No cardinal correct inner model elementarily embeds into the universe

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Abstract

An elementary embedding $j : M \to N$ between two inner models of ZFC is *cardinal preserving* if M and N correctly compute the class of cardinals. We look at the case $N = V$ and show that there is no nontrivial cardinal preserving elementary embedding from M into V, answering a question of Caicedo.

1. Introduction

Large cardinal axioms are typically formulated in terms of elementary embeddings from the universe V into some transitive subclass M. By demanding a stronger and stronger degree of resemblance between V and M, one obtains stronger and stronger principles of infinity. For example, one may require M to have more and more fragments of V: γ-*strong* cardinals have $V_\gamma \subseteq M$, and n-*superstrong* cardinals have $V_{j^{(n)}(\kappa)} \subseteq M$. Similarly, one may ask how close M is: γ-*supercompact* cardinals have $M^\gamma \subseteq M$, and n-*huge* cardinals have $M^{j^{(n)}(\kappa)} \subseteq M$.

Straining the limits of consistency, Reinhardt [15] considered the natural extreme of this trend in which the target model is the entire universe. Few years later, Kunen [13], with his celebrated inconsistency theorem, refuted this suggestion and provided what seems to be an upper bound in the formulation of large cardinal axioms.

Theorem 1.1 (Kunen, [13]). *There is no nontrivial elementary embedding from the universe to itself.*

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Kunen's proof actually shows that if $j : V \to M$ is a nontrivial elementary embedding and λ is the supremum of the *critical sequence of* j, then $V_{\lambda+1} \not\subseteq M$.

Definition 1.2. *Let $j : M \to N$ be an elementary embedding between two transitive models of* ZFC. *The* critical sequence of j *is the sequence* $\langle \kappa_n(j) \rangle_{n<\omega}$ *defined by setting* $\kappa_n(j) = j^{(n)}(\text{crit}(j))$. *The ordinal* $\kappa_\omega(j)$ *is the supremum of the critical sequence of* j.

Looking for inconsistencies and with attention shifted upward to stronger principles, a new breed of large cardinal hypothesis was introduced, the rank-into-rank embeddings. *Axiom* $I_2(\lambda)$, for example, states that there is an elementary embedding $j : V \to M$ such that $\text{crit}(j) < \lambda$, $j(\lambda) = \lambda$ and $V_\lambda \subseteq M$. Caicedo [2] pushed this further and proposed another way to extend the large cardinal hierarchy in ZFC and to obtain axioms just at the edge of Kunen inconsistency. Following the usual template of resembling V, the idea here is to impose agreement of cardinals between the models involved.

Definition 1.3. *Let $j : M \to N$ be an elementary embedding between two transitive models of* ZFC. *Then, j is* cardinal preserving *if M, N and V have the same class of cardinals, i.e.* $\text{Card}^M = \text{Card}^N = \text{Card}$.

Question 1.4 (Caicedo)**.** *Assume $j : M \to N$ is a nontrivial elementary embedding. Can j be cardinal preserving?*

The expectation is that Question 1.4 has a negative answer. Taking the first step towards this line of research, Caicedo considered the case where either M or N is V. Both principles have significant consistency strength:

Theorem 1.5 ([11], Theorem 2.10)**.** *The existence of a cardinal preserving embedding $j : V \to N$ implies the consistency of* ZFC + *there is a strongly compact cardinal.*

Theorem 1.6 ([2], Theorem 2.11)**.** *Assume that there is a cardinal preserving embedding $j : M \to V$. Then there are inner models with strong cardinals.*

In [2, Corollary 2.10], it has been shown that PFA rules out the case $N = V$. On the other hand, the first author [10, Theorem 6.6] proved that the existence of a proper class of almost strongly compact cardinals refutes the case $M = V$. In this paper we show that cardinal preserving embeddings $j : M \to V$ are inconsistent with ZFC. It is still open whether ZFC alone can refute cardinal preserving embeddings from V to N.

It will be assumed throughout the paper that, if not otherwise specified, all embeddings are elementary and between transitive models of ZFC.

2. Forcing axioms

A first impulse for this investigation around cardinal preserving embeddings is due to the following conjecture [19, Conjecture 1].

