THE LINEARITY OF THE MITCHELL ORDER

We show from an abstract comparison principle (the Ultrapower Axiom) that the Mitchell order is linear on sufficiently strong ultrafilters: normal ultrafilters, Dodd solid ultrafilters, and assuming GCH, generalized normal ultrafilters. This gives a conditional answer to the well-known question of whether a 2^κ-supercompact cardinal κ must carry more than one normal measure of order 0. Conditioned on a very plausible iteration hypothesis, the answer is no, since the Ultrapower Axiom holds in the canonical inner models at the finite levels of supercompactness.

Keywords: Ultrapower Axiom, Mitchell order, supercompact cardinal, normal ultrafilter

1. Introduction

The subject of this paper is an axiom motivated by the theory of canonical inner models for large cardinal hypotheses and its implications for the structure of the Mitchell order. This axiom, the *Ultrapower Axiom*, holds in all known canonical inner models, and despite its simplicity, seems to distill many of the features of the class of countably complete ultrafilters typical to these models. The axiom essentially says that any two wellfounded ultrapowers of the universe have a common wellfounded ultrapower.

The Ultrapower Axiom follows from Woodin's axiom *Weak Comparison* [1], a very weak form of the comparison lemma. The argument that Weak Comparison holds in canonical inner models is so general that if one can produce a canonical inner model for a supercompact cardinal by anything like the current methodology, then this model will satisfy Weak Comparison. By Woodin's universality results [1], it therefore seems to be a reasonable conjecture that Weak Comparison and the Ultrapower Axiom are consistent with *all* large cardinal axioms. On the other hand, a refutation of the Ultrapower Axiom from any large cardinal hypothesis whatsoever would be a strong anti-inner model theorem.

The Ultrapower Axiom offers a new perspective on the structure of the Mitchell order in canonical inner models, so we begin by describing the original perspective. The Mitchell order was first isolated by Mitchell in the context of the models $L[\mathcal{U}]$ constructed from coherent sequences of ultrafilters $\mathcal{U}$. In these models, the Mitchell order is manifestly a linear order. We outline Mitchell's proof. The definition of coherence implies that the Mitchell order linearly orders the sequence $\mathcal{U}$, and Mitchell showed by a comparison argument that every normal ultrafilter in $L[\mathcal{U}]$ lies on the sequence $\mathcal{U}$. Therefore the Mitchell order is linear in $L[\mathcal{U}]$.

The linearity of the Mitchell order on normal measures is one of the simplest features of canonical inner models that have not been replicated by forcing. Therefore a key test question for the theory of inner models for large cardinal axioms is whether the linearity of the Mitchell order is compatible with very large cardinals. For example, the following question was raised very early on by Solovay-Reinhardt-Kanamori [2] in a slightly weaker form:

Question 1.1. Assume there is a cardinal κ that is 2^κ-supercompact. Can the Mitchell order linearly order the normal ultrafilters on κ?

Woodin [1] and Neeman-Steel [3] have constructed canonical inner models at the finite levels of supercompactness under iteration hypotheses, and so one would expect to dispense easily with this question. Yet Mitchell's argument cannot be generalized to these models: in order to develop a comparison theory for these models, one must *prevent* certain normal measures from appearing on their extender sequences.

It turns out, however, that the existence of a comparison theory, rather than any specific requirements about the form of a coherent extender sequence, ensures the linearity of the Mitchell order by a completely different argument. This is the alternate perspective on the linearity of the Mitchell order we referred to above. More precisely, we show in Theorem 2.5 that the Ultrapower Axiom alone is sufficient to prove the linearity of the Mitchell order on normal ultrafilters. In fact, the proof is in certain respects simpler than even Mitchell's original proof.

Since the Ultrapower Axiom holds in the Woodin and Neeman-Steel models (by [4]), this answers Question 1.1 positively under a very plausible iteration hypothesis.

