves that the Ketonen order is a well-founded partial order [12, Theorem 3.7] and UA is equivalent to the linearity of the Ketonen order [12, Theorem 3.8].

The linearity of the Ketonen order allows us to define minimum measures:

Definition 3.3 (UA). For each cardinal δ , $\mathcal{K}_\delta$ denotes the Ketonen least uniform measure on δ , if one exists.

Lemma 3.4 ([7, Theorem 7.2.14]). *If δ is regular and $\mathcal{K}_\delta$ exists, then $\mathcal{K}_\delta$ is the unique weakly normal measure on δ concentrating on ordinals α such that $\mathcal{K}_{\text{cf}(\alpha)}$ does not exist.* $\square$

Recall that a measure U on δ is *weakly normal* if every regressive function on δ is bounded on a U -large set. We use this notion even in the case that δ is a singular ordinal, noting, however, that for singular cardinals, a different notion of weak normality is also in use [7, Definition 4.4.14].

The argument for Lemma 3.4 yields:

Lemma 3.5. *Assume $\text{AD} + \text{DC}_{\mathbb{R}}$. Every ordinal $\delta < \Theta$ of uncountable cofinality carries a weakly normal measure that concentrates on S_ω^δ .*

Proof. Let W be a Ketonen minimal measure extending the tail filter on δ . If $f : \delta \rightarrow \delta$ is regressive on a W -large set, then $f_*(W) <_{\mathbb{K}} W$, and so $f_*(W)$ is not uniform, which means that $f_*(W)$ concentrates on a bounded set; in other words, f is bounded on a W -large set. This proves W is weakly normal.

If $S_\omega^\delta \notin W$, let $S = \delta \setminus S_\omega^\delta$. For each $\alpha \in S$, choose $U_\alpha \in \text{meas}(\alpha)$ extending the tail filter on α . (Recall Kunen’s theorem that the measures are well-ordered.) Put $A \in U$ if and only if $\{\alpha \in S : A \cap \alpha \in U_\alpha\} \in W$. Then $U <_{\mathbb{K}} W$, but clearly U extends the tail filter, contrary to the minimality of W . $\square$

Definition 3.6. For any family of sets F , κ_F denotes the least cardinal γ such that for some sequence $\langle A_\xi : \xi < \gamma \rangle \subseteq F$, $\bigcap_{\xi < \gamma} A_\xi = \emptyset$.

If F is an ultrafilter, then κ_F is the usual completeness of F ; that is, $\kappa_F = \text{crit}(j_F)$. We will employ this more general notion in the case that F is an ultrafilter in some transitive model of set theory.

A measure on a regular cardinal is *seminormal* if it extends the closed unbounded filter. The following theorem is one of the main results in the analysis of supercompactness and strong compactness under UA.

Theorem 3.7 (UA, [7, Proposition 8.4.20(5)–(6), Theorem 7.3.34]). *Suppose δ is a regular cardinal that carries distinct seminormal measures, and let $\kappa = \kappa_{\mathcal{K}_\delta}$. Then $\mathcal{K}_\delta$ witnesses that κ is δ -strongly compact and $<\delta$ -supercompact. Moreover, κ is δ -supercompact.* $\square$

We will analyze measures in Nairian models by connecting them to canonical measures in HOD_s .

Definition 3.8. If $s \in \text{Ord}^\omega$ and η is a cardinal in HOD_s , then $\mathcal{K}_\eta^s$ denotes $(\mathcal{K}_\eta)^{\text{HOD}_s}$, if $(\mathcal{K}_\eta)^{\text{HOD}_s}$ exists.

The following lemma traces back to Ketonen [19].

Lemma 3.9. *Suppose $s \in \text{Ord}^\omega$, κ is a cardinal in HOD_s , and $\eta \in (\kappa, \Theta)$ is regular in HOD_s . If HOD_s has the $(\aleph_1, \kappa)$ -cover property at η , then $(\kappa_{\mathcal{K}_\eta})^{\text{HOD}_s} \leq \kappa$.*

Proof. The $(\aleph_1, \kappa)$ -cover property at η implies η has uncountable cofinality, and hence $\mathcal{K}_\eta^s$ exists. Note that $S_\omega^\eta \in \mathcal{K}_\eta^s$ by Lemma 3.4: by the ordinal definability of measures, if $\alpha < \Theta$ and $\text{cf}^{\text{HOD}_s}(\alpha)$ carries no uniform measure in HOD_s , then $\text{cf}(\alpha) = \omega$. The $(\aleph_1, \kappa)$ -cover property then implies $(S_{<\kappa}^\eta)^{\text{HOD}_s} \in \mathcal{K}_\eta^s$.

Now let $j : \text{HOD}_s \rightarrow M$ be the ultrapower embedding associated to $\mathcal{K}_\eta^s$. We must show $\text{crit}(j) \leq \kappa$. By Łoś's theorem, $[\text{id}]_{\mathcal{K}_\eta^s} \in \left(S_{<j(\kappa)}^{j(\eta)} \right)^M$, or in other words:

$$\text{cf}^M(\sup j[\eta]) < j(\kappa)$$

Since $\text{cf}^{\text{HOD}_s}(\sup j[\eta]) = \eta$, we must have $j(\kappa) \geq \eta$, and hence $\text{crit}(j) \leq \kappa$. $\square$

3.2 Measurable cardinals in Nairian models

In this subsection, we assume:

- $\text{AD}^+ + V = L(P(\mathbb{R}))$.
- $\xi \leq \Theta$ is strongly inaccessible in $\mathcal{N} = \mathcal{N}_\xi$.

Our next few observations, up to Proposition 3.21, concern a fixed ordinal δ with the following properties:

- δ is a regular cardinal in $\mathcal{N}$.
- $\Theta^\mathcal{N} \leq \delta < \xi$.

The first of these lemmas allows us to apply Theorem 3.7 to δ :

Lemma 3.10. *For all sequences $s \in I_\xi$, δ carries distinct seminormal measures in HOD_s .*

Proof. Since ξ is a strong limit cardinal in $\mathcal{N}$, $(2^\delta)^{\text{HOD}_s} < \xi$. In particular, every ω -sequence of OD_s -subsets of δ belongs to $\mathcal{N}$. Let $S_0 = S_\omega^\delta$ and $S_1 = \delta \setminus S_0$; since $\delta \geq \omega_2$, both are stationary in $\mathcal{N}$. Let F be the club filter of HOD_s and let $F_0 = F|S_0$ and $F_1 = F|S_1$. Then F_0 and F_1 extend to countably complete filters in V , and thus extend to measures U_0 and U_1 in V . By Kunen's theorem, these measures are OD, and therefore $U_0 \cap \text{HOD}_s$ and $U_1 \cap \text{HOD}_s$ witness that in HOD_s , there are distinct seminormal measures on δ . $\square$

Proposition 3.11. *Suppose $s \in I_\xi$, $\bar{\kappa} = (\kappa_{\mathcal{K}_\delta^s})^\mathcal{N}$, and $\kappa = (\kappa_{\mathcal{K}_\delta^s})^{\text{HOD}_s}$.*

- (1) HOD_s has the $(\bar{\kappa}, \kappa)$ -cover property at δ in $\mathcal{N}$.
- (2) $\mathcal{K}_\delta^s$ witnesses that κ is δ -strongly compact and $<\delta$ -supercompact in HOD_s .
- (3) κ is δ -supercompact in HOD_s .

Proof. Since δ carries distinct seminormal measures, we can apply Theorem 3.7 to conclude that $\mathcal{K}_\delta^s$ witnesses that κ is δ -strongly compact and $<\delta$ -supercompact in HOD_s , and moreover κ is δ -supercompact in HOD_s . Since $\mathcal{K}_\delta^s$ witnesses that κ is δ -strongly compact in HOD_s , there is a function $p : \delta \rightarrow P_\kappa(\delta)$ in HOD_s such that $\mathcal{U} = p_*(\mathcal{K}_\delta^s)$ is a fine measure on $P_\kappa(\delta)$.

The $(\bar{\kappa}, \kappa)$ -cover property now follows. Fix $\sigma \in P_{\bar{\kappa}}(\delta) \cap \mathcal{N}$. For $\alpha < \delta$, define

$$A_\alpha = \left\{ \tau \in (P_\kappa(\delta))^{\text{HOD}_s} : \alpha \in \tau \right\}$$

so $A_\alpha \in \mathcal{U}$ by fineness. Since $\mathcal{K}_\delta^s$ is $\bar{\kappa}$ -complete, $\bigcap_{\alpha \in \sigma} A_\alpha \neq \emptyset$, and if τ belongs to this intersection, then $\tau \in \text{HOD}_s$, $\sigma \subseteq \tau$ and $|\tau|^{\text{HOD}_s} < \kappa$. $\square$

Proposition 3.12. *For any $s \in I_\xi$, $\mathcal{K}_\delta^s$ is $\Theta^\mathcal{N}$ -complete in HOD_s .*

Proof. Otherwise, by Proposition 3.11, for some $\kappa < \Theta^\mathcal{N}$, HOD_s has the $(\aleph_1, \kappa)$ -cover property at δ . To get a contradiction, fix $\gamma \in (\kappa, \Theta^\mathcal{N})$ such that $\gamma \rightarrow (\gamma)^{<\omega_1}$. By a standard argument, there is an ω -club of $\alpha < \gamma$ such that α is regular in HOD_s .³ This contradicts that HOD_s has the $(\aleph_1, \kappa)$ -cover property at γ . $\square$

