Measurable cardinals and choiceless axioms

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Abstract

Kunen refuted the existence of an elementary embedding from the universe of sets to itself assuming the Axiom of Choice. This paper concerns the ramifications of this hypothesis when the Axiom of Choice is not assumed. For example, the existence of such an embedding implies that there is a proper class of cardinals λ such that λ^+ is measurable.

1 Introduction

1.1 The Kunen inconsistency

One of the most influential ideas in the history of large cardinals is Scott's reformulation of measurability in terms of elementary embeddings [7]: the existence of a measurable cardinal is equivalent to the existence of a nontrivial elementary embedding from the universe of sets V into a transitive submodel M. In the late 1960s, Solovay and Reinhardt realized that by imposing stronger and stronger closure constraints on the model M, one obtains stronger and stronger large cardinal axioms, an insight which rapidly led to the discovery of most of the modern large cardinal hierarchy. Around this time, Reinhardt formulated the ultimate large cardinal principle of this kind: there is an elementary embedding from the universe of sets to itself.[1] Soon after, however, Kunen [5] showed that this principle is inconsistent:

Theorem (Kunen)**.** *There is no elementary embedding from the universe of sets to itself.*

Kunen's proof relies heavily on the Axiom of Choice, however, and the question of whether this is necessary immediate arose.[2] Decades later, Woodin returned to this question and discovered that although the traditional large cardinal hierarchy stops short at Kunen's bound, there lies beyond it a further realm of large cardinal axioms incompatible with the Axiom of Choice, axioms so absurdly strong that Reinhardt's so-called ultimate axiom appears tame by comparison. Yet since their discovery, despite significant efforts of many researchers, no one has managed to prove the inconsistency of a single one of these choiceless large cardinal axioms. "The difficulty," according to Woodin [9], "is that without the Axiom of Choice it is extraordinarily difficult to prove anything about sets."

One remedy to this difficulty, proposed by Woodin himself [8, Theorem 227], is to *simulate* the Axiom of Choice using auxiliary large cardinal hypotheses, especially extendible

[1] Of course, the identity is such an elementary embedding. Whenever we write "elementary embedding," we will really mean "nontrivial elementary embedding."

[2] The question was first raised by the anonymous referee of Kunen's paper.

cardinals. Cutolo [2] expanded on this idea to establish the striking result that the successor of a singular Berkeley limit of extendible cardinals is measurable. While the strength of Cutolo's large cardinal hypothesis far surpasses that of Reinhardt's ultimate axiom, Asperó [1] showed that a Reinhardt cardinal alone implies the existence of cardinals that resemble extendible cardinals. Combining Woodin, Cutolo, and Asperó's ideas, we show here that one can simulate the Axiom of Choice using large cardinal axioms that can be derived from Reinhardt's principle. This allows us to establish some vast generalizations of Cutolo's results. We are optimistic that these ideas will bring clarity to the question of the consistency of the choiceless hierarchy.

1.2 Main results

Gitik showed that it is consistent with ZF that there are no regular uncountable cardinals. On the other hand, if there is an elementary embedding j from the universe of sets to itself, its critical point κ is measurable and hence regular.[3] By elementarity, the cardinals $\kappa_1(j) = j(\kappa)$, $\kappa_2(j) = j(j(\kappa))$, $\kappa_3(j) = j(j(j(\kappa)))$, and so on are all regular as well. Asperó asked whether there must be any regular cardinals larger than their supremum $\kappa_\omega(j) = \sup_{n<\omega} \kappa_n(j)$. The first theorem of this paper answers his question positively:

Theorem 2.16. *Suppose there is an elementary embedding from the universe of sets to itself. Then there is a proper class of regular cardinals.*

This theorem is a consequence of the *wellordered collection lemma*, a weak choice principle that we show follows from choiceless cardinals:

Theorem 2.20. *Suppose there is an elementary embedding from the universe of sets to itself. For every cardinal κ, there is a set I such for any sequence $\langle A_\alpha : \alpha < \kappa\rangle$ of nonempty sets, there is a set $\sigma = \{a_i : i \in I\}$ such that $A_\alpha \cap \sigma \neq \emptyset$ for all $\alpha < \kappa$.*

Having answered Asperó's question, it is natural to wonder whether an Reinhardt's principle in fact implies the existence a proper class of *measurable* cardinals. Given Cutolo's result, one would also like to know whether any of these measurable cardinals are successor cardinals.

Theorem 3.14. *Suppose there is an elementary embedding from the universe of sets to itself. Then for a closed unbounded class of cardinals κ, either κ or κ^+ is measurable.*

In particular, for every regular cardinal γ, there are arbitrarily large cardinals λ of cofinality γ such that λ^+ is measurable.

There is really only one other principle that is known to imply the existence of measurable successor cardinals: the Axiom of Determinacy (AD). Famously, Solovay showed that under AD, $\aleph_1$ is measurable. Moreover, there is a unique normal ultrafilter on $\aleph_1$: the closed unbounded filter. Later, Martin showed that $\aleph_2$ is measurable, and finally Kunen showed:

Theorem (Kunen)**.** *$\boldsymbol{\delta}^1_n$ is measurable for all $n < \omega$.*

All the projective ordinals are successor cardinals by results of Kechris, Kunen, and Martin. In each case, the ω-closed unbounded filter is a normal ultrafilter.

[3] The critical point of $j : V \to M$, denoted $\text{crit}(j)$, is the least ordinal α such that $j(\alpha) > \alpha$. Scott [7] showed that $\text{crit}(j)$ is measurable.

This raises the question of whether the structure of measures on ordinals assuming choiceless large cardinal axioms is analogous to their structure assuming AD. An easy forcing argument shows that it is consistent with choiceless axioms that for all cardinals δ of uncountable cofinality, the ω-closed unbounded filter is *not* an ultrafilter, so we must temper our aspirations accordingly. If F is a filter, a set S is an *atom of* F if $F \cup \{S\}$ generates an ultrafilter. A filter F is atomic if every F-positive set contains an atom.

Theorem 3.12. *Suppose there is an elementary embedding from the universe of sets to itself. Then for all sufficiently large regular cardinals δ, the closed unbounded filter on δ is atomic.*

One might hope to establish in this way that all sufficiently large regular cardinals are measurable. What is missing is a proof that the closed unbounded filter on δ is δ-complete. Instead, one can only show:

Theorem 3.3. *Suppose there is an elementary embedding from the universe of sets to itself. Then for all cardinals κ, for all sufficiently large regular cardinals δ, the closed unbounded filter on δ is κ-complete.*

In a sense, the previous two theorems are stronger than what is known to follow from AD. Assuming AD^+, a theorem of Steel and Woodin shows that every regular cardinal less than Θ is measurable, but it is open, for example, whether the ω_1-club filter on a regular cardinal greater than ω_1 must be an ultrafilter, or even just an atomic filter.

The rest of this paper is centered around an order on ultrafilters known as the *Ketonen order*, whose tortuous history we now describe. The discovery of the Ketonen order was precipitated by Kunen's construction of a normal ultrafilter that concentrates on nonmeasurable cardinals. Ketonen, then completing his dissertation under Kunen, realized that implicit in this proof was a natural order on weakly normal ultrafilters. He and Kunen collaborated to prove the wellfoundedness of this order, which Ketonen needed to answer a question posed by Kunen.[4] The next year, also inspired by Kunen's construction, Mitchell [6] independently discovered Ketonen's order, or rather its restriction to normal ultrafilters. Mitchell proved that this order is linear in canonical inner models of large cardinal axioms. Since then, the *Mitchell order* has become a fundamental object of study in large cardinal theory.

Ketonen's order, on the other hand, was forgotten completely until almost half a century later, the author independently discovered a generalization of his order to *all* countably complete ultrafilters on ordinals: if U and W are countably complete ultrafilters on ordinals, set $U <_{\Bbbk} W$ if there is a sequence ultrafilters U_α on α, defined for W-almost all ordinals α, such that $A \in U$ if and only if $A \cap \alpha \in U_\alpha$ for W-almost all α. Like Ketonen's original order, $<_{\Bbbk}$ is wellfounded, although this requires an argument that is completely different from Ketonen and Kunen's. Like the Mitchell order, $<_{\Bbbk}$ is linear in all known canonical inner models of large cardinal axioms, although again the proof of this is completely different from Mitchell's.

The order $<_{\Bbbk}$ is now known as the Ketonen order. In the context of the Axiom of Choice, the linearity of the Ketonen order is equivalent to the Ultrapower Axiom [3], a principle with many consequences in large cardinal theory, but in this paper, we will apply linearity properties of the Ketonen order to the theory of choiceless cardinals. The key phenomenon is that choiceless cardinals imply that the Ketonen order is *almost* linear:

[4]If every regular cardinal above κ carries a κ-complete uniform ultrafilter, then κ is strongly compact.

Theorem 4.7. *Suppose there is an elementary embedding j from the universe of sets to itself. Then for some cardinal κ, any set of Ketonen incomparable κ-complete ultrafilters has cardinality at most $\kappa_\omega(j)$.*

In other words, choiceless large cardinal axioms almost imply the Ultrapower Axiom. One cannot hope to prove such a theorem from traditional large cardinal axioms. It is open whether something analogous follows from the Axiom of Determinacy

This linearity phenomenon is a component of the proof of one of the main theorems of this paper. In the context of ZFC, an uncountable cardinal κ is said to be *strongly compact* if every κ-complete filter extends to a κ-complete ultrafilter. Kunen realized, however, that this *filter extension property* occurs in nature:

Theorem (Kunen)**.** *Assume* $\text{AD} + \text{DC}_\mathbb{R}$*. If β is an ordinal that can be enumerated as $\{\alpha_x : x \in \mathbb{R}\}$ then every countably complete filter on β extends to a countably complete ultrafilter.*

In order to prove a version of Kunen's theorem from choiceless large cardinal axioms, we extend the Ketonen order to a wellfounded partial order on filters. We then prove the filter extension property by induction on this order.

Theorem 4.9. *Assume there is an elementary embedding from the universe of sets to itself. Then for a closed unbounded class of cardinals κ, every κ-complete filter on an ordinal extends to a κ-complete ultrafilter.*

The semi-linearity of the Ketonen order on ultrafilters ends up being used here to avoid using certain properties of filters that require the Axiom of Choice.

2 Choicelike consequences of choiceless axioms

2.1 Notation

We write $|X| \leq |Y|$ to mean that there is an injection from X to Y and $|X| \leq^* |Y|$ to mean that there is a partial surjection from Y to X. The *Scott rank of* X, denoted by $\text{scott}(X)$, is the least ordinal α such that there is an injection from X to V_α, while $|X|$ denotes the family of sets of rank $\text{scott}(X)$ that are in bijection with X. The *dual Scott rank of* X, denoted by $\text{scott}^*(X)$, is the least ordinal α such that there is a partial surjection from V_α to X.