Conjecture 2.1 (Caicedo, Veličković)**.** *Assume $W \subseteq V$ are models of* MM *with the same cardinals. Then W and V have the same ω_1-sequences of ordinals.*

The conjecture is motivated by a result of Caicedo and Veličković [3, Theorem 1] which asserts that, if $W \subseteq V$ are transitive models of $\mathsf{ZFC}+\mathsf{BPFA}$ and $\omega_2^W = \omega_2^V$, then $P(\omega_1) \subseteq W$. In light of this, one may ask whether any two models $W \subseteq V$ of some strong forcing axiom with the same cardinals have the same ω_1-sequences of ordinals. Thus, Caicedo and Veličković's conjecture is a possible way to formalize this idea.

However, there is a tension between conjecture 2.1 and cardinal preserving embeddings $j : M \to N$ with $M \vDash \mathsf{MM}$. Indeed, under MM, the conjecture would refute these embeddings:

Proposition 2.2. *If $M \subseteq V$ are models of* MM *and $j : M \to N$ is a cardinal preserving embedding, then conjecture 2.1 implies that j is the identity map.*

Proof. Suppose towards a contradiction that $\kappa = \mathrm{crit}(j)$ exists and let λ be $\kappa_\omega(j)$. Conjecture 2.1 ensures that the sequence $\langle \kappa_n(j)\rangle_{n<\omega}$ belongs to M. Therefore,

$$j(\lambda) = j(\sup_{n<\omega} \kappa_n(j)) = \sup_{n<\omega} \kappa_{n+1}(j) = \lambda.$$

As $\mathrm{cof}^M(\lambda) = \omega$, we may fix in M a sequence cofinal in λ consisting of successor cardinals $\vec{\delta} = \langle \delta_n\rangle_{n<\omega}$, and a *scale* $\langle f_\alpha\rangle_{\alpha<\lambda^+}$ at λ relative to the sequence $\vec{\delta}$. A scale here is a sequence of functions from $\prod_{n<\omega} \delta_n$ which is increasing and cofinal with respect to $<^*$. For functions f and g in $\prod_{n<\omega} \delta_n$, $f <^* g$ means that there exists $n < \omega$ such that for every $m > n$, $f(m) < g(m)$. Since MM implies SCH and $2^{\aleph_0} = \aleph_2$, it holds that $\lambda^\omega = \lambda^+$. Note that by closure under countable sequences and cardinal preservation, the statement $\lambda^\omega = \lambda^+$ is absolute between M, N and V. So, using $\lambda^\omega = \lambda^+$, one can easily construct by induction such a sequence of f_α's.

Without loss of generality, we may assume that each δ_n lives in the interval (κ, λ). By elementarity,

$$N \vDash j(\langle f_\alpha\rangle_{\alpha<\lambda^+}) = \langle g_\beta\rangle_{\beta<\lambda^+} \text{ is a scale at } \lambda \text{ relative to } j(\vec{\delta}) = \langle j(\delta_n)\rangle_{n<\omega}.$$

But since N is closed under countable sequences, being a scale is upwards absolute to V. As $j[\lambda^+]$ is cofinal in λ^+, $\langle g_\beta\rangle_{\beta\in j[\lambda^+]}$ is also a scale relative to $\langle j(\delta_n)\rangle_{n<\omega}$. Of course $g_{j(\alpha)} = j(f_\alpha)$ for all $\alpha < \lambda^+$, and so $\langle j(f_\alpha)\rangle_{\alpha<\lambda^+}$ is a scale relative to $\langle j(\delta_n)\rangle_{n<\omega}$ as well. Now we reach a contradiction following Zapeltal's proof of Kunen inconsistency [21]. Let $h = \langle \sup j[\delta_n]\rangle_{n<\omega}$. Since λ is the first fixed point above κ, $\delta_n < j(\delta_n)$. Moreover, δ_n is a successor cardinal, say $\delta_n = \nu_n^+$, and so $j(\delta_n) = (j(\nu_n)^+)^N = j(\nu_n)^+$. This means that $j(\delta_n)$ is a regular cardinal larger than δ_n, and so $\sup j[\delta_n] < j(\delta_n)$. Hence $h \in \prod_{n<\omega} j(\delta_n)$. Therefore there is $\alpha < \lambda^+$ such that $h <^* j(f_\alpha)$. But for all $n < \omega$, $j(f_\alpha)(n) = j(f_\alpha)(j(n)) = j(f_\alpha(n))$ and since $f_\alpha(n) < \delta_n$, we have that $j(f_\alpha(n)) \leq \sup j[\delta_n] = h(n)$. Therefore, for all n big enough, we get