The paper is organized as follows. In the first section we state the Ultrapower Axiom and quickly prove Theorem 2.5. In the next section, we generalize it to a much wider class of ultrafilters, the Dodd solid ultrafilters. Solovay's lemma ([5], Theorem 2) implies without much work that for regular cardinals λ satisfying $2^{<\lambda} = \lambda$, normal fine ultrafilters on $P_\lambda(\lambda)$ are equivalent to Dodd solid ultrafilters. We therefore obtain that if λ is a regular cardinal such that $2^{<\lambda} = \lambda$, then the Mitchell order wellorders the normal fine ultrafilters on $P_\lambda(\lambda)$. (Regarding the assumption $2^{<\lambda} = \lambda$, in a separate paper we show that if κ is strongly compact and the Ultrapower Axiom holds, then for all $\delta \geq \kappa$, $2^\delta = \delta^+$. Using local variants of this theorem we can make do without the assumption $2^{<\lambda} = \lambda$ in many cases.)

This leaves us with the question of normal fine ultrafilters on $P(\lambda)$ for λ singular. It seems that not much is known about such ultrafilters. We begin by proving a generalization of Solovay's lemma to singular cardinals. We remark that this is the main ZFC result in this paper, and requires new ideas, especially the analysis of cofinalities of reduced products. In the final section, we show that this implies the Dodd solidity of generalized normal ultrafilters on $P(\lambda)$ under a cardinal arithmetic assumption (again $2^{<\lambda} = \lambda$) that ismnecessary.

For purely technical reasons (which are discussed after Theorem 4.4), to state the strongest theorem we prove about the Mitchell order, we first introduce in Definition 5.8 a variant of the Mitchell order called the *internal relation*. For the purposes of this paper, the internal relation simply serves as a version of the Mitchell order that is invariant under Rudin-Keisler equivalence. Theorem 5.12 then reads: if $2^{<\lambda} = \lambda$, then the internal relation wellorders the normal fine ultrafilters on $P(\lambda)$.

2. The Ultrapower Axiom

In this section we introduce the Ultrapower Axiom (UA) as briefly as possible.

Definition 2.1. Suppose M is an inner model. An *internal ultrafilter* of M is an M-countably complete M-ultrafilter that is an element of M.

Definition 2.2. Suppose U_0 and U_1 are countably complete ultrafilters. A *comparison* of $\langle U_0, U_1\rangle$ by internal ultrafilters is a pair $\langle W_0, W_1\rangle$ such that the following hold.

(1) W_0 is an internal ultrafilter of M_{U_0}.
(2) W_1 is an internal ultrafilter of M_{U_1}.
(3) $M^{M_{U_0}}_{W_0} = M^{M_{U_1}}_{W_1}$.
(4) $j^{M_{U_0}}_{W_0} \circ j_{U_0} = j^{M_{U_1}}_{W_1} \circ j_{U_1}$.

Letting $N = M^{M_{U_0}}_{W_0}$, we call $\langle W_0, W_1\rangle$ a *comparison of* $\langle U_0, U_1\rangle$ *to* N.

Given this definition, we can state the Ultrapower Axiom, a hypothesis of nearly every theorem in this paper.

Definition 2.3 (Ultrapower Axiom). Every pair of countably complete ultrafilters admits a comparison by internal ultrafilters.

Again, the Ultrapower Axiom holds in all known canonical inner models for large cardinal hypotheses. For example, it holds in the largest canonical inner models that have been unconditionally constructed from a large cardinal hypothesis, in the realm of Woodin limits of Woodin cardinals.

The more countably complete ultrafilters there are, the more interesting the Ultrapower Axiom becomes, which explains our focus on the Ultrapower Axiom in the context of supercompact cardinals. The constructions of canonical inner models conditioned on iteration hypotheses reach the finite levels of supercompactness. For example, Woodin ([1], Theorem 13.1) shows that the Ultrapower Axiom is almost certainly compatible with the GCH and the existence of a κ^{+n}-supercompact cardinal for any n:

Theorem 2.4 (Woodin). *Suppose that for all $n < \omega$, there is a coarse premouse $(M, \mathcal{E}, \delta)$ that is $(\omega_1{+}1)$-iterable for (coarse) non-overlapping $(+1)$-iteration trees by $\mathcal{E}$, and for all $A \subseteq \delta$, there is a $\kappa < \delta$ that is witnessed to be (n, A)-extendible by $\mathcal{E}$. Then for all $n < \omega$, there is an ω-short pm satisfying* ZFC *with a κ^{+n}-supercompact cardinal.*

An ω-short pm satisfying ZFC satisfies the Ultrapower Axiom by a result of [4]; the argument is essentially folklore.