2.2 Almost extendibility and supercompactness

A cardinal λ is *rank Berkeley* if for all $\alpha < \lambda \leq \beta$, there is an elementary embedding $j : V_\beta \to V_\beta$ with $\alpha < \text{crit}(j) < \lambda$. First introduced by Schlutzenberg, rank Berkeley cardinals are a weakening of Reinhardt cardinals that have the advantage of being first-order. This enables us to work in ZF rather than a class theory like NBG, and this is how we proceed for the rest of the paper.

Although the relationship between Reinhardt cardinals and rank Berkeley cardinals is not entirely understood, a folklore argument shows that above every Reinhardt cardinal lies a rank Berkeley cardinal. If $j : M \to N$ is a nontrivial elementary embedding between transitive models of set thery, the *critical sequence* $\langle \kappa_n : n \leq \omega \rangle$ *of* j is defined by recursion, setting $\kappa_0 = \text{crit}(j)$, and for $n < \omega$, $\kappa_{n+1} = j(\kappa_n)$. Finally, we set $\kappa_\omega = \sup_{n<\omega} \kappa_n$. Of course, κ_0 will be undefined if j has no critical point, and κ_{n+1} will be undefined if $\kappa_n \notin M$. If κ_n is defined, we denote it by $\kappa_n(j)$.

Proposition 2.1 (NBG)**.** *If $j : V \to V$ is a nontrivial elementary embedding, then $\kappa_\omega(j)$ is rank Berkeley.*

Proof. Let $\lambda = \kappa_\omega(j)$. Assume the proposition fails, and consider the least α such that there is no elementary $k : V_\alpha \to V_\alpha$ such that $\kappa_\omega(k) = \lambda$. Then α is definable from λ, and hence $j(\alpha) = \alpha$. But the embedding $k = j \restriction V_\alpha$ contradicts the definition of α. $\square$

The rest of this section deduces the existence of many pseudo-extendible and pseudo-supercompact cardinals from the existence of a rank Berkeley cardinal. This is similar to work of Asperó [1].

Proposition 2.2. *Suppose α is a limit ordinal and $j : V_\alpha \to V_\alpha$ is elementary. Suppose $\mathcal{S}$ is a subset of V_α consisting of structures in a fixed finite signature, $j(\mathcal{S}) = \mathcal{S}$,*[5] *and $\theta(\mathcal{S}) \geq \kappa_\omega(j)$. Then there exist distinct M_0 and M_1 in $\mathcal{S}$ such that M_0 elementarily embeds into M_1.*

Proof. Suppose not. Let $\kappa = \text{crit}(j)$ and let $\lambda = \kappa_\omega(j)$. For $\beta < \alpha$, let $\mathcal{S}_\beta = \mathcal{S} \cap V_\beta$. We claim that for some $\beta < \alpha$, $\theta(\mathcal{S}_\beta) > \text{crit}(j)$. Let $f : \mathcal{S} \to \kappa$ be a surjection. For each $\beta < \alpha$, let $\xi_\beta = \text{ot}(f[\mathcal{S}_\beta])$. Then $\sup_{\beta<\alpha} \xi_\beta = \kappa$. Since j extends to a Σ_0-elementary embedding from $V_{\alpha+1}$ to $V_{\alpha+1}$, $\text{cf}(\alpha) \neq \kappa$, and hence $\xi_\beta = \kappa$ for some $\beta < \alpha$.

Fix β such that $\theta(\mathcal{S}_\beta) > \kappa$. Let $g : \mathcal{S}_\beta \to \kappa$ be a surjection. Note that $j(g)$ is a surjection from $j(\mathcal{S}_\beta)$ to $j(\kappa)$, while $j(g)(j(M)) = j(g(M)) = g(M) < \kappa$ for all $M \in \mathcal{S}_\beta$. It follows that there is some $M_0 \in j(\mathcal{S}_\beta)$ such that $M_0 \notin j[\mathcal{S}_\beta]$. As a consequence, letting $M_1 = j(M_0)$, we have $M_0 \neq M_1$. Moreover, j restricts to an elementary embedding from M_0 to M_1. $\square$

The *lightface Vopěnka principle* states that for all parameter-free definable classes $\mathcal{S}$ of structures in a fixed finite signature, there exist distinct structures M_0 and M_1 in $\mathcal{S}$ such that M_0 elementarily embeds into M_1.

Corollary 2.3. *If λ is rank Berkeley, then for any ordinal definable set of structures $\mathcal{S}$ in a finite signature such that $\theta(\mathcal{S}) \geq \lambda$, there exist distinct structures M_0 and M_1 in $\mathcal{S}$ such that M_0 elementarily embeds into M_1. As a consequence, the lightface Vopěnka principle holds.* $\square$

A cardinal η is (γ, ∞)-*extendible* if for all $\nu > \eta$, there is an elementary embedding $\pi : V_\nu \to V_{\nu'}$ such that $\pi(\eta) > \nu$ and $\pi(\gamma) = \gamma$.

Lemma 2.4. *Assume the lightface Vopěnka principle. Then for all ordinals γ, there is a (γ, ∞)-extendible cardinal.*

Proof. Assume not. Define a continuous sequence of ordinals $\langle \eta_\xi : \xi \in \text{Ord} \rangle$ by transfinite recursion, letting $\eta_{\xi+1}$ be the least ordinal $\nu > \eta_\xi$ such that there is no $\pi : V_\nu \to V_{\nu'}$ such that $\pi(\eta_\xi) \geq \nu$. Let $\mathcal{S}$ be the class of structures $\mathcal{M}_\xi = (V_{\eta_{\xi+1}}, \eta_\xi, \gamma)$. Applying the lightface Vopěnka principle to the ordinal definable class $\mathcal{S}$, we obtain ordinals $\xi_0 < \xi_1$ and an elementary embedding $\pi : \mathcal{M}_{\xi_0} \to \mathcal{M}_{\xi_1}$. This means $\pi : V_{\eta_{\xi_0+1}} \to V_{\eta_{\xi_1+1}}$ is elementary, $\pi(\eta_{\xi_0}) = \eta_{\xi_1}$, and $\pi(\gamma) = \gamma$. This contradicts the definition of η_{ξ_0+1} since $\pi(\eta_{\xi_0}) = \eta_{\xi_1} \geq \eta_{\xi_0+1}$. $\square$

[5]More formally, we mean that $\bigcup_{x \in V_\beta} j(\mathcal{S} \cap x) = \mathcal{S}$.

A cardinal η is *almost extendible* if it is (γ, ∞)-extendible for all $\gamma < \eta$.

Corollary 2.5. *Assume the lightface Vopěnka principle. Then there is a club class of almost extendible cardinals.* □

A cardinal η is (γ, ∞)-*supercompact* if for all $\nu > \eta$, for some $\bar{\nu} < \eta$, there is an elementary embedding $\pi : V_{\bar{\nu}} \to V_\nu$ such that $\pi(\gamma) = \gamma$. A cardinal η is *almost supercompact* if it is (γ, ∞)-supercompact for all $\gamma < \eta$.

Lemma 2.6. *Any almost extendible cardinal is a limit of almost supercompact cardinals.* □

For one of our applications (the filter extension property, proved in Section 4), we require a slight strengthening of these notions. A cardinal λ is X-*closed rank Berkeley* if for all $\gamma < \lambda < \alpha$, there is an elementary embedding $j : V_\alpha \to V_\alpha$ such that $\gamma < \text{crit}(j) < \lambda$ and $j(X) = j[X]$. If κ is a cardinal and $X \in V_\kappa$, then κ is X-*closed almost extendible* if for all $\gamma < \kappa < \alpha$ there is an elementary embedding $j : V_\alpha \to V_{\alpha'}$ such that $j(\gamma) = \gamma$ and $j(X) = j[X]$.

Theorem 2.7. *For any cardinal λ, for a club class of κ, κ is X-closed almost extendible for every X such that λ is X-closed rank Berkeley.*

The proofs above easily yield the following proposition:

Proposition 2.8. *If there is an X-closed rank Berkeley cardinal, then there is a club class of X-closed almost extendible cardinals.* □

Theorem 2.7 is a trivial consequence of Proposition 2.8 once one realizes that there is essentially just a set of X such that λ is X-closed almost extendible. We will use the following lemma, which will also be important later.

Lemma 2.9. *Suppose $j : V_\alpha \to V_\alpha$ is an elementary embedding, κ is almost supercompact, and $\text{cf}(\kappa) \geq \text{crit}(j)$. Suppose $A \in V_\alpha$ is a set such that $j(A) = j[A]$. Then there is an injection from A into V_β for some $\beta < \kappa$.*

Proof. Let S be the set of Scott ranks of subsets of A. Then $j(S) = S$ since A and $j(A)$ are in bijection. Moreover for all $\nu \in S$, $j(\nu) = \nu$ since for all $B \subseteq A$, $j(B) = j[B]$. Hence $|S| < \text{crit}(j)$.

Let $\xi = \sup(S \cap \kappa)$, and note that $\xi < \kappa$ since $\text{cf}(\kappa) \geq \text{crit}(j)$.

Let $\pi : V_{\bar{\alpha}} \to V_\alpha$ such that $\xi < \bar{\alpha} < \kappa$, $A \in \text{ran}(\pi)$, and $\pi(\xi) = \xi$. Let $\bar{A} = \pi^{-1}(A)$ and let ν be the Scott rank of $\bar{A}$. Since $\pi[\bar{A}] \subseteq A$, $\nu \in S$. Therefore $\nu \in S \cap \kappa$, which implies that $\nu < \xi$, and hence $\pi(\nu) < \pi(\xi) = \xi < \kappa$. This completes the proof, noting that $\pi(\nu)$ is the Scott rank of A. □

Proof of Theorem 2.7. We may assume that λ is rank Berkeley, since otherwise the theorem is vacuous. Applying Corollary 2.5 and Lemma 2.6, for each regular γ, let ρ_γ be the least almost supercompact cardinal of cofinality γ, and let $\rho = \sup\{\rho_\gamma : \gamma \in \text{Reg} \cap \lambda\}$.

Let Γ denote the class of X such that λ is X-closed rank Berkeley. By Lemma 2.9, for each $X \in \Gamma$, there is some $Y \in V_\rho$ such that $|X| \leq |Y|$. If $|X| \leq |Y|$ and κ is Y-closed almost extendible, then κ is X-closed almost extendible, so it suffices to show that there is some κ that is Y-closed almost extendible for all $Y \in \Gamma \cap V_\rho$. This is an immediate consequence of Proposition 2.8 and the closure of club classes under set-sized intersections. □

2.3 Regular cardinals

Definition 2.10. If Γ is family of sets, then $\delta(\Gamma)$ denotes the supremum of the ranks of all prewellorders in Γ.