Proposition 3.13. *Suppose that in $\mathcal{N}$, W is a weakly normal measure on δ that concentrates on S_ω^δ . Then for all $s \in I_\xi$, $W \cap \text{HOD}_s = \mathcal{K}_\delta^s$.*

Proof. This follows immediately from Lemma 3.4 since $W \cap \text{HOD}_s$ is weakly normal in HOD_s and by the HOD_s -ultrafilter hypothesis (Theorem 2.9), $W \cap \text{HOD}_s$ concentrates on ordinals α such that $\mathcal{K}_\alpha^s$ does not exist where $\bar{\alpha} = \text{cf}^{\text{HOD}_s}(\alpha)$. $\square$

Corollary 3.14. *Suppose that in $\mathcal{N}$, W is a weakly normal measure on δ that concentrates on S_ω^δ . Then $W = \bigcup_{s \in I_\xi} \mathcal{K}_\delta^s$ and W is $\Theta^\mathcal{N}$ -complete.*

Proof. This follows from Proposition 3.12 and Proposition 3.13 since every subset of δ in $\mathcal{N}$ belongs to HOD_s for some $s \in I_\xi$. $\square$

Corollary 3.15. *In $\mathcal{N}$, there is a unique weakly normal measure concentrating on S_ω^δ .*

³Let U be the ω -club filter on γ . The partition property implies that U is a normal measure, and hence $U \cap \text{HOD}_s$ is a normal measure in HOD_s . Since HOD_s satisfies ZFC, $\text{Reg}^{\text{HOD}_s} \cap \gamma \in U$.

Proof. Existence follows from Lemma 3.5: since δ is regular in $\mathcal{N}$, $\text{cf}(\delta) > \omega$; therefore Lemma 3.5 provides a weakly normal measure W on δ concentrating on S_ω^δ ; since W is OD, $W \cap \mathcal{N} \in \mathcal{N}$. Uniqueness is immediate from Corollary 3.14. $\square$

Definition 3.16. Let W_δ denote the unique weakly normal measure of $\mathcal{N}$ with $S_\omega^\delta \in W_\delta$.

The proof of Corollary 3.15 yields the following criterion for completeness of W_δ .

Corollary 3.17. *Suppose $\kappa \leq \delta$ and for all $s \in I_\xi$, $(\kappa_{\mathcal{H}_\delta^s})^{\text{HOD}_s} \geq \kappa$. Then $(\kappa_{W_\delta})^\mathcal{N} \geq \kappa$.* $\square$

Before continuing to our general results, we state some immediate corollaries under anti-inner model hypotheses.

Proposition 3.18. *Let $N = \mathcal{N} \cap V_\xi$ and assume the following holds in N : for all cardinals λ , there is no inner model M of ZFC+UA in which some $\kappa < \lambda$ is λ -supercompact and such that the M -ultrafilter hypothesis holds. Then in N , every regular cardinal is measurable.*

Proof. This follows from Proposition 3.11 and Corollary 3.17. $\square$

It is unclear that it is consistent for the anti-inner model hypothesis of Proposition 3.18 to hold in a Nairian model $\mathcal{N}_\xi$ where ξ is strongly inaccessible. The question that arises is the following:

Question 3.19. Which of the following large cardinal hypotheses has higher consistency strength over ZFC?

- There is a cardinal κ that is κ^+ -supercompact.
- There is a Woodin limit of Woodin cardinals.

Clearly if κ is 2^κ -supercompact, then κ is a Woodin limit of Woodin cardinals, so we state the following proposition, whose anti-large cardinal hypothesis is consistent:

Proposition 3.20. *Let $N = \mathcal{N} \cap V_\xi$, and assume that in N there is no inner model of ZFC containing a cardinal κ that is 2^{2^κ} -supercompact. Then in N , every regular cardinal δ carries a measure that is λ -complete for all cardinals $\lambda < \delta$.*

Proof. The proof is the same as that of Proposition 3.18 using the fact that under UA, if κ is κ^{++} -supercompact, then $2^{2^\kappa} = \kappa^{++}$. $\square$

We now continue to the theorems that require no anti-inner model hypothesis.

Proposition 3.21. *Let $\kappa = (\kappa_{W_\delta})^\mathcal{N}$, and suppose $s \in I_\xi$ is such that $(\kappa_{\mathcal{H}_\delta^s})^{\text{HOD}_s} = \kappa$. Then HOD_s has the κ -cover and approximation properties at δ in $\mathcal{N}$. Moreover, every $S \in P(\delta) \cap \mathcal{N}$ is in $\text{HOD}_{s,x}$ for some $x \in P(\kappa) \cap \mathcal{N}$.*

Proof. The κ -cover property follows immediately from Proposition 3.11. For the approximation property, fix $\mathcal{A} = \langle A_\alpha : \alpha < \delta \rangle \in \text{HOD}_s$ such that HOD_s satisfies that $\mathcal{A}$ is a κ -independent family of subsets of δ . Suppose $T \in P(\delta) \cap \mathcal{N}$ is κ -approximated by HOD_s . Then the basis

$$\mathcal{B} = \{A_\alpha : \alpha \in T\} \cup \{\delta \setminus A_\alpha : \alpha \notin T\}$$

extends to a κ -complete filter F in $\mathcal{N}$. In particular, F is countably complete in V , and thus F extends to a measure U in V . By Kunen's theorem, U is OD, and so $U \cap \text{HOD}_s \in \text{HOD}_s$. Thus $T = \{\alpha < \delta : A_\alpha \in U\} \in \text{HOD}_s$.

To show every $S \in P(\delta) \cap \mathcal{N}$ is in $\text{HOD}_{s,x}$ for some $x \in P(\kappa) \cap \mathcal{N}$, fix $t \in I_\xi$ such that s and S belong to HOD_t . Since $\mathcal{K}_\delta^s \subseteq \mathcal{K}_\delta^t$, $(\kappa_{\mathcal{K}_\delta^t})^{\text{HOD}_t} = \kappa$. In particular, κ is δ -strongly compact in HOD_t . Let $x \subseteq \kappa$ code a well-order of $P_\kappa(\kappa) \cap \text{HOD}_t$. Note that the models $\text{HOD}_{s,x}$ and HOD_t compute $P_\kappa(\delta)$ the same way: of course, $\text{HOD}_{s,x} \subseteq \text{HOD}_t$, and in the other direction, if $\sigma \in (P_\kappa(\delta))^{\text{HOD}_t}$, then σ is covered by $\tau \in (P_\kappa(\delta))^{\text{HOD}_s}$ by the κ -cover property, $P(\tau) \cap \text{HOD}_t \subseteq \text{HOD}_{s,x}$ since $P_\kappa(\kappa) \cap \text{HOD}_t \subseteq \text{HOD}_{s,x}$. The same argument shows that

$$(\delta^{<\kappa})^{\text{HOD}_{s,x}} \leq (\delta^{<\kappa})^{\text{HOD}_s} \cdot (\kappa^{<\kappa})^{\text{HOD}_{s,x}} = \delta \cdot \kappa = \delta$$

This cardinal arithmetic yields $\mathcal{A} = \langle A_\alpha : \alpha < \delta \rangle \in \text{HOD}_{s,x}$ such that $\text{HOD}_{s,x}$ satisfies that $\mathcal{A}$ is a κ -independent family of subsets of δ . Since $\text{HOD}_{s,x}$ and HOD_t have the same $P_\kappa(\delta)$, $\mathcal{A}$ is also κ -independent in HOD_t . By Proposition 3.11, κ is δ -strongly compact in HOD_t . Repeating the argument of the previous paragraph, we can find a κ -complete measure U of HOD_t such that $S = \{\alpha < \delta : A_\alpha \in U\}$, and using the HOD_t -ultrafilter hypothesis (Theorem 2.9), we obtain that $U \cap \text{HOD}_{s,x} \in \text{HOD}_{s,x}$, and therefore $S \in \text{HOD}_{s,x}$. $\square$

Theorem 3.22. *If $\delta_0 \leq \delta_1 < \xi$ are regular cardinals in $\mathcal{N}$, then $\kappa_{W_{\delta_0}} \leq \kappa_{W_{\delta_1}}$.*

Proof. By the Steel–Woodin theorem, we may assume $\delta_0 \geq \Theta^\mathcal{N}$. By Corollary 3.17, it suffices to show that for all $s \in I_\xi$, $\mathcal{K}_{\delta_1}^s$ is κ -complete in HOD_s where $\kappa = \kappa_{W_{\delta_0}}$.

Fix $s \in I_\xi$. A consequence of [7, Lemma 7.4.49] is that under $\text{ZFC} + \text{UA}$, if $\gamma_0 \leq \gamma_1$ are regular non-isolated Fréchet cardinals, then $\kappa_{\mathcal{K}_{\gamma_0}} \leq \kappa_{\mathcal{K}_{\gamma_1}}$. By Lemma 3.10, both δ_0 and δ_1 carry distinct seminormal measures in HOD_s , and so by [7, Theorem 8.4.20], they are regular non-isolated Fréchet cardinals in HOD_s . By Proposition 3.13, it follows that $\kappa \leq \kappa_{\mathcal{K}_{\delta_0}^s} \leq \kappa_{\mathcal{K}_{\delta_1}^s}$. $\square$

We now prove our main theorem on measurable cardinals in Nairian models.