The following theorem therefore conditionally answers the old question (see [2]) of whether the Mitchell order can linearly order the normal ultrafilters on κ when κ is 2^κ-supercompact. This theorem is vastly improved by Theorem 3.9, but we give the proof as motivation for Definition 3.2 (and as a service to the reader who does not want to know what a Dodd solid ultrafilter is).

Theorem 2.5 (UA). *The Mitchell order on normal ultrafilters is linear.*

Proof. Suppose U_0 and U_1 are normal ultrafilters. Let $M_0 = M_{U_0}$ and $M_1 = M_{U_1}$, and let $j_0 : V \to M_0$ and $j_1 : V \to M_1$ be the ultrapower embeddings. Without loss of generality, assume that $\text{CRT}(j_0) = \text{CRT}(j_1)$. Let us call this cardinal κ. Let $\langle W_0, W_1\rangle$ be a comparison of $\langle U_0, U_1\rangle$ by internal ultrafilters to a common model N. Thus there are internal ultrapower embeddings $k_0 : M_0 \to N$ and $k_1 : M_1 \to N$ given by W_0 and W_1 such that $k_0 \circ j_0 = k_1 \circ j_1$. We may assume by symmetry that $k_0(\kappa) \leq k_1(\kappa)$.

Suppose first that $k_0(\kappa) = k_1(\kappa)$. We claim that $U_0 = U_1$. This is a consequence of the following calculation:

$$\begin{aligned} X \in U_0 &\iff \kappa \in j_0(X) \\ &\iff k_0(\kappa) \in k_0(j_0(X)) \\ &\iff k_1(\kappa) \in k_1(j_1(X)) \\ &\iff \kappa \in j_1(X) \\ &\iff X \in U_1 \end{aligned}$$

The third equivalence uses that $k_0 \circ j_0 = k_1 \circ j_1$ and $k_0(\kappa) = k_1(\kappa)$.

Suppose instead that $k_0(\kappa) < k_1(\kappa)$. We claim that $U_0 \in M_1$. This is a consequence of the following calculation: for any $X \subseteq \kappa$,

$$\begin{aligned} X \in U_0 &\iff \kappa \in j_0(X) \\ &\iff k_0(\kappa) \in k_0(j_0(X)) \\ &\iff k_0(\kappa) \in k_1(j_1(X)) \\ &\iff k_0(\kappa) \in k_1(j_1(X)) \cap k_1(\kappa) \\ &\iff k_0(\kappa) \in k_1(j_1(X) \cap \kappa) \\ &\iff k_0(\kappa) \in k_1(X) \end{aligned}$$

The fourth equivalence follows from the fact that $k_0(\kappa) < k_1(\kappa)$. Since k_1 is definable over M_1, this calculation shows how to compute U_0 within M_1. □

We remark that this answers a similar question from [2] and [6]: must a strongly compact cardinal κ carry more than one normal ultrafilter? The answer is no, if the Ultrapower Axiom is consistent with the existence of a measurable cardinal that is a limit of strongly compact cardinals: the least such cardinal is strongly compact yet has Mitchell order 1, and therefore carries at most one normal ultrafilter if the Ultrapower Axiom holds.

3. Dodd solid ultrafilters

The following definition is somewhat nonstandard, but it is useful in our context.

Definition 3.1. Suppose α is an ordinal. An ultrafilter U on α is *uniform* if for all $\beta < \alpha$, $\beta \notin U$. In this case, α is called the *space* of U and is denoted by $\textsc{sp}(U)$.