Proposition 2.11. *Suppose Γ is a set such that there is a γ-descendingly closed fine filter F on $P(\Gamma)$ that concentrates on $\sigma \subseteq \Gamma$ such that $\delta(\sigma) < \delta(\Gamma)$. Then $\mathrm{cf}(\delta(\Gamma)) \neq \gamma$.*

Proof. Let $f : \gamma \to \delta(\Gamma)$ be an increasing function. We will show that $f[\gamma]$ is bounded below $\delta(\Gamma)$. For $\alpha < \gamma$, let A_α denote the set of all $\sigma \in Y$ such that $\mathrm{rank}(E) \geq f(\alpha)$ for some $E \in \sigma$. Notice that $A_\alpha \subseteq A_\beta$ for $\beta \leq \alpha$ and by fineness, $A_\alpha \in F$. Since F is γ-descendingly closed, $A = \bigcap_{\alpha<\gamma} A_\alpha \in F$; note that A is the set of $\sigma \subseteq \Gamma$ such that $\delta(\sigma) \geq \sup f[\gamma]$. By our assumptions on F, any F-large set contains some σ such that $\delta(\sigma) < \delta(\Gamma)$, so we may fix such a σ belongs to A. Now $\sup f[\gamma] \leq \delta(\sigma) < \delta(\Gamma)$, so $f[\gamma]$ is bounded below $\delta(\Gamma)$, as desired. $\square$

Let $\theta(X) = \delta(P(X \times X))$. Note that $\theta(X)$ is the least ordinal that is the surjective image of a subset of X, which is also known as the Lindenbaum number of X.

Corollary 2.12. *Suppose that $\gamma \leq \eta$ are cardinals, X is a set such that $\eta \leq^* X \times X \leq^* X$, and there is a γ-descendingly closed fine filter on $P(P(X))$ that concentrates on the set of $\sigma \subseteq P(X)$ such that $\theta(\sigma) < \eta$. Then $\mathrm{cf}(\theta(X)) \neq \gamma$.*

Proof. Let $\Gamma = P(X \times X)$. We will show that for all $\sigma \subseteq P(X)$ such that $\theta(\sigma) < \eta$, $\sup_{E\in\sigma} \mathrm{rank}(E) < \theta(X)$. Then Proposition 2.11 implies the desired conclusion.

Fix $\sigma \subseteq P(X)$ such that $\theta(\sigma) < \eta$, and let $\rho = \sup_{E\in\sigma} \mathrm{rank}(E)$. For each $x \in X$, let $g_x : \sigma \to \rho$ be defined by $g_x(E) = \mathrm{rank}_E(x)$. Let $A_x = g_x[\sigma]$ and let $f_x : \alpha_x \to A_x$ be the increasing enumeration of A_x. Since A_x is the surjective image of σ, $\alpha_x < \theta(\sigma) = \eta < \theta(X)$. Let $g : X \to \theta(\sigma)$ be a surjection. Then define a partial surjection $F : X \times X \to \rho$ by setting $F(x, y) = f_x(g(y))$ whenever $g(y) < \alpha_x$. It follows that $\rho < \theta(X)$, as claimed. $\square$

A standard argument shows that if η is almost supercompact, then for all $\gamma < \eta$ and all sets X, there is a γ-descendingly closed normal fine ultrafilter on $P(P(X))$ that concentrates on the set of $\sigma \subseteq P(X)$ such that $\theta(\sigma) < \eta$.

Corollary 2.13. *If η is almost supercompact and X is a set such that $\eta \leq^* X \times X \leq^* X$, then $\mathrm{cf}(\theta(X)) \geq \eta$.* $\square$

Corollary 2.14. *If η is almost supercompact, then every successor cardinal greater than or equal to η has cofinality at least η.* $\square$

Corollary 2.15. *Assume there is an almost extendible cardinal. Then there is a proper class of regular cardinals. In particular, if there is a rank Berkeley cardinal, there is a proper class of regular cardinals.* $\square$

Theorem 2.16. *Assume there is a rank Berkeley cardinal. Then there is a proper class of regular cardinals.*

2.4 The wellordered collection lemma

Theorem 2.17. *Suppose X is a set and Λ is a family of subsets of X such that for all $\gamma \leq \beta$, there is a γ-descendingly closed fine filter on $P(\Lambda)$ concentrating on the set of $\sigma \subseteq \Lambda$ such that $\bigcup \sigma \in \Lambda$. Then for any sequence $\langle S_\xi : \xi < \beta \rangle$ of nonempty subsets of X, there is some $\tau \in \Lambda$ such that $S_\xi \cap \tau \neq \emptyset$ for all $\xi < \beta$.*

Proof. By induction, assume the theorem is true for all $\alpha < \beta$. We may assume that β is a limit ordinal. Let F be a β-descendingly closed fine filter on $P(\Lambda)$ concentrating on $\sigma \subseteq \Lambda$ such that $\bigcup \sigma \in \Lambda$. Let T_α denote the set of $\tau \in \Lambda$ such that $S_\xi \cap \tau \neq \emptyset$ for all $\xi < \alpha$. Let $B_\alpha \subseteq P(\Lambda)$ be the set of $\sigma \subseteq \Lambda$ such that $T_\alpha \cap \sigma \neq \emptyset$. Note that the sequence $\langle B_\alpha : \alpha < \beta \rangle$ is descending (since $\langle T_\alpha : \alpha < \beta \rangle$ is), and by fineness and our induction hypothesis, $B_\alpha \in F$ for all $\alpha < \beta$. Therefore $B = \bigcap_{\alpha < \beta} B_\alpha$ belongs to F since F is β-descendingly closed. Fix $\sigma \in B$ such that $\bigcup \sigma \in \Lambda$. Then $\tau = \bigcup \sigma$ is as desired. $\square$

Corollary 2.18 (Wellordered collection lemma)**.** *Suppose $\beta \leq \eta$ are ordinals such that η is (γ, ∞)-supercompact for all regular $\gamma \leq \beta$. Then for any sequence $\langle S_\xi : \xi < \beta \rangle$ of nonempty sets, there is a set τ such that* $\text{scott}^*(\tau) \leq \eta$ *and $S_\xi \cap \tau \neq \emptyset$ for all $\xi < \beta$. If η is regular, one can find such a τ with* $\text{scott}(\tau) < \eta$*.*

Proof. Let $X = \bigcup_{\xi < \beta} S_\xi$. Let Λ be the set of $\tau \subseteq X$ such that $\text{scott}^*(\tau) \leq \eta$. Fix a regular cardinal $\gamma \leq \beta$, and we will show that there is a γ-descendingly closed normal fine ultrafilter on $P(\Lambda)$ concentrating on $\sigma \subseteq \Lambda$ such that $\bigcup \sigma \in \Lambda$. Applying Theorem 2.17 then yields the first part of the theorem, and the second part is similar.

Since η is (γ, ∞)-supercompact, there is an elementary embedding $\pi : V_{\bar{\nu}} \to V_\nu$ with $\bar{\nu} < \eta$, $\pi(\gamma) = \gamma$, and $P(\Lambda) \in \text{ran}(\pi)$. Let $\bar{\Lambda} = \pi^{-1}(\Lambda)$, let $\bar{\mathcal{U}}$ be the normal fine ultrafilter on $P(\bar{\Lambda})$ derived from π using $\pi[\bar{\Lambda}]$. We claim that $\bar{\mathcal{U}}$ concentrates on σ such that $\bigcup \sigma \in \bar{\Lambda}$. To see this, it suffices to show that $\bigcup j[\bar{\Lambda}] \in \Lambda$. Define a partial function $F : V_{\bar{\nu}} \times V_\eta \to X$ by setting $F(g, x) = \pi(g)(x)$ if g is a function and $x \in \text{dom}(\pi(g))$. Then $\bigcup j[\bar{\Lambda}] \subseteq \text{ran}(F)$, which shows that $\text{scott}^*(\bigcup j[\bar{\Lambda}]) \leq \eta$. Finally let $\mathcal{U} = \pi(\bar{\mathcal{U}})$. Then $\mathcal{U}$ is a γ-descendingly closed normal fine ultrafilter on $P(\Lambda)$ concentrating on $\sigma \subseteq \Lambda$ such that $\bigcup \sigma \in \Lambda$. $\square$

Proposition 2.19. *Suppose κ is almost extendible. Then for any $\beta < \kappa$, for any sequence $\langle S_\xi : \xi < \beta \rangle$, there is a set σ with* $\text{scott}(\sigma) < \kappa$ *such that $S_\xi \cap \sigma \neq \emptyset$ for all $\xi < \beta$.*

Proof. This follows from the fact that almost extendible cardinals are limits of almost supercompact cardinals (Lemma 2.6). Note here that if $|X| \leq^* |Y|$ then $|X| \leq |P(Y)|$. $\square$

Theorem 2.20. *Suppose there is a rank Berkeley cardinal. For every cardinal κ, there is a set I such for any sequence $\langle A_\alpha : \alpha < \kappa \rangle$ of nonempty sets, there is a set $\sigma = \{a_i : i \in I\}$ such that $A_\alpha \cap \sigma \neq \emptyset$ for all $\alpha < \kappa$.*

3 Filters, saturation, and atoms

3.1 Terminology

A *filter base* is a family of nonempty sets $\mathcal{B}$ with the finite intersection property. If X is a set, a filter base $\mathcal{B}$ is *X-closed* if for any $\langle A_x : x \in X \rangle \subseteq \mathcal{B}$, there is some $A \in \mathcal{B}$ such that $A \subseteq \bigcap_{x \in X} A_x$. If Y is a family of sets, $\mathcal{B}$ is *Y-complete* if it is X-closed for all $X \in Y$. A filter base F is a *filter* if it is closed upwards under inclusion. The *filter generated by a filter base* $\mathcal{B}$ is the family of sets that contain some element of $\mathcal{B}$.

If F is a filter on X, a set A is *F-null* if $X \setminus A$ belongs to F. The dual ideal of F, denoted by F^*, is the set of $A \subseteq X$ such that $X \setminus A \in F$. A set S is *F-positive* if $S \cap A \neq \emptyset$ for all $A \in F$, or equivalently, if S is not F-null. The set of F-positive subsets of X is denoted by F^+. If F is a filter and $S \in F^+$, $F \restriction S$ denotes the filter generated by $F \cup \{S\}$.

3.2 Filter bases

A *filter base* is a family of nonempty sets $\mathcal{B}$ such that for any two sets in $\mathcal{B}$, there is a third set in $\mathcal{B}$ contained in their intersection. If X is a set, a filter base $\mathcal{B}$ is *X-closed* if for any $\langle A_x : x \in X\rangle \subseteq \mathcal{B}$, there is some $A \in \mathcal{B}$ such that $A \subseteq \bigcap_{x\in X} A_x$. If Y is a family of sets, $\mathcal{B}$ is *Y-complete* if it is X-closed for all $X \in Y$. The *filter generated by a filter base* $\mathcal{B}$ is the family of sets that contain some element of $\mathcal{B}$.

The following theorem on the completeness of the filter generated by a filter base is almost a restatement of Theorem 2.17.

