

On products of ultrafilters

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Abstract

Assuming the Generalized Continuum Hypothesis, this paper answers the question: when is the tensor product of two ultrafilters equal to their Cartesian product? It is necessary and sufficient that their Cartesian product is an ultrafilter; that the two ultrafilters commute in the tensor product; that for all cardinals λ , one of the ultrafilters is both λ -indecomposable and λ^+ -indecomposable; that the ultrapower embedding associated to each ultrafilter restricts to a definable embedding of the ultrapower of the universe associated to the other.

1 Introduction

There are two binary operations on the class of ultrafilters commonly referred to as the product, each of which fails to live up to the name. The first, the Cartesian product, truly is a product in the category of filters. The subcategory of ultrafilters, however, is not closed under Cartesian products. The other, the tensor product, is at least an operation on ultrafilters. But it is not a product. It is not even commutative (up to isomorphism). This raises the following questions:

- Question 1.1.** (1) When is the Cartesian product of two ultrafilters an ultrafilter?
(2) On which pairs of ultrafilters is the tensor product commutative?
(3) When do the Cartesian and tensor products coincide?

For ultrafilters on ω , the answer to all three questions is essentially *never*. The general question, however, is much more subtle and involves large cardinals. The main conjecture motivating this paper is that all three questions have the same answer:

Conjecture 1.2. *The Cartesian product of two ultrafilters is an ultrafilter if and only if the ultrafilters commute under the tensor product if and only if their Cartesian product is equal to their tensor product.*

We now define the two products in order to state this conjecture more precisely.

Definition 1.3. If F and G are filters on sets X and Y , the *Cartesian product* of F and G , denoted $F \times G$, is the filter on $X \times Y$ generated by sets of the form $A \times B$ where $A \in F$ and $B \in G$.

The usual notation for the tensor product of ultrafilters is \otimes , but the following notation is more convenient here:

Definition 1.4. Suppose F and G are filters on sets X and Y . Their *left and right tensor products* are defined as follows:

$$\begin{aligned} F \ltimes G &= \{R \subseteq X \times Y : \{x \in X : (R)_x \in G\} \in F\} \\ F \rtimes G &= \{R \subseteq X \times Y : \{y \in Y : (R)^y \in F\} \in G\} \end{aligned}$$

Here $(R)_x = \{y \in Y : (x, y) \in R\}$ and $(R)^y = \{x \in X : (x, y) \in R\}$.

Our main conjecture now reads:

Conjecture 1.5. *For any ultrafilters U and W , the following are equivalent:*

- (1) $U \times W$ is an ultrafilter.
- (2) $U \ltimes W = U \rtimes W$.
- (3) $U \ltimes W = U \times W$.

The implication from (1) to (2) and the equivalence of (1) and (3) are easy to see: $U \times W$ is certainly contained in both $U \ltimes W$ and $U \rtimes W$, so if $U \times W$ is an ultrafilter, its maximality implies that $U \ltimes W = U \times W = U \rtimes W$. There is really just one open question:

Question 1.6. Suppose U and W are ultrafilters such that $U \ltimes W = U \rtimes W$. Must $U \times W$ be an ultrafilter?

We will answer this question positively assuming the Generalized Continuum Hypothesis:

Theorem 1.7 (GCH). *Suppose U and W are ultrafilters. Then $U \ltimes W = U \rtimes W$ if and only if $U \times W$ is an ultrafilter.*

It is *not* true that if $U \ltimes W$ and $W \ltimes U$ are Rudin-Keisler equivalent, then $U \times W$ is an ultrafilter. For example, if $U = W$, then $U \times W$ is never an ultrafilter, but obviously $U \ltimes W = W \ltimes U$. So it is not enough that $U \ltimes W \equiv_{\text{RK}} W \ltimes U$; it is also necessary that this equivalence be witnessed by the coordinate-swapping bijection from $X \times Y$ to $Y \times X$ where X and Y are the underlying sets of U and W .

The theorem is proved by isolating a combinatorial criterion that arguably answers Question 1.1 under GCH.

Definition 1.8. A filter F is (λ, η) -*indecomposable* if for any family of sets \mathcal{P} such that $|\mathcal{P}| \leq \eta$ and $\bigcup \mathcal{P} \in F$, there is some $\mathcal{Q} \subseteq \mathcal{P}$ of cardinality less than λ such that $\bigcup \mathcal{Q} \in F$.

Theorem 2.14. *Suppose U and W are ultrafilters such that for all cardinals λ , either U or W is $(\lambda, 2^\lambda)$ -indecomposable. Then $U \times W$ is an ultrafilter.*

Theorem 2.15. *Suppose U and W are ultrafilters such that $U \ltimes W = U \rtimes W$. Then for all cardinals λ , either U or W is (λ, λ^+) -indecomposable.*

In addition, we will provide a third, somewhat more surprising condition equivalent to the statement that $U \times W$ is an ultrafilter. This involves a variant of the Mitchell order called the *internal relation*. Recall that the Mitchell order is defined on countably complete ultrafilters U and W by setting $U \triangleleft W$ if U belongs to the ultrapower of the universe by W , which will be denoted by M_W . The question of whether $U \in M_W$ only makes sense when W is countably complete, so that M_W can be identified with a transitive class. The internal relation is a combinatorial proxy for the Mitchell order that in particular makes sense regardless of the completeness of the ultrafilters in question.

Definition 1.9. The *internal relation* is defined on ultrafilters U and W by setting $U \sqsubset W$ if $j_U \restriction M_W$ is isomorphic to an internal ultrapower embedding of M_W .¹

Here $j_U : V \rightarrow M_U$ denotes the ultrapower embedding, as it will throughout the paper.

One advantage of the internal relation over the Mitchell order is that the former relation is more amenable to algebraic or diagrammatic techniques. This is exploited in the author's thesis to give a complete analysis of the internal relation on countably complete ultrafilters assuming the *Ultrapower Axiom*. The Ultrapower Axiom (UA) is a combinatorial principle motivated by inner model theory that enables one to develop a theory of countably complete ultrafilters that is far more detailed than the theory available assuming ZFC alone. The theory of countably complete ultrafilters encompasses a major part of the theory of large cardinals, and so UA is a natural setting for large cardinal theory.²

Conjecture 1.5 was originally motivated by the following consequence of UA:

Theorem (UA). *For any countably complete ultrafilters U and W , $U \times W$ is an ultrafilter if and only if $U \times W = U \times W$ if and only if $U \sqsubset W$ and $W \sqsubset U$.*

We show here that this theorem can be proved for arbitrary ultrafilters using GCH instead of UA:

Theorem 3.18 (GCH). *For any ultrafilters U and W , $U \times W$ is an ultrafilter if and only if $U \sqsubset W$ and $W \sqsubset U$.*

The proof of Theorem 3.18 is motivated by ideas developed in the context of UA, and more specifically it makes use of the two main tools from [2], the Ketonen order and the factorization of ultrafilters. Arguably, the theorem could not have proved without the help of UA, even though UA makes no appearance in the proof.

