ON THE OPTIMALITY OF THE HOD DICHOTOMY

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ABSTRACT. In the first part of the manuscript, we establish several consistency results concerning Woodin's HOD hypothesis and large cardinals around the level of extendibility. First, we prove that the first extendible cardinal can be the first strongly compact in HOD. We extend a former result of Woodin by showing that under the HOD hypothesis the first extendible cardinal is $C^{(1)}$-supercompact in HOD. We also show that the first cardinal-correct extendible may not be extendible, thus answering a question by Gitman and Osinski [GO24, §9].

In the second part of the manuscript, we discuss the extent to which weak covering can fail below the first supercompact cardinal δ in a context where the HOD hypothesis holds. Answering a question of Cummings et al. [CFG15], we show that under the HOD hypothesis there are many singulars $\kappa < \delta$ where $\mathrm{cf}^{\mathrm{HOD}}(\kappa) = \mathrm{cf}(\kappa)$ and $\kappa^{+\mathrm{HOD}} = \kappa^+$. In contrast, we also show that the HOD hypothesis is consistent with δ carrying a club of HOD-regulars cardinals κ such that $\kappa^{+\mathrm{HOD}} < \kappa^+$. Finally, we close the manuscript with a discussion about the HOD hypothesis and ω-strong measurability.

1. INTRODUCTION

When is a mathematical object definable? Can this intuitive concept be formalized at all? On first thought, the answer is no. Suppose the concept of definability could itself be precisely defined. Consider the least undefinable ordinal number. Since there are uncountably many ordinals and only countably many definitions, this ordinal must exist. However the number is definable, contrary to its definition!

In a 1946 Princeton lecture [Göd46], Gödel set out to circumvent this paradox, arriving at the concept of ordinal definability. Intuitively, a set X is *ordinal definable* if it is the unique object satisfying some set-theoretic property that can be defined in terms of finitely many ordinal numbers. A set X is *hereditarily ordinal definable* (in symbols, $X \in \mathrm{HOD}$) if X is ordinal

definable, every element of X is ordinal definable, every element of every element of X is ordinal definable, and so on; see §2 for a formal definition. After Gödel, subsequent investigation by Myhill–Scott [Myh71], Vopěnka [VH72], and others began to outline the theory of HOD. In last few years, the study of HOD has experienced renewed interest after groundbreaking findings due to Woodin [Woo10, Woo11, WDR$^+$12, Woo17].

HOD is a transitive class containing all the ordinals and satisfying ZFC, the Zermelo-Fraenkel axioms of set theory together with the Axiom of Choice. In modern set-theoretic terminology, HOD is an *inner model*. *Inner model theory* is the major area of research in set theory launched by Gödel's discovery of the *constructible universe*, L [Göd40]. Ever since set theorist like Jensen, Mitchell, Neeman, Steel and Woodin (see [Mit10]) have produced a hierarchy of fine-structural inner models extending Gödel's L. This hierarchy is organized according to the extent to which an inner model captures the large cardinal hierarchy of the mathematical universe V. Although HOD is not regarded as a canonical inner model —for example, it is malleable under the *method of forcing*—it contains all canonical inner models constructed to date and is sufficiently rich to accommodate all known large cardinal, thereby embodying the greatest mathematical richness. *Large cardinals axioms* postulates the existence of infinite cardinals whose existence cannot be proved by ZFC. These axioms form a hierarchy of exhaustive principles which permit for a classification of virtually all mathematical theories –extending ZFC– according to their *consistency strength*. Large cardinals have also helped to tackle deep problems concerning *down-to-earth* mathematical objects; such as whether every subset of the real line can be Lebesgue measurable [Sol70], or whether every almost free abelian group must be necessarily free [MS94]. These axioms play a pivotal role in the foundations of mathematics by testing the limits of mathematical theories and providing deep insights into the independence phenomenon [Kan09, Koe14, Mad11].

Woodin's work on HOD [Woo10, Woo17] concerns the question of whether HOD is close to resembling the true mathematical universe. In the presence of strong enough large cardinals, his *HOD hypothesis* implies that the answer is yes: the two models closely resemble one another in terms of their large cardinal structure. This paper examines the limits on this resemblance, the extent to which HOD and V can differ even under the assumption of the HOD hypothesis. Before turning to this, let us define the HOD hypothesis.

Definition ([Woo17])**.** *The* HOD *hypothesis states that there is a proper class of regular cardinals that are not ω-strongly measurable in* HOD.[1]

The above hypothesis was formulated in the wake of the discovery of groundbreaking HOD dichotomy:

Theorem (HOD Dichotomy, [Woo17])**.** *If δ is an extendible cardinal then exactly one of the following holds:*

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[1]For relevant definitions, see §2.

(1) HOD *is a weak extender model for the supercompactness of δ.*
(2) *Every regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in* HOD*.*

The HOD dichotomy has placed the foundations of mathematics at a critical crossroad. In simple terms, the dichotomy states that, assuming sufficiently strong large cardinals exist (i.e., extendibles), either there are reasonable prospects for completing Gödel's inner model program, or there is no fine-structural insight into the mathematical universe V [BKW19]. A natural question thus emerges: which of these scenarios prevails? The HOD hypothesis predicts that the prevailing scenario is the first one.

A corollary of the HOD dichotomy is that, under the HOD hypothesis, every extendible cardinal is supercompact in HOD. This raises an interesting question: could one perhaps prove that under the HOD hypothesis every extendible cardinal is extendible in HOD? In fact, it is a consequence of *Woodin's Universality Theorem* (see Theorem 2.10 below) that every extendible cardinal except perhaps the first one is extendible in HOD. Woodin's theorem is in stark contrast to classical fine-structural inner models, where the extent of similarity between these models and the mathematical universe is contingent upon so-called *anti-large cardinal hypotheses*.

Our first theorem shows that Woodin's theorem is optimal:

Theorem 3.1. *It is consistent with the* HOD *hypothesis that the least extendible cardinal is the least strongly compact cardinal in* HOD*.*

Thus the first extendible is genuinely different from the rest.[2]

The second theorem is that under the HOD hypothesis, even though the first extendible may be the first supercompact cardinal of HOD, there is still a sense in which it must be a rather large supercompact cardinal:

Theorem 3.7. *Under the* HOD *hypothesis, if a cardinal δ is extendible then it must be $C^{(1)}$-supercompact in* HOD*.*

The class of $C^{(n)}$-supercompacts was introduced by Bagaria [Bag12] and it has been extensively studied by Poveda in [HMP22, Pov23, Pov24]. Theorem 3.7 above improves a former result by Woodin saying that under the HOD conjecture, the first extendible cardinal is supercompact in HOD. Incidentally, Theorems 3.1 and 3.7 provide an alternative proof of the main identity crisis theorem of Hayut, Magidor and Poveda [HMP22].

Following up on this issue, in §3.2 we show that under the HOD hypothesis the classes of $C^{(n)}$-supercompact/$C^{(n)}$-extendible cardinals above the first extendible transfer down to HOD. More precisely, we show the following:

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[2]In Goldberg's paper [Gol23], it is claimed that Woodin proved that under the HOD hypothesis, every extendible cardinal must be extendible in HOD. This is the negation of the theorem stated above. In fact, Woodin never claimed to prove such a thing, and this was Goldberg's mistake alone.

Theorem 3.11. *Suppose δ is the first extendible and that the HOD hypothesis holds. If $\kappa > \delta$ is $C^{(n)}$-supercompact (resp. $C^{(n)}$-extendible) then κ is $C^{(n)}$-supercompact (resp. $C^{(n)}$-extendible) in HOD.*

We conclude this first part of the manuscript discussing a related problem. Inspired by previous work of Magidor and Väänänen [MV11] and Galeotti, Khomskii and Väänänen [GKV20], Gitman and Osinski [GO24] introduced cardinal-correct extendible cardinals: A cardinal δ is *cardinal-correct extendible* if for each ordinal $\alpha > \delta$ there is an elementary embedding $j \colon V_\alpha \to M$ such that $\mathrm{crit}(j) = \delta$, $j(\delta) > \alpha$ and M is *cardinal-correct* (i.e., $\mathrm{Card}^M = \mathrm{Card} \cap M$). It will be evident from the definition of extendibility (see page 6) that every extendible is cardinal-correct extendible. It is implicitly asked in work by Gitman and Osinski [GO24, §9] whether the opposite implication is true as well. We show that the answer is negative.

Theorem 3.14. *It is consistent with the HOD hypothesis that the first cardinal-correct extendible is the first strongly compact cardinal.*

This is a consequence of Theorem 3.1 and the following general result:

Theorem 3.13. *Suppose that HOD is cardinal-correct. Then every extendible cardinal is cardinal-correct extendible in HOD.*

In the second part of the paper we analyze the optimality of the HOD dichotomy for strong compact cardinals proved by the first author in [Gol23]. Namely, if δ is strong compact and the HOD hypothesis holds, every singular cardinal $\lambda > \delta$ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$ (i.e., *weak covering holds* at λ). A natural question is whether the assumption "δ is strong compact + HOD hypothesis" is consistent with many singular cardinals $\lambda < \delta$ violating weak covering. Building upon previous work of Cummings, Friedman and Golshani [CFG15] we improve their main result showing that:

Theorem 4.6. *The following configuration is consistent with the HOD hypothesis: The first supercompact δ carries a club $C \subseteq \delta$ of consisting of HOD-regular cardinals κ such that $\kappa^{+\mathrm{HOD}} < \kappa^+$.*

Answering a question in [CFG15], we show that this is arguably optimal:

Theorem 4.4. *If δ is supercompact and the HOD hypothesis holds then there are unboundedly many singular cardinals $\kappa < \delta$ that are singular in HOD and $\kappa^{+\mathrm{HOD}} = \kappa^+$.*

We conclude the paper with a brief discussion on the HOD dichotomy and ω-strongly measurability in HOD. In [Gol23], Goldberg showed that if the HOD hypothesis fails and there is a strongly compact cardinal δ, then all sufficiently large regular cardinals are ω-strongly measurable in HOD. Let η_0 be the least ordinal greater than or equal to δ such that all regular cardinals $\theta \geq \eta_0$ are ω-strongly measurable in HOD. Is it possible to bound η_0? We show that the answer is no. This shows that the refinement of Woodin's HOD dichotomy proved in [Gol23] cannot be significantly improved:

Theorem 4.9. *Suppose δ is supercompact and the* HOD *hypothesis fails. Then for any ordinal α, there is a forcing extension $V[G]$ in which δ is supercompact, the* HOD *hypothesis fails, and $\eta_0^{V[G]} \geq \alpha$.*

The manuscript is fairly self-contained and the notations are standard in set theory. In §2, we provide all the pertinent preliminaries. In §§3.1, 3.2 and 3.3 we prove Theorems 3.1 through 3.13 above. The first part of Section 4 is devoted to proving Theorems 4.6 and 4.4, while in the second part we will prove Theorem 4.9. We also collect a few open problems through the paper.

2. Preliminaries and notations

Recall that a set X is *ordinal definable* (in symbols, $X \in$ OD) if there is a formula $\varphi(x, x_0, \ldots, x_n)$ of the language of set theory which, together with ordinals $\alpha_0, \ldots, \alpha_n$, defines X; namely, $X = \{x : \varphi(x, \alpha_0, \ldots, \alpha_n)\}$. Additionally, X is *Hereditarily Ordinal Definable* (in symbols, $X \in$ HOD) if $X \in$ OD and the transitive closure of $\{X\}^3$ consists of OD-sets only. A cardinal κ is called ω-*strongly measurable in* HOD if for some η with $(2^\eta)^{\text{HOD}} < \kappa$, there is no partition in HOD of $S_\omega^\kappa := \{\xi < \kappa : \text{cf}(\xi) = \omega\}$ into η-many stationary sets. An inner model M of ZFC is a *weak extender model* for the supercompactness of a cardinal κ if for all $X \in M$, there is a normal fine κ-complete ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(X) \cap M$ with $\mathcal{U} \cap M \in M$. In particular, in this situation κ is a supercompact cardinal in both M and V. Woodin has showed that if δ is extendible and the HOD hypothesis holds then HOD is a weak extender model for the supercompactness of δ [Woo10].

2.1. **Some large cardinal notions.** Here we collect a few relevant large-cardinal notions. For further information we refer to Kanamori's [Kan09].

Definition 2.1 ([Bag12])**.** For each $n < \omega$, denote by $C^{(n)}$ the class of all ordinals θ that are Σ_n-*correct* (in symbols, $V_\theta \prec_{\Sigma_n} V$); namely, the ordinals for which given $\varphi(\bar{x})$ a Σ_n formula in the language of set theory and $\bar{a} \in V_\theta$,

$$V_\theta \models \varphi(\bar{a}) \iff V \models \varphi(\bar{a}).$$

$C^{(0)}$ and $C^{(1)}$ are, respectively, the collections of all ordinals and $\beth$-fixed points. For each $n < \omega$, $C^{(n)}$ is a Π_n definable club class. For additional information on these classes we refer the reader to [Bag12, §2].