Theorem 3.23. *Let $\mathcal{H} = \text{HOD}$. Inside $\mathcal{N}$, suppose $\eta < \xi$ is a regular cardinal satisfying:*

- (1) *For all $\alpha < \eta$, $P(\alpha) \subseteq \mathcal{H}(\eta^\omega)$.*
- (2) *For all $\alpha < \eta$, $\Theta(\alpha^\omega) \leq \eta$.*

Then either W_η is η -complete in $\mathcal{N}$ or $\mathcal{H}$ correctly computes all finite successors of η^{++} .

Proof. By (2), $\Theta^\mathcal{N} \leq \eta < \xi$, which puts us in a position to apply the previous results of this section with $\delta = \eta$.

Assume towards a contradiction that the conclusion fails. In other words, $\kappa < \eta$ where $\kappa = (\kappa_{W_\eta})^\mathcal{N}$. Fix $s \in I_\xi$ such that $(\kappa_{\mathcal{K}_\eta^s})^{\text{HOD}_s} = \kappa$.

By Proposition 3.21, HOD_s has the κ -cover property at δ in $\mathcal{N}$ and every $S \in P(\eta) \cap \mathcal{N}$ is in $\text{HOD}_{s,x}$ for some $x \in P(\kappa) \cap \mathcal{N}$. Combining (1) and (2), for any $x \in P(\kappa) \cap \mathcal{N}$, $\text{HOD}_{s,x}$ is a η -cc generic extension of HOD_s . Since every subset of η in $\mathcal{N}$ belongs to some such $\text{HOD}_{s,x}$, it follows that $(\eta^+)^{\text{HOD}_s} = (\eta^+)^\mathcal{N}$.

Since HOD_s has the κ -cover property at η and $(\eta^+)^{\text{HOD}_s} = (\eta^+)^\mathcal{N}$, HOD_s has the κ -cover property at $(\eta^+)^\mathcal{N}$. Let $\gamma = (\eta^+)^\mathcal{N}$. Since γ is a cardinal in $\mathcal{N}$ that is seen to be a successor cardinal by an inner model of choice, γ is regular in $\mathcal{N}$. Applying Lemma 3.9 and Theorem 3.22,

$$\kappa = (\kappa_{\mathcal{K}_\eta^s})^{\text{HOD}_s} \leq (\kappa_{\mathcal{K}_\gamma^s})^{\text{HOD}_s} \leq \kappa$$

Thus we can apply Proposition 3.21 again, to obtain that every $S \in P(\gamma) \cap \mathcal{N}$ is in $\text{HOD}_{s,x}$ for some $x \in P(\kappa) \cap \mathcal{N}$, and thus $(\gamma^+)^{\text{HOD}_s} = (\gamma^+)^\mathcal{N}$.

Iterating this argument, we have that HOD_s correctly computes $(\eta^{+n})^\mathcal{N}$ for all $n < \omega$. Let $\lambda = (\eta^{+\omega})^\mathcal{N}$. By Proposition 3.11, κ is $<\lambda$ -supercompact in HOD_s and HOD_s has the κ -cover and approximation properties below λ . Indeed, this holds in $\text{HOD}_{s,u}$ for all $u \in I_\xi$.

We claim that $\Theta^\mathcal{N}(P(\eta)) \leq (\gamma^+)^\mathcal{N}$. Note first that for all since UA implies GCH in regions of supercompactness, $(2^\eta)^{\text{HOD}_{s,u}} = \gamma$ for all $u \in I_\xi$. For each $u \in \eta^\omega$, let $\langle S_\alpha^u : \alpha < \gamma \rangle$ be an $\text{OD}_{s,u}$ -enumeration of $P(\eta) \cap \text{HOD}_{s,u}$. Note that each $S \subseteq \eta$ is of the form S_α^u for some $u \in \eta^\omega$ and $\alpha < \gamma$, since $S \in \text{HOD}_{s,x}$ for some $x \in P(\kappa) \cap \mathcal{N}$ and $x \in \text{HOD}_u$ for some $u \in \eta^\omega$ by (1).

Suppose $\nu < \Theta^\mathcal{N}(P(\eta))$, and let us show $\nu < (\gamma^+)^\mathcal{N}$. Suppose $f : P(\eta) \rightarrow \nu$ is a surjection. Define $g : \eta^\omega \times \gamma \rightarrow \nu$ by setting $g(u, \alpha) = f(S_\alpha^u)$. Then g is also a surjection. Note however that for each fixed $\alpha < \gamma$, if $A_\alpha = \{g(u, \alpha) : u \in \eta^\omega\}$, then $|A_\alpha| \leq \eta$ by (2). (Here we use that $\Theta^\mathcal{N}(\eta^\omega) = (\eta^+)^\mathcal{N}$ as a consequence of (2).) Since $\nu = \bigcup_{\alpha < \gamma} A_\alpha$, $\nu < (\gamma^+)^\mathcal{N}$.

Fix $y \subseteq \eta$ such that $\text{HOD}_s \cap P_\kappa(\eta)$ is coded by y . Since $\Theta^\mathcal{N}(P(\eta)) \leq (\gamma^+)^\mathcal{N}$, HOD_y is contained in a $(\gamma^+)^\mathcal{N}$ -cc extension of HOD . Moreover, applying the Hamkins uniqueness theorem [10, Theorem 4.7] at each finite successor of η , we see that $Y = P_{\text{bd}}(\lambda) \cap \text{HOD}_s$ is OD_y . In particular, $\text{HOD}_y^{L(Y)} \subseteq \text{HOD}_y$. Note that $L(Y)$ is a model of $\text{ZFC} + \text{UA}$ in which κ is $<\lambda$ -supercompact; the fact that $L(Y) \models \text{AC}$ follows from [7, Theorem 6.2.19] applied in HOD_s , which implies that there is a well-order of Y that is definable from parameters over Y . By [7, Theorem 6.2.19], it follows that $Y \subseteq \text{HOD}_y^{L(Y)} \subseteq \text{HOD}_y$. This implies that Y is contained in a $(\gamma^+)^\mathcal{N}$ -cc extension of HOD . It follows that HOD correctly computes all finite successors of $(\gamma^+)^\mathcal{N} = (\eta^{++})^\mathcal{N}$. $\square$

Note that if HOD correctly computes all finite successors of $(\eta^{++})^\mathcal{N}$, then there is an ordinal $\kappa \leq (\eta^{++})^\mathcal{N}$ that is $(\eta^{+n})^\mathcal{N}$ -supercompact in HOD for all $n < \omega$. In particular, this is impossible under HPC.

3.3 Measurable cardinals in minimal Nairian models

Throughout this subsection, assume

$$\text{ZF} + \text{AD}_\mathbb{R} + \Theta_{\text{reg}} + \text{HPC} + V = L(P(\mathbb{R})).$$

Fix $\xi < \Theta$ such that ξ is strongly inaccessible in $\mathcal{N}_\xi$. We also assume that

$$N = \mathcal{N}_\xi \cap V_\xi$$

is a minimal Nairian model in the sense of [3, Definition 1.1]. We let $\mathcal{H}$ denote HOD as computed in the ambient determinacy model.

Lemma 3.24. *In N , if $\alpha \geq \omega$ and $\eta = \Theta(V_{\alpha+1})$, then for all $\beta < \eta$, $P(\beta) \subseteq \mathcal{H}(\eta^\omega)$.*

Proof. If $\alpha = \omega$, the lemma follows from $\text{AD}_\mathbb{R} + \text{DC}$. Assume $\alpha > \omega$. Fix β such that $\Theta \leq \beta < \eta$, and let $\bar{\eta}$ be the least $<\xi$ -strong of $\mathcal{H}$ above β . By [3, Theorem 19.1], $\bar{\eta} = \Theta(\beta^\omega) \leq \eta$. By [2, Theorem 10.13], $P(\beta) \subseteq L(\mathcal{H}[\bar{\eta}, I_{\bar{\eta}}]) \subseteq \mathcal{H}(\eta^\omega)$. $\square$

Lemma 3.25. *In N , for all $\alpha \geq \omega$, $|V_\alpha| =^* |I_\lambda|$ where $\lambda = \sup_{\beta < \alpha} \Theta(V_\beta)$.*

Proof. Work in N . The nontrivial direction is $|V_\alpha| \leq^* |I_\lambda|$. We produce the required surjections uniformly in α by induction. This uniformity handles the limit steps. For the successor step, suppose $|V_\alpha| \leq^* |I_\lambda|$, and let us show that $|V_{\alpha+1}| \leq^* |\Theta(V_\alpha)^\omega|$.

First, assume $\text{cf}(\alpha) = \omega$. Then $|V_{\alpha+1}| =^* |V_\alpha^\omega| \leq^* |(I_\lambda)^\omega| =^* |\lambda^\omega| \leq^* |\Theta(V_\alpha)^\omega|$.