Thus for the purposes of this paper a uniform ultrafilter has an ordinal as its underlying set, and we do not require that every element of a uniform ultrafilter U have the same cardinality, but rather that every element be unbounded in $\textsc{sp}(U)$ (though this distinction is only relevant when $\textsc{sp}(U)$ is a singular ordinal).

We now put down the most important definition in the context of the Ultrapower Axiom. We will use the definition only in a superficial way. We note that it is motivated by the attempt to generalize Theorem 2.5: recall that the key to the proof was to consider, given a comparison $\langle W_0, W_1\rangle$ of a pair of normal ultrafilters $\langle U_0, U_1\rangle$, whether $j_{W_0}(\kappa) \leq j_{W_1}(\kappa)$.

Definition 3.2. The *seed order* is the binary relation $\leq_S$ defined on uniform countably complete ultrafilters U_0 and U_1 by $U_0 \leq_S U_1$ if and only if there exists a comparison $\langle W_0, W_1\rangle$ of $\langle U_0, U_1\rangle$ by internal ultrafilters such that $j_{W_0}([\text{id}]_{U_0}) \leq j_{W_1}([\text{id}]_{U_1})$.

Theorem 3.3. *The following are equivalent.*

(1) The Ultrapower Axiom.
(2) The seed order wellorders the class of uniform countably complete ultrafilters.

The fact that one can *define* a wellorder of all uniform countably complete ultrafilters assuming a principle as general as the Ultrapower Axiom is quite surprising. For example, it has the following immediate consequence, which is not obvious from the statement of the Ultrapower Axiom.

Corollary 3.4 (UA). *Every uniform countably complete ultrafilter is ordinal definable.*

We now define the notion of Dodd solidity, which is a strong form of the initial segment condition.

Definition 3.5. Suppose U is an ultrafilter. The *Dodd initial segment* of U is the function $E_U : P(\textsc{sp}(U)) \to P^{M_U}([\text{id}]_U)$ defined by

$$E_U(X) = j_U(X) \cap [\text{id}]_U$$

We remark E_U is not in general an extender in the usual sense: one cannot form its ultrapower since $[\text{id}]_U$ may not be closed under pairing. In the case that U is a weakly normal ultrafilter (i.e. $[\text{id}]_U = \sup j_U[\textsc{sp}(U)]$), E_U *is* an extender.

Definition 3.6. Suppose U is a nonprincipal uniform countably complete ultrafilter. Then U is *Dodd solid* if $E_U \in M_U$.

Thus an ultrafilter is Dodd solid if its ultrapower contains the longest possible initial segment of its extender. For example, every normal ultrafilter U on κ is trivially Dodd solid, since E_U is the identity on $P(\kappa)$ and therefore $E_U \in M_U$. We will show in this paper that if $2^{<\lambda} = \lambda$, then any normal fine ultrafilter on $P(\lambda)$ is Rudin-Keisler equivalent to a Dodd solid ultrafilter. This is Theorem 5.7, which along with our next theorem, Theorem 3.9, is the key to the proof of our main theorem Theorem 5.12. Since the existence of a Dodd solid ultrafilter on λ implies $2^{<\lambda} = \lambda$, Theorem 5.7 is optimal.

Dodd solidity also admits a simple combinatorial characterization.

Proposition 3.7. *A nonprincipal uniform countably complete ultrafilter U on δ is Dodd solid if and only if there is a sequence $\langle \mathcal{S}_\alpha : \alpha < \delta\rangle$ of families $\mathcal{S}_\alpha \subseteq P(\alpha)$ such that for any sequence $\langle A_\alpha : \alpha < \delta\rangle$ of sets $A_\alpha \subseteq \alpha$,*

$$\{\alpha < \delta : A_\alpha \in \mathcal{S}_\alpha\} \in U \iff \exists A \subseteq \delta \ \{\alpha < \delta : A_\alpha = A \cap \alpha\} \in U$$

Proof. We first remark that a sequence $\langle \mathcal{S}_\alpha : \alpha < \delta\rangle$ has the property in the statement of the proposition if and only if $[\langle \mathcal{S}_\alpha : \alpha < \delta\rangle]_U = E_U[P(\delta)]$.