2 Preliminaries

2.1 Ultrafilters and the Rudin-Keisler order

If U is an ultrafilter on a set X and $f : X \rightarrow Y$ is a function, the *pushforward* of U by f is the ultrafilter $f_*(U) = \{A \subseteq Y : f^{-1}[A] \in U\}$. If $W = f_*(U)$, then there is a unique elementary embedding $k : M_W \rightarrow M_U$ such that $k \circ j_W = j_U$ and $k([id]_W) = [f]_U$. Conversely, if there is an elementary embedding $k : M_W \rightarrow M_U$ such that $k \circ j_W = j_U$, then $W = f_*(U)$ for some f . In this case (that is, if W is a pushforward of U), we say that W precedes U in the *Rudin-Keisler order*, and write $W \leq_{RK} U$.

Suppose M is a model of ZFC and $X \in M$. An M -ultrafilter on X is an ultrafilter over the Boolean algebra $P^M(X)$. Given an M -ultrafilter U on X , one can form the associated ultrapower, denoted M_U^M , by taking the quotient $M^X \cap M$ (the class of functions on X as computed in M) under the equivalence relation of U -almost everywhere equality (which may not be definable over M). The equivalence class of $f \in M^X \cap M$ is denoted by $[f]_U^M$. The associated ultrapower embedding $j_U^M : M \rightarrow M_U^M$ is given by $j(s) = [c_s]_U^M$ where $c_s : X \rightarrow \{s\}$ is the constant function.

¹This means that there is an ultrafilter \tilde{U} of M_W and an isomorphism $k : (M_{\tilde{U}})^{M_W} \rightarrow j_U(M_W)$ such that $k \circ (j_{\tilde{U}})^{M_W} = j_U$.

²It is unknown, however, whether UA is consistent with large cardinals beyond a superstrong cardinal, or more generally with large cardinals for which a canonical inner model theory has not been developed. The author conjectures that UA is consistent with all large cardinal axioms.

Suppose $j : M \rightarrow N$ is an elementary embedding. For any $X \in M$ and $a \in N$ such that $N \models a \in j(X)$, the M -ultrafilter on X derived from j using a , is the ultrafilter $D_X(j, a) = \{A \in P^M(X) : N \models a \in j(A)\}$. Letting $D = D_X(j, a)$, there is again a canonical factor embedding $k : M_D^M \rightarrow N$, denoted $k_{a,j}$, uniquely determined by requiring $k \circ j_D = j$ and $k([\text{id}]_D) = a$. Now if $j : M \rightarrow N$ is an ultrapower embedding, there is some $X \in M$ and $a \in j(X)$ such that every element of N is definable in N from parameters in $j[M] \cup \{a\}$. Let $D = D_X(j, a)$. Then $k_{a,j} : M_D^M \rightarrow N$ is surjective since its range contains $j[M] \cup \{a\}$, and so k is an isomorphism. Note that if $f_*(U) = W$, then W is the ultrafilter derived from j_U using $a = [f]_U$, and k_{a,j_U} is the canonical factor embedding.

2.2 The symmetry of regularity

Part of the motivation for this work was to explain the connection between a theorem from Blass's thesis [1, Theorem 3.6] and Kunen's Commuting Ultrapowers Lemma [4, Lemma 1.1.25]. These lemmas concern two binary relations on ultrafilters that despite appearances turn out to be symmetric.

Definition 2.1. Suppose F and G are filters on sets X and Y .

- G is F -regular if there is a sequence $\langle B_x : x \in X \rangle \subseteq G$ such that for any $A \in F^+$, $\bigcap_{x \in A} B_x = \emptyset$.
- G is F -closed if for any sequence $\langle B_x : x \in X \rangle \subseteq G$, there is some $A \in F$ such that $\bigcap_{x \in A} B_x \in G$.

In this section, we show that our main question (Question 1.1) reduces to the problem of whether, given an ultrafilter U , every U -nonregular ultrafilter is U -closed.

The concept of regularity is a natural generalization of the classical concept of a regular ultrafilter:

Definition 2.2. If $\kappa \leq \lambda$ are cardinals, a filter G is (κ, λ) -regular if there is a sequence $\langle B_\alpha : \alpha < \lambda \rangle \subseteq G$ such that for any $\sigma \subseteq \lambda$ with $|\sigma| \geq \kappa$, $\bigcap_{\alpha \in \sigma} B_\alpha = \emptyset$.

In other words, G is (κ, λ) -regular if it is $F_\kappa(\lambda)$ -regular where $F_\kappa(\lambda)$ is the dual of the ideal $P_\kappa(\lambda) = \{\sigma \subseteq \lambda : |\sigma| < \kappa\}$.

The concept of closure is extracted from the following theorem from Blass's thesis:

Theorem 2.3 (Blass, [1, Theorem 3.6]). *Suppose U and W are ultrafilters on sets X and Y . Then the following are equivalent:*

- (1) $U \times W$ is an ultrafilter.
- (2) W is U -closed.
- (3) U is W -closed.

Although the symmetry expressed in Theorem 2.3 (2) and (3) does not extend to filters, one can deduce Blass's Theorem from a more general filter-theoretic fact:

Lemma 2.4. *Suppose F and G are filters. Then G is F -closed if and only if $F \times G = F \times G$.*

Proof. In general, $F \times G \subseteq F \times G$, so it suffices to prove that the reverse inclusion is equivalent to the F -closure of G . This is a routine matter of applying the correspondence between binary relations $R \subseteq X \times Y$ and X -indexed sequences $\langle B_x : x \in X \rangle \subseteq Y$. \square

Proof of Theorem 2.3. It suffices to prove the equivalence of (1) and (2), since the obvious symmetry of (1) in U and W then immediately implies the equivalence of (2) and (3). If $U \times W$ is an ultrafilter, then since $U \times W$ is a maximal filter contained in $U \times W$, $U \times W = U \times W$, and hence W is U -closed by Lemma 2.4. Conversely if W is U -closed, then $U \times W = U \times W$ by Lemma 2.4, and so since $U \times W$ is an ultrafilter, $U \times W$ is. \square

It turns out that regularity, like closure, is also a symmetric relation, but this time the symmetry does extend to all filters. Despite the extensive literature on regular ultrafilters, as far as the author knows, this symmetry has never been observed.