Two large-cardinal notions in which we will be interested are:

Definition 2.2 (Bagaria [Bag12])**.** A cardinal δ is called λ-$C^{(n)}$-*supercompact* for $\lambda > \delta$ if there is an elementary embedding $j \colon V \to M$ such that $\text{crit}(j) = \delta$, $j(\delta) > \lambda$, M is a transitive class with $M^\lambda \subseteq M$ and $j(\delta) \in C^{(n)}$.
δ is $C^{(n)}$-*supercompact* if it is λ-$C^{(n)}$-supercompact for all $\lambda > \delta$.

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[3]Recall that the transitive closure of $\{X\}$ is the smallest transitive set containing $\{X\}$.

Definition 2.3 (Bagaria [Bag12])**.** A cardinal δ is called λ-$C^{(n)}$-*extendible* for $\lambda > \delta$ if there is $\theta \in \mathrm{Ord}$ and an elementary embedding $j \colon V_\lambda \to V_\theta$ such that $\mathrm{crit}(j) = \delta$, $j(\delta) > \lambda$ and $j(\delta) \in C^{(n)}$.

δ is $C^{(n)}$-*extendible* if it is λ-$C^{(n)}$-extendible for all $\lambda > \delta$.

A cardinal δ is supercompact if it witnesses the property described in Definition 2.2 except for $j(\delta) \in C^{(1)}$ (cf. [Kan09]). Likewise, δ is extendible if and only if it is $C^{(1)}$-extendible. $C^{(n)}$-extendibility yields a strong hierarchy in the sense that the first $C^{(n+1)}$-extendible is a limit of $C^{(n)}$-extendibles. In contrast, this does not apply to the class of $C^{(n)}$-supercompact cardinals (see [HMP22, Pov24] or Corollary 3.6 below).

The next theorems characterize both $C^{(n)}$-supercompactness and $C^{(n)}$-extendibility in terms of "tallness" [Ham09]:

Theorem 2.4 (Poveda, [Pov24, §3])**.** *The following are equivalent:*

- *(1) δ is $C^{(n)}$-supercompact;*
- *(2) δ is supercompact and for each regular $\lambda > \delta$ there is $i \colon V \to N$ such that $\mathrm{crit}(i) = \delta$, $N^{<\delta} \subseteq N$, $\mathrm{cf}(i(\delta)) > \lambda$, and $i(\delta) \in C^{(n)}$.*

Theorem 2.5 (Bagaria and Goldberg, [BG24])**.** *A cardinal δ is extendible if and only if for arbitrary large $\lambda > \delta$, there is a normal fine δ-complete ultrafilter on $T_{\delta,\lambda} := \{x \in \mathcal{P}_\delta(\lambda) : \mathrm{otp}(x) \text{ is } \Sigma_2\text{-correct}\}$.*

It is not hard to show that if δ is extendible and $\lambda > \delta$ is an *indestructibly Σ_2-correct cardinal* (i.e., $\mathbb{1} \Vdash_{\mathbb{P}}$ "$\lambda \in C^{(2)}$" for all λ-directed-closed poset $\mathbb{P}$) then there is a normal ultrafilter on the following set

$$T^*_{\delta,\lambda} := \{x \in \mathcal{P}_\delta(\lambda) : \mathrm{otp}(x) \text{ is indestructibly } \Sigma_2\text{-correct}\}.$$

An interesting question in regards to the HOD dichotomy is whether a large cardinal hypothesis weaker than extendibility can be employed to prove that HOD is a weak extender model for the supercompactness of some cardinal. A solution to this problem was provided by Woodin:

Definition 2.6 (Woodin [Woo10, §7.1])**.** A cardinal δ is HOD-*supercompact* if for all cardinals λ, there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \delta$, $M^\lambda \subseteq M$, $j(\delta) > \lambda$, and $\mathrm{HOD}^M \cap V_\lambda = \mathrm{HOD} \cap V_\lambda$.

Theorem 2.7 ([Woo10, Theorem 193])**.** *Under the* HOD *hypothesis, if a cardinal δ is* HOD-*supercompact, then* HOD *is a weak extender model for the supercompactness of δ (in particular, δ is supercompact in* HOD*).*

Every extendible cardinal is HOD-supercompact [Woo10, §7], but the consistency strength of HOD-supercompactness is the same as supercompactness[4]. However how do these two notions relate implication-wise? For instance, a priori it is not far-fetched that every extendible cardinal is a limit of HOD-supercompacts. We will answer this question in the negative in the forthcoming Corollary 3.6.

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[4]One can force $V = \mathrm{HOD}$ while preserving all supercompacts, and if $V = \mathrm{HOD}$, then every supercompact cardinal is trivially HOD-supercompact.

2.2. **The Universality Theorems.** For future convenience, in this section we formulate the Universality Theorems for inner models N containing the ordinals and having the so-called *covering* and *approximation* properties.

Definition 2.8 (Hamkins, [Ham03a]). Suppose that $\delta > \omega$ is a regular cardinal and that N is a transitive inner model of ZFC containing the ordinals.

(1) N has the δ-*cover property* if for all $\sigma \subseteq N$ with $|\sigma| < \delta$, there exists $\tau \in N$ such that $\sigma \subseteq \tau$ and $|\tau| < \delta$.
(2) N has the δ-*approximation property* if for all $X \subseteq N$, $X \in N$ if and only if $X \cap \tau \in N$ for all $\tau \in N$ with $|\tau| < \delta$.

The next theorem due to Hamkins shows that if N has the δ-cover and δ-approximation properties then N is easily definable. Thus, the theory of such inner models is part of the first order theory of V.

Theorem 2.9 (Definability Theorem, Hamkins [FHR15]). *Suppose δ is a regular uncountable cardinal and that N is an inner model of ZFC containing all the ordinals and having the δ-covering and δ-approximation properties. Then N is Σ_2 definable with $N \cap H(\delta^+)$ as a parameter.*

Hamkins and Woodin have showed that inner models N with the approximation and cover properties inherit most of the large cardinal structure of V. A precise formulation of this fact is provided by the *Universality Theorems*:

Theorem 2.10 (Universality Theorem I, Woodin). *Suppose δ is a regular uncountable cardinal and that N is an inner model of ZFC containing all the ordinals and having the δ-covering and δ-approximation properties. Suppose that E is an N-extender of length η with critical point $\kappa_E \geq \delta$. Let*

$$j_E \colon N \to N_E$$

be the ultrapower embedding. Then the following are equivalent:

(1) *For each $A \in N \cap \mathcal{P}(\eta^{<\omega})$, $j_E(A) \cap \eta^{<\omega} \in N$.*
(2) *$E \in N$.*

Theorem 2.11 (Universality Theorem II, Hamkins [Ham03b]). *Suppose δ is a regular uncountable cardinal and that N is an inner model of ZFC containing all the ordinals and having the δ-covering and δ-approximation properties. Suppose $\gamma > \delta$ is a strong limit cardinal with $\mathrm{cf}(\gamma) \geq \delta$ and that*

$$j \colon V \to M$$

is an elementary embedding with $\mathrm{crit}(j) > \delta$ and such that $H(\gamma) \subseteq M$. Letting E the N-extender of length γ given by j and

$$j_E \colon N \to N_E$$

the corresponding ultrapower, then

(1) *$N_E \cap H(\gamma) = N \cap H(\gamma)$.*
(2) *$E \in N$.*

Under appropriate assumptions the above theorems apply to HOD:

Theorem 2.12 (Woodin, [Woo10])**.** *Assume the* HOD *hypothesis holds and that δ is* HOD-*supercompact. Then,*

    (1) HOD *is a weak extender model for the supercompactness of δ.*
    (2) HOD *has the δ-covering and δ-approximation properties.*

2.3. **Some forcing posets.** For regular cardinals $\gamma < \mu$ we denote

$$S_\gamma^\mu := \{\theta < \mu : \operatorname{cf}(\theta) = \gamma\}.$$

A stationary set $S \subseteq S_\gamma^\mu$ is called non-reflecting if $S \cap \alpha$ is non-stationary in α for all $\alpha < \mu$. There is a natural forcing order to add such an object.

Definition 2.13. Conditions in $\mathbb{NR}(S_\gamma^\mu)$ are functions $p\colon \alpha \to \{0,1\}$ for some $\alpha < \mu$ and letting $S_p = \{\xi < \alpha : p(\xi) = 1\}$, $S_p \subseteq S_\gamma^\mu$ and for every limit $\beta \le \alpha$ there is a closed unbounded set $c \subseteq \beta$ such that $S_p \cap c = \emptyset$.

Conditions in $\mathbb{NR}(S_\gamma^\mu)$ are ordered by reversed inclusion.

Fact 2.14 (Folklore, see [Cum10, §6])**.** *Assume $\mu = \mu^{<\mu}$. Then,*

    (1) If $G \subseteq \mathbb{NR}(S_\gamma^\mu)$ is generic over V then

$$S_G := \{\xi < \mu : \exists p \in G,\, p(\xi) = 1\}$$

    is a non-reflecting stationary such that $S_G \subseteq S_\gamma^\mu$.
    (2) $\mathbb{NR}(S_\gamma^\mu)$ is a μ^+-cc partial order.
    (3) $\mathbb{NR}(S_\gamma^\mu)$ is a μ-strategically-closed partial order.

Suppose that $S \subseteq \mu^+$ is a stationary set. There is a standard forcing, due to Abraham and Shelah, to *shoot* a club through S:

Definition 2.15 (Abraham–Shelah, [AS83])**.** Conditions in $\mathbb{C}(S)$ consists of closed bounded sets $c \subseteq S$. The order relation of $\mathbb{C}(S)$ is $\sqsupseteq$-extension.

Fact 2.16 ([AS83])**.** *Suppose that $\mu^{<\mu} = \mu$ and $2^\mu = \mu^+$.*

    (1) $\mathbb{C}(S)$ forces a club $C \subseteq S$.
    (2) $\mathbb{C}(S)$ is μ^{++}-cc.

The poset $\mathbb{C}(S)$ is in general ill-behaved unless $\mu = \aleph_0$ or S is chosen carefully; for example, $\mathbb{C}(S)$ may collapse μ^+. Abraham and Shelah show in [AS83, Theorem 1] that if $S \subseteq \mu^+$ is *fat* then $\mathbb{C}(S)$ is μ^+-distributive, hence it preserves cardinals $\ge\mu^+$.[5] In our particular circumstances we will be able to ignore this technical aspect thanks to the following fact:

Fact 2.17 (Folklore)**.** *Suppose that μ is a cardinal, $\gamma \le \mu$ is regular and that $\dot{S}$ is the canonical $\mathbb{NR}(S_\gamma^{\mu^+})$-name for the generic non-reflecting stationary set. Then,* $\operatorname{Add}(\mu^+, 1)$ *is equivalent to the two step iteration*

$$\mathbb{NR}(S_\gamma^{\mu^+}) * \dot{\mathbb{CS}}(\mu^+ \setminus \dot{S}).$$

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[5]A stationary set $S \subseteq \mu^+$ is called fat if there is a club $C \subseteq \mu^+$ such that $C \cap S$ contains closed sets of arbitrarily large order types below κ.

Given a set $X \subseteq \alpha$ and a cardinal κ, there is a standard forcing that *codes* X within the power-set-function-pattern in the interval $[\kappa^+, \kappa^{+\alpha+1}]$. This poset is originally due to McAloon [McA71] and is defined as follows.

Let $\mathcal{E}_X \colon [\kappa^+, \kappa^{+\alpha}] \to [\kappa^+, \kappa^{+\alpha}]$ be the partial function defined as

$$\mathcal{E}_X(\kappa^{+\beta+1}) := \begin{cases} \kappa^{+\beta+2}, & \text{if } \beta \in X; \\ \kappa^{+\beta+3}, & \text{if } \beta \notin X. \end{cases}$$

Definition 2.18 (Coding)**.** $\mathrm{Code}(X, \kappa)$ is the Easton-product

$$\prod_{\beta < \alpha} \mathrm{Add}(\kappa^{+\beta+1}, \mathcal{E}_X(\kappa^{+\beta+1})).$$

Clearly, $\mathrm{Code}(X, \kappa)$ is a κ^+-directed-closed forcing. Also, if the GCH holds in $[\kappa^+, \kappa^{+\alpha}]$, $\mathrm{Code}(X, \kappa)$ codes X: For a V-generic filter $G \subseteq \mathrm{Code}(X, \kappa)$,

$$X = \{\beta < \alpha : V[G] \models 2^{\kappa^{+\beta+1}} = \kappa^{+\beta+2}\}.$$

Thus, $X \in \mathrm{HOD}^{V[G]}$. A classical argument due to McAloon [McA71] shows that "$V = \mathrm{HOD}$" can be forced using a class iteration of the coding posets $\mathrm{Code}(X, \kappa)$ (see [Rei07]). In fact, this poset forces the stronger axiom denoted "$V = g\mathrm{HOD}$", where $g\mathrm{HOD}$ stands for the *generic* HOD; namely,

$$g\mathrm{HOD} := \{x : \text{For all set-sized posets } \mathbb{P}, \ \mathbf{1} \Vdash_{\mathbb{P}} \check{x} \in \mathrm{HOD}\}.$$

We note that this coding forcing can be arranged to start above any given regular cardinal χ, thereby yielding a χ-directed-closed class product forcing.