In one direction, if U is Dodd solid then $E_U \in M_U$ so $E_U[P(\delta)] \in M_U$, and so there is such a sequence $\langle \mathcal{S}_\alpha : \alpha < \delta\rangle$.

In the other direction, suppose there is such a sequence $\langle \mathcal{S}_\alpha : \alpha < \delta\rangle$. Then $E_U[P(\delta)] \in M_U$, and this implies $E_U \in M_U$, since E_U is the inverse of the transitive collapse of $E_U[P(\delta)]$. (Note that $P(\delta)$ is transitive and extensional.) □

Our definition of Dodd solidity is essentially equivalent to that of [7], except that Dodd solidity is defined there for arbitrary extenders:

Proposition 3.8. *If U is a nonprincipal uniform countably complete ultrafilter, the following are equivalent:*

(1) U is Dodd solid.
(2) Letting p be the least descending sequence of ordinals such that $M_U = H^{M_U}(j_U[V] \cup p)$, for each $i \in \text{dom}(p)$, the extender

$$E_i = \{(a, X) : a \in [p_i]^{<\omega}, X \subseteq [\text{SP}(U)]^{<\omega}, p[i] \cup a \in j_U(X)\}$$

is an element of M_U.

The notion of Dodd solidity was introduced by Steel in the case of short extenders. Theorem 3.8(2) is a simplification of the definition that Steel used that is valid for ultrafilters. Steel showed that if E is an extender on the sequence of an iterable Mitchell-Steel model satisfying ZFC, then E is Dodd solid. Most of the proof appears in [7], but see also [8]. Schlutzenberg [9] later showed that conversely any Dodd solid ultrafilter in an iterable Mitchell-Steel model satisfying ZFC lies on the extender sequence. (Similar but more complicated results hold for extenders.) Thus in the Mitchell-Steel models, the Mitchell order is linear on Dodd solid ultrafilters.

This is the most one could hope to prove using Mitchell's original argument for the linearity of the Mitchell order in $L[\mathcal{U}]$ since it exhausts the class of ultrafilters that lie on the extender sequence.

Surprisingly, the linearity of the Mitchell order on these ultrafilters is a consequence of the Ultrapower Axiom alone.

Theorem 3.9 (UA). *The Mitchell order on Dodd solid ultrafilters is linear.*

We will use in the proof the following easily verified fact about the seed order.

Lemma 3.10. *Suppose U_0 and U_1 are uniform countably complete ultrafilters such that $U_0 \leq_S U_1$. Then $\textsc{sp}(U_0) \leq \textsc{sp}(U_1)$.*

We now prove Theorem 3.9. What we actually show is a bit stronger:

Theorem 3.11. *Suppose U_0 is a uniform countably complete ultrafilter, U_1 is a Dodd solid ultrafilter, and $U_0 <_S U_1$. Then $U_0 <_M U_1$.*

Proof. Let $\alpha_0 = [\text{id}]_{U_0}$ and $\alpha_1 = [\text{id}]_{U_1}$. Let $\langle W_0, W_1\rangle$ be a comparison of $\langle U_0, U_1\rangle$ by internal ultrafilters to a common model N witnessing $U_0 <_S U_1$. Thus there are internal ultrapower embeddings $k_0 : M_{U_0} \to N$ and $k_1 : M_{U_1} \to N$ given by W_0 and W_1 such that $k_0 \circ j_{U_0} = k_1 \circ j_{U_1}$ and $k_0(\alpha_0) < k_1(\alpha_1)$.