Lemma 2.5. *Suppose F and G are filters. Then F is G -regular if and only if $F \times G$ is incompatible with $F \times G$.*

Proof. First assume F is G -nonregular. Fix $R, S \subseteq X \times Y$ with $R \in F \times G$ and $S \in F \times G$, and let us show that $R \cap S$ is nonempty. Take $A \in F$ such that for all $x \in A$, $(R)_x \in G$. Note that since $S \in F \times G$, for G -almost all $y \in Y$, $(S)^y \in F$, and hence $(S)^y \cap A \in F$. Consider the sequence $\langle (S)^y \cap A : y \in Y \rangle$. Since F is G -nonregular, there is a G -positive set B such that $\bigcap_{y \in B} (S)^y \cap A \neq \emptyset$. Fix $x \in \bigcap_{y \in B} (S)^y \cap A$. Since $x \in A$, $(R)_x \in G$, and therefore $(R)_x \cap B \neq \emptyset$. Since $x \in \bigcap_{y \in B} (S)^y$, for any $y \in (R)_x \cap B$, $x \in (S)^y$; but then $(x, y) \in R \cap S$.

Conversely, suppose F is G -regular, and let $\langle A_y : y \in Y \rangle \subseteq F$ witnesses this. Let $B_x = \{y \in Y : x \notin A_y\}$. We claim that $B_x \in G$ for all $x \in X$. This is because $x \in \bigcap_{y \in Y \setminus B_x} A_y$, so $Y \setminus B_x \notin G^+$, and hence $B_x \in G$. But let $\mathcal{A} = \{(x, y) : x \in A_y\}$ and $\mathcal{B} = \{(x, y) : y \in B_x\}$. Then $\mathcal{A} \in F \times G$, $\mathcal{B} \in F \times G$, and $\mathcal{A} = (X \times Y) \setminus \mathcal{B}$. This shows $F \times G$ is incompatible with $F \times G$. \square

Theorem 2.6. *Suppose F and G are filters. Then F is G -regular if and only if G is F -regular.* \square

For ultrafilters, we of course get more.

Theorem 2.7. *If U and W are ultrafilters, then U is W -nonregular if and only if $U \times W = U \times W$.* \square

Let us just mention one more familiar characterization of regularity.

Definition 2.8. If F is a family of sets, the *dual of F* is the family F^* of sets of the form $(\bigcup F) \setminus A$ for $A \in F$. The *fine filter on the dual of F* , denoted $\mathbb{F}(F)$, is the filter on F^* generated by sets of the form $\{\sigma \in F^* : x \in \sigma\}$ where $x \in \bigcup F$.

The prototypical special case of this concept arises when $F = F_\kappa(\lambda)$ is the filter of subsets of λ with complement of cardinality less than κ , so that $\mathbb{F}(F)$ is the fine filter on $P_\kappa(\lambda)$. Note that if $\bigcap F \neq \emptyset$, then $\mathbb{F}(F)$ is improper.

The Katětov order is the natural generalization of the Rudin-Keisler order to filters:

Definition 2.9. Suppose F is a filter on X and G is a filter on Y . The *Katětov order* is defined by setting $G \leq_{\text{kat}} F$ if there is a function $f : X \rightarrow Y$ such that $G \subseteq f_*(F)$.

Proposition 2.10. *A filter F is G -regular if and only if $\mathbb{F}(F) \leq_{\text{kat}} G$.*

Proof. Let X be the underlying set of F . Notice that $\langle A_y : y \in Y \rangle$ witnesses that F is G -regular if and only if the function $f(y) = X \setminus A_y$ pushes G forward to $\mathbb{F}(F)$. \square

An immediate consequence of Proposition 2.10 is that the F -regular filters are closed upwards in the Katětov order, or dually:

Lemma 2.11. *Suppose F, G and H are filters such that F is G -regular and $G \leq_{\text{kat}} H$. Then F is H -regular.* \square

Put another way, if F is H -nonregular, then the class of F -regular filters contains no Katětov predecessors of H .

Finally, we explain the connection between the combinatorial structure we have been exploring and Kunen's Commuting Ultrapowers Lemma.

Definition 2.12. If U is an ultrafilter, then $\text{size}(U)$ denotes the minimum cardinality of a set in U .

Kunen's Commuting Ultrapowers Lemma is the following fact, which we have already considerably generalized.

Theorem 2.13 (Kunen, [4, Lemma 1.1.25]). *Suppose U and W are countably complete ultrafilters such that W is $\text{size}(U)^+$ -complete. Then $j_W(j_U) = j_U \restriction M_W$ and $j_U(j_W) = j_W \restriction M_U$.*

Proof. By Lemma 3.5, $j_U \restriction M_W$ is the ultrapower of M_W by $s_W(U)$, while $j_W(j_U)$ is clearly the ultrapower of M_W by $j_W(U)$. So to show $j_W(j_U) = j_U \restriction M_W$, it suffices to show $s_W(U) = j_W(U)$, and similarly for $j_U(j_W) = j_W \restriction M_U$. But since W is $\text{size}(U)^+$ -complete, W is U -closed. Hence $U \times W$ is an ultrafilter, which implies $U \times W = U \rtimes W$, and so by Theorem 2.7, $s_W(U) = j_W(U)$ and $s_U(W) = j_U(W)$. \square

2.3 Indecomposable ultrafilters

In this subsection, we prove the following theorems:

Theorem 2.14. *Suppose U and W are ultrafilters such that for all cardinals λ , either U or W is $(\lambda, 2^\lambda)$ -indecomposable. Then $U \times W$ is an ultrafilter.*

Theorem 2.15. *Suppose U and W are ultrafilters such that $U \times W = U \rtimes W$. Then for all cardinals λ , either U or W is (λ, λ^+) -indecomposable.*

As an immediate corollary of Theorem 2.14 and Theorem 2.15, we obtain our main theorem:

Theorem 2.16 (GCH). *For any ultrafilters U and W , $U \times W = U \rtimes W$ if and only if $U \times W$ is an ultrafilter.* \square

Recall the notion of indecomposability introduced above (Definition 1.8).

Definition 2.17. If λ is a cardinal, F is λ -indecomposable if F is (λ, λ) -indecomposable.

It is easy to see that U is (λ, η) -indecomposable (see Definition 1.8) if and only if U is γ -indecomposable whenever $\lambda \leq \gamma \leq \eta$. We will use the term λ -decomposable in the obvious way. (We avoid the notion of (λ, η) -decomposability in general, however, to avoid potential ambiguity.) We will use the following straightforward lemma:

Lemma 2.18. *A filter F is λ -decomposable if and only if $F_\lambda(\lambda) \leq_{\text{kat}} F$ where $F_\lambda(\lambda)$ is the Fréchet filter on λ .* \square

We start with the proof of Theorem 2.14.