Otherwise, let $\theta = \Theta(\lambda^\omega)$, and note that $\theta \leq \Theta(V_\alpha)$ since $|(V_\alpha)^\omega| = |V_\alpha|$. We use that $P(\lambda^\omega) \subseteq L_\theta(\mathcal{H}|\theta, I_\theta)$ by [2, Theorem 10.6 and 10.13]. Since every subset of V_α can be coded by a subset of I_λ , it follows that $V_{\alpha+1} \subseteq L_\theta(\mathcal{H}|\theta, I_\theta)$. Thus $|V_{\alpha+1}| \leq^* |I_\theta| \leq^* |I_{\Theta(V_\alpha)}|$, as desired. $\square$

We can now connect the cardinals $\Theta(V_\alpha)$ to the $<\xi$ -strong cardinals of HOD:

Proposition 3.26. *The set of $<\xi$ -strong cardinals of HOD is equal to*

$$S = \{\Theta(V_\alpha) : \alpha \geq \omega \text{ and } \alpha \in \xi \setminus S_\omega^\xi\}$$

Proof. Fix $\alpha \geq \omega$ and let $\lambda = \sup_{\beta < \alpha} \Theta(V_\beta)$. We will show that the least $<\xi$ -strong of HOD above λ is $\min(S \setminus (\lambda + 1))$. By [3, Theorem 19.1], the least $<\xi$ -strong of HOD above λ is $\Theta(\lambda^\omega)$. If $\text{cf}(\alpha) = \omega$, Lemma 3.25 implies $\Theta(V_{\alpha+1}) = \Theta(\lambda^\omega)$, and clearly $\Theta(V_{\alpha+1}) = \min(S \setminus (\lambda + 1))$. Otherwise, $\Theta(V_\alpha) = \Theta(\lambda^\omega)$, and clearly $\Theta(V_\alpha) = \min(S \setminus (\lambda + 1))$. $\square$

Theorem 3.27. *In N , if $\alpha \geq \omega$ and $\kappa = \Theta(V_\alpha)$, then W_κ is κ -complete so κ is measurable.*

Proof. We first consider the case that α is a successor ordinal. Let $\lambda = \Theta(V_{\alpha-1})$. Since λ has uncountable cofinality, Lemma 3.25 implies $\Theta(V_\alpha) = \Theta(\lambda^\omega)$. Since $\Theta(V_\alpha)$ is a $<\xi$ -strong cardinal of HOD, it is regular by [2, Theorem 10.20].

Apply Theorem 3.23 in $\mathcal{N}_\xi$ to $\eta = \kappa$. We check the three criteria from that theorem. It is easy to see that for all $\beta < \kappa$, $\Theta(\beta^\omega) \leq \kappa$. By Lemma 3.24, for all $\beta < \kappa$, $P(\beta) \subseteq \mathcal{H}(\kappa^\omega)$. Finally, HOD does not correctly compute any successor cardinals by our HPC assumption.

For the limit case, we use the following claim:

Claim 1. *In N , for every limit ordinal α , $\Theta(V_\alpha) = (\sup_{\beta < \alpha} \Theta(V_\beta))^+$ and $\Theta(V_\alpha)$ is regular.*

Proof. It is a consequence of ZF that $\Theta(V_\alpha) = (\sup_{\beta < \alpha} \Theta(V_\beta))^+$; see [8, Lemma 2.1]. Work in N . Let $\lambda = \sup_{\beta < \alpha} \Theta(V_\beta)$. Since N is a minimal Nairian model, it contains no strongly inaccessible cardinals, and so λ is singular.

Assume towards a contradiction that λ^+ is singular, say of cofinality $\mu < \lambda$. Let ν be a sufficiently large ordinal. By Blue–Larson–Sargsyan [3, Theorem 1.17 and Definitions 6.2 and 8.6], there is an elementary substructure $X \preceq V_\nu$ such that $\mu \subseteq X$, $\lambda \in X$, and $|X| \leq^* |V_\beta|$ for some $\beta < \alpha$. (We use here that one can bound the sizes of levels of the N -hierarchy from [3] by levels of the cumulative hierarchy; for example, $|N_\gamma| \leq^* |V_{\omega+\gamma+2}|$, which is easy to prove by induction.)

We now use the argument of [24, Lemma 3]. Since $\mu \subseteq X$, $X \cap \lambda^+$ is cofinal in λ^+ . By elementarity, evaluating the surjections in X from λ onto ordinals below λ^+ gives a partial surjection $F : X \times \lambda \rightarrow \lambda^+$. For each $\nu < \lambda$, the set $R_\nu = \{F(h, \nu) : h \in X\}$ has order type below λ , since $\Theta(X) \leq \lambda$. The increasing enumerations of the R_ν 's therefore give a partial surjection from $\lambda \times \lambda$ onto λ^+ , a contradiction. $\square$

We now consider the case that α is a limit ordinal. Let $\lambda = \sup_{\beta < \alpha} \Theta(V_\beta)$. By Claim 1, $\Theta(V_\alpha) = \lambda^+$ is regular. By Theorem 3.22, the completeness of W_{λ^+} is at least the completeness of W_δ for any regular $\delta < \lambda$, and so W_{λ^+} is λ -complete as a consequence of the successor case. The minimality of N implies that λ is singular, and hence W_{λ^+} is λ^+ -complete. $\square$

Corollary 3.28. *The theory “ZF + there are infinitely many consecutive measurable cardinals” is consistent relative to a Woodin limit of Woodin cardinals.*

Proof. Blue–Larson–Sargsyan construct minimal Nairian models starting with a Woodin limit of Woodin cardinals, and in these models they prove that $\Theta(V_{\omega+n+1}) = \Theta^{+n}$ for all $n < \omega$. By Theorem 3.27, this yields infinitely many consecutive measurable cardinals. $\square$

We now show minimal Nairian models satisfy $\aleph(V_{\alpha+1}) = \Theta(V_\alpha)$ for all infinite ordinals α . In other words, there is no $\Theta(V_\alpha)$ -sequence of distinct subsets of V_α .

We will use the following abstract forcing lemma [14, Lemma 5.2].⁴

Lemma 3.29. *Assume the Axiom of Choice. Suppose κ is weakly compact and $\mathbb{B}$ is a κ -cc complete Boolean algebra completely generated by fewer than κ elements. Then $|\mathbb{B}| < \kappa$. $\square$*

Theorem 3.30. *Suppose X is a transitive set and N is an inner model of ZFC such that $N = \text{HOD}_{N \cup \{X\}}$. Let $g \subseteq \text{Col}(\omega, X)$ be V -generic and let $\kappa = \Theta(X^{<\omega})$. Then g belongs to a κ -cc forcing extension of N .*

Proof. We verify that N has the uniform κ -cover property in $M = \text{HOD}_{N \cup \{g\}}$ and then cite Bukovsky’s criterion [11, Proposition 4.37]. Suppose $I \in N$ and $f : I \rightarrow \text{Ord}$ is a function in M . For each $i \in I$, let

$$F(i) = \{\alpha \in \text{Ord} : \exists p \in \text{Col}(\omega, X) (p \Vdash f(i) = \alpha)\}$$

Then $F \in N$, and for all $i \in I$, $f(i) \in F(i)$ and $|F(i)| \leq^* |X^{<\omega}|$, so $\text{ot}(F(i)) < \kappa$. $\square$

Proposition 3.31. *Suppose κ is measurable and X is a transitive set with $\Theta(X) \leq \kappa$. Then, for any inner model M of ZFC, the following hold:*

- (1) κ is strongly inaccessible in $M(X)$.
- (2) For every normal measure W on κ , W -almost every $\rho < \kappa$ is regular in $M(X)$.

Proof. Fix a nonprincipal κ -complete measure W on κ . Let N be the class of sets hereditarily ordinal definable from parameters in $M \cup \{X, W\}$. Then $N = \text{HOD}_{N \cup \{X\}}$. Since $M \subseteq N$, it suffices, by absoluteness, to prove (1) and (2) for $N(X)$.

Applying Theorem 3.30, let $\mathbb{A}$ be a κ -cc complete Boolean algebra and $G \subseteq \mathbb{A}$ be an N -generic filter such that $N[G]$ contains a V -generic collapse $g : \omega \rightarrow X$. Let $\dot{E}$ be an $\mathbb{A}$ -name for a binary relation on ω such that $(X, \in) \cong (\omega, \dot{E}_G)$. Let $\mathbb{B}$ be the complete subalgebra of $\mathbb{A}$ generated by the Boolean values of $n \dot{E} m$ for $n, m < \omega$. Then in N , $\mathbb{B}$ is a countably generated κ -cc Boolean algebra and $N(X)$ is contained in a $\mathbb{B}$ -generic extension of N .

Since κ is measurable in N , Lemma 3.29 implies $|\mathbb{B}| < \kappa$. Therefore κ remains strongly inaccessible in any $\mathbb{B}$ -generic extension of N , and so κ is strongly inaccessible in $N(X)$, which proves (1). For (2), note that if W is normal, then W concentrates on regular cardinals of N , and forcing with $\mathbb{B}$ preserves all sufficiently large N -regular cardinals below κ . $\square$

For the next corollary, we return to the setting of Section 3.3: assume

$$\text{ZF} + \text{AD}_{\mathbb{R}} + \Theta_{\text{reg}} + V = L(P(\mathbb{R})) + \text{HPC}.$$

Let N be a minimal Nairian model, let $\xi = \text{Ord} \cap N$, and let $\mathcal{N} = \mathcal{N}_\xi$. Thus $N = \mathcal{N} \cap V_\xi$.

Corollary 3.32. *In $\mathcal{N}$, if $\omega \leq \alpha < \xi$, there is no $\Theta(V_\alpha)$ -sequence of distinct subsets of V_α .*

⁴The result is due to Jech [18, p. 5] and Harrington–Shelah [15, p. 186, Lemma 1]. The suggestion to use this result to bound the size of the Vopěnka algebra is due to GPT-6.

Proof. Let $\kappa = \Theta(V_\alpha)$, which is measurable by Theorem 3.27.