$$h(n) < j(f_\alpha)(n) = j(f_\alpha(n)) \leq h(n),$$

which is absurd. $\square$

All we used to get the contradiction in the proof above is the fact that the cardinal correct models M, N and V have the same ω-sequences of ordinals, together with $\lambda^\omega = \lambda^+$. In [20, Theorem 2.4], Foreman proved that if $j : M \to V$ is a nontrivial elementary embedding ($\mathrm{Card}^M \neq \mathrm{Card}$ is possible), then ${}^\omega M \nsubseteq M$. The argument above shows that this is also the case for embeddings of the form $j : M \to N$ that are cardinal preserving up to $\kappa_\omega(j)^+$.

Proposition 2.3. *Let $j : M \to N$ be a nontrivial elementary embedding between two transitive models of* ZFC*, and let $\lambda = \kappa_\omega(j)$. Suppose $\lambda^\omega = \lambda^+$ and $\mathrm{Card}^M \cap \lambda^+ = \mathrm{Card}^N \cap \lambda^+ = \mathrm{Card} \cap \lambda^+$. Then either $j(\lambda) > \lambda$ or ${}^\omega\lambda \nsubseteq N$. In particular, ${}^\omega\lambda \nsubseteq M \cap N$.*

The first author observed in [10] that N cannot be closed under ω-sequences, whenever N is the target model of a cardinal preserving elementary embedding whose domain is V. Exploiting the following result by Viale one can show that, under PFA, the same conclusion holds in a more general case.

Lemma 2.4 ([19], Corollary 28)**.** *If* PFA *holds and M is an inner model with the same cardinals, then M computes correctly all ordinals of cofinality ω.*

Corollary 2.5 (PFA)**.** *Let $j : M \to N$ be a cardinal preserving embedding. Then ${}^\omega\lambda \nsubseteq N$, where $\lambda = \kappa_\omega(j)$.*

Proof. By Lemma 2.4, $\mathrm{cof}^M(\lambda) = \omega$ and so $j(\lambda) = \lambda$. Hence ${}^\omega\lambda \nsubseteq N$, by Proposition 2.3. $\square$

3. Singular cardinal combinatorics

The proof of nonexistence of $j : M \to V$ with $\mathrm{Card}^M = \mathrm{Card}$ involves singular cardinal combinatorics. More specifically, it relies on some results concerning square principles (Magidor-Sinapova [14]), good scales, Jónsson cardinals (Shelah [17]), ω_1-strongly compact cardinals (Bagaria-Magidor [1]) and basic facts from Shelah's pcf theory.

Accordingly, with a view to the proof of Theorem 5.9 below, we collect some definitions and results, with the further intent of fixing the notation. In the following, unless stated otherwise, all embeddings we will consider are supposed to be nontrivial.

Weak forms of square. The notion of square principle, denoted $\square_\kappa$, is a central concept in Jensen's [12] fine structure analysis of L. For a cardinal κ, $\square_\kappa$ states that there exists a sequence $\langle C_\alpha : \alpha < \kappa^+\rangle$ such that each C_α is a club subset of α, $\mathrm{ot}(C_\alpha) \leq \kappa$, and if $\delta \in \lim C_\alpha$, then $C_\alpha \cap \delta = C_\delta$. In his study of core models for Woodin cardinals, Schimmerling [16] isolated a spectrum of square principles $\square_{\lambda,\kappa}$, for $1 \leq \lambda \leq \kappa$, that form a natural hierarchy below $\square_\kappa$. We are interested in a specific weakening of $\square_\kappa$ that allows at most countably many guesses for the clubs at each point.

Definition 3.1. *Let κ be a cardinal. The principle $\square_{\kappa,\omega}$ asserts that there exists a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+\rangle$ such that, for all $\alpha < \kappa^+$,*

(1) $1 \leq |\mathcal{C}_\alpha| \leq \omega$,
(2) every $C \in \mathcal{C}_\alpha$ is a club subset of α with $\mathrm{ot}(C) \leq \kappa$, and
(3) if $C \in \mathcal{C}_\alpha$ and $\delta \in \lim(C)$, then $C \cap \delta \in \mathcal{C}_\delta$.