Proof of Theorem 2.14. Suppose $U \times W$ is not an ultrafilter. We will show that there is some cardinal λ such that neither U nor W is $(\lambda, 2^\lambda)$ -decomposable. Let X be a set of minimal cardinality carrying an ultrafilter $\tilde{U} \leq_{\text{RK}} U$ such that $\tilde{U} \times W$ is not an ultrafilter. Let Y be a set of minimal cardinality carrying an ultrafilter $\tilde{W} \leq_{\text{RK}} W$ such that $\tilde{U} \times \tilde{W}$ is not an ultrafilter.

Let $\langle A_y : y \in Y \rangle \subseteq \tilde{U}$ witness that \tilde{U} is not \tilde{W} -closed. Let $f : Y \rightarrow P(X)$ be the function $f(y) = A_y$. Then \tilde{U} is not $f_*(\tilde{W})$ -closed: indeed, setting $A_B = B$ if $B \in \text{ran}(f)$ and $A_B = X$ otherwise, the sequence $\langle A_B : B \in P(X) \rangle$ witnesses that \tilde{U} is not $f_*(\tilde{W})$ -closed. The minimality of the cardinality of Y therefore implies $|Y| \leq 2^{|X|}$.

Similarly, $|X| \leq 2^{|Y|}$. Let $\lambda = \min\{|X|, |Y|\}$. Then \tilde{U} and \tilde{W} are both uniform ultrafilters on sets whose cardinalities lie in the interval $[\lambda, 2^\lambda]$. Since $\tilde{U} \leq_{\text{RK}} U$ and $\tilde{W} \leq_{\text{RK}} W$, this implies (by Lemma 2.18) that neither U nor W is $(\lambda, 2^\lambda)$ -indecomposable. \square

We now turn to Theorem 2.15, which follows from a simple combinatorial fact about the relationship between regularity and decomposability.

Corollary 2.19. *Suppose F and G are filters such that F is G -nonregular and λ is a cardinal such that G is λ -decomposable. Then F is (λ, λ) -nonregular.*

Proof. Assume F is (λ, λ) -regular, towards a contradiction. In other words, F is $F_\lambda(\lambda)$ -regular where $F_\lambda(\lambda)$ denotes the Fréchet filter on λ . But G is λ -decomposable if and only if $F_\lambda(\lambda) \leq_{\text{kat}} G$. So by Lemma 2.11, F is G -regular, contrary to hypothesis. \square

Theorem 2.15 will be a consequence of Corollary 2.19 using the following theorem, considerably generalized by Lipparini [5, Theorem 2.2]:

Lemma 2.20. *If U is λ^+ -decomposable, either U is $\text{cf}(\lambda)$ -decomposable or U is (κ, λ^+) -regular for some $\kappa < \lambda$. In particular, if λ is regular, then U is λ -decomposable.* \square

Lemma 2.21. *Suppose U is (λ, λ) -nonregular. Then U is (λ, λ^+) -indecomposable.*

Proof. Clearly U is $\text{cf}(\lambda)$ -indecomposable (and in particular, U is λ -indecomposable). Therefore by Lemma 2.20, if U were λ^+ -decomposable, then U would be (γ, λ^+) -regular for some $\gamma < \lambda$, but this of course would imply that U is (λ, λ) -regular, contrary to assumption. Therefore U is λ^+ -indecomposable. It follows that U is λ -indecomposable and λ^+ -indecomposable, so U is (λ, λ^+) -indecomposable. \square

Theorem 2.15 is now immediate.

Proof of Theorem 2.15. Assume that W is not (λ, λ^+) -indecomposable, and we will show that U is. Since $U \times W = U \times W$, U is W -nonregular. Therefore by Corollary 2.19, if W is λ -decomposable, then U is (λ, λ) -nonregular. It follows that U is (λ, λ^+) -indecomposable by Lemma 2.21. \square

3 The internal relation

3.1 Amalgamations

Our main questions concern the relationship between $U \times W$ and its extensions $U \ltimes W$ and $U \rtimes W$. More generally, one can consider arbitrary ultrafilters extending of $U \times W$, which we will call *amalgamations* of (U, W) . An amalgamation D of (U, W) is an upper bound of U and W in the Rudin-Keisler order, since $U = (\pi_0)_*(D)$ and $W = (\pi_1)_*(D)$ where π_n denotes projection to the n -th coordinate.

There is a second way to view amalgamations: as iterated ultrapowers. Suppose U is an ultrafilter on X and W is an ultrafilter on Y . If D is an amalgamation of (U, W) , then there is a unique elementary embedding $k : M_U \rightarrow M_D$ such that $k \circ j_U = j_D$ and $k([\text{id}]_U) = [\pi_0]_D$. This embedding is isomorphic to the ultrapower of M_U by an M_U -ultrafilter, defined in this context as follows:

Definition 3.1. Suppose U and W are ultrafilters on sets X and Y and D is an amalgamation of (U, W) . Then $(D)_0$ denotes the M_U -ultrafilter on $j_U(Y)$ derived from k using $[\pi_1]_D$ where $k : M_U \rightarrow M_D$ is the canonical factor embedding.

The ultrafilter $(D)_1$ is defined similarly, so $(D)_1 = (\check{D})_0$ where \check{D} denotes the amalgamation of (W, U) given by D . One can compute:

Lemma 3.2. *If D is an amalgamation of (U, W) , then*

$$(D)_0 = \{[f_R]_U : R \in D\}$$

where $f_R : X \rightarrow P(Y)$ is given by $f_R(x) = (R)_x = \{y \in Y : (x, y) \in R\}$. \square

To prove that the canonical factor embedding $k : M_U \rightarrow M_D$ is isomorphic to the ultrapower embedding of M_U associated to $(D)_0$, one must show that the canonical factor embedding $h : M_{(D)_0}^{M_U} \rightarrow M_D$ is an isomorphism. It is enough to show h is surjective, and this holds because every element of M_D is definable from parameters among $j_D[V]$, $[\pi_0]_D$, and $[\pi_1]_D$, and all these parameters lie in the range of h : $j_D[V] = h \circ j_{(D)_0}^{M_U} \circ j_U[V]$, $[\pi_0]_D = h(j_{(D)_0}^{M_U}([\text{id}]_U))$, and $[\pi_1]_D = h([\text{id}]_{(D)_0}^{M_U})$.

3.2 Amalgamations and the internal relation

The internal relation is closely related to the ultrafilters $U \ltimes W$ and $U \rtimes W$, viewed as amalgamations of (U, W) . This subsection is devoted to this relationship.

Lemma 3.3. *If U and W are ultrafilters, then $(U \ltimes W)_0 = j_U(W)$.* \square

The ultrafilter $(U \ltimes W)_1$ is more interesting.

Definition 3.4. If U and W are ultrafilters, then $s_W(U) = (U \ltimes W)_1$.

The following lemma shows that $s_W(U)$ is in a sense the canonical M_W -ultrafilter giving rise to the ultrapower embedding $j_U \upharpoonright M_W$.