Theorem 3.1. *Suppose $\mathcal{B}$ is a filter base on a set X and β is an ordinal such that for all regular $\gamma \leq \beta$, there is a γ-descendingly closed fine filter on $P(\mathcal{B})$ concentrating on the set of $\sigma \subseteq \mathcal{B}$ such that $\bigcap \sigma \in \mathcal{B}$. Then the filter generated by $\mathcal{B}$ is β-closed.* □

The following consequence of supercompactness bears a similar relationship to Corollary 2.18.

Theorem 3.2. *Suppose $\beta < \eta$ are ordinals such that η is (γ, ∞)-supercompact for all regular $\gamma \leq \beta$. Then any V_η-complete filter base generates a β-closed filter.* □

Theorem 3.3. *Suppose η is almost supercompact. Then for all ordinals δ of cofinality at least η, the club filter on δ is η-complete.*

Proof. By Theorem 3.2, it suffices to show that the set of club subsets of δ is a V_η-complete filter base. The proof of this fact is familiar from the standard theory of clubs. Suppose $X \in V_\eta$ and $\langle C_x : x \in X\rangle$ is a sequence of club subsets of δ. The set $\bigcap_{x\in X} C_x$ is clearly closed, so it suffices to show it is unbounded.

Fix $\alpha_0 < \delta$, and we will exhibit an ordinal $\alpha_\omega > \alpha_0$ that belongs to $\bigcap_{x\in X} C_x$. For $n < \omega$ and $x \in X$, let $\alpha_{n+1}(x)$ be the least element of C_x above α_n, and let $\alpha_{n+1} = \sup_{x\in X} \alpha_{n+1}(x)$. Since $\mathrm{cf}(\delta) \geq \eta > \theta(X)$, α_n is defined for all $n < \omega$. Let $\alpha_\omega = \sup_{n<\omega} \alpha_n$. Then α_ω is a limit point of C_x for all $x \in X$, and therefore $\alpha_\omega \in \bigcap_{x\in X} C_x$. □

3.3 Wellfounded filters

A filter F on a set X is γ-wellfounded if the reduced product $\mathcal{O} = \gamma^X/F$ is wellfounded. We say that F is *wellfounded* if it is γ-wellfounded for all ordinals γ. If F is a γ-wellfounded filter on X and $\langle \mathbb{P}_x : x \in X\rangle$ is a sequence of wellfounded structures of rank at most γ, then the reduced product $\mathbb{P} = \prod_{x\in X} \mathbb{P}_x/F$ is again wellfounded. Indeed, define $o : \prod_{x\in X} \mathbb{P}_x \to \gamma^X$ by setting $o(f)(x) = \mathrm{rank}_{\mathbb{P}_x}(f(x))$. Then o descends to an order-preserving function from $\mathbb{P}$ to $\mathcal{O}$, which easily implies that $\mathbb{P}$ is wellfounded.

A set $A \subseteq P(X)$ lies below a set $B \subseteq P(Y)$ in the *Katětov order*, denoted $A \leq_{\text{kat}} B$, if there is a function $f : Y \to X$ such that for all $S \in A$, $f^{-1}[S] \in B$. If $F \leq_{\text{kat}} G$ where G is a γ-wellfounded filter, then F is a γ-wellfounded filter as well.

The following universality fact for fine filters, due to Kunen, allows us to conclude that under large cardinal hypotheses all sufficiently complete filters are wellfounded. If $\mathcal{B}$ is a filter base on X and σ is a subset of $P(X)$, let $A_{\mathcal{B}}(\sigma) = \bigcap_{A \in \mathcal{B} \cap \sigma} A$. If X is wellordered, define a partial funcion $\chi_{\mathcal{B}} : P(P(X)) \to X$ by $\chi_{\mathcal{B}}(\sigma) = \min(A_{\mathcal{B}}(\sigma)))$.

Theorem 3.4 (Kunen)**.** *Assume δ is an ordinal, $\mathcal{B}$ is a filter base on δ, and $\mathcal{W}$ is a filter on $P(P(\delta))$ concentrating on the set Γ of all σ such that $\bigcap(\mathcal{B} \cap \sigma) \neq \emptyset$. Then $\mathcal{B}$ lies below $\mathcal{W}$ in the Katětov order.*

Proof. Note that $\chi_{\mathcal{B}}$ is defined on the $\mathcal{W}$-large set Γ, and for all $A \in \mathcal{B}$, $\chi_{\mathcal{B}}^{-1}[A] \in \mathcal{W}$ since $\mathcal{W}$ is fine and $\{\sigma \in P(P(\delta)) : A \in \sigma\} \subseteq \chi_{\mathcal{B}}^{-1}[A]$. $\square$

The following lemma on the completeness of filters on ordinals is useful to keep in mind.

Lemma 3.5. *Suppose κ is an ordinal, X is a set, and there is no κ-sequence of distinct subsets of X. If F is a κ-complete filter on an ordinal δ, then F is X-closed.*

Proof. Suppose $\langle S_x : x \in X\rangle$ is a sequence of sets that $\bigcup_{x\in X} S_x$ is F-positive, and we will show that S_x is F-positive for some x.

Let $S = \bigcup_{x\in X} S_x$. For each $\alpha \in S$, $D_\alpha = \{x \in X : \alpha \in S_x\}$. Then $\{D_\alpha : \alpha < \delta\}$ is a wellorderable family of subsets of X, and hence it has cardinality less than κ.

Let $A_\alpha = \bigcap f[D_\alpha]$. Then $|\{A_\alpha : \alpha \in S\}| < \kappa$ and $\bigcup_{\alpha\in S} A_\alpha = S$ since $\alpha \in A_\alpha$. into fewer than κ-many sets, and so since F is κ-complete there is some $\alpha \in S$ such that A_α is F-positive. Fix $x \in X$ such that $\alpha \in S_x$, and note that $A_\alpha \subseteq S_x$, and hence S_x is F-positive, as desired. $\square$

A filter F is *weakly x-closed* if the intersection of an x-indexed family of sets in F is F-positive, and *weakly X-complete* if it is x-closed for all $x \in X$. In the context of ZFC, if κ is an infinite cardinal and F is a weakly κ-closed filter, then $\{\bigcap \sigma : \sigma \in [F]^\kappa\}$ generates a κ-closed filter, but this is not clear in ZF.

Lemma 3.6. *Suppose there is a wellfounded ν-complete fine ultrafilter $\mathcal{W}$ on $P(P(\delta))$ that concentrates on the set of $\sigma \subseteq P(\delta)$ such that $\aleph(P(\sigma)) < \nu$. Then every weakly ϵ-complete filter F on δ extends to a ν-complete ultrafilter. Moreover the set of ϵ-complete ultrafilters on δ can be wellordered.*

Proof. Lemma 3.5 puts us in a position to apply Theorem 3.4, which implies that F lies below $\mathcal{W}$ in the Katětov order. Let $f : P(P(\delta)) \to \delta$ be such that for all $A \in F$, $f^{-1}[A] \in \mathcal{W}$. Then the ultrafilter $\{A \subseteq \delta : f^{-1}[A] \in \mathcal{W}\}$ extends F.

One can wellorder the set of ϵ-complete ultrafilters on δ by setting $U_0 < U_1$ if $\chi_{U_0}(\sigma) < \chi_{U_1}(\sigma)$ for $\mathcal{W}$-almost all σ. This is linear because $\mathcal{W}$ is an ultrafilter and wellfounded because $\mathcal{W}$ is wellfounded. $\square$

The proof of this theorem derives from Kunen's theorem on ultrafilters under AD:

Theorem (Kunen)**.** *Assume* AD + $\text{DC}_{\mathbb{R}}$. *Then the set of all ultrafilters on ordinals less than $\theta(\mathbb{R})$ is wellorderable. In fact, every ultrafilter on an ordinal less than $\theta(\mathbb{R})$ is ordinal definable.*

3.4 Saturated filters in normal measures

Lemma 3.7. *Suppose $\mathcal{U}$ is a normal fine ultrafilter on $P(X)$ and $\mathcal{E} \in \mathcal{U}$. Suppose there is an I-indexed family of disjoint $\mathcal{E}$-positive sets. Then every function $f : P(X) \to I$ is constant on a set in $\mathcal{U}$.*

Proof. Let $f : P(X) \to I$ be a function. Suppose $\langle S_i : i \in I\rangle$ is a family of $\mathcal{E}$-positive sets. For $\mathcal{U}$-almost all σ, $\sigma \cap S_{f(\sigma)} \neq \emptyset$, and so since $\mathcal{U}$ is normal, there is some $x \in X$ such that for $\mathcal{U}$-almost all σ, $x \in S_{f(\sigma)}$. Since the sets $\langle S_i : i \in I\rangle$ are disjoint, there is a unique $i \in I$ such that $x \in S_i$. Thus for $\mathcal{U}$-almost all σ, $f(\sigma) = i$. □

For any ultrafilter $\mathcal{U}$, let $\kappa_{\mathcal{U}}$ denote the largest cardinal κ such that $\mathcal{U}$ is κ-complete. If $\mathcal{U}$ is a normal fine ultrafilter on $P(\delta)$, either $\kappa_{\mathcal{U}}$ is defined or $\mathcal{U}$ is principal.

A family of sets $\mathcal{B}$ is γ-*weakly saturated* if any collection of disjoint $\mathcal{B}$-positive sets has cardinality less than γ.

Corollary 3.8. *Suppose δ is an ordinal, $\mathcal{U}$ is a nonprincipal normal fine ultrafilter on $P(\delta)$, and $\mathcal{E} \in \mathcal{U}$. Then $\mathcal{E}$ is $\kappa_{\mathcal{U}}$-weakly saturated.*

Proof. If P is a family of disjoint $\mathcal{E}$-positive sets, then P is wellorderable since there is an injection $f : P \to \delta$ defined by $f(S) = \min(S \cap \delta)$. By Lemma 3.7, no family of disjoint $\mathcal{E}$-positive sets has cardinality $\kappa_{\mathcal{U}}$, and therefore every family of disjoint $\mathcal{E}$-positive sets has cardinality less than $\kappa_{\mathcal{U}}$. □

A filter F is (κ, ϵ)-*indecomposable* if any family P of disjoint sets with $|P| < \epsilon$ has a subfamily Q such that $|Q| < \kappa$ and $\bigcup(P \setminus Q)$ is F-null. Note that if F is (κ, ϵ)-indecomposable, so is every extension of F.