Cummings and Schimmerling [5] showed that after Prikry forcing at κ, $\square_{\kappa,\omega}$ holds in the generic extension. Magidor and Sinapova [14] observed that arguments in Gitik [9] and independently in Dźamonja-Shelah [6] yield a more general result for $\square_{\kappa,\omega}$:

Lemma 3.2. *Let $V \subseteq W$ be transitive class models of* ZFC*. Suppose that κ is an inaccessible cardinal in V, singular of countable cofinality in W, and $(\kappa^+)^V = (\kappa^+)^W$. Then $W \models \square_{\kappa,\omega}$.*

There is a connection between this square principle and a pcf-theoretic object called *good scale.*

Definition 3.3. *Suppose $\langle \kappa_n\rangle_{n<\omega}$ is an increasing sequence of regular cardinals with $\sup_{n<\omega} \kappa_n = \lambda$. Let $\vec{f} = \langle f_\alpha\rangle_{\alpha<\lambda^+}$ be a sequence of functions from $\prod_{n<\omega} \kappa_n$. The sequence $\vec{f}$ is a* good scale at λ *if it is a scale relative to the sequence $\langle \kappa_n\rangle_{n<\omega}$, and for all $\nu < \lambda^+$ with $\mathrm{cof}(\nu) > \omega$, there is a cofinal subset C of ν such that, for some $n < \omega$, and for all $k \geq n$, the sequence $\langle f_\alpha(k) : \alpha \in C\rangle$ is strictly increasing.*

Lemma 3.4 ([4], Theorem 3.1)**.** *Let λ be a singular cardinal. Then $\square_{\lambda,\omega}$ implies that there is a good scale at λ.*

Jónsson cardinals. Another crucial role for our argument will be played by Jónsson cardinals, and their influence on the pcf structure: the presence of successor Jónsson cardinals implies the failure of SSH, which in turn ensures the existence of good scales.

Definition 3.5. *(1) Suppose that λ is a singular cardinal, and let $\mathrm{PP}(\lambda)$ be the set of all cardinals of the form $\mathrm{cof}(\prod A/D)$, where A is a set of regular cardinals cofinal in λ of order-type $\mathrm{cof}(\lambda)$ and D is an ultrafilter on A containing no bounded subsets. We define the* pseudopower *of λ, denoted $\mathrm{pp}(\lambda)$, as the supremum of $\mathrm{PP}(\lambda)$. In symbols,*

$$\mathrm{pp}(\lambda) = \sup \mathrm{PP}(\lambda).$$

(2) Shelah's Strong Hypothesis, *denoted by* SSH, *states that* $\mathrm{pp}(\lambda) = \lambda^+$ *for every singular cardinal* λ.

Definition 3.6. *A cardinal* κ *is called* Jónsson *if every first-order structure of cardinality* κ *whose language is countable possesses a* Jónsson substructure, *i.e. a proper elementary substructure of the same cardinality.*

The following characterization of Jónsson cardinals in terms of elementary embeddings will be useful.

Proposition 3.7 ([18], Lemma 1)**.** *Let* κ *be an uncountable cardinal. Then the following are equivalent:*

(1) κ *is Jónsson.*
(2) For some $\alpha > \kappa$*, there is an elementary embedding* $j : N \to V_\alpha$ *such that* N *is a transitive set,* $j(\kappa) = \kappa$ *and* $\mathrm{crit}(j) < \kappa$.

Erdős and Hajnal [8] proved that, under GCH, a successor cardinal cannot be Jónsson. Tryba showed in ZFC alone a special case of this result.

Theorem 3.8 ([18], Theorem 2)**.** *If a regular cardinal* κ *is Jónsson, then every stationary* $S \subseteq \kappa$ *reflects. In particular, if* κ *is a regular cardinal, then* κ^+ *is not Jónsson.*

Proposition 3.7 and Theorem 3.8 provide a new proof of the result by Kunen quoted in the Introduction.

Theorem 3.9 (Kunen)**.** *There is no elementary embedding from the universe to itself.*

Proof. Suppose $j : V \to V$ is an elementary embedding. Then $\kappa_\omega(j)^{++}$ is fixed by j. Let α be a fixed point of j above $\kappa_\omega(j)^{++}$, and consider the elementary embedding $j \restriction V_\alpha : V_\alpha \to V_\alpha$. By Proposition 3.7, $\kappa_\omega(j)^{++}$ is Jónsson. On the other hand, $\kappa_\omega(j)^{++}$ is the successor of a regular cardinal and so, by Theorem 3.8, it cannot be Jónsson. □

Among other things, Theorem 3.8 says that, if λ^+ is Jónsson, λ has to be singular and GCH fails. Therefore it makes sense to ask whether SSH holds at λ.