A poset $\mathbb{Q}$ is said to be *cone homogeneous* if for each pair of conditions $q_0, q_1 \in \mathbb{Q}$ there are $q_0^* \leq q_0$ and $q_1^* \leq q_1$ such that $\mathbb{Q}/q_0^*$ is isomorphic to $\mathbb{Q}/q_1^*$. The next is the key standard fact about cone homogeneous posets.

Fact 2.19 ([Jec03, Theorem 26.12])**.** *If $\mathbb{Q}$ is an ordinal definable and cone-homogeneous forcing then* $\mathrm{HOD}^{V[G]} \subseteq V$ *for all V-generics $G \subseteq \mathbb{Q}$.* $\square$

All the posets mentioned so far (except for McAloon class product forcing) are cone-homogeneous. The above fact will be used both in §3 and §4.1 to control the HOD of the relevant generic extensions.

3. The HOD Dichotomy and extendible-like cardinals

This section consists of three subsections: §3.1, §3.2 and §3.3. In §3.1 we prove Theorem 3.1 which says that (even under the HOD hypothesis) the first extendible cardinal can be the first strongly compact cardinal in HOD. In §3.2 we complement this result by showing that, again under the HOD hypothesis, the first extendible cardinal must be $C^{(1)}$-supercompact in HOD, therefore a "big" supercompact cardinal in HOD. Finally, in §3.3 we discuss *cardinal-correct extendible* cardinals and show that in HOD of the model of §3.1 the first cardinal-correct extendible is not extendible in HOD.

3.1. **The first extendible can be the first strongly compact of** HOD**.**
The basic idea is to prove this by forcing with an iteration that shoots non-reflecting stationary sets and immediately after kills them. Our arguments combine ideas from [Gol24] and Cheng–Friedman–Hamkins [CFH15]. We will also employ the characterization of extendibility via *reflecting measures* due to Bagaria and Goldberg. (See Theorem 2.5 above.)

Theorem 3.1. *Let δ be an extendible cardinal. Then there is a generic extension of the universe of sets where the* HOD *hypothesis holds, δ remains extendible, and δ is the least strongly compact cardinal of* HOD.

Proof. By virtue of [Gol24, Theorem 3.8 and Proposition 3.9], we may start with a model V where the following properties hold:

   (1) δ is an extendible cardinal.
   (2) Σ_2-correct cardinals are Σ_2-indestructibly correct.

Claim 3.1.1. *For each ordinal $\alpha < \delta$, there is an interval of cardinals $I \subseteq \delta$ with $\mathrm{otp}(I) = \alpha$ such that $2^\theta = \theta^+$ for all $\theta \in I$. Also $V = \mathrm{gHOD}$ holds.*

Proof of claim. Fix an ordinal $\alpha < \delta$ and let $\lambda > \delta$ be a Σ_2-indestructibly correct cardinal. We may force, using λ-directed-closed forcing, the GCH within an interval of cardinals of order-type α above λ. In particular, we can force this configuration while keeping λ a Σ_2-correct cardinal. By Σ_2-correctness of λ in the generic extension there must be an interval of cardinals below λ of order-type α where the GCH holds. But this poset does not change V_λ, so the said interval of cardinals existed back in the ground model V. Finally, since δ is extendible (hence Σ_3-correct) in V and $\alpha < \delta$, an interval of cardinals I as in the claim must exist below δ.

The argument for $V = \mathrm{gHOD}$ is similar. $\square$

Notice that the reflection argument presented above allows for the selection of an interval $I \subseteq \delta$ with a minimal element greater than any fixed $\beta < \delta$.

We will make δ become the first strongly compact in $\mathrm{HOD}^{V[G]}$ by adding many non-reflecting stationary sets below δ. Since we wish to keep δ extendible in $V[G]$, we plan to kill these stationary sets right after they are introduced employing the club shooting poset. By Fact 2.17, each of these two step iterations will be equivalent to Cohen forcing, and as a result they will be nicely behaved. The key point is that the non-reflecting stationaries will be coded into $\mathrm{HOD}^{V[G]}$ while their club counterparts will not.

Let us start by fixing an *extendible Laver function* $f : \delta \to \delta$. Specifically, f has the following guessing property: For each Σ_2-indestructibly correct cardinal $\lambda > \delta$ there is a normal fine δ-complete ultrafilter $\mathcal{U}$ on $T_{\delta,\lambda}$ such that $j_{\mathcal{U}}(f)(\delta) = \lambda$. Such a function exist as per the argument in [Gol24, Theorem 3.10, p.10]. Tsaprounis previously observed in [Tsa18] that extendible-like Laver functions, similar to those considered here, exist.

Let $\mathbb{P}$ be the Easton-supported product $\prod_{\alpha<\delta} \mathbb{Q}_\alpha$ where $\mathbb{Q}_\alpha$ is defined recursively follows: Suppose that $\langle \mathbb{Q}_\beta : \beta < \alpha \rangle$ has been defined and that

$|\mathbb{Q}_\beta| < \delta$ for all $\beta < \alpha$. If α is an inaccessible cardinal with $f[\alpha] \subseteq \alpha$, then we take λ_α the least cardinal above $\max(f(\alpha), \sup_{\beta<\alpha} \lambda_\beta)$ and stipulate that

$$\mathbb{Q}_\alpha := \mathbb{NR}(S_\omega^{\lambda_\alpha^+}) * \dot{\mathbb{CS}}(\lambda_\alpha^+ \setminus \dot{S}_\alpha) * \dot{\text{Code}}(\dot{S}_\alpha, \min(I_\alpha)),$$

where $I_\alpha \subseteq \delta$ is an interval of cardinals above λ_α^+ with order-type λ_α^+ such that $2^\theta = \theta^+$ for all $\theta \in I^+$. (This choice of I_α guarantees that forcing with $\dot{\text{Code}}(\dot{S}_\alpha, \min(I_\alpha))$ codes $\dot{S}_\alpha$ into the power-set function above $\min(I_\alpha)$.)

In any other circumstances $\mathbb{Q}_\alpha$ is declared to be the trivial and $\lambda_\alpha := 0$.

Note that $\{\mathbb{Q}_\alpha : \alpha < \delta\} \subseteq V_\delta$. Since we are taking Easton supports this implies that $\mathbb{P} \subseteq V_\delta$.

To streamline the forthcoming discussion we shall denote:

- $\mathbb{NR}_\alpha := \mathbb{NR}(S_\omega^{\lambda_\alpha^+})$;
- $\dot{\mathbb{CS}}_\alpha := \mathbb{CS}(\lambda_\alpha^+ \setminus \dot{S}_\alpha)$;
- $\dot{\text{Code}}_\alpha := \dot{\text{Code}}(\dot{S}_\alpha, \min(I_\alpha))$.

In what follows $G \subseteq \mathbb{P}$ is a fixed V-generic filter.

Claim 3.1.2. *δ remains extendible in $V[G]$.*

Proof. We use Theorem 2.5—that is, the Bagaria–Goldberg ultrafilter-based characterization of extendibility. Let $\lambda > \delta$ be a Σ_2-indestructibly correct cardinal such that $\text{cf}(\lambda) > \delta$. (Note that such a cardinal exist by our departing assumption.). By Solovay's theorem the SCH holds at λ, hence $\lambda^{<\delta} = \lambda$. Let $\mathcal{U}$ be a normal fine δ-complete ultrafilter on $T_{\delta,\lambda}^*$ such that $j_{\mathcal{U}}(f)(\delta) = \lambda$ where $j_{\mathcal{U}} \colon V \to M_{\mathcal{U}}$ denotes the corresponding ultrapower embedding.

Thus one obtains the following factoring:

$$j_{\mathcal{U}}(\mathbb{P}) \simeq \mathbb{P} \times \mathbb{Q} = \mathbb{P} \times (\textstyle\prod_{\delta \leq \alpha < j_{\mathcal{U}}(\delta)} \mathbb{Q}_\alpha)^{M_{\mathcal{U}}}.$$

Let us take a closer look at the tail forcing $\mathbb{Q}$. Since δ is inaccessible and a closure point of $j_{\mathcal{U}}(f)$ (i.e., $j_{\mathcal{U}}(f)[\delta] = f[\delta] \subseteq \delta$) the definition of $j_{\mathcal{U}}(\mathbb{P})$ in $M_{\mathcal{U}}$ guarantees that $\mathbb{Q}_\delta$ is a non-trivial partial order. Thus,

$$(\mathbb{Q}_\delta)^{M_{\mathcal{U}}} = (\mathbb{NR}_\delta * \dot{\mathbb{CS}}_\delta * \dot{\text{Code}}_\delta)^{M_{\mathcal{U}}}.$$

By our choice, $(\lambda_\delta)^{M_{\mathcal{U}}} > j_{\mathcal{U}}(f)(\delta) = \lambda$, so that $(\mathbb{NR}_\delta * \dot{\mathbb{CS}}_\delta)^{M_{\mathcal{U}}}$ contains in $M_{\mathcal{U}}$ a dense $(\lambda^+)^{M_{\mathcal{U}}}$-directed-closed forcing, $\mathbb{D}_\delta$ (see Fact 2.17). Since $M_{\mathcal{U}}$ is closed under λ-sequences it follows that $\mathbb{D}_\delta$ is λ^+-directed-closed in V.

The same argument shows that the tail forcing $\mathbb{Q}$ has a dense subset of conditions that is λ^+-directed-closed in V. The reason is that $(\lambda_\alpha)^{M_{\mathcal{U}}} > (\lambda_\delta)^{M_{\mathcal{U}}} > j_{\mathcal{U}}(f)(\delta)^+ = \lambda^+$ for all $\alpha \in (\delta, j_{\mathcal{U}}(\delta))$ where $(\mathbb{Q}_\alpha)^{M_{\mathcal{U}}}$ is non-trivial.

Notice that any V-generic for

$$\mathbb{C} := \textstyle\prod_{\delta \leq \alpha < j_{\mathcal{U}}(\delta)} \mathbb{D}_\alpha * (\dot{\text{Code}}_\alpha)^{M_{\mathcal{U}}}$$

induces (by density) a V-generic filter for $\mathbb{Q}$. Since $\mathbb{C}$ is a λ^+-directed-closed forcing in V and there are at most λ^+-many dense open sets for it in $M_{\mathcal{U}}$[6]

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[6]Here we use that $\lambda^{<\delta} = \lambda$, $2^\lambda = \lambda^+$ and the usual extender representation of dense open subsets $D \subseteq \mathbb{Q}$ in $M_{\mathcal{U}}$ (i.e., $D = j(f)(j``\lambda)$ for a function $f \colon \mathcal{P}_\delta(\lambda) \to \mathcal{P}(\mathbb{P})$).

we can construct (inside V) a V-generic filter for $\mathbb{C}$, hence also a V-generic filter $H \subseteq \mathbb{Q}$. Moreover, since $|\mathbb{P}| < \lambda$, a mutual genericity argument shows that any $M_{\mathcal{U}}$-generic for $\mathbb{Q}$ is in fact $M_{\mathcal{U}}[G]$-generic. Thereby we obtain a $M_{\mathcal{U}}[G]$-generic filter $H \in V[G]$ for the poset $\mathbb{Q}$.

At this stage we are almost done with the preservation of extendibility. Thanks to the Easton support $j_{\mathcal{U}}{}^{\prime\prime}G \subseteq G \times H$, hence $j_{\mathcal{U}} \colon V \to M_{\mathcal{U}}$ lifts to

$$j_{\mathcal{U}} \colon V[G] \to M_{\mathcal{U}}[G \times H] = M_{\mathcal{U}}[H \times G].$$

Since λ was Σ_2-indestructibly correct in $M_{\mathcal{U}}$ it remains Σ_2-correct in $M_{\mathcal{U}}[H]$. Also, G is a generic for a small forcing relative to λ, so the Σ_2-correctness of the cardinal is preserved in $M_{\mathcal{U}}[H \times G]$. To conclude define

$$\mathcal{W} := \{x \in \mathcal{P}_\delta(\lambda)^{V[G]} : j{}^{\prime\prime}\lambda \in j_{\mathcal{U}}(X)\}$$

which is easily shown to be a 2-reflecting measure on $\mathcal{P}_\delta(\lambda)$.