Lemma 3.5. *Suppose U and W are ultrafilters on sets X and Y . Then $s_W(U)$ is the M_W -ultrafilter on $j_W(X)$ derived from $j_U \upharpoonright M_W : M_W \rightarrow j_U(M_W)$ using $[j_W]_U$, and the canonical factor embedding from $M_{s_W(U)}^{M_W}$ to $j_U(M_W)$ is an isomorphism. Finally, $s_W(U)$ is given by the formula*

$$s_W(U) = \{A \in j_W(P(X)) : j_W^{-1}[A] \in U\}$$

Proof. By definition, $s_W(U)$ is the M_W -ultrafilter on $j_W(X)$ derived from the canonical factor embedding $i : M_W \rightarrow M_{U \times W}$ using $[\pi_0]_{U \times W}$. In general, the factor embedding $p : M_{s_W(U)}^{M_W} \rightarrow M_{U \times W}$ is an isomorphism; see the remarks following Lemma 3.2.

We first show $s_W(U)$ is the M_W -ultrafilter on $j_W(X)$ derived from $j_U \upharpoonright M_W$ using $[j_W]_U$. Let $k : M_U \rightarrow M_{U \times W}$ denote the canonical factor embedding. By Lemma 3.3, there is an isomorphism $h : M_{U \times W} \rightarrow M_{j_U(W)}^{M_U}$ such that $h \circ k$ is equal to the ultrapower embedding of M_U associated to $j_U(W)$ and $h([\pi_1]_{U \times W}) = [\text{id}]_{j_U(W)}$.

We claim that

$$h \circ i = j_U \upharpoonright M_W$$

Note that by elementarity $j_U(M_W) = M_{j_U(W)}^{M_U}$, so $h \circ i$ and $j_U \upharpoonright M_W$ at least have the same codomain. It suffices to show that $h \circ i$ and j_U agree on the parameters $j_W[V] \cup \{[\text{id}]_W\}$, from which all other elements of M_W are definable in M_W . First,

$$h \circ i \circ j_W = h \circ j_{U \times W} = j_{j_U(W)} \circ j_U = j_U \circ j_W$$

and this implies $h \circ i \upharpoonright j_W[V] = j_U \upharpoonright j_W[V]$. Second,

$$h(i([\text{id}]_W)) = h([\pi_1]_{U \times W}) = [\text{id}]_{j_U(W)} = j_U([\text{id}]_W)$$

This proves $h \circ i = j_U \upharpoonright M_W$.

Now since $s_W(U)$ is the M_W -ultrafilter on $j_W(X)$ derived from the canonical factor embedding $i : M_W \rightarrow M_{U \times W}$ using $[\pi_0]_{U \times W}$, $s_W(U)$ is the M_W -ultrafilter on $j_W(X)$ derived from $h \circ i$ using $h([\pi_0]_{U \times W})$. But $h \circ i = j_U \upharpoonright M_W$ and

$$h([\pi_0]_{U \times W}) = h(k([\text{id}]_U)) = j_U(j_W)([\text{id}]_U) = [j_W]_U$$

This proves $s_W(U)$ is the M_W -ultrafilter on $j_W(X)$ derived from $j_U \upharpoonright M_W$ using $[j_W]_U$.

The canonical factor embedding $p : M_{s_W(U)}^{M_W} \rightarrow M_{U \times W}$ is an isomorphism, and the embedding $h : M_{U \times W} \rightarrow j_U(M_W)$ is an isomorphism, so $h \circ p : M_{s_W(U)}^{M_W} \rightarrow j_U(M_W)$ is an isomorphism, and by uniqueness, $h \circ p$ is the canonical factor embedding from $M_{s_W(U)}^{M_W}$ to $j_U(M_W)$. This proves that the canonical factor embedding from $M_{s_W(U)}^{M_W}$ to $j_U(M_W)$ is an isomorphism.

The pushforward $(j_W)_*(U) = \{A \subseteq j_W(X) : j_W^{-1}[A] \in U\}$ is as usual equal to the ultrafilter derived from j_U using $[j_W]_U$. Therefore $s_W(U)$, being the ultrafilter derived from $j_U \upharpoonright M_W$ using $[j_W]_U$, is equal to $(j_W)_*(U) \cap M_W$. In other words,

$$s_W(U) = \{A \in j_W(P(X)) : j_W^{-1}[A] \in U\} \quad \square$$

From Lemma 3.5, we immediately obtain:

Lemma 3.6. *For any ultrafilters U and W , $U \sqsubset W$ if and only if $s_W(U) \in M_W$.* \square

Given this equivalence, the internal relation looks quite a bit like the Mitchell order, the difference being that instead of demanding that the V -ultrafilter U belong to M_W , one demands that the corresponding M_W -ultrafilter $s_W(U)$ does.

The basic relationship between the internal relation and products of ultrafilters is the following:

Lemma 3.7. *If U and W are ultrafilters, then U is W -nonregular if and only if $s_W(U) = j_W(U)$.*

Proof. Note that an amalgamation D of (U, W) is uniquely determined by U and $(D)_0$ (or W and $(D)_1$): by Lemma 3.2,

$$D = \{R \subseteq X \times Y : [f_R]_U \in (Z)_0\}$$

Now $(U \rtimes W)_1 = (W \rtimes U)_0 = j_W(U)$ by Lemma 3.3 and $(U \rtimes W)_1 = s_W(U)$ by definition, so $s_W(U) = j_W(U)$ if and only if $U \rtimes W = U \rtimes W$. \square

3.3 The internal relation and the Ketonen order

The following theorem is almost proved in [3], but here we are interested in ultrafilters that may not be countably complete, so a new argument is needed.

Theorem 3.8. *Suppose U and W are ultrafilters such that $U \sqsubset W$ and $W \sqsubset U$. Then for any cardinal δ , either U or W is δ -indecomposable.*

Theorem 3.17 below improves this theorem in the context of GCH, but we do not know how to prove Theorem 3.17 in ZFC.

To prove Theorem 3.8, we introduce a version of the Ketonen order [3, ?] on countably incomplete ultrafilters:

Definition 3.9. If X is a set, a *normed ultrafilter on X* is a pair (U, f) where U is an ultrafilter on a set X and $f : X \rightarrow \text{Ord}$ is a function. If δ is an ordinal, (U, f) is δ -*normed* if $f(x) < \delta$ for U -almost all $x \in X$.

Definition 3.10. The *Ketonen order* is defined on normed ultrafilters (U, f) and (W, g) by setting $(U, f) <_{\mathbf{k}} (W, g)$ if there is some $U_* \in M_W$ extending $j_W[U]$ such that $(U_*, j_W(f))$ is a $[g]_W$ -normed ultrafilter in M_W .