The following lemma is due to Shelah. The first assertion can be found in [7, Corollary 5.9], while the second one is proved in [7, Theorem 4.78] (see also [17, Chapter 2, Claim 1.3]).

Lemma 3.10. *Let* λ *be a singular cardinal.*

(1) If λ^+ *is Jónsson then* $\mathrm{pp}(\lambda) > \lambda^+$.
(2) If $\mathrm{pp}(\lambda) > \lambda^+$ *then* λ *carries a good scale.*

4. Discontinuities

Using the alternative definition of Jónsson cardinal (Proposition 3.7), we provide a different proof of the following lemma due to Caicedo.

Lemma 4.1 ([2], Theorem 2.5)**.** *If $j : M \to V$ is a cardinal preserving embedding, then j has no fixed points above its critical point. In particular, for all $\lambda > \operatorname{crit}(j)$, if $j[\lambda] \subseteq \lambda$, then $\operatorname{cof}^M(\lambda) \geq \operatorname{crit}(j)$.*

Proof. Suppose towards a contradiction that there is some ordinal α above $\operatorname{crit}(j)$ such that $j(\alpha) = \alpha$. By cardinal correctness, $j(\alpha^{++}) = \alpha^{++}$. Pick an ordinal $\beta > \alpha^{++}$ and stipulate $N = (V_\beta)^M$. Then $j \restriction N$ is an elementary embedding from N to $V_{j(\beta)}$. To see this, note that for b in N and ψ a first-order formula, $N \vDash \psi(b)$ is equivalent to $M \vDash \psi^N(b)$. By elementarity, $M \vDash \psi^N(b)$ if and only if $V \vDash \psi^{V_{j(\beta)}}(j(b))$. Finally, $V \vDash \psi^{V_{j(\beta)}}(j(b))$ is equivalent to $V_{j(\beta)} \vDash \psi(j(b))$. Therefore $j \restriction N$ witnesses that α^{++} is Jónsson, contradicting the fact that the successor of a regular cardinal cannot be Jónnson (Theorem 3.8).

To prove the last assertion suppose $\lambda > \operatorname{crit}(j)$, $j[\lambda] \subseteq \lambda$ and $\operatorname{cof}^M(\lambda) < \operatorname{crit}(j)$. Let A be a cofinal subset of λ in M with $|A| < \operatorname{crit}(j)$. By elementarity, $j(A)$ is a cofinal subset of $j(\lambda)$. Hence

$$j(\lambda) = \sup j(A) = \sup j[A] \leq \sup j[\lambda] \leq \sup \lambda = \lambda,$$

contradicting the fact that $\lambda < j(\lambda)$. $\square$

An immediate corollary is that, if $j : M \to V$ is a cardinal preserving embedding, then M cannot closed under sequences of size less than $\operatorname{crit}(j)$.

Despite this discontinuity, we still have a proper class of *closure points.*

Definition 4.2. *Let $j : M \to N$ be an elementary embedding between two transitive models of* ZFC*. A cardinal λ is* closure point of j *if $j[\lambda] \subseteq \lambda$.*

So for example if $j : M \to V$ is cardinal preserving, then $\kappa_\omega(j)$ is a closure point. More generally, one can easily find an ω-club class consisting of strong limit closure points of j of countable cofinality above $\operatorname{crit}(j)$. We will show that such cardinals are either M-regular cardinals or predecessors of a Jónsson cardinal.

5. The main result

Factor embeddings. The key step in the proof of Lemma 5.3 below deals with ultrafilters derived from an embedding, and applied to a model to which they do not belong. Accordingly, we review some basic definitions and facts about relativized ultrapowers and factor embeddings.