This completes the verification that δ is extendible in $V[G]$. $\qquad\square$

Let us now argue that δ is the first strong compact cardinal in $\mathrm{HOD}^{V[G]}$.

Denote $\pi \colon \mathbb{P} \to \prod_{\alpha < \delta} \mathbb{NR}_\alpha$ and $\pi_\alpha \colon \prod_{\alpha < \delta} \mathbb{NR}_\alpha \to \mathbb{NR}_\alpha$ the natural projections. Let G_0 be the V-generic induced by G and π, and $G_{0,\alpha}$ the V-generic induced by π_α and G_0, respectively. The next claim is key:

Claim 3.1.3. $\mathrm{HOD}^{V[G]} = V[G_0]$.

Proof of claim. To prove the more difficult inclusion, that $\mathrm{HOD}^{V[G]} \subseteq V[G_0]$, it suffices to show that in $V[G_0]$, $\mathbb{P}/G_0$ is cone homogeneous and definable from G_0 and parameters in V. It is tempting to say that the quotient forcing $\mathbb{P}/G_0$ is the Easton product $\prod_{\alpha < \delta}(\mathbb{CS}_\alpha * \dot{\mathrm{Code}}_\alpha)_{G_{0,\alpha}}$ as computed in $V[G_0]$, but we do not know whether this is true. Instead, we show the following:

Lemma 3.2. *Suppose $\mathbb{P}$ is the Easton product $\prod_{\alpha<\delta} \mathbb{A}_\alpha * \dot{\mathbb{B}}_\alpha$ and for each $\alpha < \delta$, $\mathbb{A}_\alpha$ forces that $\dot{\mathbb{B}}_\alpha$ is cone homogeneous. Then if $H \subseteq \prod_{\alpha<\delta} \mathbb{A}_\alpha$ is V-generic, $\mathbb{P}/H$ is cone homogeneous.*

Proof. Let $H_\alpha \subseteq \mathbb{A}_\alpha$ be the generic filter induced by H. Let $\mathbb{B}_\alpha = (\dot{\mathbb{B}}_\alpha)_{H_\alpha}$. Working in $V[H]$, say a condition $(b_\alpha)_{\alpha<\delta} \in \prod_{\alpha<\delta} \mathbb{B}_\alpha$ is *relevant* if there is an Easton-supported sequence $(\dot{b}_\alpha)_{\alpha<\delta} \in (\prod_{\alpha<\delta} \dot{\mathbb{B}}_\alpha)^V$ such that $b_\alpha = (\dot{b}_\alpha)_{H_\alpha}$ for each $\alpha < \delta$. An easy calculation shows that the separative quotient of $\mathbb{P}/H$ is isomorphic to the suborder $\mathbb{B} \subseteq \prod_{\alpha<\delta} \mathbb{B}_\alpha$ consisting of relevant conditions.[7] Given a condition $b \in \mathbb{B}$ we denote by $\mathbb{B}_b$ the subposet of $\mathbb{B}$ whose conditions are all stronger than b.

Suppose $(b_\alpha)_{\alpha<\delta}$ and $(c_\alpha)_{\alpha<\delta}$ are relevant conditions, and we will define, in $V[H]$, relevant conditions $(b_\alpha^*)_{\alpha<\delta} \leq (b_\alpha)_{\alpha<\delta}$ and $(c_\alpha^*)_{\alpha<\delta} \leq (c_\alpha)_{\alpha<\delta}$ and an isomorphism $\pi : \mathbb{B}_{b^*} \to \mathbb{B}_{c^*}$. Let $(\dot{b}_\alpha)_{\alpha<\delta}$ and $(\dot{c}_\alpha)_{\alpha<\delta}$ witness the relevance of $(b_\alpha)_{\alpha<\delta}$ and $(c_\alpha)_{\alpha<\delta}$. Working in V, for each $\alpha < \delta$ in the union of the

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[7]Recall here that $\mathbb{P}/H$ is equal to the set of conditions $(p_\alpha, \dot{q}_\alpha)_{\alpha<\delta} \in \mathbb{P}$ such that $(p_\alpha)_{\alpha<\delta} \in H$. The forcing equivalence of $\mathbb{P}/H$ and $\mathbb{B}$ is witnessed by the map $f : \mathbb{P}/H \to \mathbb{B}$ defined by $f((p_\alpha, \dot{q}_\alpha)_{\alpha<\delta}) = ((\dot{q}_\alpha)_{G_\alpha})_{\alpha<\delta}$, and moreover, f is the separative quotient map.

supports of $(\dot{b}_\alpha)_{\alpha<\delta}$ and $(\dot{c}_\alpha)_{\alpha<\delta}$, fix $\mathbb{A}_\alpha$-names $\dot{b}_\alpha^*$, $\dot{c}_\alpha^*$, and $\dot{\pi}_\alpha$ such that $\mathbb{A}_\alpha$ forces that $\dot{b}_\alpha^* \leq \dot{b}_\alpha$, $\dot{c}_\alpha^* \leq \dot{c}_\alpha$, and $\dot{\pi}_\alpha^* \colon (\dot{\mathbb{B}}_\alpha)_{\dot{b}_\alpha^*} \to (\dot{\mathbb{B}}_\alpha)_{\dot{c}_\alpha^*}$ is an isomorphism.

Let $\pi_\alpha = (\dot{\pi}_\alpha)_{H_\alpha}$, $b_\alpha^* = (\dot{b}_\alpha^*)_{H_\alpha}$, and $c_\alpha^* = (\dot{c}_\alpha^*)_{H_\alpha}$. Note that b^* and c^* are relevant conditions. Define $\pi \colon \mathbb{B}_{b^*} \to \mathbb{B}_{c^*}$ by $\pi((u_\alpha)_{\alpha<\delta}) = (\pi_\alpha(u_\alpha))_{\alpha<\delta}$. Then π is the desired isomorphism. $\qquad\square$

Clearly, $(\mathbb{CS}_\alpha * \dot{\mathrm{Code}}_\alpha)_{G_{0,\alpha}}$ is a cone homogeneous forcing in $V[G_{0,\alpha}]$, since in $V[G_{0,\alpha}]$, finite iterations of $\mathrm{OD}_{G_{0,\alpha}}$ cone homogeneous forcings are cone homogeneous; this is a relativized version of [BNH23, Lemma 8.3], which in turn follows from results of [DF08]. Therefore we can apply the lemma above to obtain that the quotient $\mathbb{P}/G_0$ is cone homogeneous. It follows that

$$\mathrm{HOD}^{V[G]} \subseteq V[G_0].$$

Furthermore, we claim that $G_0 \in \mathrm{HOD}^{V[G]}$ and so

$$\mathrm{HOD}^{V[G]} = V[G_0].$$

This is because G was constructed so as to code all the non-reflecting stationary sets into the continuum function. Specifically, G_0 is ordinal definable in $V[G]$ taking $\prod_{\alpha<\delta} \mathbb{NR}_\alpha$ and $\langle \lambda_\alpha : \alpha < \delta \rangle$ as parameters. Since $V \models$ "$V = g\mathrm{HOD}$" (in fact, V satisfies Reitz's *Continuum Coding Axiom* [Rei07], which implies $V = g\mathrm{HOD}$) both of these parameters belong to $\mathrm{HOD}^{V[G]}$ which yields $G_0 \in \mathrm{HOD}^{V[G]}$. $\qquad\square$

The HOD hypothesis holds in $V[G]$ because $\mathrm{HOD}^{V[G]}$ is a generic extension of HOD^V by a set-size forcing. As a result, $\mathrm{HOD}^{V[G]}$ is a weak extender model for the supercompactness of δ and therefore δ is supercompact in this model. Finally, δ must be the first strongly compact cardinal in $\mathrm{HOD}^{V[G]}$ precisely because $\mathrm{HOD}^{V[G]} = V[G_0]$. $\qquad\square$

Remark 3.3. Employing standard techniques one can also arrange the *mirror image configuration* of Theorem 3.1. Namely, one can prove the consistency (with the HOD hypothesis) of the first strongly compact cardinal being extendible in HOD. The idea is simple: Start with a model for $V = g\mathrm{HOD}$ accomodating an extendible cardinal δ (see [BP23]). Force with a *Magidor product* of Prikry forcings as in [HMP22]. This forcing preserves the strong compactness of δ and it is not hard to check that it is cone homogeneous. As a result, in the generic extension $V[G]$, δ becomes the first strongly compact while remains extendible (non-necessarily the first one!) in $\mathrm{HOD}^{V[G]}$.

Remark 3.4. Note that the GCH fails in the model $V[G]$ from the previous construction, which raises the question of whether this failure is necessary. The answer is no. One can achieve the configuration of Theorem 3.1 while preserving the GCH as follows: Start with a model of GCH and prepare it so that every inaccessible Σ_2-correct cardinal λ becomes indestructible under λ-directed-closed forcing that preserve the GCH. Lemma 3.5 below shows that this preparation is feasible, and that it preserves both the GCH

and extendible cardinals. In the resulting model, instead of coding the stationary sets S_α into the power-set-function pattern, use $\diamondsuit^*$ as in [BT09a] for the coding. The charm of this alternate coding is that it preserves the GCH. Finally force with the modified Easton support product $\mathbb{P}$.

Lemma 3.5 (GCH-preserving preparation). *There is a class forcing $\mathbb{Q}$ such that every Σ_2-correct cardinal becomes Σ_2-indestructibly correct under posets that preserve the GCH. Moreover, this forcing preserves extendible cardinals and if the GCH holds in V, then so does it in $V^{\mathbb{Q}}$.*

Proof. The proof is a variation of the argument preceding [Gol24, Theorem 3.7] aimed at guaranteeing that the GCH is preserved. For an ordinal η, a set y of rank at most η and a Σ_2 formula in the language of set theory $\varphi(x)$, we denote by $\Phi(\eta, y, \varphi)$ the sentence "There is an η-directed-closed forcing $\mathbb{P}$ that preserves the GCH such that for some $\beta \in \mathrm{Ord}$ and $p \in \mathbb{P}$, $p \Vdash_{\mathbb{P}} V[\dot{G}]_\beta \models \varphi(\check{y})$". If $\Phi(\eta, y, \varphi)$ holds we let $F(\eta, y, \varphi)$ be the first $\beth$-fixed point β such that $\Phi(\eta, y, \varphi)$ is witnessed by β and some $\mathbb{P} \in V_\beta$. Otherwise, if $\Phi(\eta, y, \varphi)$ fails we declare $F(\eta, y, \varphi) := 0$. Set

$$f(\eta) := \sup_{y \in V_\eta, \varphi \in \Sigma_2} F(\eta, y, \varphi).$$

We define an Easton-supported class iteration

$$\mathbb{Q} := \varinjlim \langle \mathbb{Q}_\alpha, \dot{\mathbb{P}}_\alpha : \alpha \in \mathrm{Ord} \rangle$$

together with a countinuous increasing sequence of ordinals $\langle \eta_\alpha : \alpha \in \mathrm{Ord} \rangle$ as follows: Suppose that both $\mathbb{Q}_\alpha$ and η_α have been defined. Then we let $\dot{\mathbb{P}}_\alpha$ be a $\mathbb{Q}_\alpha$-name for the lottery sum (see [Ham00, §3]) of all the GCH-preserving η_α-directed-closed forcings in $V_{f(\eta_\alpha)}$ (this computation takes place in $V^{\mathbb{Q}_\alpha}$).

The argument of [Gol24, Theorem 3.8] shows that forcing with $\mathbb{Q}$ makes every Σ_2-correct cardinal of V become indestructibly Σ_2-correct under posets preserving the GCH. The same conclusion applies to the preservation of extendible cardinal from the ground model V [Gol24, Proposition 3.9].

Claim 3.5.1. *If the GCH holds in V then so does in $V^{\mathbb{Q}}$.*

Proof of claim. Let $G \subseteq \mathbb{Q}$ a V-generic filter and fix λ a cardinal in $V[G]$. Since $\langle \eta_\alpha : \alpha \in \mathrm{Ord} \rangle$ is an increasing continuous sequence there is an ordinal α such that $\eta_\alpha \leq \lambda < \eta_{\alpha+1}$. By the closure properties of $\mathbb{Q}/G_{\alpha+1}$,

$$\mathcal{P}(\lambda)^{V[G]} = \mathcal{P}(\lambda)^{V[G_{\alpha+1}]}.$$

By the construction of $\mathbb{Q}$ it follows that $|\mathbb{Q}_\alpha| \leq \eta_\alpha$. Standard nice name arguments together with "$V \models$ GCH" yield $V[G \restriction \alpha] \models 2^\lambda = \lambda^+$. By assumption, $(\dot{\mathbb{P}}_\alpha)_{G_\alpha}$ preserves the GCH so $V[G] \models 2^\lambda = \lambda^+$. $\square$

The above finishes the proof of the proposition. $\square$

Let us conclude this section with an interesting corollary of Theorem 3.1.