Lemma 3.9. *Suppose F is a filter that is ϵ-complete and κ-weakly saturated. Then F is (κ, ϵ)-indecomposable.*

Proof. Suppose P is a family of disjoint sets with $|P| < \epsilon$. Let $Q \subseteq P$ be the collection of F-positive sets in P. Since F is κ-weakly saturated, $|Q| < \kappa$. Since F is ϵ-complete, $\bigcup(P \setminus Q)$ is F-null, being the union of fewer than ϵ-many F-null sets. □

Corollary 3.10. *Suppose δ is an ordinal and $\mathcal{U}$ is a nonprincipal normal fine ultrafilter on $P(\delta)$. Suppose F is an ϵ-complete filter in $\mathcal{U}$. Then every $\kappa_{\mathcal{U}}$-complete filter extending F is ϵ-complete.*

Proof. Suppose G is a $\kappa_{\mathcal{U}}$-complete filter extending F. Since $F \in \mathcal{U}$, F is $\kappa_{\mathcal{U}}$-weakly saturated, and therefore F is $(\kappa_{\mathcal{U}}, \epsilon)$-indecomposable, and hence so is G. Since G is $(\kappa_{\mathcal{U}}, \epsilon)$-indecomposable and $\kappa_{\mathcal{U}}$-complete, G is ϵ-complete. □

If F is a filter, an F-positive set A is an *atom of* F if $F \restriction A$ is an ultrafilter. In other words, A cannot be partitioned into distinct F-positive sets.

Theorem 3.11. *Suppose δ is an ordinal, $\mathcal{U}$ is a nonprincipal normal fine ultrafilter on $P(\delta)$, and $F \in \mathcal{U}$ is an ϵ-complete filter. Suppose there is a wellfounded $\kappa_{\mathcal{U}}^+$-complete fine ultrafilter $\mathcal{W}$ on $P(P(\delta))$ that concentrates on the set of $\sigma \subseteq P(\delta)$ such that $\aleph(P(\sigma)) < \epsilon$. Then δ can be partitioned into fewer than $\kappa_{\mathcal{U}}$-many atoms of F.*

Proof. Let $\mathcal{S}$ be the set of ϵ-complete ultrafilters extending F. We start by showing that for any $\mathcal{T} \subseteq \mathcal{S}$ such that $|\mathcal{T}| \leq \kappa_{\mathcal{U}}$, there is a family $\mathcal{P}$ of disjoint subsets of δ such that each $A \in \mathcal{P}$ belongs to exactly one $U \in \mathcal{T}$.

Since $\mathcal{W}$ is fine, for any $U_0, U_1 \in \mathcal{S}$, for $\mathcal{W}$-almost all σ, there is some $A \in \sigma$ such that $A \in U_0 \setminus U_1$. Since $\mathcal{W}$ is $\kappa_{\mathcal{U}}^+$-complete and $|\mathcal{T}| \leq \kappa_{\mathcal{U}}$, for $\mathcal{W}$-almost all σ, for all $U_0, U_1 \in \mathcal{T}$, there is an $A \in \sigma$ such that $A \in U_0 \setminus U_1$. Fix such a set σ such that $\aleph(P(\sigma)) < \epsilon$. Then $\mathcal{P} = \{A_U(\sigma) : U \in \mathcal{T}\}$ is as desired.

By Lemma 3.6, $\mathcal{S}$ can be wellordered. It follows that $|\mathcal{S}| < \kappa_{\mathcal{U}}$. Assume not. Then there is some $\mathcal{T} \subseteq \mathcal{S}$ such that $|\mathcal{T}| = \kappa_{\mathcal{U}}$. Fix a family $\mathcal{P}$ of disjoint subsets of δ such that each $A \in \mathcal{P}$ belongs to exactly one $U \in \mathcal{T}$. Then $\mathcal{P}$ is a family of $\kappa_{\mathcal{U}}$-many disjoint F-positive sets, contrary to Lemma 3.7.

Thus $|\mathcal{S}| < \kappa_{\mathcal{U}}$, and so there is a family $\mathcal{P}$ of disjoint subsets of δ such that each $A \in \mathcal{P}$ belongs to exactly one $U \in \mathcal{T}$. We claim that $\mathcal{P}$ witnesses that F is atomic.

We first show that there is some $\sigma \subseteq P(\delta)$ such that $A_F(\sigma) \in F$ and $A_F(\sigma) \subseteq \bigcup_{A\in\tau} A$. Suppose not. Let $S = \delta \setminus \bigcup_{A\in\tau} A$. Then $F \restriction S$ is an ϵ-complete filter and so as a consequence of Lemma 3.6 extends to a $\kappa_{\mathcal{U}}^+$-complete ultrafilter U. By Corollary 3.10, U is ϵ-complete. Therefore for some $A \in \tau$, $A \in U$. Since U is a proper filter, $S \cap A \neq \emptyset$, which contradicts that $S \subseteq \delta \setminus A$.

By a similar argument, we show that each $A \in \tau$ is an F-atom. Assume towards a contradiction that $S \subseteq A$ has the property that both S and $A \setminus S$ are F-positive. Let U and W be ϵ-complete ultrafilters extending $F \restriction S$ and $F \restriction (A \setminus S)$ respectively. Fix B and C in τ such that $B \in U$ and $C \in W$. Since $S \subseteq A$ and $S \in U$, $S \cap B \neq \emptyset$, and it follows that $A = B$ since the sets in τ are disjoint. Similarly $A = C$. Since each $D \in \tau$ belongs to exactly one ultrafilter in $\mathcal{T}$, $U = W$, which contradicts that $A \in U$ and $A \setminus S \in W$. $\square$

3.5 Atoms of the club filter

A filter F on X is said to be *atomic* if every F-positive subset of X contains an atom of F. For example, the conclusion of Theorem 3.11 implies a strong form of atomicity.

Theorem 3.12. *Suppose there is a rank Berkeley cardinal. Then for all sufficiently large regular cardinals δ, the club filter on δ is atomic.*

Before turning to Theorem 3.12, we consider a special case in which one can prove a stronger result. (To be clear, the proof of Theorem 3.12 does not depend on this result.) For any stationary set $T \subseteq \delta$, the *club filter restricted to* T is the filter consisting of subsets of δ that contain all but nonstationarily many elements of T. Thus T is an atom of the club filter if and only if the club filter restricted to T is an ultrafilter.

If S is a stationary subset of δ, let S^- denote the set of ordinals $\alpha \in S$ such that $S \cap \alpha$ is nonstationary. Clearly $(S^-)^- = S^-$, and slightly less obviously, S^- is again stationary. Note that if S is an atom of the club filter, then $S = S^-$ modulo a nonstationary set. In the context of the Axiom of Determinacy there is a partial converse to this result: a theorem of Kechris-Kleinberg-Woodin [4] states that if δ is a strong partition cardinal, then for any stationary $S \subseteq \delta$, the club filter restricted to S^- is an ultrafilter. This motivates the following theorem.

Theorem 3.13. *If λ is rank Berkeley, δ is a sufficiently large regular cardinal, and $S \subseteq \delta$ is stationary, then S^- can be partitioned into fewer than λ-many atoms of the club filter.*

Proof. Let $\alpha > \delta$ be an ordinal such that $V_\alpha \preceq_{\Sigma_2} V$ and let $j : V_\alpha \to V_\alpha$ be an elementary embedding such that $\mathrm{crit}(j) < \lambda$ and $j(S) = S$. Let $\mathcal{U}$ be the normal fine ultrafilter on $P(\delta)$ derived from j using $j[\delta]$. Let F be the filter generated by S^- along with club subsets of δ. We will show that $F \in \mathcal{U}$, and then applying Theorem 3.11, if δ is sufficiently large, there is a partition of S^- into fewer than λ-many stationary sets T such that the club filter restricted to T is an ultrafilter.

To show that $F \in \mathcal{U}$, it suffices to show that $j[\delta] \in j(F)$. Since $j(S) = S$, $j(F) = F$. In fact, we will show that if $\xi \in S^-$ is a closure point of j, then $j(\xi) = \xi$. This shows that the set of fixed points of j contains the intersection of S^- with a club, which easily implies that $j[\delta] \in F$.

Suppose $\xi \in S^-$ is a closure point of j. Assume towards a contradiction that $j(\xi) > \xi$. We will show that $S^- \cap \xi$ is stationary. Fix a club $C \subseteq \xi$. Since $j(C)$ is club in $j(\xi)$ and ξ is an accumulation point of $j(C)$ below $j(\xi)$, ξ belongs to $j(C)$. But then $j(S^-) \cap j(C) \neq \emptyset$, and hence $S^- \cap C \neq \emptyset$. This shows that $S^- \cap \xi$ is stationary, and this contradicts that for all $\xi \in S^-$, S does not reflect to ξ. $\square$

To generalize this theorem to an arbitrary stationary set, we must contend with the case where the stationary set S is not fixed by any elementary embedding. (If λ is Berkeley rather than merely rank Berkeley, this issue evaporates since one can obtain embeddings fixing any set one wants.)

Proof of Theorem 3.12. For the proof, we need the existence of a countably complete ultrafilter U on δ extending the club filter with $S^- \in U$ along with an ordinal $\alpha > \delta$ such that $V_\alpha \preceq_{\Sigma_2} V$ and an elementary embedding $j : V_\alpha \to V_\alpha$ such that $j(U) = U$. We defer the proof that these objects exist to Lemma 4.6, after the basic theory of the Ketonen order has been set up.

Let

$$T = \bigcap_{n<\omega} j^n(S^-)$$

Then $j(T) = \bigcap_{n<\omega} j^{n+1}(S^-)$ and so $T \subseteq j(T)$. Moreover, since $S^- \in U$ and $j^n(U) = U$ for all $n < \omega$, $j^n(S^-) \in U$ for all $n < \omega$. Therefore since U is countably complete, $T \in U$. In particular, T is stationary. Note that for all $\xi \in T$, $T \cap \xi$ is nonstationary, simply because $T \subseteq S^-$. By the elementarity of j, for all $\xi \in j(T)$, $j(T) \cap \xi$ is nonstationary. This will be important below.

To finish, we show that the club filter restricted to T belongs to the normal fine ultrafilter on $P(\delta)$ derived from j using $j[\delta]$. By Theorem 3.11, if δ is sufficiently large, this implies that there is a stationary subset of T on which the club filter is an ultrafilter.

Proceeding as in Theorem 3.13, fix a closure point ξ of j such that $\xi \in j(T)$, and we will show that $j(\xi) = \xi$. Assume towards a contradiction that $j(\xi) > \xi$. We will show that $j(T) \cap \xi$ is stationary, contradicting the fact that for all $\xi \in j(T)$, $j(T) \cap \xi$ is nonstationary.

Fix a club $C \subseteq \xi$. Since $j(C)$ is club in $j(\xi)$ and ξ is an accumulation point of $j(C)$ below $j(\xi)$, ξ belongs to $j(C)$. But then $j(T) \cap j(C) \neq \emptyset$, and hence $T \cap C \neq \emptyset$. This proves that $T \cap \xi$ is stationary. Since $T \subseteq j(T)$, $j(T) \cap \xi$ is stationary, as claimed, which leads to a contradiction as explained above. $\square$

Theorem 3.14. *Suppose there is a rank Berkeley cardinal. Then for a club class of cardinals ϵ, either ϵ or ϵ^+ is measurable.* $\square$

4 The filter extension property

4.1 The Ketonen order

The *Ketonen order* is a wellfounded partial order of countably complete filters introduced in the author's thesis. The wellfoundedness of the Ketonen order cannot be proved without appeal to the Axiom of Dependent Choice (DC), and so to avoid using DC in our applications below, we work here with a restricted version of the Ketonen order whose wellfoundedness can be proved in ZF. The main idea is to use wellfounded filters instead of merely countably complete ones.