Definition 5.1. *Let $j : N \to P$ be an elementary embedding between two transitive models of* ZFC *and let x be a set in N. Suppose $a \in j(x)$.*

- *An N-*ultrafilter on x *is a set $U \subseteq P(x) \cap N$ such that*

$$(N,U) \models U \text{ is an ultrafilter.}$$

- *The* N-ultrafilter on x derived from j using a *is the* N*-ultrafilter*

$$\{A \in P(x) \cap N : a \in j(A)\}.$$

If U is an N-ultrafilter on a set x, M^N_U denotes the unique transitive collapse of the class $\mathrm{Ult}(N,U) = \{[f]^N_U : f : x \to N\}$, where $[f]^N_U$ is the set of functions $g : x \to N$ in N such that $g =_U f$ and for each function $h : x \to N$ in N, if $f =_U h$ then $\mathrm{rank}(g) \le \mathrm{rank}(h)$. The latter requirement ensures that $[f]^N_U$ is a set and it is known as Scott's trick. $\mathrm{Ult}(N,U)$ is called *relativized ultrapower of* N *by* U, and will be tacitly identified with M^N_U. N is elementarily embeddable in its ultrapower via the map $j^N_U : N \to M^N_U$, defined as

$$j^N_U : y \mapsto [c_y]^N_U.$$

We refer to j^N_U as the *the canonical embedding of* N *in* M^N_U.

The following is an useful relationship between an elementary embedding and the ultrapowers associated to its derived ultrafilters.

Lemma 5.2. *Let* $j : N \to P$ *be an elementary embedding between two transitive models of* ZFC*. Suppose* $x \in N$ *and* $a \in j(x)$*. Let* U *be the* N*-ultrafilter on* x *derived from* j *using* a*. Then there is an elementary embedding* $k : M^N_U \to P$ *such that* $k \circ j^N_U = j$ *and* $k([\mathrm{id}]^N_U) = a$*.*

Proof. For each $[f]^N_U \in M^N_U$, stipulate $k([f]^N_U) = j(f)(a)$. It is routine to verify that k fulfills the desired properties. □

We refer to the embedding k as the *factor embedding* associated to the derived N-ultrafilter U.

Good scales at $\boldsymbol{\lambda}$. In the following $j : M \to V$ is a cardinal preserving elementary embedding and λ is a strong limit closure point of j of countable cofinality above $\mathrm{crit}(j)$.

Lemma 5.3. *Either* λ *is regular in* M *or* λ^+ *is Jónsson.*

Proof. Suppose λ is singular in M. First we factor the embedding j. Let D be the M-ultrafilter on λ derived from j using λ, let $j^M_D : M \to M^M_D$ be the canonical embedding of M in M^M_D, and let $k : M^M_D \to V$ be the factor embedding associated to D. An easy argument yields $[\mathrm{id}]^M_D = \lambda$. Nevertheless, we provide the proof just to clarify where the closure of λ is used.

Claim 5.4. $[\mathrm{id}]^M_D = \lambda$.

Proof of claim. Let $\alpha < \lambda$. As $j[\lambda] \subseteq \lambda$, $j(\alpha) < \lambda$. So λ belongs to the set $\{\beta < j(\lambda) : j(\alpha) < \beta\}$. By definition of D, $\{\beta < \lambda : \alpha < \beta\} \in D$. Łoś's theorem yields $\alpha \le j^M_D(\alpha) = [c_\alpha]^M_D < [\mathrm{id}]^M_D$. Now suppose $\alpha < [\mathrm{id}]^M_D$, say $\alpha = [f]^M_D$ for some function $f : \lambda \to \lambda$ in M. By Łoś's theorem, $\{\beta < \lambda : f(\beta) < \beta\} \in D$. Hence $\lambda \in \{\beta < j(\lambda) : j(f)(\beta) < \beta\}$. Finally, $\alpha = [f]^M_D \le k([f]^M_D) = j(f)(\lambda) < \lambda$. $\square$

Therefore λ is fixed by k. A standard argument shows that λ^+ is correctly computed by M^M_D, that is $(\lambda^+)^{M^M_D} = \lambda^+$. To see this, pick a wellorder $\prec$ of λ in M. By elementarity, $\lhd = j^M_D(\prec) \cap (\lambda \times \lambda)$ is a wellorder of λ in M^M_D with length $\ge \mathrm{ot}(\prec)$. Thus $(\lambda^+)^{M^M_D} > \mathrm{ot}(\lhd) \ge \mathrm{ot}(\prec)$. If $(\lambda^+)^{M^M_D} < (\lambda^+)^M$, there are in M a bijection $f : \lambda \to (\lambda^+)^{M^M_D}$ and a wellorder $\prec$ of λ given by $\alpha \prec \beta$ iff $f(\alpha) \in f(\beta)$, leading to the following contradiction:

$$(\lambda^+)^{M^M_D} > \mathrm{ot}(j^M_D(\prec) \cap (\lambda \times \lambda)) \ge \mathrm{ot}(\prec) = (\lambda^+)^{M^M_D}.$$

So $(\lambda^+)^{M^M_D}$ has to be $(\lambda^+)^M$. In particular, $(\lambda^+)^{M^M_D} = (\lambda^+)^M = \lambda^+$ and $k(\lambda^+) = \lambda^+$.