First suppose that α is a successor ordinal. Assume towards a contradiction that $\vec{A} = \langle A_\nu : \nu < \kappa \rangle$ is a sequence of distinct subsets of V_α . By Lemma 3.25, for some $\eta < \kappa$, there is a surjection $f : \eta^\omega \rightarrow V_\alpha$. (We can take $\eta = \Theta(V_{\alpha-1})$.) Let $B_\nu = f^{-1}[A_\nu]$ for $\nu < \kappa$. Choose $s \in I_\xi$ such that f and $\vec{A}$ are OD_s . By [3, Theorem 12.11], the relation

$$\{(\nu, x) \in \kappa \times \eta^\omega : x \in B_\nu\},$$

has an ∞ -Borel code in HOD_s . Hence $\langle B_\nu : \nu < \kappa \rangle$ belongs to $\text{HOD}_s(\eta^\omega)$. By Proposition 3.31(1), κ is strongly inaccessible in $\text{HOD}_s(\eta^\omega)$. But in this model, $\nu \mapsto B_\nu$ is an injection from κ into $P(\eta^\omega)$, a set of rank less than κ since $\eta < \kappa$. This contradicts that κ is strongly inaccessible.

Now suppose that α is a limit ordinal, and fix a κ -complete nonprincipal measure U on κ . The successor case shows that for every $\beta < \alpha$, there is no κ -sequence of distinct subsets of V_β . By [9, Lemma 3.5], U is therefore closed under V_β -indexed intersections. Since $\alpha < \kappa$ and U is κ -complete, this implies that U is closed under V_α -indexed intersections as well. A standard argument now shows that there is no κ -sequence of distinct subsets of V_α ; see the proof of [17, Lemma 10.4]. $\square$

3.4 Nairian models, supercompact cardinals, and Gitik's theorem

In this subsection (except Proposition 3.37), we assume:

- $\text{AD}^+ + V = L(P(\mathbb{R}))$.
- $\xi \leq \Theta$ is strongly inaccessible in $\mathcal{N} = \mathcal{N}_\xi$.

This means we can apply the results of Section 3.2 to any δ that is regular in $\mathcal{N}$ and satisfies $\Theta^\mathcal{N} \leq \delta < \xi$. In particular, W_δ is defined for all such δ ; see Definition 3.16.

Theorem 3.33. *Exactly one of the following holds:*

- (1) $\mathcal{N} \cap V_\xi$ satisfies that for all κ , for all sufficiently large regular δ , W_δ is κ -complete.
- (2) $\text{HOD} \cap V_\xi$ satisfies that there is a supercompact cardinal.

Proof. Work in $\mathcal{N}$. Assume (1) fails. By Theorem 3.22, there is some cardinal κ_* such that for all sufficiently large regular $\delta < \xi$, $\kappa_{W_\delta} = \kappa_*$.

By Proposition 3.11 and Proposition 3.21, for all regular $\delta \in [\kappa_*, \xi)$, there is some $s \in I_\xi$ such that κ_* is δ -supercompact in HOD_s and HOD_s has the κ_* -cover and approximation properties at δ . Let $\gamma = \kappa_*^+$. By the Hamkins uniqueness theorem [10, Theorem 4.7], if M_0 and M_1 are inner models of ZFC with the κ_* -approximation and cover properties at δ such that $M_0 \cap H(\gamma) = M_1 \cap H(\gamma)$, then $M_0 \cap P(\delta) = M_1 \cap P(\delta)$. By a pigeonhole argument, using that ξ is inaccessible, we can find a single transitive model $M \subseteq V_\xi$ with the κ_* -cover and approximation properties such that for arbitrarily large regular δ , there is some $s \in I_\xi$ such that $P(\delta) \cap M = P(\delta) \cap \text{HOD}_s$ and κ_* is δ -supercompact in HOD_s .

It follows easily that κ_* is supercompact in M and M satisfies UA. Moreover M is ordinal definable from $x = M \cap H(\gamma)$. Thus $\text{HOD}^M \subseteq \text{HOD}_x$. Note also that HOD^M has the κ_* -cover and approximation properties in M since $M \models \text{UA}$ and UA implies that HOD is a weak extender model at each supercompact cardinal [7, Theorem 6.2.15].

We now leave $\mathcal{N}$ and work in the ambient determinacy model. We have that $\text{HOD}^M \subseteq \text{HOD}_x^{\mathcal{N}} \subseteq \text{HOD}_x$. By Vopěnka's theorem, HOD_x is contained in a ρ -cc generic extension of HOD for some $\rho < \xi$. (A coarse upper bound is $\rho = \Theta_0(V_{\gamma+1}^{\mathcal{N}})$.) Since HOD^M has the κ_* -cover property below ξ and is contained in a ρ -cc generic extension of HOD , it follows that HOD has the (κ_*, ρ) -cover property below ξ . This covering property implies that for all regular $\delta \in (\rho, \xi)$, HOD satisfies $\kappa_{\mathcal{H}_\delta} \leq \rho$. Applying Proposition 3.11, it follows immediately that there is a supercompact cardinal in $\text{HOD} \cap V_\xi$.

Finally, we sketch the proof that (1) and (2) are mutually exclusive. Assume (2) holds. We will show that ξ is a limit of regular cardinals in $\mathcal{N}$. Granting this, let us show that (1) fails. Let κ_0 be the least supercompact of $\text{HOD} \cap V_\xi$. By [7, Lemma 7.4.49] if $\kappa_0 \leq \delta < \xi$ and δ is a regular cardinal in HOD , then $\mathcal{H}_\delta^0 = (\mathcal{H}_\delta)^{\text{HOD}}$ has completeness κ_0 in HOD . If $\delta > \kappa_0$ is regular in $\mathcal{N}$, then by Proposition 3.13 $\kappa_{W_\delta} \leq \kappa_{\mathcal{H}_\delta^0} = \kappa_0$, which rules out (1).

We now show that ξ is a limit of regular cardinals in $\mathcal{N}$. First, note that Proposition 3.11 shows that HOD has the $(\aleph_1, \kappa_0)$ -cover property below ξ . Thus the same is true for HOD_s for any $s \in I_\xi$, and it follows from Lemma 3.9 that there is a supercompact cardinal in $\text{HOD}_s \cap V_\xi$. Let us denote by κ_s the least supercompact cardinal of $\text{HOD}_s \cap V_\xi$. Thus HOD_s has the $(\aleph_1, \kappa_s)$ -cover property below ξ .

Fix $s \in I_\xi$ with κ_s as small as possible. We claim that κ_s is regular in $\mathcal{N}$: otherwise, take $t \in I_\xi$ such that $s \in \text{HOD}_t$ and κ_s is singular in HOD_t . Since $\text{HOD}_s \subseteq \text{HOD}_t$, $\kappa_t \leq \kappa_s$, but since κ_s is singular in HOD_t , $\kappa_t < \kappa_s$, contrary to the minimality of κ_s . The proof of [10, Theorem 2.8] now shows that if $\kappa_s \leq \delta < \xi$ and δ is regular in HOD_s , then in $\mathcal{N}$, $\text{cf}(\delta) = |\delta|$. Since ξ is a limit cardinal in $\mathcal{N}$, it follows that ξ is a limit of regular cardinals. $\square$

Note that Theorem 3.33 (2) implies that one can force choice over $\mathcal{N} \cap V_\xi$ by collapsing κ^ω ; in this case, one might say $\mathcal{N}$ is not truly *Nairian*. This raises the question of whether this alternative is even possible. Theorem 3.33 (2) can be ruled out under fairly general hypotheses, including the central cases of minimal Nairian models and full Nairian models:

Proposition 3.34. *Assuming HPC, Theorem 3.33 (1) holds.*

Proof. Since we are assuming $\text{AD}_{\mathbb{R}} + \text{HPC} + V = L(P(\mathbb{R}))$, Steel's analysis of HOD as an lbr hod pair [23] shows that in HOD , there is no cardinal κ that is κ^+ -supercompact. $\square$

Proposition 3.35. *Assuming ξ is a cardinal in V , Theorem 3.33 (1) holds.*

Proof. By a theorem of Woodin [26, Theorem 7.24], assuming $\text{AD}^+ + V = L(P(\mathbb{R}))$, if ξ is a cardinal, then in HOD , no $\kappa < \xi$ is $<\xi$ -supercompact. $\square$

These propositions show that in the central cases, Theorem 3.33 (1) holds. On the other hand, Proposition 3.37 below shows that assuming the existence of very large cardinals in HOD and a natural condition on the Sargsyan sequence, it is possible for Theorem 3.33 (2) to hold in a Nairian model $\mathcal{N} = \mathcal{N}_\xi$ such that ξ is inaccessible in $\mathcal{N}$.

Conjecture 3.36. *Assume $\text{AD}^+ + V = L(P(\mathbb{R}))$. Suppose $\gamma < \Theta$ and the following hold:*

- (1) γ is a cardinal in HOD .
- (2) No ordinal $\kappa < \gamma$ is $<\gamma$ -supercompact in HOD .
- (3) $\text{cf}(\gamma) = \omega$.

Then $\Theta_0(\gamma^\omega)$ is the least Woodin cardinal of HOD above γ .

In Proposition 3.37, we drop the assumption that ξ is inaccessible in $\mathcal{N}_\xi$, since our conclusion is simply that this must hold.

Proposition 3.37. *Assume Conjecture 3.36. If ξ is inaccessible in HOD and $\text{HOD} \cap V_\xi$ satisfies that there is a supercompact cardinal, then ξ is inaccessible in $\mathcal{N}_\xi$.*

Proof. Let κ be the least supercompact cardinal of $\text{HOD} \cap V_\xi$. We begin by computing $\Theta_0(\kappa^\omega)$. Note that κ is a limit of cardinals γ satisfying the hypotheses of Conjecture 3.36. Since κ is a limit of Woodins in HOD , this implies that $\Theta_0(\gamma^\omega) < \kappa$ for all $\gamma < \kappa$. Moreover by the HOD -ultrafilter hypothesis (Theorem 2.9), $\text{cf}(\kappa) \geq \omega_1$. Therefore $\Theta_0(\kappa^\omega) = \kappa^{+\text{HOD}}$.

