

A note on cardinal preserving embeddings

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1 Introduction

An elementary embedding between two transitive models of set theory is *cardinal preserving* if its domain and target models have the same class of cardinals. The purpose of this note is to extend some ideas due to Caicedo and Woodin, leading to a proof that the critical point κ of a cardinal preserving embedding from the universe into an inner model is strongly compact in V_γ for some inaccessible cardinal $\gamma > \kappa$.

In particular, we have the following consistency result:

Theorem 2.10. *The existence of a cardinal preserving embedding from the universe into an inner model implies the consistency of $\text{ZFC} + \text{there is a strongly compact cardinal}$.*

We also show that the existence of a cardinal preserving embedding from the universe into M implies much stronger resemblance properties of V and M than are immediately apparent, especially assuming the Singular Cardinals Hypothesis:

Theorem 2.11 (SCH). *Suppose there is a cardinal preserving elementary embedding from the universe into an inner model M . Then for any ordinal α , $\text{cf}^M(\alpha) = \text{cf}(\alpha)$ and for any cardinal ρ , $(2^\rho)^M = 2^\rho$.*

2 Some proofs

If $j : V \rightarrow M$ is an elementary embedding, then the critical sequence $\kappa_n(j)$ is defined by recursion, setting $\kappa_0(j) = \text{crit}(j)$ and $\kappa_{n+1}(j) = j(\kappa_n(j))$. Finally, $\kappa_\omega(j) = \sup_{n < \omega} \kappa_n(j)$ denotes the supremum of the critical sequence of j .

The following is straightforward using an observation of Caicedo:

Proposition 2.1. *Suppose there is a cardinal preserving embedding $j : V \rightarrow M$. Let $\kappa = \text{crit}(j)$ and let $\lambda = \kappa_\omega(j)$. Then κ is λ -strongly compact.*

Proof. By Ketonen's Theorem [3], it suffices to show that every regular cardinal in the interval $[\kappa, \lambda]$ carries a κ -complete uniform ultrafilter. By Prikry's Theorem [4], it suffices to show that every successor cardinal in the interval $[\kappa, \lambda]$ carries a κ -complete uniform ultrafilter.

Suppose δ is an ordinal with $\kappa \leq \delta < \lambda$. We claim there is a κ -complete uniform ultrafilter on δ^+ . First, since $\kappa \leq \delta^+ < \lambda$, $\delta^+ < j(\delta^+)$. Also, note that $j(\delta^+) = j(\delta)^+$ since M correctly computes successor cardinals. This means that the cardinal $j(\delta^+)$ is regular, and in particular it has cofinality strictly larger than δ^+ . As a consequence, $\sup j[\delta^+] < j(\delta^+)$. Therefore one can derive an ultrafilter on δ^+ from j using $\sup j[\delta^+]$, and this ultrafilter is clearly uniform; it is κ -complete since $\text{crit}(j) = \kappa$. \square

The proposition does not obviously yield the consistency of a strongly compact cardinal from a cardinal preserving embedding since it is not clear that the interval (κ, λ) contains an inaccessible cardinal. In fact, it is not even clear that $2^\kappa < \lambda$. We will prove that λ is a limit of inaccessible cardinals, and in fact $\kappa_n(j)$ is inaccessible for every $n < \omega$.

The key lemma is not hard, but it seems interesting in its own right. The main concept involved is the *tightness function* of an elementary embedding.

Definition 2.2. If $j : V \rightarrow M$ is an elementary embedding, then for any cardinal λ , $t_j(\lambda)$ denotes the least M -cardinality of a set in M that covers $j[\lambda]$.

Note that for any set X , $t_j(|X|)$ is the least M -cardinality of a set in M that covers $j[X]$. The following theorem, implicit in the work of Ketonen [3], appears explicitly in the author's thesis [2, Theorem 7.2.12].

Theorem 2.3 (Ketonen, [3]). *If $j : V \rightarrow M$ is an elementary embedding and δ is a regular cardinal, then $t_j(\delta) = \text{cf}^M(\sup j[\delta])$.* \square

Lemma 2.4. *Suppose $j : V \rightarrow M$ is an elementary embedding with critical point κ and λ is a singular cardinal of cofinality $\iota < \kappa$. Then for some M -cardinal η of M -cofinality ι , $\eta < t_j(\lambda^\iota) \leq (\eta^\iota)^M$.*

Proof. Let U be the ultrafilter on λ^+ derived from j using $\sup j[\lambda^+]$. Let $i : V \rightarrow N$ be the ultrapower associated to U .