Claim 5.5. $\mathrm{crit}(k) < \lambda$.

Proof of claim. Note that since λ is singular in M, there is a set $C \in D$ such that $|C|^M < \lambda$. In fact, let $C \in M$ be a closed unbounded subset of λ of ordertype $\mathrm{cof}^M(\lambda)$. Then $j(C)$ is closed in $j(\lambda) > \lambda$, and $j(C) \cap \lambda$ is unbounded in λ since it contains $j[C]$. Therefore $\lambda \in j(C)$, so $C \in D$ by the definition of a derived ultrafilter.

Now j^M_D is continuous at every M-regular $\gamma < \lambda$ such that $\gamma > |C|^M$. On the other hand, j is discontinuous at every M-regular cardinal by Lemma 4.1. Therefore letting $\gamma < \lambda$ be any M-regular cardinal such that $\gamma > |C|^M$, we have that $k(j^M_D(\gamma)) = j(\gamma) > j^M_D(\gamma)$, and so since $j^M_D(\gamma) < \lambda$, $\mathrm{crit}(k) < \lambda$. $\square$

As argued in the proof of Lemma 4.1, we get that λ^+ is Jónsson: pick $\beta > \lambda^+$ and let N be $(V_\beta)^{M^M_D}$. Then $k \restriction N : N \to V_{k(\beta)}$ is an elementary embedding witnessing the Jónssonness of λ^+. $\square$

We have already pointed out that both cases lead to some kind of incompactness. Suppose λ is regular in M. Then λ, being strong limit, is inaccessible in M. So we can apply Lemma 3.2 to infer that $\square_{\lambda,\omega}$ holds. Moreover, Lemma 3.4 ensures that λ carries a good scale. On the other hand, if λ^+ is Jónsson, we get the same conclusion by Lemma 3.10. Altogether we deduce a more quotable corollary:

Corollary 5.6. *There is a good scale at λ.*

Now we aim to show that this cannot be the case.

Embeddings into V. In [1], Bagaria and Magidor proved that SCH holds above an ω_1-strongly compact cardinal. They essentially used the following theorem, together with a result due to Shelah [17] asserting that the failure of SCH implies the existence of a good scale.

Theorem 5.7 (Bagaria-Magidor)**.** *If κ is ω_1-strongly compact, then there is no $\lambda > \kappa$ of countable cofinality carrying a good scale.*

We will not need Bagaria and Magidor's theorem but rather its proof. Indeed, it carries over exactly to our context:

Lemma 5.8. *If $j : M \to V$ is cardinal preserving, then no singular cardinal $\lambda > j(\mathrm{crit}(j))$ of countable cofinality carries a good scale.*

Proof. Towards a contradiction, suppose there is a singular cardinal greater than $j(\mathrm{crit}(j))$ of countable cofinality carrying a good scale. By elementarity, this is true in M, namely

$$M \models \text{there is a cardinal } \lambda > \mathrm{crit}(j) \text{ with } \mathrm{cof}(\lambda) = \omega \text{ carrying a good scale.}$$

Working in M, let $\langle f_\alpha \rangle_{\alpha<\lambda^+}$ be a good scale, relative to an increasing sequence of regular cardinals $\langle \lambda_n \rangle_{n<\omega}$ with limit λ. By Lemma 4.1 $\lambda^+ < j(\lambda^+)$. Passing to V, let $\beta = \sup j[\lambda^+]$. Since $\lambda^{+M} = \lambda^+$, we have $j(\lambda^+) = j(\lambda^{+M}) = j(\lambda)^+$. In particular, $j(\lambda^+)$ is regular. On the other hand, β is the supremum of a subset of $j(\lambda^+)$ of cardinality less than $j(\lambda^+)$, and therefore bounded in $j(\lambda^+)$. Thus, $\beta < j(\lambda^+)$. Elementarity of j leads to

$$V \models j(\langle f_\alpha \rangle_{\alpha<\lambda^+}) \text{ is a good scale relative to } \langle j(\lambda_n) \rangle_{n<\omega}.$$