If F is a filter on X, $A \in F$, and $\langle G_x : x \in A\rangle$ is a sequence of filters on Y, then $F\text{-}\lim_{x\in A} G_x$ denotes the filter on Y consisting of all $B \subseteq Y$ such that for F-almost all x, $B \in G_x$.

Fix an ordinal δ. For each $\alpha < \delta$, let $\mathscr{G}_\alpha(\delta)$ denote the set of filters F on δ such that $\alpha \in F$. We define a set of filters $\mathscr{F}_\alpha(\delta) \subseteq \mathscr{G}_\alpha(\delta)$ and binary relations $<_{\Bbbk}^{\alpha,\delta}$ and $\leq_{\Bbbk}^{\alpha,\delta}$ on $\mathscr{G}_\alpha(\delta)$ by induction on α. To ameliorate notation, we suppress the fixed parameter δ. Let $\mathscr{F}_0 = \{\emptyset\}$, and let $<_{\Bbbk}^0$ and $\leq_{\Bbbk}^0$ be the unique strict and nonstrict orders on $\mathscr{F}_0$ respectively. Assume by induction that $(\mathscr{F}_\alpha, <_{\Bbbk}^\alpha)$ has been defined for $\alpha < \beta$. Then a filter $F \in \mathscr{G}_\beta$ belongs to $\mathscr{F}_\beta$ if $\prod_{\alpha<\beta}(\mathscr{F}_\alpha, <_{\Bbbk}^\alpha)/F$ is wellfounded. If $F \in \mathscr{G}_\delta$ and $G \in \mathscr{G}_\beta$,

- $F <_{\Bbbk}^\beta G$ if $F \subseteq G\text{-}\lim_{\alpha<\beta} F_\alpha$ where $\langle F_\alpha : \alpha < \beta\rangle \in \prod_{\alpha<\beta} \mathscr{F}_\alpha$.
- $F \leq_{\Bbbk}^\beta G$ if $F \subseteq G\text{-}\lim_{\alpha<\beta} F_\alpha$ where $\langle F_\alpha : \alpha < \beta\rangle \in \prod_{\alpha<\beta} \mathscr{F}_{\alpha+1}$.

Theorem 4.1. *For all ordinals $\alpha \leq \delta$, $(\mathscr{F}_\alpha, <_{\Bbbk}^\alpha)$ is a wellfounded partial order. Moreover, for all $G \in \mathscr{F}_\alpha$ and $\langle F_\nu : \nu < \alpha\rangle \in \prod_{\nu<\alpha} \mathscr{F}_\nu$, $G\text{-}\lim_{\nu<\alpha} F_\nu \in \mathscr{F}_\alpha$.*

Proof. Fix an ordinal $\beta \leq \delta$, and assume by induction that for all $\alpha < \beta$, $(\mathscr{F}_\alpha, <_{\Bbbk}^\alpha)$ is wellfounded and if $G \in \mathscr{F}_\alpha$ and $\langle F_\nu : \nu < \alpha\rangle \in \prod_{\nu<\alpha} \mathscr{F}_\nu$, then $G\text{-}\lim_{\nu<\alpha} F_\nu \in \mathscr{F}_\alpha$.

For each filter $F \in \mathscr{G}_\delta$, let $\mathbb{P}_F$ denote the reduced product $\prod_{\alpha<\delta} \mathscr{F}_\alpha/F$. Note that by definition, $F \in \mathscr{F}_\beta$ if and only if $\mathbb{P}_F$ is wellfounded and $\beta \in F$.

For each sequence $\vec{F} = \langle F_\alpha : \alpha < \beta\rangle \in \prod_{\alpha<\beta} \mathscr{F}_\alpha$, define a function

$$i_{\vec{F}} : \prod_{\alpha<\delta} \mathscr{F}_\alpha \to \prod_{\alpha<\beta} \mathscr{F}_\alpha$$

as follows. Fix $\vec{H} = \langle H_\nu : \nu < \delta\rangle \in \prod_{\nu<\delta} \mathscr{F}_\nu$. For $\alpha < \beta$, set $E_\alpha = F_\alpha\text{-}\lim_{\nu<\alpha} H_\nu$. By our induction hypothesis, $E_\alpha \in \mathscr{F}_\alpha$ for all $\alpha < \beta$. Finally, let $i_{\vec{F}}(\vec{H}) = \langle E_\alpha : \alpha < \beta\rangle$.

Fix $G \in \mathscr{F}_\beta$ and $\langle F_\alpha : \alpha < \beta\rangle \in \prod_{\alpha<\beta} \mathscr{F}_\alpha$. Then for any $F \subseteq G\text{-}\lim_{\alpha<\beta} F_\alpha$, $i_{\vec{F}}$ descends to an order-preserving function $\tilde{i} : \mathbb{P}_F \to \mathbb{P}_G$ such that for all $x \in \mathbb{P}_F$, $\tilde{i}(x)$ lies below $[\langle F_\alpha : \alpha < \delta\rangle]_G$. As a consequence, $\mathbb{P}_F$ is wellfounded and has rank strictly less than the rank of $\mathbb{P}_G$. Define a function $r : \mathscr{F}_\beta \to \text{Ord}$ by $r(H) = \text{rank}(\mathbb{P}_H)$. Then r is order-preserving, which proves that $(\mathscr{F}_\beta, <_{\Bbbk}^\beta)$ is wellfounded.

Moreover, consider the filter $H = G\text{-}\lim_{\alpha<\beta} F_\alpha$. The partial order $\mathbb{P}_H$ is wellfounded by the previous paragraph, and $\beta \in H$ since $\beta \in F_\alpha$ for all $\alpha < \beta$. It follows that $H \in \mathscr{F}_\beta$. $\square$

Notice that $(\mathscr{F}_\alpha(\delta), <_{\Bbbk}^{\alpha,\delta}, \leq_{\Bbbk}^{\alpha,\delta}) \cong (\mathscr{F}(\alpha), <_{\Bbbk}, \leq_{\Bbbk})$. We let

$$(\mathscr{F}(\delta), <_{\Bbbk}, \leq_{\Bbbk}) = (\mathscr{F}_\delta(\delta), <_{\Bbbk}^{\delta,\delta}, \leq_{\Bbbk}^{\delta,\delta})$$

and we refer to this partial order as the *Ketonen order on* $\mathscr{F}(\delta)$.

For any filter $F \in \mathscr{F}(\delta)$, $|F|_{\Bbbk}$ denotes the rank of F in the Ketonen order on $\mathscr{F}(\delta)$. We let $|\delta|_{\Bbbk} = |\mathscr{F}(\delta)|_{\Bbbk}$ and $|{<}\delta|_{\Bbbk} = \sup_{\alpha<\delta} |\alpha|_{\Bbbk}$.

Corollary 4.2. *For any δ, $\mathscr{F}(\delta)$ includes the set of all $|{<}\delta|_{\Bbbk}$-wellfounded filters.* $\square$

The following lemmas clarify the relationship between $\leq_{\Bbbk}$ and $<_{\Bbbk}$ to a certain extent.

Lemma 4.3. *Suppose $F, G \in \mathscr{F}(\delta)$ satisfy $F \leq_{\Bbbk} G$. Then either $F <_{\Bbbk} G$ or there is some $S \in G^+$ such that $F \subseteq G \restriction S$.*

Proof. Let $F \subseteq G\text{-}\lim_{\alpha<\delta} F_\alpha$ where $F_\alpha \in \mathscr{F}(\alpha+1)$ for $\alpha < \delta$. Since $F \not<_{\Bbbk} G$, the set $S = \{\alpha < \delta : \alpha \notin F_\alpha\}$ is G-positive. For all $\alpha \in S$, $F_\alpha \subseteq p_\alpha$ where p_α is the principal ultrafilter on δ concentrated at α, and so

$$F \subseteq G\text{-}\lim_{\alpha<\delta} F_\alpha \subseteq (G \restriction S)\text{-}\lim_{\alpha<\delta} F_\alpha \subseteq (G \restriction S)\text{-}\lim_{\alpha<\delta} p_\alpha = G \restriction S$$

$\square$

Corollary 4.4. *If $U \leq_{\Bbbk} W$ are ultrafilters in $\mathscr{F}(\delta)$, either $U <_{\Bbbk} W$ or $U = W$.* $\square$

Lemma 4.5. *Suppose $U \in \mathscr{F}(\delta)$ is an ultrafilter, $\alpha > |U|_{\Bbbk}$, and $j : V_\alpha \to V_\alpha$ is an elementary embedding fixing δ and $|U|_{\Bbbk}$. Then $j(U) = U$.*

Proof. Note that $U \leq_{\Bbbk} j(U)$: we have $U = j(U)\text{-}\lim_{\beta<\delta} U_\beta$ where U_β is the ultrafilter on δ derived from j using β, and for U-almost all β, U_β belongs to $\mathscr{F}_{\beta+1}(\delta)$: U_β is α-wellfounded and hence $|U|_{\Bbbk}$-wellfounded and hence $|{<}\beta|_{\Bbbk}$-wellfounded assuming $\beta \notin U$. On the other hand, $U \not<_{\Bbbk} j(U)$ since the Ketonen rank of $j(U)$ is $j(|U|_{\Bbbk}) = |U|_{\Bbbk}$, the same as the Ketonen rank of U. Therefore by Corollary 4.4, $j(U) = U$. $\square$

Lemma 4.6. *Let κ be the least $(0,\infty)$-supercompact cardinal. Then for any κ-complete filter F on δ and any ordinal $\alpha > \delta + 1$, there is an ultrafilter $U \in \mathscr{F}(\delta)$ extending F along with an elementary embedding $j : V_\alpha \to V_\alpha$ such that $j(U) = U$.*

Proof. The existence of U follows from Lemma 3.6 and the existence of j follows from Lemma 4.5. $\square$

4.2 The semi-linearity of the Ketonen order

In the context of the Axiom of Choice, the *Ultrapower Axiom* (UA) roughly states that any two ultrapowers of the universe have a common internal ultrapower. UA is equivalent to the linearity of the Ketonen order on ultrafilters. The principle has found a number of applications in the theory of supercompact cardinals. Here we will show that in the context of a rank Berkeley cardinal, the Ketonen order is *almost linear* in the sense that it contains no large antichains. This fact will find an application in the proof of the filter extension property below.

Theorem 4.7. *Suppose λ is the least rank Berkeley cardinal, κ is almost supercompact, $\mathrm{cf}(\kappa) > \lambda$, and A is a set of pairwise Ketonen incomparable ultrafilters in $\mathscr{F}(\delta)$.*

(1) There is a bijection from A to a set in V_κ.

(2) If A can be wellordered, then $|A| \leq \lambda$.