Let $\mathbb{P} \in \text{HOD}$ be the product Vopěnka forcing adding a sequence of elements of κ^ω to HOD . By counting ordinal definable ∞ -Borel codes, one can show $\text{HOD} \models |\mathbb{P}| \leq 2^{<\theta}$ where $\theta = \Theta_0(\kappa^\omega)$. In particular, $|\mathbb{P}| < \xi$. Since $\text{HOD}_{\kappa^\omega}$ is an inner model of a $\mathbb{P}$ -generic extension of HOD , it follows that ξ remains inaccessible in $\text{HOD}_{\kappa^\omega}$.

By Proposition 3.11, HOD has the $(\aleph_1, \kappa)$ -cover property below ξ . Therefore $\mathcal{N}_\xi = \text{HOD}_{\kappa^\omega}$, and so ξ remains inaccessible in $\mathcal{N}$, proving the proposition. $\square$

We use the following theorem, abstracted from work of Adkisson–Ben-Neria [1], to connect Theorem 3.33 to Gitik’s theorem that ZF does not prove the existence of a regular uncountable cardinal.

Theorem 3.38 (Adkisson–Ben-Neria). *Suppose that there is a class function that assigns to each regular cardinal δ a uniform measure $U_\delta \in \text{meas}(\delta)$ so that the function $\Phi(\delta) = \kappa_{U_\delta}$ is monotone and unbounded. Then there is a class symmetric extension satisfying ZF in which every uncountable cardinal is singular.*

Proof. Let C be the closure of the class $\{\kappa_{U_\delta} : \delta \in \text{Reg}\}$, and let $\langle \kappa_i : i \in \text{Ord} \rangle$ enumerate the Beth fixed-points in C in increasing order. (An ordinal κ is a *Beth fixed-point* if for all $a \in V_\kappa$, $\Theta(a) < \kappa$.) Let $\vec{U} = \langle U_\delta : \delta \in \text{Reg} \rangle$. Let $\mathbb{P} = \mathbb{P}_{\text{fin}}(\vec{U})$ be Adkisson–Ben-Neria’s class forcing from [1, Definition 56]. The proofs of [1, Lemma 57 and Proposition 64], showing that Adkisson–Ben-Neria’s symmetric submodel of $V^\mathbb{P}$ satisfies ZF plus all uncountable cardinals are singular, ostensibly require the existence of a rank Berkeley cardinal. In fact, their proofs go through in our context, with the sequence of almost supercompact cardinals replaced by the sequence $\langle \kappa_i : i \in \text{Ord} \rangle$ defined above. $\square$

Corollary 3.39. *Assuming HPC or that ξ is a V -cardinal, $\mathcal{N} \cap V_\xi$ has a symmetric extension satisfying ZF in which every uncountable cardinal is singular.* $\square$

Corollary 3.40. *The theory “ZF + every uncountable cardinal is singular” is consistent relative to a Woodin limit of Woodin cardinals.* $\square$

4 Comparing Nairian models

We work in this section under the hypotheses:

- $\text{AD}^+ + \text{AD}_\mathbb{R} + V = L(P(\mathbb{R}))$.
- $\xi < \Theta$ is strongly inaccessible in $\mathcal{N} = \mathcal{N}_\xi$.

Our next theorem is a Nairian generalization of a key consequence of determinacy. Assuming $\text{AD}_{\mathbb{R}}$, Woodin [25] showed that if $|X| \leq^* |\mathbb{R}|$, then there is a unique normal measure μ_X on $P_{\omega_1}(X)$.

If $X \in \mathcal{N} \cap V_\xi$, then the restriction of μ_X to $\mathcal{N}$ belongs to $\mathcal{N}$ and witnesses that ω_1 is X -supercompact there. In analogy with Woodin's theorem, we will prove that this is the *unique* normal measure on $P_{\omega_1}(X)$ in $\mathcal{N}$.

The main idea of the proof is that the “iteration maps” from Theorem 2.6 can be lifted from their original domains, which are countable hulls of HOD , to act on the Nairian models associated to these hulls.

We need the following strengthening of Theorem 2.4.

Theorem 4.1 (Woodin). *In the context of Theorem 2.4, for any $\gamma < \Theta$, the model M and the embedding j can be chosen so that $j \restriction \gamma \in M$.*

Note that by homogeneity, the conclusion of Theorem 4.1 implies that $j \restriction \gamma$ belongs to *any* derived model of M at η . In the corresponding symmetric extension $M(\mathbb{R}^*)$, $j \restriction \gamma$ is coded by a bounded countable subset of Θ^D , and hence belongs to D . The same need not hold for $j[\sigma]$, where $\sigma \in [\text{Ord}]^{<\omega_1}$, since σ may not belong to M . Finally, $j \restriction \gamma \in M$ implies that $M \cap P(\gamma) = \text{HOD} \cap P(\gamma)$.

Theorem 4.2. *In $\mathcal{N} \cap V_\xi$, for all X , there is a unique normal measure on $P_{\omega_1}(X)$.*

Proof. Suppose $H \in \text{HOD}$ is a countable transitive model of ZF admitting an elementary embedding $\rho : H \rightarrow \mathcal{H}$ with $\xi \in \text{ran}(\rho)$, and let $\bar{\xi} = \rho^{-1}(\xi)$. Applying $\text{AD}_{\mathbb{R}}$, let $\gamma \in (\xi, \Theta)$ be least such that every $s \in I_\xi$ is $<\gamma$ -generic over HOD , and let $\bar{\gamma} = \rho^{-1}(\gamma)$.

We will use the construction from Theorem 2.6. Let $\tilde{M}$ denote the model M from Theorem 4.1, and let $M = \rho^{-1}[\tilde{M}]$. Let D be a derived model of M at the appropriate limit of Woodin cardinals and let $i : H \rightarrow \mathcal{H}^D$ be the embedding corresponding to the one from Theorem 4.1. We may assume that $i \restriction \bar{\gamma} \in M \cap D$. In particular, $M \cap P(\bar{\gamma}) = H \cap P(\bar{\gamma})$.

Suppose $\bar{\mathcal{N}} \in \text{HOD}$ is an outer model of H such that ρ is partial elementary from $(\bar{\mathcal{N}}, H)$ to $(\mathcal{N}, \mathcal{H})$. Let $\bar{I} = I_{\bar{\xi}} \cap \bar{\mathcal{N}}$. By our choice of $\bar{\gamma}$, every $s \in \bar{I}$ is $<\bar{\gamma}$ -generic over H , and since $M \cap P(\bar{\gamma}) = H \cap P(\bar{\gamma})$, it follows that the same is true for M .

Let $\mathbb{R}^* = \mathbb{R} \cap \tilde{M}$, and as in Theorem 2.6, let $\pi : M \rightarrow M^*$ be a Neeman-style $\mathbb{R}^*$ -genericity iteration with critical point above $\bar{\gamma}$. Let D^* be the derived model of M^* computed in $M^*(\mathbb{R}^*)$, and recall as in Theorem 2.6 that $P(\mathbb{R}^*) \cap D^* \leq \text{Hom}^{\tilde{M}}$.

Note that $k = \pi \circ i$ is an elementary embedding from H to $H^* = \mathcal{H}^{D^*}$. Moreover, since $i \restriction \bar{\xi} \in M \cap D$ and $\text{crit}(\pi) > \bar{\xi}$, we have $k \restriction \bar{\xi} = \pi(i \restriction \bar{\xi}) \in D^*$. We claim that k lifts to an elementary embedding

$$\hat{k} : (\bar{\mathcal{N}}, H) \rightarrow (\mathcal{N}^*, H^*)$$

where $\mathcal{N}^* = (\mathcal{N}_{k(\bar{\xi})})^{D^*}$. Namely, if $a \in \bar{\mathcal{N}}$ is definable in $(\bar{\mathcal{N}}, H)$ from $s \in \bar{I}$, set $\hat{k}(a)$ equal to the element of $\mathcal{N}^*$ defined in $(\mathcal{N}^*, H^*)$ from $k[s]$ in the same way. Note here that $k[s] \in D^*$ since $k \restriction \bar{\xi} \in D^*$. The fact that $\hat{k}$ is well-defined and elementary follows from the fact that $\pi \circ i$ extends, for each $s \in \bar{I}$, to an elementary embedding from H_s to $\mathcal{H}_{k[s]}^{D^*}$, using Theorem 4.1 and the fact that $\text{crit}(\pi) > \bar{\gamma}$.

Since it will be relevant in Theorem 4.3, we note that the embedding $\hat{k}$ is close to $\bar{\mathcal{N}}$. Fix ordinals δ and $\alpha < k(\delta)$, and let U be the $\bar{\mathcal{N}}$ -measure on δ derived from $\hat{k}$ using α . Let $\tilde{U}$ be the M -ultrafilter on $i(\delta)$ derived from π using α . Since π is close to M , $\tilde{U} \in M$.

Moreover, since $\text{crit}(\pi) > \bar{\gamma}$, $\kappa_{\tilde{U}} > \bar{\gamma}$, and therefore for each $s \in \bar{I}$, Levy–Solovay implies $\tilde{U}$ generates a measure $\tilde{U}_s$ in $M[s] = M[i[s]]$. Finally,

$$U = \{A \in P(\delta) \cap \bar{\mathcal{N}} : \exists s \in \bar{I} (A \in H[s] \wedge i_s(A) \in \tilde{U}_s)\}$$

and this is definable over $(\bar{\mathcal{N}}, H)$ from the parameter $\tilde{U} \in M \subseteq H$.