Let $\langle \delta_\alpha : \alpha < \iota \rangle$ be an increasing sequence of regular cardinals cofinal in λ . Let $\delta'_\alpha = t_i(\delta_\alpha)$. Then since N is closed under ι -sequences, $t_i(\lambda) = \sup \delta'_\alpha$. Moreover, Theorem 2.3 easily implies that $\delta'_\alpha < \delta'_\beta$ whenever $\alpha < \beta$. Therefore $t_i(\lambda)$ has cofinality ι (in N). Let $\nu = t_i(\lambda)$. Fix $A \in N$ covering $i[\lambda]$ with $|A|^N = \nu$. Since N is closed under ι -sequences and $\text{crit}(j) > \iota$, $i[\iota\lambda] \subseteq ({}^\iota A)^N$. It follows that $t_i(\lambda^\iota) \leq \nu^\iota$.

Let $k : N \rightarrow M$ be the factor embedding, and set $\eta = k(\nu)$. Trivially, $t_j(\lambda^\iota) \leq (\eta^\iota)^M$. Since $k(\sup i[\lambda^+]) = \sup j[\lambda^+]$, Theorem 2.3 implies that $k(t_i(\lambda^+)) = t_j(\lambda^+)$. Therefore since $\nu = t_i(\lambda) < t_i(\lambda^+)$, $\eta < t_j(\lambda^+) \leq t_j(\lambda^\iota)$. This finishes the proof. \square

Here is one intriguing consequence, which is very similar to a consequence of the Ultrapower Axiom [2, Lemma 8.4.14] originally proved in the analysis of cardinal preservation under UA.

Theorem 2.5 (SCH). *Suppose $j : V \rightarrow M$ is an elementary embedding and δ is the successor of a singular cardinal of cofinality $\iota < \text{crit}(j)$. Then in M , the cofinality of $\sup j[\delta]$ is the successor of a singular cardinal of cofinality ι .* \square

We will avoid the SCH in our main theorem by citing a local form of Solovay's Theorem.

Theorem 2.6 (Solovay, [5]). *If κ is λ -strongly compact, then for any singular cardinal η with $\kappa \leq \eta \leq \lambda$, $\eta^{\text{cf}(\eta)} = \eta^+ \cdot 2^{\text{cf}(\eta)}$.* \square

The author tentatively conjectures that the existence of a cardinal preserving embedding of the universe of sets is inconsistent with ZFC. The existence of the partially cardinal preserving embeddings involved in the hypotheses of the next few theorems, however, follows from the axiom I_2 , and therefore is very likely to be consistent.

Theorem 2.7. *Suppose $j : V \rightarrow M$ is an elementary embedding. Let $\kappa = \text{crit}(j)$, and let $\lambda = \kappa_\omega(j)$. Assume $j \upharpoonright V_\lambda$ is cardinal preserving, or in other words $\text{Card}^M \cap \lambda = \text{Card}^V \cap \lambda$. Then for any ordinal $\alpha \leq \lambda$, $\text{cf}^M(\alpha) = \text{cf}(\alpha)$.*

Proof. First, fix a singular cardinal ρ of cofinality ω with $\rho \leq \lambda$. We will show that $t_j(\rho^+) = \rho^+$. If $j(\rho^+) = \rho^+$, this is trivial, so we may assume $\kappa \leq \rho < \lambda$. By Lemma 2.4, there is a cardinal η of M -cofinality ω such that $\eta < t_j(\lambda^+) \leq (\eta^\omega)^M$.

We claim $(\eta^\omega)^M = \eta^+$. Note that $\kappa \leq \eta < \lambda$, so since κ is λ -strongly compact by Proposition 2.1, $\eta^\omega = \eta^+$ by Solovay's Theorem. Therefore $(\eta^\omega)^M < \eta^{++}$, and so by cardinal correctness, $(\eta^\omega)^M \leq \eta^+$. By König's Theorem, $(\eta^\omega)^M \geq (\eta^+)^M = \eta^+$. It follows that $(\eta^\omega)^M = \eta^+$, as claimed.

Thus $\eta < t_j(\rho^+) \leq \eta^+$, or in other words, $t_j(\rho^+) = \eta^+$. Hence $\text{cf}^M(\sup j[\rho^+]) = \eta^+$, and so η^+ and ρ^+ have the same cofinality. Since successor cardinals are regular, $\eta^+ = \rho^+$.

Now let $\gamma < \lambda$ be a regular cardinal. We have

$$\gamma \leq t_j(\gamma) < t_j(\gamma^{+\omega+1}) = \gamma^{+\omega+1}$$

where the final equality follows from the previous paragraphs with $\rho = \gamma^{+\omega}$. Thus $t_j(\gamma) = \gamma^{+n}$ for some $n < \omega$. But by Theorem 2.3, $t_j(\gamma)$ has cofinality γ , and therefore $t_j(\gamma) = \gamma$.