Equivalently, $(U, f) <_{\mathbf{k}} (W, g)$ if there is a sequence $\langle U_y : y \in Y \rangle$ of ultrafilters such that $U = W\text{-}\lim_{y \in Y} U_y$ and for W -almost all y , for U_y -almost all x , $f(x) < g(y)$.

If M and N are models of set theory and $j : M \rightarrow N$ is an elementary embedding, then j is *close to M* if for all $A \in N$, $j^{-1}[A]$ is the extension of a set in M .

Lemma 3.11. *For normed ultrafilters (U, f) and (W, g) , $(U, f) <_{\mathbf{k}} (W, g)$ if and only if there is a model of set theory N admitting an elementary embedding $k : M_U \rightarrow N$ and a close embedding $j : M_W \rightarrow N$ such that $k \circ j_U = j \circ j_W$ and $k([f]_U) < j([g]_W)$.*

Sketch. Assuming $(U, f) <_{\mathbf{k}} (W, g)$, let $j : M_W \rightarrow N$ be the ultrapower embedding of M_W associated to U_* , and let $k : M_U \rightarrow N$ map $[h]_U$ to $[j_W(h)]_{U_*}$.

Conversely, given such embeddings j and k , the ultrafilter U_* derived from j using $k([f]_U)$ witnesses $(U, f) <_{\mathbf{k}} (W, g)$. \square

The Ketonen order is a strict partial order of the normed ultrafilters. We leave the proof of transitivity to the reader, but we do show that it is irreflexive.

Theorem 3.12. *$<_{\mathbf{k}}$ is irreflexive.*

Proof. Suppose (U, φ) is a normed ultrafilter on X , and suppose $\langle U_x : x \in X \rangle$ satisfies $U = U\text{-}\lim_{x \in X} U_x$. We must show that for U -almost all x , for U_x -almost all y , $\varphi(x) \leq \varphi(y)$.

Let B be the set of $x \in X$ such that for U_x -almost all $y \in Y$, $\{y \in X : \varphi(y) < \varphi(x)\} \in U_x$. Let \prec be the wellfounded relation on X defined by setting $y \prec x$ if $\varphi(y) < \varphi(x)$. We define a

set $A \subseteq B$ by recursion on \prec , putting $x \in A$ if and only if $\{y \in X : y \prec x \text{ and } y \in A\} \notin U_x$. By the definition of B , for all $x \in B$, $\{y \in X : y \prec x\} \in U_x$, and therefore by the definition of A , $x \in A$ if and only if $A \notin U_x$. That is, $A = \{x \in B : A \notin U_x\}$. As a consequence,

$$\begin{aligned}
A \in U &\iff \{x \in X : A \in U_x\} \in U \\
&\iff \{x \in X : A \notin U_x\} \notin U \\
&\implies \{x \in B : A \notin U_x\} \notin U \\
&\iff A \notin U
\end{aligned} \tag{1}$$

The implication (1) cannot be reversed on pain of contradiction, so $B \notin U$. \square

We will not use the following corollary, but it seems to have been unknown.

Corollary 3.13. *Suppose U is an ultrafilter, $j, k : M_U \rightarrow N$ are elementary embeddings with $j \circ j_U = k \circ j_U$, and j is close to M_U . Then for every ordinal a of M_U , $j(a) \leq k(a)$. \square*

The analog of Theorem 3.12 proved in [3] applies to arbitrary directed systems of ultrafilters whose limit ultrapower is wellfounded. Theorem 3.12, on the other hand, applies to a single ultrafilter but requires no assumption of wellfoundedness, or equivalently of countable completeness. The two theorems cannot be mutually generalized: it simply is not true that the Ketonen order on arbitrary directed systems of ultrafilters, defined following [3], is irreflexive. For example, there is a directed system of ultrafilters E whose limit ultrapower M_E admits a nontrivial elementary self embedding $k : M_E \rightarrow M_E$ such that $k \circ j_E = j_E$, yet k is isomorphic to an internal ultrapower embedding of M_E . This means Corollary 3.13 does not generalize to E , and hence Theorem 3.12 cannot either.

We now explain the connection between the Ketonen order and the internal relation.

Definition 3.14. If (W, g) is a normed ultrafilter, then $\delta(W, g)$ denotes the least ordinal δ such that (W, g) is δ -normed.

Theorem 3.15. *If $U \sqsubset W$, $\delta = \delta(W, g)$ is a limit ordinal, and (U, f) is δ -normed, then $(U, f) <_{\mathbf{k}} (W, g)$.*

Proof. For any $\epsilon < \delta$, for W -almost all y , $g(y) > \epsilon$. Therefore in M_U , for any $\epsilon < j_U(\delta)$, for $j_U(W)$ -almost all y , $j_U(g)(y) > \epsilon$. It follows that for $j_U(W)$ -almost all y , $j_U(g)(y) > [f]_U$. In other words, for U -almost all x , for W -almost all y , $f(x) < g(y)$. This means $\{x : j_W(f)(x) < [g]_W\} \in U$, or equivalently, $\{x : j_W(f)(x) < [g]_W\} \in s_W(U)$. This means that $(s_W(U), j_W(f))$ is a $[g]_W$ -normed ultrafilter in M_W , and hence $s_W(U)$ witnesses that $(U, f) <_{\mathbf{k}} (W, g)$. \square

We now prove our first indecomposability theorem for the internal relation.

Proof of Theorem 3.8. Assume towards a contradiction that the theorem fails, and neither U nor W is δ -indecomposable. Let X and Y be the underlying sets of U and W . The δ -decomposability of U yields a function $f : X \rightarrow \delta$ such that f is not bounded below δ on a set in U . In other words, $\delta(U, f) = \delta$. Similarly, for some $g : X \rightarrow \delta$, $\delta(W, g) = \delta$. By Theorem 3.15, since $U \sqsubset W$ and (U, f) is δ -normed, $(U, f) <_{\mathbf{k}} (W, g)$. The same logic shows $(W, g) <_{\mathbf{k}} (U, f)$. By the transitivity of the Ketonen order, $(U, f) <_{\mathbf{k}} (U, f)$, and this contradicts Theorem 3.12. \square

The following corollary is proved more directly in [2].

Corollary 3.16. *Any ultrafilter U such that $U \sqsubset U$ is principal.*

Proof. Suppose U is nonprincipal. Let κ be the completeness of U , the supremum of all cardinals δ such that U is δ -complete. Then κ is regular and U is κ -decomposable. But $U \sqsubset U$, so Theorem 3.8 implies U is not κ -decomposable, and this is a contradiction. \square

3.4 (λ, λ^+) -indecomposability for mutually internal ultrafilters

This final subsection is devoted to the following theorem:

Theorem 3.17 (GCH). *Suppose U and W are countably complete ultrafilters such that $U \sqsubset W$ and $W \sqsubset U$. Then for all cardinals λ , either U or W is (λ, λ^+) -indecomposable.*

Combining this with Theorem 2.14 immediately yields the last of our main results:

Theorem 3.18 (GCH). *Suppose U and W are ultrafilters such that $U \sqsubset W$ and $W \sqsubset U$. Then $U \times W$ is an ultrafilter.* \square

Under UA, one can easily prove that if $U \sqsubset W$ and $W \sqsubset U$, then U is W -nonregular. This fact is unique among UA results in that it applies to arbitrary ultrafilters, so we sketch a proof for the reader familiar with [2].