Corollary 3.6. *The following are consistent with the* HOD *hypothesis:*

(1) *The first extendible is the first* HOD-*supercompact.*

(2) $C^{(1)}$-*supercompact cardinals may not be strongly compact in* HOD.

Proof. Let $V[G]$ be the model constructed in Theorem 3.1.

(1) Since δ is extendible in $V[G]$ it is $\text{HOD}^{V[G]}$-supercompact by [Woo10, Lemma 188]. Note that there are no $\text{HOD}^{V[G]}$-supercompact cardinals $\kappa < \delta$ as those κ would be supercompact in HOD, (here we used the HOD hypothesis in $V[G]$ and Theorem 2.7) but δ is the first supercompact in HOD.

(2) Poveda showed in [Pov23] that every extendible cardinal is a limit of $C^{(1)}$-supercompacts. Since the first extendible in the model of Theorem 3.1 is the first strongly compact in HOD, many $C^{(1)}$-supercompact cardinals (in $V[G]$) are not strongly compacts in HOD. $\qquad\square$

3.2. **Transferring large cardinals down to HOD.**

Woodin showed in [Woo10] that if δ is an extendible cardinal and the HOD hypothesis holds then δ is supercompact in HOD. The next theorem offers an improvement of this fact by showing that δ is $C^{(1)}$-supercompact in HOD.

Theorem 3.7. *Under the* HOD *hypothesis, every extendible cardinal is $C^{(1)}$-supercompact in* HOD.

Proof. Let δ be an extendible cardinal. The interesting case is when δ is the first extendible: In any other case, Hamkins Universality Theorem 2.11 implies that δ is extendible in HOD, and by [Pov23] $C^{(1)}$-supercompact.

So going forward we assume that δ is the first extendible. We would like to employ Poveda's characterization of $C^{(1)}$-supercompactness (Theorem 2.4) to show that δ is $C^{(1)}$-supercompact. First, note that that δ is supercompact in HOD, by Woodin's theorem. (This yields Clause (1) of Theorem 2.4.)

To verify Clause (2) of Theorem 2.4 we argue as follows. Let $j \colon V \to M$ be an extender embedding with $\text{crit}(j) = \delta$, $M^\delta \subseteq M$, $V_{j(\delta)} \subseteq M$ and $\lambda < j(\delta)$ being inaccessible. (This embedding exists by Tsaprounis' theorem [Tsa18].)

Note that

$$(*) \qquad \text{HOD}^M \cap V_{j(\delta)} = \text{HOD}^{V_{j(\delta)}} \subseteq \text{HOD}.$$

The first equality follows from Σ_2-correctness of $j(\delta)$ in M and $V_{j(\delta)} \subseteq M$.

Let E^* be the extender of length $j(\delta)$ inducing $j \colon V \to M$. Denote by E the HOD-extender of length $j(\delta)$ derived from j. Clearly, $\text{crit}(E) = \delta$ and $j_E(\delta) = j(\delta)$ is HOD-inaccessible. So we are left with showing two things:

(a) $E \in \text{HOD}$;

(b) $\text{Ult}(\text{HOD}, E)$ is closed under $<\delta$-sequences in HOD.

Claim 3.7.1. *Clause (a) holds.*

Proof. We will use Woodin's Universality Theorem 2.10. This says that $E \in \text{HOD}$ is equivalent to saying that for all sets $A \in \text{HOD} \cap \mathcal{P}(j(\delta)^{<\omega})$,

$$j_E(A) \cap j(\delta)^{<\omega} \in \text{HOD}.$$

By equation $(*)$ above, for each $\tau \in \mathrm{HOD}$ with $|\tau|^{\mathrm{HOD}} < j(\delta)$,

$$j_E(A) \cap j(\delta)^{<\omega} \cap \tau \in \mathrm{HOD}^M \cap V_{j(\delta)} \subseteq \mathrm{HOD}.$$

Since HOD has the δ-cover and δ-approximation properties (by Theorem 2.12) it also has the $j(\delta)$-cover and $j(\delta)$-approximation properties. (For this one uses the fact that $j(\delta)$ is regular.) Therefore, $j_E(A) \cap j(\delta)^{<\omega} \in \mathrm{HOD}$. $\square$

Claim 3.7.2. *Clause (b) holds.*

Proof of claim. We use that HOD has the γ-cover property for all sufficiently large strong limit cardinals $\gamma < \delta$. This follows from Goldberg's version of the HOD dichotomy theorem for strongly compact cardinals [Gol23]. Using this observation, the elementarity of j_{E^*}, and the fact that $\mathrm{crit}(j_{E^*}) = \delta$, we obtain that $j_{E^*}(\mathrm{HOD})$ has the γ-cover property in $\mathrm{Ult}(V, E^*)$ for all sufficiently large strong limit cardinals $\gamma < \delta$. Since $\mathrm{Ult}(V, E^*)$ is closed under $<\delta$-sequences, this means that $j_{E^*}(\mathrm{HOD})$ has the δ-cover property.

Suppose that $\sigma \in \mathcal{P}_\delta(j_{E^*}(\delta)) \cap \mathrm{HOD}$, and we will show that $\sigma \in j_{E^*}(\mathrm{HOD})$. Let $\tau \in \mathcal{P}_\delta(j_{E^*}(\delta)) \cap j_{E^*}(\mathrm{HOD})$ cover σ. Since $E \in \mathrm{HOD}$ and $j_{E^*}(\mathrm{HOD}) = \mathrm{Ult}(\mathrm{HOD}, E)$, we have $j_{E^*}(\mathrm{HOD}) \subseteq \mathrm{HOD}$ and therefore $\tau \in \mathrm{HOD}$. Since $\mathrm{HOD} \cap V_\delta = j_{E^*}(\mathrm{HOD}) \cap V_\delta$, for any set $\pi \in \mathrm{HOD} \cap j_{E^*}(\mathrm{HOD})$ with $|\pi| < \delta$, $\mathcal{P}(\pi) \cap \mathrm{HOD} = \mathcal{P}(\pi) \cap j_{E^*}(\mathrm{HOD})$. In particular,

$$\mathcal{P}(\tau) \cap \mathrm{HOD} = \mathcal{P}(\tau) \cap j_{E^*}(\mathrm{HOD})$$

and therefore $\sigma \in j_{E^*}(\mathrm{HOD})$, since $\sigma \in \mathcal{P}(\tau) \cap \mathrm{HOD}$.

It follows that $\mathcal{P}_\delta(j_E(\delta)) \cap \mathrm{Ult}(\mathrm{HOD}, E) = \mathcal{P}_\delta(j_E(\delta)) \cap \mathrm{HOD}$, and since $\mathrm{cf}(j_E(\delta)) \geq \delta$, $\mathrm{Ult}(\mathrm{HOD}, E)$ is closed under $<\delta$-sequences in HOD. $\square$

After this we have that $j_E \colon \mathrm{HOD} \to \mathrm{Ult}(\mathrm{HOD}, E)$ witnesses (in HOD) the properties in the characterization of $C^{(1)}$-supercompactness. $\square$

Corollary 3.8. *In the* HOD *of the model of Theorem 3.1 the first supercompact is $C^{(1)}$-supercompact.*

Remark 3.9. The above yields another proof of the main identity crises theorem proved by Hayut, Magidor and Poveda in [HMP22].

Next, we would like to show that the hierarchies of $C^{(n)}$-supercompact/$C^{(n)}$-extendible cardinals above the first extendible transfer down to HOD. We remind our readers that since we will be working under the assumptions of Theorem 2.12, HOD will have the δ-covering and δ-approximation.

Theorem 3.10. *Assume δ is* HOD*-supercompact and that the* HOD *hypothesis holds. For each $n \geq 1$ and $\lambda \in C^{(n)}$ with $\lambda > \delta$, $\mathrm{HOD} \models$ "$\lambda \in C^{(n)}$".*

Proof. The claim is evident for Σ_1-correct cardinals (i.e., $\beth$-fixed points). So we assume that $n \geq 2$. Fix $\lambda > \delta$ a Σ_n-correct cardinal. By our assumption, HOD is a weak extender model for the supercompactness of δ and every such model has the δ-cover and δ-approximation properties.

By the Definability Theorem (Theorem 2.9), HOD is Σ_2 definable (in V) taking $P := \mathrm{HOD} \cap H(\delta^+)$ as parameter. In other words, $\mathrm{HOD} = (W_P)^V$

where W_P is a predicate for the unique inner model with the δ-covering and δ-approximation properties that is Σ_2 definable via P. In particular,

$$\mathrm{HOD}^{V_\lambda} = \mathrm{HOD} \cap V_\lambda =: \mathrm{HOD}_\lambda.$$

We will show that

$$\mathrm{HOD}_\lambda \prec_{\Sigma_n} \mathrm{HOD}.$$

Let us fix a Σ_n formula $\varphi(x)$ and a set $a \in \mathrm{HOD}_\lambda$.

Claim 3.10.1. *"$\mathrm{HOD} \models \varphi(a)$" is a Σ_n formula with parameters a, P.*

Proof of claim. The above is equivalent to saying

$$\exists \alpha \in C^{(n-1)}(V_\alpha \models \text{"}W_P \models \varphi(a)\text{"}).$$

Thus "$\mathrm{HOD} \models \varphi(a)$" has the claimed complexity. □

Suppose first that $\mathrm{HOD}_\lambda \models \varphi(a)$. This amounts to saying

$$V_\lambda \models \text{"}\mathrm{HOD} \models \varphi(a)\text{"}.$$

By Σ_n-correctness of λ and the previous claim, $\mathrm{HOD} \models \varphi(a)$.

Conversely, suppose that $\mathrm{HOD} \models \varphi(a)$ holds. By the claim this is a true $\Sigma_n(a, P)$ sentence. By Σ_n-correctness of λ there is $\alpha \in C^{(n-1)} \cap \lambda$ with

$$V_\alpha \models \mathrm{HOD} \models \varphi(a),$$

with $a, P \in V_\alpha$. However, V_α and V_λ agree on Σ_n sentences with parameters in V_α, so $V_\lambda \models \text{"}\mathrm{HOD} \models \varphi(a)\text{"}$, which is the same as $\mathrm{HOD}_\lambda \models \varphi(a)$. □

Theorem 3.11. *Assume δ is HOD-supercompact and that the HOD hypothesis holds. If $\kappa > \delta$ is $C^{(n)}$-extendible (resp. $C^{(n)}$-supercompact) then it is $C^{(n)}$-extendible (resp. $C^{(n)}$-supercompact) in HOD.*

Proof. Fix $\kappa > \delta$ as above.

Claim 3.11.1. *If κ is $C^{(n)}$-supercompact then so it is in HOD.*

Proof of claim. We employ the characterization of $C^{(n)}$-supercompactness provided by Theorem 2.4. First, κ is supercompact in HOD by Hamkins Univerality Theorem II (Theorem 2.11). Second, for each $\lambda > \delta$ let $j \colon V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \delta$, $\mathrm{cf}(j(\delta)) > \lambda$ and $j(\delta) \in C^{(n)}$. Let E be the HOD-extender of length $j(\delta)$ induced by j. By Hamkins Universality Theorem, $j_E \colon \mathrm{HOD} \to N$ is an elementary embedding definable in HOD. By the argument in Theorem 3.7, $N^{<\delta} \cap \mathrm{HOD} \subseteq \mathrm{HOD}$. Finally, thanks to Theorem 3.10, $\mathrm{HOD} \models j(\delta) \in C^{(n)}$. □

Claim 3.11.2. *If κ is $C^{(n)}$-extendible then so it is in HOD.*

Proof of claim. We employ Tsaprounis' characterization of $C^{(n)}$-extendibility [Tsa18]; namely, δ is $C^{(n)}$-extendible if and only if for each $\lambda > \delta$ there is an elementary embedding $j \colon V \to M$ with $\mathrm{crit}(j) = \delta$, $j(\delta) > \lambda$, $M^\lambda \subseteq M$, $V_{j(\delta)} \subseteq M$ and $j(\delta) \in C^{(n)}$. Let E be the HOD-extender of length $j(\delta)$ induced by j. By the Universality Theorem II (Theorem 2.11), the embedding

$j_E \colon \mathrm{HOD} \to N$ is definable inside HOD and $\mathrm{HOD} \cap V_{j(\delta)} = N \cap V_{j(\delta)}$. In particular, $\mathrm{HOD}_{j(\delta)} \subseteq N$. Also, arguing exactly as before, $N^\lambda \cap \mathrm{HOD} \subseteq N$ and $\mathrm{HOD} \models j(\delta) \in C^{(n)}$. This shows that δ is $C^{(n)}$-extendible in HOD. $\square$

This completes the proof of the theorem. $\square$

3.3. **On cardinal-correct extendible cardinals.** In [Cai07] Caicedo introduced *cardinal-preserving embeddings*; namely, elementary embeddings $j \colon M \to N$ between transitive models such that $\mathrm{Card}^M = \mathrm{Card}^N = \mathrm{Card}$. (Here Card denotes the class of all cardinals.) More recently, Goldberg showed that if there is a proper class of strongly compact cardinals there is no cardinal-preserving elementary embedding $j \colon V \to M$ [Gol20]. The situation with set-sized transitive models is different. For instance, any elementary embedding between rank initial segments of V is cardinal-preserving. Inspired by previous work of Magidor and Väänänen on Löwenheim-Skolem-Tarski numbers of strong logics [MV11], Galeotti, Khomskii, and Väänänen considered *upward* Löwenheim–Skolem–Tarski numbers [GKV20]. To study the upward Löwenheim–Skolem–Tarski number of the equicardinality logic $\mathcal{L}(I)$, in [GO24] the notion of *cardinal-correct extendibility* was proposed:

Definition 3.12 (Gitman–Osinski)**.** A cardinal δ is *cardinal-correct extendible* (in short, δ is cce) if for each $\alpha > \delta$ there is an elementary embedding $j \colon V_\alpha \to M$ with $\mathrm{crit}(j) = \delta$, $j(\delta) > \alpha$ and $\mathrm{Card}^M = M \cap \mathrm{Card}$.