Next, suppose $\delta < \lambda$ is a cardinal that is regular in M . By our work so far, $\delta \leq t_j(\delta) < t_j(\delta^+) = \delta^+$, so $t_j(\delta) = \delta$. It follows that there is a cofinal subset of $\sup j[\delta]$ of ordertype δ in M , and so since δ is regular in M , $\text{cf}^M(\sup j[\delta]) = \delta$. Let γ be the cofinality of δ , so γ is regular and there is an increasing cofinal function $f : \gamma \rightarrow \delta$. Note that $j(f) \upharpoonright \sup j[\gamma]$ is an increasing cofinal function from $\sup j[\gamma]$ to $\sup j[\delta]$ is an increasing cofinal function. Hence $\delta = \text{cf}^M(\sup j[\delta]) = \text{cf}^M(\sup j[\gamma]) = \gamma$, and so δ is regular.

Finally, we conclude the theorem. Fix an ordinal $\alpha < \lambda$. Then $\text{cf}^M(\alpha)$ is an M -regular cardinal δ , and so $\text{cf}(\alpha) = \text{cf}(\delta) = \delta$. \square

We now argue that cardinal preserving embeddings preserve the continuum function. This extends an argument due to Woodin-Caicedo [1].

Lemma 2.8. *Suppose $j : V \rightarrow M$ is an elementary embedding. If ρ is a cardinal and $\theta = t_j(\rho)$ then $(2^\theta)^M \geq 2^\rho$.*

Proof. Let $A \in M$ be a cover of $j[\rho]$ with $|A|^M = \theta$. Note that $|P^M(A)|^M = (2^\theta)^M$. But there is an injection $f : P(\rho) \rightarrow P^M(A)$ defined by $f(B) = j(B) \cap A$. So $2^\rho = |P(\rho)| \leq |P^M(A)| \leq |P^M(A)|^M = (2^\theta)^M$. \square

Theorem 2.9. *Suppose $j : V \rightarrow M$ is an elementary embedding. Let $\kappa = \text{crit}(j)$, and let $\lambda = \kappa_\omega(j)$. Assume $j \upharpoonright V_\lambda$ is cardinal preserving, or in other words $\text{Card}^M \cap \lambda = \text{Card}^V \cap \lambda$. Then for any cardinal $\rho < \lambda$, $(2^\rho)^M = 2^\rho$.*

Proof. Since $|(2^\rho)^M| = |P^M(\rho)|$, we have $(2^\rho)^M < (2^\rho)^+$. But $(2^\rho)^M$ is a cardinal, so it follows that $(2^\rho)^M \leq 2^\rho$. On the other hand, by Theorem 2.7, $t_j(\rho) = \rho$, and so by Lemma 2.8, $(2^\rho)^M \geq 2^\rho$. \square

Theorem 2.10. *The existence of a cardinal preserving embedding from the universe into an inner model implies the consistency of $\text{ZFC} + \text{there is a strongly compact cardinal}$.*

Proof. If $j : V \rightarrow M$ is cardinal preserving, then by a simple induction using Theorem 2.9, $\kappa_n(j)$ is strongly inaccessible for all $n < \omega$. By Proposition 2.1, for any $n \geq 1$, $V_{\kappa_n(j)}$ satisfies $\text{ZFC} + \text{there is a strongly compact cardinal}$. \square

Theorem 2.9 has some surprising reflection consequences: for example, the SCH must hold for all sufficiently large cardinals below the critical point κ of a cardinal preserving embedding $j : V \rightarrow M$. (It is open whether the SCH can fail everywhere below a strongly compact cardinal.) Moreover, if GCH holds below κ , it holds up to $\kappa_\omega(j)$. Can κ be the least measurable cardinal?

The same arguments almost prove that any cardinal preserving embedding $j : V \rightarrow M$ is cofinality preserving and continuum function preserving, but now there is a problem applying Solovay's Theorem that SCH holds above a strongly compact cardinal. For example, letting $\lambda = \kappa_\omega(j)$, it is not clear how large λ^ω can be. We do obtain:

Theorem 2.11 (SCH). *Suppose there is a cardinal preserving elementary embedding from the universe into the inner model M . Then for any ordinal α , $\text{cf}^M(\alpha) = \text{cf}(\alpha)$ and for any cardinal ρ , $(2^\rho)^M = 2^\rho$.* \square

References

- [1] Andrés Eduardo Caicedo. Cardinal preserving elementary embeddings. In *Logic Colloquium 2007*, volume 35 of *Lect. Notes Log.*, pages 14–31. Assoc. Symbol. Logic, La Jolla, CA, 2010.
- [2] Gabriel Goldberg. *The Ultrapower Axiom*. PhD thesis, Harvard University, 2019.
- [3] Jussi Ketonen. Strong compactness and other cardinal sins. *Ann. Math. Logic*, 5:47–76, 1972/73.
- [4] Kenneth Kunen and Karel Prikry. On descendingly incomplete ultrafilters. *J. Symbolic Logic*, 36:650–652, 1971.
- [5] Robert M. Solovay. Strongly compact cardinals and the GCH. In *Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971)*, pages 365–372. Amer. Math. Soc., Providence, R.I., 1974.