Proof. The key observation is that if $U \in \mathscr{F}(\delta)$ and $j : V_\alpha \to V_\alpha$ is an elementary embedding where $\alpha > \delta$ and $V_\alpha \preceq_{\Sigma_2} V$, then for some $n < \omega$, $j_n(U) = U$. To see this, find $n < \omega$ such that $j_n(\xi) = \xi$ where ξ is the rank of U in the Ketonen order on $\mathscr{F}(\delta)$, and apply Lemma 4.5.

Fix α larger than the rank of A, and suppose $j : V_\alpha \to V_\alpha$ is an elementary embedding with $\kappa_\omega(j) = \lambda$. For each $n < \omega$, let A_n be the set of ultrafilters in A fixed by j_n. Then $A = \bigcup_{n<\omega} A_n$.

We claim that $j_n(A_n) = j_n[A_n]$. Suppose not, and fix $U \in j_n(A_n) \setminus j_n[A_n]$. Let

$$B = j_{n+1}(j_n(A_n)) = j_n(j_n(A_n))$$

Then $j_n(U)$ and $j_{n+1}(U)$ belong to B, and B is a set of Ketonen incomparable ultrafilters. Since every ultrafilter in A_n is fixed by j_n, every ultrafilter in $j_n(A_n)$ is fixed by j_{n+1}, and it follows that $j_{n+1}(U) = U$. On the other hand, $U \leq_{\Bbbk} j_n(U)$ since $j_n : P(\delta) \to P(\delta)$ is Ketonen. Also $U \neq j_n(U)$ since $U \notin j_n[A_n]$. Therefore $U <_{\Bbbk} j_n(U)$. But both of these ultrafilters are in B, which contradicts that B is a set of pairwise Ketonen incomparable ultrafilters.

We now prove (1). By Lemma 2.9, since $j_n(A_n) = j_n[A_n]$, there is some $\beta_n < \kappa$ such that A_n injects into V_{β_n}. Applying the wellordered collection lemma, there is some $\gamma < \kappa$ and a set $\{f_x : x \in V_\gamma\}$ such that for each n, there is some $x \in V_\gamma$ such that $f_x : A_n \to V_{\beta_n}$ is an injection. Define $g_n : A_n \to {}^{V_\gamma}V_{\beta_n}$ by $g_n(a)(x) = f_x(a)$. Then g_n is an injection from A_n into V_{ρ_n} where $\rho_n = \gamma \cdot \beta_n + 1$. Now setting $g(a) = \langle g_n(a) : n < \omega \rangle$, we obtain an injection from A to V_β where $\beta = (\sup_{n<\omega} \rho_n) + 1$ is less than κ.

Assuming A can be wellordered, one can moreover conclude from the fact that $j_n(A_n) = j_n[A_n]$ that $|A_n| < \kappa_n(j)$, and hence $|A| \leq \lambda$, proving (2). $\square$

The semi-linearity property of the Ketonen order that will actually be applied in the proof of the filter extension property is more technical but a bit easier to prove.

Proposition 4.8. *Suppose λ is a rank Berkeley cardinal, δ and ξ are ordinals, and $\mathscr{U}$ is the set of all ultrafilters of rank ξ in the Ketonen order on $\mathscr{F}(\delta)$. Then λ is $\mathscr{U}$-closed rank Berkeley.*

Proof. Suppose $\alpha > \xi$ is an ordinal and $j : V_\alpha \to V_\alpha$ is an elementary embedding that fixes δ and ξ. Then $j(\mathscr{U}) = \mathscr{U}$, and moreover for any ultrafilter $W \in \mathscr{F}(\delta)$, $j^{-1}[W] \leq_{\Bbbk} W$:

$$j^{-1}[W] = W\text{-}\lim_{\beta<\delta} D_\beta$$

where D_β is the ultrafilter on δ derived from j using β. Notice that D_β is α-wellfounded, and so $D_\beta \in \mathscr{F}_{\beta+1}(\delta)$ by Corollary 4.2. If $W \in \mathscr{U}$, then $|j(W)|_{\Bbbk} = j(\xi) = \xi = |W|_{\Bbbk}$, and hence $j(W) = W$. It follows that $j[\mathscr{U}] = \mathscr{U} = j(\mathscr{U})$. This easily implies that λ is $\mathscr{U}$-closed rank Berkeley. $\square$

4.3 The filter extension property

Theorem 4.9. *Assume there is a rank Berkeley cardinal. Then for a club class of cardinals κ, every κ-complete filter on an ordinal extends to a κ-complete ultrafilter.*

Proof. Let λ be the least rank Berkeley cardinal and let Γ be the class of all X such that λ is X-closed rank Berkeley. Let κ a cardinal that is X-closed almost extendible for all $X \in \Gamma$;

by Theorem 2.7, there is a club class of such cardinals. We will show that every κ-complete filter on an ordinal extends to a κ-complete ultrafilter.

Fix an ordinal δ and assume towards a contradiction that there is a κ-complete filter on δ that does not extend to a κ-complete ultrafilter. By Lemma 3.6, every κ-complete filter on δ extends to a $|{<}\delta|_{\Bbbk}$-wellfounded ultrafilter and hence is itself $|{<}\delta|_{\Bbbk}$-wellfounded. In particular, every κ-complete filter on δ belongs to $\mathscr{F}(\delta)$ by Corollary 4.2. Let $\mathscr{S}$ be the set of κ-complete filters on δ that do not extend to κ-complete ultrafilters. Applying Theorem 4.1, $\mathscr{S}$ has a minimal element in the Ketonen order on $\mathscr{F}(\delta)$.

Fix $\alpha > |{<}\delta|_{\Bbbk}$, and let $\mathcal{E}$ be the set of elementary embeddings $j : V_\alpha \to V_\alpha$ such that $F \in \text{ran}(j)$. Let $\kappa' = \kappa$ if κ is regular and $\kappa' = \kappa + 1$ if κ is singular. Let

$$\mathcal{B} = \left\{ \bigcap_{j \in \sigma} j[\delta] : \sigma \subseteq \mathcal{E},\, \text{scott}^*(\sigma) < \kappa' \right\}$$

Let G be the filter on δ generated by $\mathcal{B}$. The filter G is κ'-complete by the wellordered collection lemma (Proposition 2.19). The main claim of the proof is that $G \cup F$ generates a proper filter, or in other words, that every set in $\mathcal{B}$ is F-positive.

Before proving the claim, let us show how to use it to complete the proof of the theorem. Let $i : V_{\alpha+\omega} \to V_{\alpha+\omega}$ be an elementary embedding fixing δ, and note that $i[\delta] \in i(G)$. Therefore $G \in \mathcal{U}$ where $\mathcal{U}$ is the normal fine ultrafilter on $P(\delta)$ derived from i using $i[\delta]$. Since G is κ-complete, Theorem 3.11 implies that there is a partition of δ into fewer than λ-many G-positive sets $\langle S_\nu : \nu < \gamma \rangle$ such that $G \restriction S_\nu$ is an ultrafilter for all $\nu < \gamma$. Assume towards a contradiction that for all $\nu < \gamma$, $F \nsubseteq G \restriction S_\nu$. Therefore there is a set in F that is not in $G \restriction S_\nu$, and so since $G \restriction S_\nu$ is an ultrafilter, there is a set G-large set whose intersection with S_ν is in the dual ideal F^*. By the wellordered collection lemma, there is a set $\upsilon \subseteq P(\delta)$ such that $\text{scott}^*(\upsilon) < \kappa'$ and for each $\nu < \gamma$, there is a set $A \in G \cap \upsilon$ such that $A \cap S_\nu \in F^*$. Let $B = \bigcap \upsilon$. Then $B \in G$ since G is κ-complete, but $B \cap S_\nu \in F^*$ for all $\nu < \gamma$. Since $\langle S_\nu : \nu < \gamma \rangle$ is a partition of δ, $B = \bigcup_{\nu<\gamma} B \cap S_\nu$. Since F is κ-complete, it follows that $B \in F^*$. Now $B \in G$ and B is not F-positive, contrary to the claim. This contradiction establishes that for some $\nu < \gamma$, $F \subseteq G \restriction S_\nu$, which shows that F extends to a κ-complete ultrafilter.

We now proceed to the proof of the claim. Assume towards a contradiction that for some $\sigma \subseteq \mathcal{E}$ with $\text{scott}^*(\sigma) < \kappa'$, $\bigcap_{j\in\sigma} j[\delta]$ is F-null. In other words,

$$S = \bigcup_{j \in \sigma} (\delta \setminus j[\delta]) \in F$$

For each $\xi < \delta$ and $j \in \sigma$, let $D_\xi(j)$ be the ultrafilter on δ derived from j using ξ. For all $\xi \in S$ and all $j \in \sigma$, $\xi \in D_\xi(j)$ since $\xi < j(\xi)$. Also $D_\xi(j) \in \mathscr{F}_\xi(\delta)$ by Corollary 4.2 since j is an embedding of V_α and $\alpha > |{<}\delta|_{\Bbbk}$.

For $\xi < \delta$, let

$$D_\xi = \bigcap \{D_\xi(j) : j \in \sigma\}$$

and note that if $\xi \in S$, then $\xi \in (D_\xi)^+$ since for some $j \in \sigma$, $\xi \in D_\xi(j)$. For each $j \in \sigma$, let $F_j = j^{-1}(F)$. Since $F \in \text{ran}(j)$, F_j is a κ-complete filter that does not extend to a κ-complete ultrafilter. We claim that

$$\bigcap_{j\in\sigma} F_j \subseteq F\text{-}\lim_{\xi<\delta} D_\xi$$

In fact, equality holds, but we do not need this.

Suppose $A \in \bigcap_{j\in\sigma} F_j$. Then for all $j \in \sigma$, $j(A) \in F$, or in other words,

$$\{\xi < \delta : A \in D_\xi(j)\} \in F$$

Since F is κ'-complete and $\text{scott}^*(\sigma) < \kappa'$,

$$\left\{\xi < \delta : A \in \bigcap_{j\in\sigma} D_\xi(j)\right\} \in F$$

and so

$$\{\xi < \delta : A \in D_\xi\} \in F$$

which implies that $A \in F\text{-}\lim_{\xi<\delta} D_\xi$.

It follows that $\bigcap_{j\in\sigma} F_j <_\Bbbk F$. To see this, note that $S \in F$, and for all $\xi \in S$, $\xi \in (D_\xi)^+$. Let $E_\xi = D_\xi \restriction \xi$. Then $E_\xi \in \mathscr{F}_\xi(\delta)$: this follows from the fact that for any $j \in \sigma$ such that $j(\xi) < \xi$, $D_\xi(j) \in \mathscr{F}_\xi(\delta)$ and $E_\xi \subseteq D_\xi(j)$. As a consequence, $\bigcap_{j\in\sigma} F_j \subseteq F\text{-}\lim_{\xi\in S} E_\xi$, and so $\bigcap_{j\in\sigma} F_j <_\Bbbk F$.