Arguments of Goldberg showed that cce cardinals are strongly compact (see [GO24, Proposition 9.5]). Poveda also showed that the first cce cardinal is consistently greater than the first supercompact [Pov24], and later Gitman and Osinski found an alternate proof [GO24]. Clearly, every extendible cardinal is cce. The opposite question—namely, whether every cce cardinal must be extendible—is implicit in [GO24]. This very question was asked by Osinski to Poveda during a visit to Harvard in the Spring of 2024. As we will show, it turns out that the two notions are not equivalent. The following theorem provides the first key observation:

Theorem 3.13. *Suppose that* HOD *is cardinal-correct (i.e.,* $\mathrm{HOD}^M = \mathrm{HOD} \cap \mathrm{Card}$*). If* δ *is extendible then it is* cce *in* HOD.

Proof. Let $\alpha > \delta$ be a Σ_2-correct cardinal such that $\mathrm{cf}(\alpha) > \delta$. Let $j \colon V_{\alpha+1} \to V_{\beta+1}$ be an elementary embedding witnessing $(\alpha+1)$-extendibility of δ and F be the HOD-extender of length β derived from j. That is,

$$F = \langle F_a : a \in [\beta]^{<\omega} \rangle$$

where

$$F_a := \{X \subseteq [\alpha]^{|a|} : X \in \mathrm{HOD} \wedge a \in j(X)\}.$$

Consider the extender ultrapower of HOD by F (using only functions in HOD:

$$j_F \colon \mathrm{HOD} \to M_F.$$

Claim 3.13.1. $j_F \restriction \mathrm{HOD}_{\alpha+1} = j \restriction \mathrm{HOD}_{\alpha+1}$.

Proof of claim. This is a standard argument using that the factor embedding $k \colon (M_F)_{\beta+1} \to j(\mathrm{HOD}_{\alpha+1})$, defined by

$$k([a, f]_F) := j(f)(a)$$

has no critical point (since by construction $\beta \subseteq \mathrm{ran}(k)$). (To be precise, k is defined on $[a, f]_F$ whenever $f \colon [\alpha]^{|a|} \to V_{\alpha+1}$, in which case f is essentially an element of $V_{\alpha+1}$, so we can make sense of $j(f)$.)

It follows from this that $k \restriction (M_F)_\beta$ is onto $j(\mathrm{HOD}_\alpha)$, from which it immediately follows that $(M_F)_{\beta+1} \subseteq j(\mathrm{HOD}_{\alpha+1})$ and k is the inclusion map. To see $k \restriction (M_F)_\beta$ is onto $j(\mathrm{HOD}_\alpha)$, note that since β is a Beth fixed-point, $(M_F)_\beta$ satisfies that for each $\xi < \beta$, $(M_F)_\xi$ is the surjective image of an ordinal γ. Fixing in $(M_F)_\beta$ a such a surjection $g \colon \gamma \to (M_F)_\xi$, we have

$$j(\mathrm{HOD}_\alpha) \cap V_\xi = k((M_F)_\xi) = k(g)[k(\gamma)] = k(g)[k[\gamma]] = k[g[\gamma]] \subseteq \mathrm{ran}(k). \;\square$$

Next we argue that $F \in \mathrm{HOD}$. For this we use Woodin's Universality Theorem I (Theorem 2.10). We shall also employ Theorem 2.12 saying that if δ is HOD-supercompact (in our case δ is extendible) and the HOD hypothesis holds, then HOD has the δ-approximation property. Notice that since HOD is assumed to be cardinal correct, the HOD hypothesis must hold, since no successor cardinal is ω-strongly measurable in HOD.

Claim 3.13.2. $F \in \mathrm{HOD}$.

Proof of claim. By Woodin's Universality Theorem I (Theorem 2.10), it suffices to argue that

$$j_F(A) \cap \beta \in \mathrm{HOD},$$

for all $A \in \mathcal{P}(\alpha) \cap \mathrm{HOD}$. So fix such an $A \subseteq \alpha$.

Since $A \in \mathrm{HOD}$, A is amenable to HOD_α in the sense that $A \cap \xi \in \mathrm{HOD}_\alpha$ for all $\xi < \alpha$. Since α is Σ_2-correct, we have $\mathrm{HOD}_\alpha = \mathrm{HOD}^{V_\alpha}$, and so

$$j(\mathrm{HOD}_\alpha) = j(\mathrm{HOD}^{V_\alpha}) = \mathrm{HOD}^{V_\beta} \subseteq \mathrm{HOD}$$

Moreover by the elementarity of j, $j(A)$ is amenable to $j(\mathrm{HOD}_\alpha)$, and so $j(A)$ is amenable to HOD.

Since α has cofinality at least δ, $j(\alpha) = \beta$ has cofinality at least δ. Since HOD has the δ-approximation property, any $B \subseteq \beta$ that is amenable to HOD actually belongs to HOD. It follows that $j(A) \in \mathrm{HOD}$, as desired. $\square$

We deduce from the previous claims that

$$j_F \restriction \mathrm{HOD}_\alpha = j \restriction \mathrm{HOD}_\alpha \in \mathrm{HOD},$$

and recall that, by elementarity and Σ_2-correctness of α,

$$j \restriction \mathrm{HOD}_\alpha \colon \mathrm{HOD}_\alpha \to \mathrm{HOD}^{V_\beta}.$$

Thus HOD thinks that there is an elementary embedding

$$\iota \colon V_\alpha \to M := \mathrm{HOD}^{V_\beta}$$

with $\mathrm{crit}(\iota) = \delta$ and $\iota(\delta) > \alpha$.

Claim 3.13.3. HOD^{V_β} *is cardinal-correct in* HOD.

Proof. Since $\mathrm{HOD}^{V_\beta} \subseteq \mathrm{HOD}$, every HOD-cardinal is a HOD^{V_β}-cardinal.

Let us argue the converse. Since α is Σ_2-correct that yields

$$\mathrm{HOD}^{V_\alpha} = \mathrm{HOD}_\alpha.$$

If τ is a HOD_α-cardinal then it is a HOD-cardinal and thus a V-cardinal as well. (Because HOD is cardinal-correct.) Thus, τ is a cardinal in V_α.

One concludes that

$$V_\alpha \models \text{"Every HOD-cardinal is a cardinal"}.$$

By elementarity of j,

$$V_\beta \models \text{"Every HOD-cardinal is a cardinal"},$$

hence every HOD^{V_β}-cardinal is a cardinal in V and thus a cardinal in HOD (again, by cardinal-correctness of HOD). $\qquad\square$

The above argument show that in HOD the cardinal δ is α-cce for proper-class many α's, thereby we conclude that δ is cce in HOD. $\qquad\square$

The second key observation to answering Gitman–Osinski's question in the negative is precisely Theorem 3.1:

Theorem 3.14. *It is consistent with the* HOD *hypothesis that the first strongly compact cardinal is cardinal-correct extendible. In particular, it is consistent for a cardinal-correct extendible not to be extendible.*

Proof. Let $V[G]$ be the model of Theorem 3.1. Recall that in this model $V \subseteq \mathrm{HOD}^{V[G]} \subseteq V[G]$ and V and $V[G]$ have the same cardinals and cofinalities. This implies that $V[G]$ and $\mathrm{HOD}^{V[G]}$ have the same cardinals and cofinalities. Moreover δ is extendible in $V[G]$, so by Theorem 3.13, δ is cce in $\mathrm{HOD}^{V[G]}$. But by the proof of Theorem 3.1, δ is the least strongly compact cardinal of $\mathrm{HOD}^{V[G]}$. $\qquad\square$

Corollary 3.15 (Identity Crisis)**.** *The first* cce *cardinal can be either the first strongly compact or greater than the first strongly compact.*

Proof. The consistency of the first configuration has just been proved. The second follows from results of Goldberg and Poveda [Pov24, §5]. $\qquad\square$

The crux of the matter of the previous argument is Theorem 3.13. This was proved under the assumption that HOD is cardinal-correct, which is stronger than asserting the HOD hypothesis. Therefore, we ask:

Question 1. *Assume the* HOD *hypothesis. If δ is* cce *must it be so in* HOD*?*

Goldberg and Poveda conjecture that the answer is negative.

Another question highlighted by our analysis is the following:

Question 2 (Gitman–Osinski)**.** *Is it consistent for the first* cce *to be the first measurable cardinal?*

4. ON THE HOD DICHOTOMY FOR SUPERCOMPACT CARDINALS

In this section we analyze the optimality of the version of the HOD Dichotomy theorem proved by the first author in [Gol23]. First, in §4.1 we analyze the optimality of the hypothesis of Goldberg's theorem. Specifically, we construct a model where

"δ is the first supercompact + HOD hypothesis"

and there is a club of cardinals $\kappa < \delta$ such that

κ is regular in HOD and $\kappa^{+\mathrm{HOD}} < \kappa^+$.

This shows that the HOD dichotomy theorem cannot be proved from any large cardinal hypothesis rank reflected by a supercompact cardinal. For instance, it does not follow from a proper class of strong cardinals, etc. In addition, we answer a question due to Cummings, Friedman and Golshani [CFG15] relative to failure of weak covering and supercompact cardinals.

Next, in §4.2 we examine the optimality of the conclusion of Goldberg's version of the HOD dichotomy theorem. Goldberg's result implies that if δ is supercompact and the HOD hypothesis fails, then all sufficiently large regular cardinals are ω-strongly measurable in HOD. We show that this cannot be improved to the conclusion that every regular cardinal greater than or equal to δ is ω-strongly measurable in HOD. Moreover, we show that for each cardinal $\lambda \geq \delta$, there is a set-forcing extension in which δ remains supercompact and the power-set of λ is contained in HOD. In particular, since the failure of the HOD hypothesis is preserved by set-sized forcing, we obtain a model where the HOD hypothesis fails, δ is supercompact and $\eta_0 > \lambda$ where η_0 is defined as follows:

$\eta_0 := \min\{\theta \geq \delta \mid \theta \text{ is } \omega\text{-strongly measurable in HOD}\}$.

Note that by the HOD dichotomy proved by Goldberg in [Gol23], every regular cardinal above η_0 must be ω-strongly measurable in HOD.

4.1. **Where weak covering takes hold.** We begin this section addressing the following question from Cummings et al. [CFG15, p.31]:

Question (Cummings–Friedman–Golshani). *Is it consistent for δ to be supercompact and $\kappa^{+\mathrm{HOD}} < \kappa^+$ for all $\kappa < \delta$?*

We argue that, at least under the HOD hypothesis, there must be *many* agreements between V and HOD on singular cardinals and their successors. (In [CFG15], the authors mention that Woodin conjectured a negative answer to their question, granting the HOD conjecture. Thus, it seems likely to us that Woodin was aware of the result we are about to prove.)

We shall employ the following lemma, which is a slight fix of [Che17, Theorem 4.1].[8]

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[8]In [Che17, Theorem 4.1] it is claimed that $j_2(N \cap V_{\lambda_2}) = N \cap V_\lambda$, but this does not hold. For instance, if $N := \mathrm{HOD} \cap V_\delta$ then $\delta_2 \in N \cap V_{\lambda_2}$ and $j_2(\delta_2) = \delta$ yet $\delta \notin N \cap V_\lambda = N$.