Theorem 3.19 (UA). *Suppose U and W are ultrafilters such that $U \sqsubset W$ and $W \sqsubset U$. Then U is W -nonregular.*

Proof. At least one of U and W is countably complete by Theorem 3.8. So without loss of generality, assume that U is.

Applying UA in M_W , the ultrapower embeddings associated to $s_W(U)$ and $j_W(U)$ admit an internal ultrapower comparison. In other words, there exist internal ultrapower embeddings $k : j_U(M_W) \rightarrow N$ and $\ell : j_U(M_W) \rightarrow N$ such that $k \circ j_U \upharpoonright M_W = \ell \circ j_W(j_U)$. To conclude that $s_W(U) = j_W(U)$ (and hence that U is W -nonregular), it suffices to show that $k([id]_{s_W(U)}) = \ell([id]_{j_W(U)})$.

We have that $k([id]_{s_W(U)}) = k(j_U(j_W)([id]_U))$ while $\ell([id]_{j_W(U)}) = \ell(j_W([id]_U))$. Now $\ell \circ j_W$ and $k \circ j_U(j_W)$ are both internal ultrapower embeddings from M_U to N , and moreover $\ell \circ j_W \circ j_U = k \circ j_U(j_W) \circ j_U$. By the uniqueness of such embeddings ([2, Theorem ??]), $\ell \circ j_W = k \circ j_U(j_W)$ and in particular $\ell \circ j_W([id]_U) = k \circ j_U(j_W)([id]_U)$. It follows that $k([id]_{s_W(U)}) = \ell([id]_{j_W(U)})$, and this proves the theorem. \square

We now turn to the proof of Theorem 3.17 itself, where we seem to need a completely different argument. This will require some lemmas on the interaction between the Rudin-Keisler order and the internal relation:

Theorem 3.20. *Suppose $\bar{U} \leq_{RK} U$ and W are ultrafilters.*

- (1) *If $\bar{U} = f_*(U)$, then $s_W(\bar{U}) = j_W(f)_*(s_W(U))$. In particular, if $U \sqsubset W$, then $\bar{U} \sqsubset W$.*
- (2) *If $k : M_{\bar{U}} \rightarrow M_U$ is an elementary embedding such that $k \circ j_{\bar{U}} = j_U$, then $s_{\bar{U}}(W) = k^{-1}[s_U(W)]$. In particular, if $W \sqsubset U$ and $s_U(W) \in \text{ran}(k)$, then $W \sqsubset \bar{U}$.*

Proof. For (1), recall that for any Z , $s_W(Z) = (j_W)_*(Z) \cap M_W$ and observe:

$$j_W(f)_*((j_W)_*(U)) = (j_W(f) \circ f)_*(U) = (j_W \circ f)_*(U) = (j_W)_*(f_*(U)) = (j_W)_*(\bar{U})$$

For (2), note that $A \in k^{-1}[s_U(W)]$ if and only if $j_U^{-1}[k(A)] \in W$, but $j_U^{-1}[k(A)] = j_{\bar{U}}^{-1}[A]$, so $j_{\bar{U}}^{-1}[k(A)] \in W$ if and only if $j_{\bar{U}}^{-1}[A] \in W$ or equivalently $A \in s_{\bar{U}}(W)$. \square

Corollary 3.21. *If $U \times W = U \rtimes W$, then for any $\bar{U} \leq_{\text{RK}} U$ and $\bar{W} \leq_{\text{RK}} W$, $\bar{U} \times \bar{W} = \bar{U} \rtimes \bar{W}$.*

Proof. It suffices to show that $\bar{U} \times W = \bar{U} \rtimes W$. For this we must show that $j_{\bar{U}}(W) = s_{\bar{U}}(W)$. Since $\bar{U} \leq_{\text{RK}} U$, there is an elementary embedding $k : M_{\bar{U}} \rightarrow M_U$ such that $k \circ j_{\bar{U}} = j_U$. Therefore $s_U(W) = j_U(W) = k(j_{\bar{U}}(W))$. Hence $j_{\bar{U}}(W) = k^{-1}[s_U(W)] = s_{\bar{U}}(W)$, applying Theorem 3.20. Now by Lemma 3.7, $\bar{U} \times W = \bar{U} \rtimes W$. \square

We also need a lemma that will allow us to generate an ultrafilter extending an M -ultrafilter:

Lemma 3.22. *Suppose λ is a strong limit cardinal, M is a model with the λ -cover property, and U is an M - λ -complete M -ultrafilter. Then U generates a λ -complete filter.*

Proof. Suppose $\sigma \subseteq U$ and $|\sigma| < \lambda$. We must show $\bigcap \sigma \neq \emptyset$.

Using the λ -cover property, fix $\tau \in M$ such that $\sigma \subseteq \tau$ and $|\tau| < \lambda$. Notice $\tau \cap U \in M$. First, the M - λ -completeness of U implies $j_U[\tau \cap U] = \{A \in j_U(\tau) : [\text{id}]_U \in A\} \in M_U^M$. But since $2^{|\tau|} < \lambda$, the M - λ -completeness of U implies $j_U : P^M(\tau) \rightarrow j_U(P^M(\tau))$ is an isomorphism, and so $\tau \cap U = j_U^{-1}[j_U[\tau \cap U]] \in M$.

Since U is M - λ -complete, $\bigcap(\tau \cap U) \neq \emptyset$. Since $\sigma \subseteq (\tau \cap U)$, $\bigcap \sigma \neq \emptyset$. \square

Lemma 3.23 (Kunen). *Suppose W is (κ, δ) -regular and F is a κ -complete filter generated by at most δ -many sets. Then $F \leq_{\text{kat}} W$.*

Proof. Let \mathcal{B} be a basis for F with $|\mathcal{B}| \leq \delta$. Since W is (κ, δ) -regular, the filter G on $P_\kappa(\mathcal{B})$ generated by sets of the form $\{\sigma \in P_\kappa(\mathcal{B}) : A \in \sigma\}$ for $A \in \mathcal{B}$ lies below W in the Katětov order. Therefore it suffices to show that $F \leq_{\text{kat}} G$. But let $f : P_\kappa(\mathcal{B}) \rightarrow X$ be any function such that $f(\sigma) \in \bigcap \sigma$. Easily, $F \subseteq f_*(G)$, as desired. \square

Finally, we need a theorem due essentially to Silver.