The intersection $\bigcap_{j\in\sigma} F_j$ of the κ-complete filters F_j is κ-complete, so by the minimality of F, $\bigcap_{j\in\sigma} F_j$ extends to a κ-complete ultafilter W. We will show that $F_j \subseteq W$ for some $j \in \sigma$, contradicting that F_j does not extend to a κ-complete ultrafilter. This conclusion would be obvious under the Axiom of Choice: if $F_j \not\subseteq W$ for all $j \in \sigma$, then for each $j \in \sigma$, choose $A_j \in F_j \setminus W$, and let $A = \bigcup_{j\in\sigma} A_j$; then $A \in \bigcap_{j\in\sigma} F_j$ and $A \notin W$, which contradicts that $\bigcap_{j\in\sigma} F_j \subseteq W$. Since we are working in ZF, a different argument is required.

Notice that the argument of the previous paragraph combined with the wellordered collection lemma does imply that if $\gamma < \kappa'$ and $\bigcap_{\alpha<\gamma} H_\alpha \subseteq W$, then $H_\alpha \subseteq W$ for some $\alpha < \gamma$. (Indeed, the wellordered collection yields a set $\sigma \subseteq P(\delta)$ such that $\text{scott}^*(\sigma) < \kappa'$ and $\sigma \cap (H_\alpha \setminus W) \neq \emptyset$ for all $\alpha < \gamma$, and therefore $\bigcup(\sigma \setminus W)$ belongs to $\bigcap_{\alpha<\gamma} H_\alpha$ but not W.) We will use this fact repeatedly to replace $\bigcap_{j\in\sigma} F_j$ with a more manageable intersection that is still contained in W.

Our first step is to reduce to the case that $\text{scott}^*(\sigma) < \kappa$. If κ is regular, this is true by definition, so assume instead that κ is singular. Since $\text{scott}^*(\sigma) \leq \kappa$, one can write $\sigma = \bigcup_{\alpha<\iota} S_\alpha$ where $\iota = \text{cf}(\kappa)$ and $\text{scott}^*(S_\alpha) < \kappa$ for all $\alpha < \iota$. Then $\bigcap_{\alpha<\iota}\bigcap_{j\in S_\alpha} F_j = \bigcap_{j\in\sigma} F_j \subseteq W$, and so by the previous paragraph, for some $\alpha < \iota$, $\bigcap_{j\in S_\alpha} F_j \subseteq W$. By replacing σ with S_α, we may assume that $\text{scott}^*(\sigma) < \kappa$.

For $\xi < \delta$, set $j \preceq_\xi k$ if $|D_\xi(j)|_\Bbbk \leq |D_\xi(k)|_\Bbbk$ and $j \simeq_\xi k$ if $D_\xi(j) = D_\xi(k)$. Let Z be the set of pairs $(\preceq, \simeq)$ where $\preceq$ is a prewellorder of σ and $\simeq$ is an equivalence relation on σ, let

$$A(\preceq, \simeq) = \{\xi < \delta : (\preceq_\xi, \simeq_\xi) = (\preceq, \simeq)\}$$

Then $\mathcal{P} = \{A(\preceq, \simeq) : (\preceq, \simeq) \in Z\}$ is a partition of δ, and therefore it is wellorderable: set $A < B$ if $\min(A) < \min(B)$. Since $\text{scott}^*(\sigma) < \kappa$ and κ is a limit ordinal, $\text{scott}^*(Z) < \kappa$ as well, and so $|\mathcal{P}| < \kappa$. (This is the main reason we needed to ensure that $\text{scott}^*(\sigma) < \kappa$.) Let $\mathcal{Q} = \mathcal{P} \cap F^+$, and note that $\bigcup \mathcal{Q} \in F$ since its complement is $\bigcup(\mathcal{P} \cap F^*)$ which belongs to F^* by κ-completeness.

Note that in general if κ is almost extendible, F is a κ-complete filter on X, and $\mathcal{Q}$ is a partition of an F-large set into fewer than κ-many F-positive sets, then for any sequence of filters $\langle D_x : x \in X\rangle$,

$$F\text{-}\lim_{x\in X} D_x = \bigcap_{A\in\mathcal{Q}} (F \restriction A)\text{-}\lim_{x\in X} D_x$$

For the nontrivial direction, suppose $S \in \bigcap_{A\in\mathcal{Q}}(F \restriction A)\text{-}\lim D_\xi$. In other words, for each $A \in \mathcal{Q}$, $\{\xi \in A : S \in D_\xi\} \in F \restriction A$. Applying the wellordered collection lemma, one can find a set $\tau \subseteq F$ such that $\text{scott}^*(\tau) < \kappa$ for each $A \in \mathcal{Q}$, there is some $C \in \tau$ such that $C \cap A \subseteq \{\xi < \delta : S \in D_\xi\}$. Letting $T = \bigcap \tau$, we have that $T \in F$ and for each $A \in \mathcal{Q}$, $T \cap A \subseteq \{\xi < \delta : S \in D_\xi\}$. Since $\mathcal{Q}$ is a partition of X, this implies that $T \cap \bigcup \mathcal{Q} \subseteq \{\xi < \delta : S \in D_\xi\}$. Therefore $\{\xi < \delta : S \in D_\xi\} \in F$, and hence $S \in F\text{-}\lim_{x\in X} D_x$.

Given this, we now have:

$$\begin{aligned}
\bigcap_{j\in\sigma} F_j &= \bigcap_{j\in\sigma} j^{-1}[F] \\
&= \bigcap_{j\in\sigma} F\text{-}\lim_{\xi<\delta} D_\xi(j) \\
&= \bigcap_{j\in\sigma}\bigcap_{A\in\mathcal{Q}} (F \restriction A)\text{-}\lim_{\xi<\delta} D_\xi(j) \\
&= \bigcap_{A\in\mathcal{Q}}\bigcap_{j\in\sigma} (F \restriction A)\text{-}\lim_{\xi<\delta} D_\xi(j)
\end{aligned}$$

Let $H_A = \bigcap_{j\in\sigma}(F \restriction A)\text{-}\lim_{\xi<\delta} D_\xi(j)$. We have proved that $\bigcap_{A\in\mathcal{Q}} H_A \subseteq W$. Since $|\mathcal{Q}| < \kappa$, there is some $A \in \mathcal{Q}$ such that $H_A \subseteq W$.

Let $(\preceq, \simeq)$ be such that $A = A(\preceq, \simeq)$. In other words, if $j, k \in \sigma$ and $\xi \in A$, then $|D_j(\xi)|_{\Bbbk} \leq |D_k(\xi)|_{\Bbbk}$ if and only if $j \preceq k$ and $D_j(\xi) = D_k(\xi)$ if and only if $j \simeq k$.

Let $\beta = \text{rank}(\preceq)$, so $\beta < \theta(\sigma) < \kappa$. For $\nu < \beta$, let

$$\sigma_\nu = \{j \in \sigma : \text{rank}_{\preceq}(j) = \nu\}$$

Then

$$\bigcap_{j\in\sigma}(F \restriction A)\text{-}\lim_{\xi<\delta} D_\xi(j) = \bigcap_{\nu<\beta}\bigcap_{j\in\sigma_\nu}(F \restriction A)\text{-}\lim_{\xi<\delta} D_\xi(j)$$

Again applying the wellordered collection lemma, there is some $\nu < \beta$ such that

$$\bigcap_{j\in\sigma_\nu}(F \restriction A)\text{-}\lim_{\xi<\delta} D_\xi(j) \subseteq W$$

For $j \in \sigma_\nu$, let $H_j = (F \restriction A)\text{-}\lim_{\xi<\delta} D_\xi(j)$, so that $\bigcap_{j\in\sigma_\nu} H_j \subseteq W$. Let $I = \sigma_\nu/\simeq$ be the set of equivalence classes of σ_ν modulo $\simeq$. If $j, k \in \sigma_\nu$ and $j \simeq k$, then $D_\xi(j) = D_\xi(k)$ for all $\xi \in A$, and so $H_j = H_k$. For each $\simeq$-equivalence class $x \in I$, one can therefore define H_x to be the common value of H_j for all $j \in x$.

Then

$$\bigcap_{x\in I} H_x = \bigcap_{x\in I}\bigcap_{j\in x} H_x = \bigcap_{j\in\sigma_\nu} H_j \subseteq W$$

On the other hand, if $x \in I$, then for any $j \in x$, we have

$$F_j = F\text{-}\lim_{\xi<\delta} D_\xi(j) \subseteq (F \restriction A)\text{-}\lim_{\xi<\delta} D_\xi(j) = H_j \subseteq H_x$$

and so since F_j does not extend to a κ-complete ultrafilter, neither does H_x. In particular, $H_x \nsubseteq W$.

Fix $\xi \in A$, and note that the set I is in bijection with the set $\mathcal{D} = \{D_\xi(j) : j \in \sigma_\nu\}$. Since any two embeddings in σ_ν have the same rank in $\preceq$, any two ultrafilters in $\mathcal{D}$ have the same rank in the Ketonen order. Applying Proposition 4.8, the least rank Berkeley cardinal is $\mathcal{D}$-closed rank Berkeley, and hence it is I-closed rank Berkeley since $|I| = |\mathcal{D}|$.

By our choice of κ, it follows that κ is I-closed almost extendible. In particular, κ is I-closed almost supercompact, and so there is an I-closed fine filter $\mathcal{W}$ on $P(P(\delta))$ concentrating on the set of τ such that $\text{scott}^*(\tau) < \kappa$. Since $\mathcal{W}$ is fine, for each $i \in I$, for $\mathcal{W}$-almost all τ, there is some $A \in \tau$ such that $A \in H_x \setminus W$. Since $\mathcal{W}$ is I-closed, these quantifiers can be exchanged: for $\mathcal{W}$-almost all τ, for all $i \in I$, there is some $A \in \tau$ such that $A \in H_x \setminus W$. Therefore fix such a τ with $\text{scott}^*(\tau) < \kappa$. Then $\bigcup(\tau \setminus W)$ belongs to $\bigcap_{x\in I} H_x$, but it does not belong to W since W is κ-complete. This contradicts the fact that $\bigcap_{x\in I} H_x \subseteq W$. This contradiction establishes that $G \cup F$ generates a filter, completing the proof of the claim. $\square$

5 Questions

For any ordinal δ, let F_δ be the filter on $[\delta]^\omega$ generated by sets of the form $[C]^\omega$ where C is ω-club in δ.

Question 5.1. Assume there is a rank Berkeley cardinal. Is there a cardinal δ such that F_δ is atomic?

Given the atomicity of the ω-club filter, one would expect to prove such an analog of the partition property $\delta \to (\delta)^\omega$, but the techniques of this paper seem to be powerless in the face of a filter on a set that cannot be wellordered.

Question 5.2. Assume λ is a rank Berkeley cardinal. Must there be a weakly Mahlo cardinal above λ?

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