Say $j(\langle f_\alpha \rangle_{\alpha<\lambda^+}) = \langle g_\alpha \rangle_{\alpha<j(\lambda^+)}$. Since $\beta < j(\lambda^+)$ and has uncountable cofinality, we can use the goodness of $\langle g_\alpha \rangle_{\alpha<j(\lambda^+)}$ to pick a cofinal subset C of β such that, for some $n < \omega$, for all $\xi_0 < \xi_1$ in C, and for all $k \geq n$, $g_{\xi_0}(k) < g_{\xi_1}(k)$. Following the proof of [1, Theorem 4.1], we will define by induction on $\delta < \lambda^+$ a strictly increasing sequence of ordinals $\langle \gamma_\delta \rangle_{\delta<\lambda^+}$ contained in C, and an auxiliary sequence of ordinals $\langle \alpha_\delta \rangle_{\delta<\lambda^+}$ such that $\gamma_\delta < j(\alpha_\delta) < \gamma_{\delta+1}$, for all $\delta < \lambda^+$. Let γ_0 be the first ordinal in C. Let α_0 be the least ordinal such that $\gamma_0 < j(\alpha_0)$. Then let $\gamma_1 \in C$ be such that $j(\alpha_0) < \gamma_1$. Then, let α_1 be the least ordinal such that $\gamma_1 < j(\alpha_1)$. And so on. At limit stages, take the least $\gamma \in C$ greater than all the ordinals γ_δ picked so far. Clearly, $\alpha_\delta < \lambda^+$, for all $\delta < \lambda^+$. For each $\delta < \lambda^+$, we have $g_{\gamma_\delta} <^* g_{j(\alpha_\delta)} <^* g_{\gamma_{\delta+1}}$, and so we may pick some $n_\delta > n$ such that $g_{\gamma_\delta}(m) < g_{j(\alpha_\delta)}(m) < g_{\gamma_{\delta+1}}(m)$, for all $m \geq n_\delta$. By the pigeonhole principle, there is some $D \subseteq \lambda^+$ of cardinality λ^+ such that for all $\delta \in D$, the n_δ is the same, say k. In particular, if $\delta \in D$, then $g_{\gamma_\delta}(k) < g_{j(\alpha_\delta)}(k) < g_{\gamma_{\delta+1}}(k)$. On the other hand, if $\delta_0, \delta_1 \in D$ and $\delta_0 + 1 < \delta_1$, then $g_{\gamma_{\delta_0+1}}(k) < g_{\gamma_{\delta_1}}(k)$, by goodness. Therefore,

$$g_{\gamma_{\delta_0}}(k) < g_{j(\alpha_{\delta_0})}(k) < g_{\gamma_{\delta_0+1}}(k) < g_{\gamma_{\delta_1}}(k) < g_{j(\alpha_{\delta_1})}(k) < g_{\gamma_{\delta_1+1}}(k),$$

whenever $\delta_0, \delta_1 \in D$ and $\delta_0 + 1 < \delta_1$. Note that for every $\delta < \lambda^+$,

$$g_{j(\alpha_\delta)}(k) = j(f_{\alpha_\delta}(k)) \in j[\lambda_k].$$

But this is impossible since the sequence $\langle g_{\gamma_\delta}(k) \rangle_{\delta\in D}$ has ordertype λ^+, and $\mathrm{ot}(\langle g_{\gamma_\delta}(k) \rangle_{\delta\in D}) = \mathrm{ot}(\langle g_{j(\alpha_\delta)}(k) \rangle_{\delta\in D}) \leq \mathrm{ot}(j[\lambda_k])$. However, the ordertype of $j[\lambda_k]$ is $\lambda_k < \lambda^+$. This is a contradiction. □

Theorem 5.9. *There are no nontrivial cardinal preserving elementary embeddings from an inner model M into the universe of sets V.*

Proof. Suppose not and let $j : M \to V$ be a cardinal preserving elementary embedding. Let λ be the first strong limit closure point of j of countable cofinality strictly above $j(\text{crit}(j))$. By Corollary 5.6, there is good scale at λ. But Lemma 5.8 says that this is impossible. □

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