We claim that $U_0 \in M_{U_1}$. Let $\delta_0 = \textsc{sp}(U_0)$ and $\delta_1 = \textsc{sp}(U_1)$. If $X \subseteq \delta_0$,

$$\begin{aligned} X \in U_0 &\iff \alpha_0 \in j_{U_0}(X)\\ &\iff k_0(\alpha_0) \in k_0(j_{U_0}(X))\\ &\iff k_0(c_0) \in k_1(j_{U_1}(X))\\ &\iff k_0(c_0) \in k_1(j_{U_1}(X)) \cap k_1(\alpha_1)\\ &\iff k_0(c_0) \in k_1(j_{U_1}(X) \cap \alpha_1)\\ &\iff k_0(c_0) \in k_1(E_{U_1}(X)) \end{aligned}$$

The fourth equivalence follows from the fact that $k_0(\alpha_0) < k_1(\alpha_1)$. Since k_1 is definable over M_{U_1} and $E_{U_1} \in M_{U_1}$, this calculation shows that U_0 can be computed inside M_{U_1}. (This requires $P(\delta_0) \subseteq M_{U_1}$, but $P(\delta_1) \subseteq M_{U_1}$ by Dodd solidity, and $\delta_0 \leq \delta_1$ by Lemma 3.10.) This completes the proof. □

4. Solovay's Lemma and Singular Cardinals

The following remarkable theorem, due to Solovay [5], implies that if λ is a regular cardinal then any normal fine ultrafilter on $P(\lambda)$ is Rudin-Keisler equivalent to a canonical ultrafilter on λ via the sup function.

Theorem 4.1 (Solovay's Lemma). *Suppose λ is a regular uncountable cardinal and $\langle S_\alpha : \alpha < \lambda\rangle$ is a partition of $\text{cof}(\omega) \cap \lambda$ into stationary sets. If $j : V \to M$ is an elementary embedding of V into an inner model M with $j[\lambda] \in M$, then $j[\lambda]$ is definable in M from the parameters $j(\langle S_\alpha : \alpha < \lambda\rangle)$ and $\sup j[\lambda]$.*

The key corollary of Theorem 4.1 makes no mention of the arbitrary stationary partition $\langle S_\alpha : \alpha < \lambda \rangle$.

Corollary 4.2. *Suppose λ is a regular uncountable cardinal and $\mathcal{U}$ is a normal fine ultrafilter on $P(\lambda)$. Then $\mathcal{U}$ is Rudin-Keisler equivalent to the ultrafilter*

$$U = \{X \subseteq \lambda : \{\sigma \in P(\lambda) : \sup \sigma \in X\} \in \mathcal{U}\}$$

An easy corollary is the following:

Corollary 4.3. *Suppose λ is a regular cardinal such that $2^{<\lambda} = \lambda$. Suppose $\mathcal{U}$ is a normal fine ultrafilter on $P(\lambda)$. Then $\mathcal{U}$ is Rudin-Keisler equivalent to a Dodd solid ultrafilter on λ.*

We omit the proof here, and instead prove a generalization in Theorem 5.7. When it exists, we denote the (unique) Dodd solid ultrafilter associated to a normal fine ultrafilter $\mathcal{U}$ by $U_{\mathcal{U}}$. We note that we already have the following consequence of the Ultrapower Axiom and Solovay's lemma. (The restriction to normal fine ultrafilters on $P_\lambda(\lambda)$ entails no loss of generality by Kunen's inconsistency theorem [10]: every normal fine ultrafilter on $P(\lambda)$ concentrates on $P_\lambda(\lambda)$.)

Corollary 4.4 (UA). *Suppose λ is a regular cardinal such that $2^{<\lambda} = \lambda$. Then the Mitchell order wellorders the normal fine ultrafilters on $P_\lambda(\lambda)$.*

Proof. Suppose $\mathcal{U}_0$ and $\mathcal{U}_1$ are normal fine ultrafilters on $P_\lambda(\lambda)$. Let $U_0 = U_{\mathcal{U}_0}$ and $U_1 = U_{\mathcal{U}_1}$. By Theorem 3.9, either $U_0 <_M U_1$, $U_1 <_M U_0$, or $U_0 = U_1$. In the latter case, it is easy to see that $\mathcal{U}_0 = \mathcal{U}_1$. Thus assume without loss of generality that $U_0 <_M U_1$, or equivalently that $U_0 \in M_{U_1}$. Since λ is a regular cardinal and $2^{<\lambda} = \lambda$, $\lambda^{<\lambda} = \lambda$. Therefore since $M_{\mathcal{U}_1}$ is closed under λ-sequences, $P(P_\lambda(\lambda)) \subseteq M_{\mathcal{U}_1}$ and the Rudin-Keisler reduction $f : \lambda \to P_\lambda(\lambda)$ reducing $\mathcal{U}_0$ to U_0 is in $M_{\mathcal{U}_1}$. Since