Definition 3.24. Suppose U and W are ultrafilters on sets X and Y . Then $U \leq_{\text{RK}}^\gamma W$ if there is a function $p : Y \rightarrow X$ such that for all $\beta < \gamma$, for all $f : Y \rightarrow \beta$, f factors through p modulo W : there is a $g : X \rightarrow \beta$ such that $[f]_W = [g \circ p]_W$.

Theorem 3.25 (Silver, [6]). *Suppose λ is a regular cardinal and U is a (λ, η) -indecomposable ultrafilter and $2^\lambda \leq \eta$. Then there is an ultrafilter D on a cardinal less than λ such that $D \leq_{\text{RK}}^\eta U$.* \square

Proof of Theorem 3.17. We assume by induction that the theorem holds for all $\bar{\lambda} < \lambda$, and towards a contradiction that there exist ultrafilters U and W such that $U \sqsubset W$, $W \sqsubset U$, but neither U nor W is (λ, λ^+) -indecomposable. By Lemma 2.20, we may assume without loss of generality that λ is singular, W is λ -decomposable, and U is (κ, λ^+) -regular for some $\kappa < \lambda$.

We claim one can reduce to the case that W is a uniform ultrafilter on λ . Note that W is $(\lambda^+, \lambda^{+\omega})$ -indecomposable since W is λ^+ -indecomposable (by Lemma 2.20). Applying Theorem 3.25 and the GCH, let \tilde{W} be a uniform ultrafilter on λ such that $\tilde{W} \leq_{\text{RK}}^{\lambda^{+\omega}} W$. Let p witness that $\tilde{W} \leq_{\text{RK}}^{\lambda^{+\omega}} W$, so $p_*(W) = \tilde{W}$. Let \tilde{U} be the ultrafilter on $\beta(\lambda) \times P_\kappa(\lambda)$ derived from j_U using $(s_U(\tilde{W}), \sigma)$ where $\sigma \in j_U(P_\kappa(\lambda^+))$ covers $j_U[\lambda^+]$. (Here $\beta(\lambda)$ denotes the set of ultrafilters on λ .) Thus by Theorem 3.20, $\tilde{W} \sqsubset \tilde{U}$ and by Lemma 2.11, \tilde{U} is (κ, λ^+) -regular. Let f be such that $[f]_W = s_W(\tilde{U})$, and assume without loss of generality

that the range of f has cardinality strictly less than $\lambda^{+\omega}$. Since $\tilde{W} \leq_{\text{RK}}^{\lambda^{+\omega}} W$, there is a function \tilde{f} such that $\tilde{f} \circ p = f$. By Theorem 3.20, $[\tilde{f}]_{\tilde{W}} = s_{\tilde{W}}(\tilde{U})$. So $\tilde{U} \sqsubset \tilde{W}$. Replacing U and W with \tilde{U} and \tilde{W} , we reduce to the case that W is a uniform ultrafilter on λ .

Since U is (κ, λ^+) -regular, U is δ -decomposable for every regular cardinal δ such that $\kappa \leq \delta < \lambda$. (This is a well-known fact, but it is also an immediate consequence of Lemma 2.18 and Lemma 3.23.) Therefore by our induction hypothesis, W is η -indecomposable for all cardinals η such that $\kappa \leq \eta < \lambda$: we know at least one of U and W is (η, η^+) -indecomposable, and U is not (since U is η^+ -decomposable since η^+ is regular). We are assuming GCH, so we can apply Silver's Lemma (Theorem 3.25) to obtain an ultrafilter D on a cardinal γ less than κ such that $D \leq_\lambda W$. Equivalently, there is an elementary embedding $k : M_D \rightarrow M_W$ such that $k \circ j_D = j_W$ and $j_W(\eta) \subseteq \text{ran}(k)$ for all $\eta < \lambda$.

By our induction hypothesis, for any cardinal $\bar{\lambda} < \lambda$, either U or W is $(\bar{\lambda}, (\bar{\lambda})^+)$ -indecomposable. Since $D \leq_{\text{RK}} W$ and $\text{size}(D) < \lambda$, for any cardinal $\bar{\lambda} < \lambda$, either U or D is $(\bar{\lambda}, (\bar{\lambda})^+)$ -indecomposable. Applying Theorem 2.14, $D \times U$ is an ultrafilter. In particular, $j_D(U) = s_D(U)$. Let $k : M_D \rightarrow M_W$ be an elementary embedding such that $k \circ j_D = j_W$ and $j_W(\eta) \subseteq \text{ran}(k)$ for all $\eta < \lambda$. Let W_* be the M_D -ultrafilter on $j_D(\lambda)$ derived from k using $[\text{id}]_W$. Then M_W is isomorphic to the ultrapower of M_D by W_* and k to the associated ultrapower embedding. We will identify the two models via the unique isomorphism between k and $(j_{W_*})^{M_D}$, so $M_W = (M_{W_*})^{M_D}$ and $k = (j_{W_*})^{M_D}$.

Let F be the filter generated by W_* (in V). By Lemma 3.22, F is λ -complete, and since $|j_D(P(\lambda))| \leq (\lambda^+)^{\gamma} = \lambda^+$, F is generated by at most λ^+ -many sets. Therefore by Lemma 3.23, F extends to an ultrafilter G preceding the (κ, λ^+) -regular ultrafilter U in the Rudin-Keisler order. Since $G \leq_{\text{RK}} U \sqsubset D$, Theorem 3.20 yields that $s_D(G) \leq_{\text{RK}} s_D(U)$ in M_D , and in particular $G \sqsubset D$. Note that W_* is the ultrafilter derived from $j_G \upharpoonright M_D$ using $[\text{id}]_G$. Therefore $W_* \in M_D$ and $W_* \leq_{\text{RK}} s_D(G)$ in M_D .

So far we have shown that in M_D ,

$$W_* \leq_{\text{RK}} s_D(G) \leq_{\text{RK}} s_D(U)$$

Since $U \sqsubset W$, $s_W(U) \in M_W$. But $s_W(U) = s_{W_*}(s_D(U))$, so since $s_{W_*}(s_D(U)) \in M_W = (M_{W_*})^{M_D}$, $s_D(U) \sqsubset W_*$ in M_D . Now in M_D , $W_* \leq_{\text{RK}} s_D(U) \sqsubset W_*$. Applying Theorem 3.20 one last time, $W_* \sqsubset W_*$ in M_D , and it follows (by Corollary 3.16) that W_* is a principal ultrafilter of M_D . Thus $j_D = j_{W_*}^{M_D} \circ j_D = j_W$, or in other words, $D \equiv_{\text{RK}} W$, contradicting that $\text{size}(D) < \text{size}(W)$. \square

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