Lemma 4.1. *Let $N \subseteq V_\delta$. The following assertions are equivalent:*

  (1) δ is supercompact.
  (2) For each $\lambda > \delta$ and $a \in V_\lambda$ there are elementary embeddings
$$j_1 \colon V_{\lambda_1+1} \to V_{\lambda_2+1} \text{ and } j_2 \colon V_{\lambda_2+1} \to V_{\lambda+1}$$
  and sets a_1 and a_2 such that:
  (a) $\mathrm{crit}(j_1) = \delta_1$, $\mathrm{crit}(j_2) = \delta_2$, $j_1(\delta_1) = \delta_2$, $j_2(\delta_2) = \delta$.
  (b) $a_1 \in V_{\delta_1}$, $a_2 \in V_{\delta_2}$, $j_1(a_1) = a_2$ and $j_2(a_2) = a$.
  (c) $j_1(N \cap V_{\lambda_1}) = N \cap V_{\lambda_2}$.   □

Remark 4.2. For each triple $\langle a, \lambda, \delta_1 \rangle$, let us denote by $\varphi(a, \lambda, \delta_1)$ the formula asserting the existence of embeddings j_1 and j_2, cardinals λ_1, λ_2, δ_2 and sets a_1 and a_2 as in Clause (2) above. One can show that $\{\delta_1 < \delta : \varphi(a, \lambda, \delta_1)\}$ is stationary in δ. This is because this belongs to the normal measure on δ induced by the second ultrapower of a normal fine measure on $\mathcal{P}_\delta(|V_{\lambda+1}|)$.

It is well-known that if N is a weak extender model for δ supercompact then every singular cardinal $\gamma > \delta$ is singular in N and $(\gamma^+)^{\mathrm{HOD}} = \gamma^+$ ([Woo17, §2.2]). A close inspection of Lemmas 2.8, 2.9 and 2.10 in [Woo17] makes clear that this can be obtained level-by-level. Specifically:

Lemma 4.3. *Let $\delta < \sigma$ be cardinals with $\mathrm{cf}(\sigma) < \sigma$. If N is a weak extender model for δ is σ-supercompact then σ is singular in N and $(\sigma^+)^N = \sigma^+$.*   □

Theorem 4.4. *Assume the* HOD *hypothesis holds. If δ is a supercompact cardinal then there is a stationary set $S \subseteq \delta$ such that the ωth-successor of each point in S is singular in* HOD *and* HOD *is correct about its successor.*

Proof. Let $\delta < \mu < \kappa$ be cardinals that are (respectively) a $\beth$-fixed point and a regular non-ω-strongly measurable in HOD above 2^μ. (This choice is possible by virtue of the HOD hypothesis.) Thus, there is a stationary partition $\langle S_\alpha : \alpha < \mu \rangle \in \mathrm{HOD}$ of E_ω^κ. Set $a := \{\kappa, \mu, \langle S_\alpha : \alpha < \mu \rangle\}$ and let $\lambda > \mu$ be a Σ_2-correct cardinal. Applying Lemma 4.1 (and the remark following it) to $\langle \lambda, a \rangle$ we get a stationary set $S \subseteq \delta$ (which depends upon $\langle \lambda, a \rangle$) such that for each $\delta_1 \in S$ there are elementary embeddings
$$j_1 \colon V_{\lambda_1+1} \to V_{\lambda_2+1} \text{ and } j_2 \colon V_{\lambda_2+1} \to V_{\lambda+1}$$
with $\mathrm{crit}(j_1) = \delta_1$, $\mathrm{crit}(j_2) = \delta_2$, $j_1(\delta_1) = \delta_2$ and $j_2(\delta_2) = \delta$ and
$$j_1(\mathrm{HOD} \cap V_{\lambda_1}) = \mathrm{HOD} \cap V_{\lambda_2}.$$

These embeddings also do come together with respective *reflections* of the members of a; namely, $\langle \kappa_1, \mu_1, \langle S_\alpha^1 : \alpha < \mu_1 \rangle \rangle$ and $\langle \kappa_2, \mu_2, \langle S_\alpha^2 : \alpha < \mu_2 \rangle \rangle$.

Since $\langle S_\alpha : \alpha < \mu \rangle \in \mathrm{HOD} \cap V_\lambda = \mathrm{HOD}^{V_\lambda}$ (by Σ_2-correctness of λ) elementarity of j_2 implies that $\langle S_\alpha^2 : \alpha < \mu_2 \rangle \in \mathrm{HOD}^{V_{\lambda_2}} \subseteq \mathrm{HOD}$.

Next look at the normal measure on $\mathcal{P}_{\delta_1}(\delta_1^{+\omega})$ induced by j_1; namely,
$$\mathcal{U} := \{X \subseteq \mathcal{P}_{\delta_1}(\delta_1^{+\omega}) : j_1 \text{``}\delta_1^{+\omega} \in j_1(X)\}.$$

Claim 4.4.1. $\mathcal{U} \cap \mathrm{HOD} \in \mathrm{HOD}$ *and* $\mathcal{P}_{\delta_1}(\delta_1^{+\omega}) \cap \mathrm{HOD} \in \mathcal{U}$.

Proof of claim. Clearly, $\mathcal{U} \cap \mathrm{HOD}$ is definable via $j_1 \text{``} \delta_1^{+\omega}$ and $j_1 \restriction \mathrm{HOD} \cap V_{\mu_1}$, so $\mathcal{U} \cap \mathrm{HOD} \in \mathrm{HOD}$ as long as the aforementioned objects belong to HOD.

On the one hand,

$$j_1 \text{``} \mu_1 = \{\alpha < \mu_2 : S_\alpha^2 \cap (\sup j_1 \text{``} \kappa_1) \text{ is stationary}\} \in \mathrm{HOD}$$

in that $\langle S_\alpha^2 : \alpha < \mu_2 \rangle \in \mathrm{HOD}$. In particular, $j_1 \text{``} \delta_1^{+\omega} \in \mathrm{HOD} \cap V_{\lambda_2}$.

On the other hand, fix $f : \mu_1 \to \mathrm{HOD} \cap V_{\mu_1}$ a bijection in HOD (this is possible because $\mu_1 = \beth_{\mu_1}$). In fact, $f \in \mathrm{HOD} \cap V_{\lambda_1}$ (as λ_1 is strong limit) and thus $j_1(f) \in \mathrm{HOD} \cap V_{\lambda_2} \subseteq \mathrm{HOD}$. Since $j_1 \restriction \mathrm{HOD} \cap V_{\mu_1}$ is definable via $j_1 \text{``} \mu_1$ and $j_1(f)$ one concludes that it belongs to HOD.

The above shows that $\mathcal{U} \cap \mathrm{HOD} \in \mathrm{HOD}$. Finally note that $\mathcal{P}_{\delta_1}(\delta_1^{+\omega}) \cap \mathrm{HOD} \in \mathcal{U}$ because this is equivalent to saying $\mathcal{P}_{\delta_1}(\delta_1^{+\omega}) \cap \mathrm{HOD} \cap V_{\lambda_1} \in \mathcal{U}$ and this latter assertion holds in that $j_1 \text{``} \delta_1^{+\omega} \in j_1(\mathrm{HOD} \cap V_{\lambda_1}) = \mathrm{HOD} \cap V_{\lambda_2}$. $\square$

The above claim shows that HOD is a weak extender model for "δ_1 is $\delta_1^{+\omega}$-supercompact" and therefore $\delta_1^{+\omega}$ is singular in HOD and its successor is correctly computed (by Lemma 4.3). Since we have argued this for each $\delta_1 \in S$ the theorem follows. $\square$

Thereby we answer the question in [CFG15] in the negative:

Corollary 4.5. *If δ is supercompact and the* HOD *hypothesis holds then for unboundedly many cardinals κ, $\mathrm{cf}^{\mathrm{HOD}}(\kappa) = \mathrm{cf}(\kappa)$ and $\kappa^{+\mathrm{HOD}} = \kappa^+$.*

Next we would like to show that Theorem 4.4 is arguably optimal:

Theorem 4.6. *Suppose that there is a supercompact cardinal and that the* GCH *holds. Then, there is a model of* ZFC *where the following hold:*

  (1) *The* HOD *hypothesis.*
  (2) *δ is the first supercompact.*
  (3) *There is a club $D \subseteq \delta$ consisting of cardinals κ such that:*
     (a) *$\kappa^{+\mathrm{HOD}} < \kappa^+$;*
     (b) *$\mathrm{HOD} \models$ "κ is regular".*

In [CFG15], Cummings, Friedman and Golshani construct a model where the HOD hypothesis holds and a supercompact δ carries a club of cardinals κ such that $\kappa^{+\mathrm{HOD}} < \kappa^+$. Building upon this work we will be able to prove the above-mentioned theorem. The model under consideration is a generic extension via *Supercompact Radin forcing*, $\mathbb{R}_u^{\mathrm{sup}}$. The definition of $\mathbb{R}_u^{\mathrm{sup}}$ is rather technical so we refrain to reproduce it here. Instead, we refer the reader to [CFG15] (indeed, we recommend our readers to have a copy of [CFG15] close to them) for a complete account of the poset.

In [CFG15] a pair (u, A) is called *good* if $u \in \mathcal{U}_\infty^{\mathrm{sup}}$, $A \subseteq \mathcal{U}_\infty^{\mathrm{sup}} \cap S(\kappa_u, \lambda_u)$ and $A \in \mathcal{F}(u)$. Following [CFG15, §4], given a good pair (u, A) we denote

  • $\phi(u) := \langle \kappa_u \rangle^\frown \langle u(\alpha) : 1 \leq \alpha < \ell(u) \rangle$,
  • $\phi(A) := \{\phi(v) : v \in A\}$.

(Note that formally speaking ϕ is possibly a class function.) Following [CFG15, §4], denote $\mathcal{U}_\infty^{\mathrm{proj}} = \{\phi(u) : u \in \mathcal{U}_\infty^{\mathrm{sup}}\}$. Similarly, write

$$\mathcal{U}_\infty^{\mathrm{proj},-} := \{\phi(u) \upharpoonright \ell_u : u \in \mathcal{U}_\infty^{\mathrm{sup}}\}.$$

In [CFG15, §4] the authors call a pair (w, B) *good for projected forcing* if

$$w \in \mathcal{U}_\infty^{\mathrm{proj}}, \; B \subseteq \mathcal{U}_\infty^{\mathrm{proj}} \cap V_{\kappa_w} \text{ and } B \in \bigcap_{0 < \alpha < \ell(w)} \phi(w(\alpha)).$$

Let $\mathbb{R}_{\phi(u)}^{\mathrm{proj}}$ be defined as the usual Supercompact Rading forcing yet employing good pairs for projected forcing [CFG15, Definition 4.4]. Intuitively speaking, $\mathbb{R}_{\phi(u)}^{\mathrm{proj}}$ is like regular Radin forcing except for a difference—the measures $w(\alpha)$ on the measure sequences w appearing in a condition $p \in \mathbb{R}_{\phi(u)}^{\mathrm{proj}}$ are on $S(\kappa_w, \lambda_w)$ and not on V_{κ_w}. As it turns out that, if $\langle \langle \sigma_\alpha \rangle ^\frown w_\alpha \mid \alpha < \theta \rangle$ is the generic object introduced after forcing with $\mathbb{R}_u^{\mathrm{sup}}$ then $\langle w_\alpha \mid \alpha < \theta \rangle$ is the generic object introduced after forcing with the projected forcing $\mathbb{R}_{\phi(u)}^{\mathrm{proj}}$. In [CFG15, §4], the authors show that ϕ induces a (weak) projection between the supercompact Radin forcing $\mathbb{R}_u^{\mathrm{sup}}$ and its projected forcing $\mathbb{R}_{\phi(u)}^{\mathrm{proj}}$. Thus, every V-generic G for $\mathbb{R}_u^{\mathrm{sup}}$ induces a generic filter G^ϕ for $\mathbb{R}_{\phi(u)}^{\mathrm{proj}}$. Also, they show that $\mathbb{R}_{\phi(u)}^{\mathrm{proj}}$ preserves cardinals.

The following is a well-known result (see [Git10, Theorem 5.15]):

Lemma 4.7. *Let $w \in \mathcal{U}_\infty^{\mathrm{sup}}$ be such that $\langle \kappa_w \rangle ^\frown \langle \phi(w(\alpha)) : 1 \leq \alpha < \ell(w) \rangle$ has a repeat point.[9] Then, $\mathbb{R}_{\phi(w)}^{\mathrm{proj}}$ forces "κ_w is measurable".* $\qquad\square$

After the above explanations we are in a position to prove Theorem 4.6. Suppose the following properties hold in a ground model V:

(1) GCH.
(2) δ is supercompact.
(3) $V = g\mathrm{HOD}$.

The above configuration can be obtained from GCH and the existence of a supercompact cardinal δ: First, make δ indestructible under δ-directed-closed forcings that preserve the GCH. (This forcing itself preserves the GCH by Lemma 3.5) Then force with the $\diamondsuit^*$ Oracle partial forcing of [BT09b, §3] above the supercompact δ to get a model where GCH and $V = g\mathrm{HOD}$ hold.