Now fix a finitely generated structure H with an elementary $\rho : H \rightarrow \mathcal{H}$ such that $\rho(\bar{\xi}) = \xi$ and $\rho(\bar{\Theta}) = \Theta$. Construct $\bar{V} \in \text{HOD}$ by a type omission argument so that $H = \mathcal{H}^{\bar{V}}$. Let $\bar{\mathcal{N}} = (\mathcal{N}_{\bar{\xi}})^{\bar{V}}$.

Suppose that $X, \mathcal{U} \in \bar{\mathcal{N}}$ and $\bar{\mathcal{N}}$ satisfies that $\mathcal{U}$ is a normal measure on $P_{\omega_1}(X)$. We will show that $\mathcal{U}$ is the measure derived from the embedding $\hat{k} : \bar{\mathcal{N}} \rightarrow \mathcal{N}^*$ defined above using $\hat{k}[X]$.

Using Spector’s ultrapower construction, form in HOD an elementary $j : \bar{\mathcal{N}} \rightarrow \mathcal{P}$ such that $\mathcal{U}$ is the measure derived from j using $j[X]$ and $j \upharpoonright (P(\text{Ord}) \cap \bar{\mathcal{N}})$ agrees with the ultrapower embedding of $\mathcal{U}$. Let $H_{\mathcal{P}} = j(H)$. By a standard realizability argument, there is an elementary embedding $\rho_{\mathcal{P}} : H_{\mathcal{P}} \rightarrow \mathcal{H}$ such that $\rho = \rho_{\mathcal{P}} \circ j \upharpoonright H$. This embedding is automatically partial elementary from $\mathcal{P}$ to $\mathcal{N}$ since the first-order theory of $\mathcal{N}$ with ordinal parameters uniformly reduces to that of $\mathcal{H}$.

Applying the construction above to $\mathcal{P}$ instead of $\bar{\mathcal{N}}$, we obtain $\hat{k}_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{N}_{\mathcal{P}}^*$. Using that H is finitely generated, one can show as in Theorem 2.6 that the relevant derived models are equal. Moreover, $\hat{k} \upharpoonright H = \hat{k}_{\mathcal{P}} \circ j \upharpoonright H$ since H is finitely generated and these are close embeddings. We can now conclude that $\mathcal{N}_{\mathcal{P}}^* = \mathcal{N}^*$. Moreover, it follows that $\hat{k}$ and $\hat{k}_{\mathcal{P}} \circ j$ agree on $\bar{I}$, and hence on $\bar{\mathcal{N}}$. In particular, $\hat{k}_{\mathcal{P}}(j[X]) = \hat{k}_{\mathcal{P}} \circ j[X] = \hat{k}[X]$. Since $\mathcal{U}$ is derived from j using $j[X]$, we also have that $\mathcal{U}$ is derived from $\hat{k}_{\mathcal{P}} \circ j$ using $\hat{k}_{\mathcal{P}}(j[X])$. In other words, $\mathcal{U}$ is the measure derived from $\hat{k}$ using $\hat{k}[X]$. $\square$

Theorem 4.2 confirms a consequence of Blue–Larson–Sargsyan’s quasi-club conjecture. The *quasi-club filter* on $P_{\omega_1}(X)$ is the filter generated by sets of the form

$$\{\sigma \cap X : \sigma \preceq \mathcal{M} \text{ and } |\sigma| \leq \aleph_0\}$$

where $\mathcal{M}$ is a structure in a countable language whose underlying set contains X . The *quasi-club conjecture* states that assuming $\text{AD}_{\mathbb{R}} + \Theta$ is regular $+ \text{HPC} + V = L(P(\mathbb{R}))$, if $\mathcal{N}$ is a minimal Nairian model and $X \in \mathcal{N} \cap V_{\bar{\xi}}$, then the quasi-club filter on $P_{\omega_1}(X)$ is an ultrafilter. This conjecture probably cannot be proved as easily as Theorem 4.2. For one thing, the conclusion of the quasi-club conjecture is false in the model of Proposition 3.37.

Next, we show that the Ultrapower Axiom holds in Nairian models. See [12] for the statement of the Ultrapower Axiom in the context of $\text{ZF} + \text{DC}$; note that it only applies to measures on ordinals.

Theorem 4.3. *The Ultrapower Axiom holds in $\mathcal{N}$.*

Proof. Fix a finitely generated structure H with an elementary $\rho : H \rightarrow \mathcal{H}$ such that $\rho(\bar{\xi}) = \xi$ and $\rho(\bar{\Theta}) = \Theta$. Construct $\bar{V} \in \text{HOD}$ by a type omission argument so that $H = \mathcal{H}^{\bar{V}}$. We will prove that UA holds in $\bar{\mathcal{N}} = (\mathcal{N}_{\bar{\xi}})^{\bar{V}}$.

Suppose $j_0 : \bar{\mathcal{N}} \rightarrow \mathcal{P}_0$ and $j_1 : \bar{\mathcal{N}} \rightarrow \mathcal{P}_1$ are Spector ultrapowers of $\bar{\mathcal{N}}$ by $\bar{\mathcal{N}}$ -internal measures U_0 and U_1 on ordinals below $\bar{\xi}$. The construction from Theorem 4.2 produces close embeddings $k_0 : \mathcal{P}_0 \rightarrow M$ and $k_1 : \mathcal{P}_1 \rightarrow M$ such that $k_0 \circ j_0 = k_1 \circ j_1$. The arguments from [12] show this implies that UA holds for U_0 and U_1 . $\square$

4.1 UA and the filter extension property

The reflection structure of Nairian models developed in [3] allows us to extend to $\mathcal{N}$ more of the theory of ZFC + UA than can be recovered in the pure AD^+ context of [12].

Theorem 4.4. *Assume $\mathcal{N}$ satisfies NT_{base} . Suppose $\alpha > 0$ is a limit ordinal and $\lambda = \sup_{\beta < \alpha} \Theta(V_\beta)$. Then one of the following holds:*

- *Some $\kappa < \lambda$ is λ -supercompact in HOD .*
- *Every λ -complete filter on an ordinal below ξ extends to a λ -complete ultrafilter.*

In the case that λ is singular, these two alternatives are mutually exclusive. Also, as in Proposition 3.34 and Proposition 3.35, if HPC holds or if λ is a cardinal in the ambient determinacy model, then the first alternative cannot hold.

For the rest of the section we assume $\mathcal{N}$ satisfies NT_{base} and work in $\mathcal{N}$. We fix throughout a limit ordinal $\alpha > 0$ such that $\lambda = \eta_\alpha$.

If U is an ultrafilter on X and P and Q are partitions of X , then P refines Q modulo U , denoted $P \preceq_U Q$, if there is some $A \in U$ such that $P \restriction A$ refines $Q \restriction A$.

An ultrafilter U is γ -decomposable if there is a partition P of its underlying set such that $|P| = \gamma$ and whenever $Q \preceq_U P$, $|Q| \geq \gamma$. The *decomposability spectrum* of U consists of all γ such that U is γ -decomposable.

Lemma 4.5. *Suppose U is a measure on an ordinal $\delta < \xi$. If the decomposability spectrum of U is unbounded below λ , then HOD satisfies that some $\kappa < \lambda$ is λ -supercompact.*

Proof. Suppose $\gamma \leq \lambda$ is a limit point of the decomposability spectrum of U of cofinality ω . We show that some $\kappa < \gamma$ is γ -supercompact in HOD . Fix $\eta < \xi$ such that $P(\gamma) \subseteq \text{HOD}_{\eta^\omega}$. Let $s \in \eta^\omega$ be a UA-sequence such that in HOD_s , $\text{cf}(\gamma) = \omega$ and γ is a limit of the decomposability spectrum of $U \cap \text{HOD}_s$. The proof of [11, Lemma 5.4] shows that in HOD_s , there is a countably complete uniform ultrafilter on γ^+ . Therefore by [7, Corollary 7.4.10] some $\kappa < \gamma$ is γ -supercompact in HOD_s . The proof shows moreover that $\gamma^{+\text{HOD}_s} = \gamma^+$. By the approximation property argument from Theorem 3.33, it follows that $\gamma^{+\text{HOD}} = \gamma^+$, and so HOD satisfies that some $\kappa < \gamma$ is γ -supercompact.

If λ has countable cofinality we are done. Finally, if λ has uncountable cofinality, a regressive function argument implies that some $\kappa < \lambda$ is $<\lambda$ -supercompact in HOD . If λ is regular, it is easy to finish using Theorem 3.7. If λ is singular of uncountable cofinality, a Silver-style argument shows that $\lambda^{+\text{HOD}} = \lambda^+$, and therefore again some $\kappa < \lambda$ is λ -supercompact. $\square$

A *finest partition of U below λ* is a $\preceq_U$ -minimum element of the set of $<\lambda$ -sized partitions of the underlying set of U . The following is based on a result of Silver [21].

Lemma 4.6. *Suppose U is a measure on an ordinal $\delta < \xi$ and the decomposability spectrum of U is bounded below λ . Then U has a finest partition below λ .*

Proof. Fix an infinite cardinal $\rho < \lambda$ bounding the decomposability spectrum of U below λ , so that every $<\lambda$ -sized partition of δ is equivalent modulo U to a $<\rho$ -sized partition.