Let $u \in \mathcal{U}_\infty^{\mathrm{sup}}$ be a (δ, δ^+)-supercompact measure sequence with length $\delta^{++} < \ell(u) < \delta^{+3}$ such that forcing with $\mathbb{R}_u^{\mathrm{sup}}$ preserves supercompactness of δ (see [CFG15, Remark 5.1]). Fix $G \subseteq \mathbb{R}_u^{\mathrm{sup}}$ a V-generic filter and denote

$$\mathcal{S}_G := \{w \in \mathcal{U}_\infty^{\mathrm{sup}} : \exists p \in G \, (w \text{ appears in } p)\}$$

and

$$C = \langle \kappa_\alpha \mid \alpha < \delta \rangle$$

the increasing enumeration of $\{\kappa_w \mid \exists p \in G \, (w \text{ appears in } p \text{ and } w \neq u)\}$.

---

[9]Recall that a *repeat point* point for $\langle \kappa_w \rangle ^\frown \langle \phi(w(\alpha)) : 1 \leq \alpha < \ell(w) \rangle$ is an ordinal $\gamma < \ell(w)$ for which $\bigcap_{1 \leq \beta < \gamma} \phi(u(\beta)) = \bigcap_{1 \leq \beta < \ell(u)} \phi(u(\beta))$.

Remark 4.8. Since $\lambda_u = \delta^+$ by forcing below an appropriate condition we may assume without loss of generality that $\lambda_w = \kappa_w^+$ for all supercompact measure sequence w appearing in a typical condition in $\mathbb{R}_u^{\mathrm{sup}}$.

We claim that $V[G]$ is the model witnessing Theorem 4.6:

Proof of Theorem 4.6. Clause (1) is obvious. To establish Clause (2) (i.e., that δ is the first supercompact cardinal in $V[G]$) we use a theorem of Magidor and Sinapova (see Theorem 3.1 in [MS17]). The argument goes as follows: Let $\kappa \in C$ be a cardinal with $V[G]$-cofinality ω. Since

$$\mathrm{cf}^{V[G]}(\kappa^{+V}) = \omega \text{ and } (\kappa^{++V}) = (\kappa^{+V[G]})$$

(by the properties of $\mathbb{R}_u^{\mathrm{sup}}$) the assumptions of [MS17, Theorem 3.1] are met. Thus $\square_{\kappa,\omega}$ holds in $V[G]$ for all such κ. Since this comment applies for unboundedly-many cardinals $\kappa < \delta$, there are no strongly compact cardinals below δ. (This is a well-known fact about square-like sequences and strongly compact cardinals, c.f. [CFM01].)

The basic idea behind (3) is that the supercompact Radin forcing $\mathbb{R}_u^{\mathrm{sup}}$ introduces many supercompact sequences w whose associated measure sequence $\langle\kappa_w\rangle^\frown\langle\phi(w(\alpha)) : 1 \le \alpha < \ell(w)\rangle$ on V_{κ_w} contains a repeat point. Employing Lemma 4.7, we will infer that forcing with $\mathbb{R}_{\phi(w)}^{\mathrm{proj}}$ preserves the regularity of κ_w. Finally, using the factoring lemma of the projected forcing, we will conclude that the same applies to the generic extension by $\mathbb{R}_{\phi(u)}^{\mathrm{proj}}$.

Claim 4.8.1. *The set $D := \{\kappa_w : w \in \mathcal{S}_G \wedge \ell(w) \ge \kappa_w^{++}\}$ is a club in δ.*

Proof. Unboundedness follows by density. The key observation is that for each $\alpha < \delta$, $\mathcal{D}_\alpha := \{p \in \mathbb{R}_u^{\mathrm{sup}} : p \text{ mentions } w \text{ with } \ell(w) \ge \kappa_w^{++} \text{ and } \kappa_w > \alpha\}$. (Recall that our supercompact measure sequence u has $\ell(u) > \delta^{++}$.)

For closure we argue as usual. Let $\dot{D}$ be a name for the above-displayed set and suppose that $\alpha < \delta$ and $p \in \mathbb{R}_u^{\mathrm{sup}}$ forces "$\alpha \notin \dot{D}$". By definition, there is no w appearing in p such that both $\ell(w) \ge \kappa_w^{++}$ and $\kappa_w = \alpha$.

Case 1: Suppose that there is w appearing in p with $\kappa_w = \alpha$. Say,

$$p = \langle(u_0, A_0), \ldots, (w, B), \ldots, (u, A)\rangle.$$

Then it must be that $\ell(w) < \kappa_w^{++}$. Let X be the set of supercompact measure sequences $v \in \mathcal{U}_\infty^{\mathrm{sup}}$ for which there is C such that (v, C) is a good pair and the sequence

$$\langle(u_0, A_0), \ldots, (v, C), (w, B), \ldots, (u, A)\rangle$$

is an extension of p. It turns out that $\pi_w[X] \in \mathcal{F}(w)$ ([CFG15, Lemma 3.7]), where π_w is the unique order isomorphism between $w(0)$ and $\lambda_w := \mathrm{otp}(w(0))$. In particular, we can let $p^* \le^* p$ be the condition resulting from replacing in p the measure one set B by

$$B^* := B \cap \pi_w[X] \cap \{v \in S(\kappa_w, \kappa_w^+) : \ell(v) < \kappa_v^{++}\}.$$

This makes sure that p^* forces "$\sup(\dot{D} \cap \alpha) < \alpha$", as needed.

<u>Case 2:</u> Suppose that there is no w appearing in p with $\kappa_w = \alpha$. Let w be the first measure sequence in p with $\kappa_w > \alpha$. As in the previous argument, set $B^* := B \cap \pi_w[X] \cap \{v \in S(\kappa_w, \kappa_w^+) : \kappa_v > \alpha\}$ and let p^* be defined as before. The resulting condition $p^* \leq^* p$ forces "$\sup(\dot{D} \cap \alpha) < \alpha$". $\qquad \square$

Note that the V-successor cardinal of each $\theta \in D$ is collapsed in $V[G]$: Let $\theta \in D$ and $w \in \mathcal{S}_G$ be the unique sequence such that $\kappa_w = \theta$. Let $p \in G$ be a condition in the generic of the form $\langle (w, B), (u, A) \rangle$. By the factoring property of supercompact Radin forcing,

$$\mathbb{R}_u^{\sup}/p \simeq \mathbb{R}_w^{\sup}/\langle (w, B) \rangle \times \mathbb{R}_u^{\sup}/\langle (u, A^*) \rangle$$

where $A^* := \{v \in A : w \prec v\}$. By standard properties of the Supercompact Radin forcing, the first of these factors forces "θ is a cardinal", "$|\theta^{+V}| = \theta$" and "θ^{++V} remains a cardinal". Note that the second factor is distributive enough so as to not alter these properties. By the analysis of [CFG15, §4],

$$V \subseteq \mathrm{HOD}^{V[G]} \subseteq V[G^\phi],$$

and $V[G^\phi]$ is a cardinal-preserving generic extension of V.

Therefore,

$$V[G] \models \text{``}\theta^{+\mathrm{HOD}^{V[G]}} = \theta^{+V} < \theta^+\text{''}.$$

The above argument disposes with the first bullet of (3) of the theorem. As for the θ being regular in HOD we argue as follows: Since the GCH holds below θ and $\ell(w) \geq \theta^{++}$ the measure sequence

$$\langle \theta \rangle^\frown \langle \phi(w(\alpha)) : 1 \leq \alpha < \ell(w) \rangle$$

on V_θ has a repeat point. Thus, by Lemma 4.7 above, θ remains measurable in the generic extension by $\mathbb{R}_w^{\mathrm{proj}}$. Forcing with the projected forcing below a condition of the form $\langle (\phi(w), B), (\phi(u), A) \rangle$ we know that

$$\mathbb{R}_{\phi(u)}^{\mathrm{proj}}/\langle (\phi(w), B), (\phi(u), A) \rangle \simeq \mathbb{R}_{\phi(w)}^{\mathrm{proj}}/\langle (\phi(w), B) \rangle \times \mathbb{R}_{\phi(u)}^{\mathrm{proj}}/\langle (\phi(u), A^*) \rangle$$

(see [CFG15, Lemma 3.6]). Note that the second factor is distributive enough to preserve regularity of θ. Therefore we deduce that θ is regular. In particular, θ is regular in $V[G^\phi]$, hence in $\mathrm{HOD}^{V[G]}$ as well. $\qquad \square$

Question 3. *Assume the* HOD *hypothesis holds and that δ is the first supercompact cardinal. Can δ be ω-strongly measurable in* HOD*?*

By [Gol23, Theorem 2.11] if δ is regular and N is an inner model such that $\mathrm{Reg}^N \cap \delta$ contains an ω-club then it actually contains a club. Therefore, a positive answer to Question 3 yields a stronger configuration than the one arranged in Theorem 4.6. In contrast, by [Gol23, Propositive 2.3], if the HOD hypothesis holds and δ is strongly compact no regular cardinal strictly above δ is ω-strongly measurable in HOD.

A related open question is whether Theorem 4.6 can be established below the first HOD-supercompact cardinal δ. This is particularly interesting because, under the HOD hypothesis and the existence of a HOD-supercompact,

HOD exhibits a behavior similar to that under the existence of an extendible cardinal (e.g., HOD satisfies δ-covering and δ-approximation). Nonetheless, it is known that a configuration like that of Theorem 4.6 is impossible below the first extendible cardinal. Therefore, it is reasonable to ask the following:

Question 4. *Is the configuration described in Theorem 4.6 consistent with δ being* HOD-*supercompact?*

Also, it is not difficult to produce a model in which δ is $C^{(n)}$-supercompact, the HOD hypothesis holds, and there are unboundedly many $\kappa < \delta$ such that $\kappa^{+\mathrm{HOD}} < \kappa^+$. The idea is to force with a Magidor product of supercompact Prikry forcings on a sufficiently sparse set of cardinals $\kappa < \delta$ that are κ^+-supercompact. The resulting forcing is cone-homogeneous and preserves $C^{(n)}$-supercompacts by the arguments in [HMP22]. The situation becomes less clear if one attempts to arrange this configuration on a club of cardinals κ. In other words, it remains open whether the Cummings–Friedman–Golshani configuration [CFG15] can be extended to the case where δ is $C^{(n)}$-supercompact. In fact, more can be said: in the model of [CFG15], δ is not $C^{(1)}$-supercompact cardinal per the argument in [Pov24, Theorem 5.2].

Question 5. *Is it consistent for the first $C^{(1)}$-supercompact δ to carry a club $C \subseteq \delta$ where $\kappa^{+\mathrm{HOD}} < \kappa^+$?*

4.2. **Where ω-strong measurability takes hold.** This section pertains to Goldberg's result [Gol23] that if the HOD hypothesis fails and there is a strongly compact cardinal, then all sufficiently large regular cardinals are ω-strongly measurable in HOD.

For the purposes of this subsection, let us assume that κ is the least strongly compact cardinal and the HOD hypothesis fails. Let η_0 be the least ordinal greater than or equal to κ such that all regular cardinals greater than or equal to η_0 are ω-strongly measurable in HOD. What can be said about the size of η_0? Recall that Woodin's HOD dichotomy shows that if δ is extendible, then all regular cardinals greater than or equal to δ are ω-strongly measurable in HOD [Woo17]. Therefore $\eta_0 \le \delta$. In fact, definability considerations imply that $\eta_0 < \delta$. This is because η_0 is Σ_3-definable from the parameter κ, and so in fact η_0 is strictly less than the least Σ_3-reflecting cardinal above κ, which is well below the first extendible cardinal δ.

The analysis of [Gol23] shows that HOD is quite close to V between κ and η_0. For example, no regular cardinal between κ and η_0 is ω-strongly measurable in HOD. Moreover, HOD has the κ-cover property below η_0 and correctly computes all successors of singular cardinals between κ and η_0.

Our next proposition shows that our assumptions provide essentially no insight into the size of η_0 relative to κ. In particular, this shows that the strongly compact (or supercompact) HOD dichotomy theorem from [Gol23] cannot be significantly improved.

Theorem 4.9. *Suppose δ is supercompact and the* HOD *hypothesis fails. Then for any cardinal λ, there is a forcing extension $V[G]$ in which δ*

is indestructibly supercompact, the HOD hypothesis fails, and $\mathcal{P}^{V[G]}(\lambda) \subseteq$ $\mathrm{HOD}^{V[G]}$. In particular, $\eta_0^{V[G]} \geq \lambda$.

Proof. First force using the Laver preparation to produce a generic extension $V[g]$ in which the supercompactness of δ is indestructible under δ-directed closed forcings. Then over $V[g]$, use a λ^+-directed-closed forcing to produce an extension $V[G] = V[g][h]$ such that every subset of λ in $V[g]$ is coded into the continuum function of $V[G]$ above λ (recall the coding poset from p.9). Then $\mathcal{P}^{V[G]}(\lambda) = \mathcal{P}^{V[g]}(\lambda) \subseteq \mathrm{HOD}^{V[G]}$, and in particular $\eta_0^{V[G]} \geq \alpha$. □

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