Choose a successor ordinal $\beta < \alpha$ large enough that $|\rho| \leq^* |V_{\omega+\beta}|$, and let $\theta = \Theta(V_{\omega+\beta})$. Since we are assuming NT_{base} , $H_{V_{\omega+\beta}}$ is H -like and $\theta > \rho$ is regular. Let ζ be a sufficiently large ordinal, and applying NT_{base} , fix a $V_{\omega+\beta}$ -closed $Y \preceq V_{\zeta+1}$ containing all the relevant

parameters and satisfying $|Y| \leq^* |V_{\omega+\beta+1}|$. Since $H_{V_{\omega+\beta}}$ is hierarchical, we can write Y as an increasing union $\bigcup_{i<\theta} Y_i$ with $|Y_i| \leq^* |V_{\omega+\beta}|$; by $V_{\omega+\beta}$ -closure, each Y_i belongs to Y .

Let P be the least common refinement of all $<\rho$ -sized partitions of δ that belong to Y . Then $|P| < \lambda$ and P is $\preceq_U$ -below every $<\lambda$ -sized partition of δ that belongs to Y . Choose $A \in U$ such that $|P \restriction A| < \rho$. For distinct cells S, T of $P \restriction A$, let $f(S, T)$ be the least $i < \theta$ such that some $<\rho$ -sized partition in Y_i separates S and T . By the regularity of $\theta > \rho$, these indices are bounded by some $i < \theta$. Let Q be the least common refinement of all $<\rho$ -sized partitions of δ in Y_i . Then $Q \in Y$, $|Q| \leq |P| < \lambda$, and $Q \restriction A = P \restriction A$. The latter implies that Q is $\preceq_U$ -below every $<\lambda$ -sized partition of δ that belongs to Y , so in Y , Q is a finest partition of U below λ . The lemma follows by elementarity. $\square$

Definition 4.7. Suppose U and W are measures and X is the underlying set of U . Then

$$s_W(U) = \{A \in j_W(P(X)) : j_W^{-1}[A] \in U\}$$

In the ZFC context, $s_W(U) \in M_W$ if and only if $j_U \restriction M_W$ is definable over M_W . Under UA, there is a useful inaccessibility criterion for W that implies $s_W(U) \in M_W$:

Lemma 4.8 (ZF + DC + UA). *Suppose U and W are measures on ordinals ϵ and δ , respectively. Suppose W is not the U -limit of a sequence $\langle W_\xi : \xi < \epsilon \rangle$ of measures on δ such that $W_\xi <_{\mathbb{K}} W$ for all ξ . Then $s_W(U) \in M_W$.*

Sketch. This is a choiceless analog of [7, Lemma 5.5.15]. The details are contained in the proofs of [12, Theorem 3.8] and [6, Lemma 3.19]. $\square$

Proof of Theorem 4.4. Let F be a λ -complete filter on an ordinal $\delta < \xi$. Let U be the Ketonen minimum ultrafilter extending F . Assume that no $\kappa < \lambda$ is λ -supercompact in HOD. We will show that U is λ -complete.

First, we show that if D is a measure on a cardinal $\gamma < \lambda$, then $D \sqsubset U$. In fact, if $\langle U_\xi : \xi < \gamma \rangle$ is a sequence of measures on δ and $U_\xi <_{\mathbb{K}} U$ for all $\xi < \gamma$, we show that $D\text{-}\lim_{\xi < \gamma} U_\xi \neq U$. This is proved by an elementary substructure argument which yields X with $\aleph(P(X)) \leq \lambda$ and $X \cap (F \setminus U_\xi) \neq \emptyset$ for all $\xi < \gamma$. Letting $A = \bigcap (F \cap X)$, since F is λ -complete and there is no λ -sequence of distinct subsets of X , we have $A \in F$ by [9, Lemma 3.5], and clearly $A \notin U_\xi$ for any $\xi < \gamma$, so $A \notin D\text{-}\lim_{\xi < \gamma} U_\xi$.

By Theorem 4.3 and Lemma 4.8, $s_U(D) \in M_U$ for every $D \in \bigcup_{\gamma < \lambda} \text{meas}(\gamma)$. We show that this implies that U is λ -complete. By Lemma 4.6, U has a finest partition Q below λ . Let D be the projection of U to Q . Then $s_U(D) \in M_U$, but since the critical point of the factor embedding $k : M_D \rightarrow M_U$ is at least λ , we have $s_U(D) \in \text{ran}(k)$ and $k^{-1}[s_U(D)] = s_D(D) \in M_D$. If D were nonprincipal, $s_D(D) \in M_D$ would yield a strict Lipschitz reduction of D to itself, contradicting [7, Corollary 3.4.29]. Thus D is principal, and hence U is λ -complete. $\square$

References

- [1] William Adkisson and Omer Ben-Neria. On the strength of ultrafilters above choiceless large cardinals and their Prikry forcings, 2025.
- [2] Douglas Blue, Paul Larson, and Grigor Sargsyan. Nairian models, 2025.
- [3] Douglas Blue, Paul Larson, and Grigor Sargsyan. The failure of square at all uncountable cardinals is weaker than a Woodin limit of Woodin cardinals, 2026.

- [4] Ali Enayat. Models of set theory with definable ordinals. *Archive for Mathematical Logic*, 44(3):363–385, 2005. doi:10.1007/s00153-004-0256-9.
- [5] Moti Gitik. All uncountable cardinals can be singular. *Israel Journal of Mathematics*, 35(1–2):61–88, 1980. doi:10.1007/BF02760939.
- [6] Gabriel Goldberg. Even ordinals and the Kunen inconsistency. *arXiv preprint arXiv:2006.01084*, 2020.
- [7] Gabriel Goldberg. *The Ultrapower Axiom*, volume 10 of *De Gruyter Series in Logic and Its Applications*. De Gruyter, Berlin, 2022.
- [8] Gabriel Goldberg. Choiceless cardinals and the continuum problem. Preprint, 2022. arXiv:2201.11557.
- [9] Gabriel Goldberg. Measurable cardinals and choiceless axioms. *Annals of Pure and Applied Logic*, 175(1):103323, 2024.
- [10] Gabriel Goldberg. Strongly compact cardinals and ordinal definability. *Journal of Mathematical Logic*, 24(1):2250010, 2024.
- [11] Gabriel Goldberg. The uniqueness of elementary embeddings. *The Journal of Symbolic Logic*, 89(4):1430–1454, 2024.
- [12] Gabriel Goldberg. The Ultrapower Axiom from determinacy. Preprint, 2025.
- [13] Gabriel Goldberg. Generalizations of the Ultrapower Axiom. Preprint, March 6, 2026. <https://math.berkeley.edu/~goldberg/Papers/GeneralizationsOfUA.pdf>.
- [14] Joel David Hamkins and W. Hugh Woodin. The necessary maximality principle for c.c.c. forcing is equiconsistent with a weakly compact cardinal. *Mathematical Logic Quarterly*, 51(5):493–498, 2005.
- [15] Leo A. Harrington and Saharon Shelah. Some exact equiconsistency results in set theory. *Notre Dame Journal of Formal Logic*, 26(2):178–188, 1985.
- [16] James M. Henle, A. R. D. Mathias, and W. Hugh Woodin. A barren extension. In Carlos A. Di Prisco, editor, *Methods in Mathematical Logic (Caracas, 1983)*, volume 1130 of *Lecture Notes in Mathematics*, pages 195–207. Springer, Berlin, 1985. doi:10.1007/BFb0075312.
- [17] Thomas Jech. *Set Theory: The Third Millennium Edition, revised and expanded*. Springer Monographs in Mathematics. Springer, Berlin, Heidelberg, 2003.
- [18] Thomas J. Jech. Simple complete Boolean algebras. *Israel Journal of Mathematics*, 18(1):1–10, 1974.
- [19] Jussi Ketonen. Strong compactness and other cardinal sins. *Annals of Mathematical Logic*, 5(1):47–76, 1972.
- [20] Jeffrey B. Paris. Minimal models of ZF. In *Proceedings of the Bertrand Russell Memorial Logic Conference (Uldum, Denmark, 1971)*, pages 327–331. Leeds University Press, Leeds, 1973.

- [21] Jack H. Silver. Indecomposable ultrafilters and $0^\#$. In *Proceedings of the Tarski Symposium*, volume 25 of *Proceedings of Symposia in Pure Mathematics*, pages 357–363. American Mathematical Society, Providence, R.I., 1974.
- [22] John R. Steel. A stationary-tower-free proof of the derived model theorem. In Su Gao, Steve Jackson, and Yi Zhang, editors, *Advances in Logic*, volume 425 of *Contemporary Mathematics*, pages 1–8. American Mathematical Society, Providence, RI, 2007.
- [23] John R. Steel. *A Comparison Process for Mouse Pairs*, volume 51 of *Lecture Notes in Logic*. Cambridge University Press, 2022.
- [24] Toshimichi Usuba. Choiceless Löwenheim–Skolem property and uniform definability of grounds, 2020. Version 4, Lemma 3.
- [25] W. Hugh Woodin. AD and the uniqueness of the supercompact measures on $P_{\omega_1}(\lambda)$. In Alexander S. Kechris, Donald A. Martin, and Yiannis N. Moschovakis, editors, *Cabal Seminar 79–81*, volume 1019 of *Lecture Notes in Mathematics*, pages 67–71. Springer, Berlin, 1983.
- [26] W. Hugh Woodin. In search of Ultimate-L: The 19th Midrasha Mathematicae Lectures. *Bulletin of Symbolic Logic*, 23(1):1–109, 2017.