$$\mathcal{U}_0 = \{X \in P(P_\lambda(\lambda)) : f^{-1}(X) \in U_0\}$$

and $P(P_\lambda(\lambda))$, f, and U_0 are in $M_{\mathcal{U}_1}$, $\mathcal{U}_0$ is in $M_{\mathcal{U}_1}$. □

In the case that λ is singular, the seemingly trivial issue of whether the powerset of the underlying set of $\mathcal{U}_0$ lies in $M_{\mathcal{U}_1}$ will actually block the attempt to easily state some of our theorems about normal measures on $P(\lambda)$ in terms of the Mitchell order. There is a slightly deeper issue here, which is that in general when λ is a singular cardinal, given a normal fine ultrafilter $\mathcal{U}$ on $P(\lambda)$, there seems to be no canonical choice of a subset X of $P(\lambda)$ to which one can restrict $\mathcal{U}$ in order to ensure that $\mathcal{U} \restriction X$ is a uniform ultrafilter (in the standard sense of the word uniform); see Corollary 5.5 and the comments following it. In the regular case, $P_\lambda(\lambda)$ works, but for singular cardinals $P_\lambda(\lambda)$ is usually too large.

Theorem 4.9 generalizes Solovay's lemma to all cardinals, regular or not, though the following lemma shows that Corollary 4.2 does not generalize naively to the singular case.

Lemma 4.5. *Suppose λ has cofinality ι and $j : V \to M$ is an elementary embedding. Let $\iota_* = \sup j[\iota]$ and $\lambda_* = \sup j[\lambda]$. Let $g_0 : \iota \to \lambda$ be the increasing enumeration of any closed cofinal subset λ of order type ι. Then the ordinals ι_* and λ_* are interdefinable in M from the parameter $j(g_0)$.*

Proof. Note that $\lambda_* = j(g_0)(\iota_*)$ since $j \circ g_0[\iota]$ is cofinal in $j(g_0)[\iota_*]$ and $j(g_0)$ is continuous. Clearly this defines λ_* from ι_* using $j(g_0)$, but it also defines ι_* from λ_* using $j(g_0)$, since ι_* is the unique ordinal α such that $j(g_0)(\alpha) = \lambda_*$. □

Thus if λ is a singular cardinal and $\mathcal{U}$ is a normal fine ultrafilter on $P(\lambda)$, the ultrafilter derived from $j_{\mathcal{U}}$ using λ_* is equivalent to the ultrafilter W derived from $j_{\mathcal{U}}$ using ι_*, which is *not* Rudin-Keisler equivalent to $\mathcal{U}$, since j_W is continuous at ι^+ while $j_{\mathcal{U}}$ is not. In fact, W is Rudin-Keisler equivalent to the projection of $\mathcal{U}$ to $P(\iota)$, again by Solovay's lemma.

We state a lemma that is an immediate consequence of Solovay's lemma, just because we will apply it many times in the proof of Theorem 4.9.

Lemma 4.6. *Suppose $i : V \to N$ is an elementary embedding, ι is a regular cardinal, and $i[\iota] \in N$. Then for any $f : \iota \to V$, $i \circ f$ is in N and is definable in N from $\sup i[\iota]$ and a point in the range of i.*

Proof. By Solovay's lemma, $i[\iota]$ is definable in N from $\sup i[\iota]$ and a point in the range of i. But $i \circ f = i(f) \circ i \restriction \iota$. □

We now prove the correct generalization of Solovay's lemma. This involves the notion of a *generator* of an elementary embedding, from the theory of extenders:

Definition 4.7. Suppose $j : V \to M$ is an elementary embedding. An ordinal θ is a generator of j if there is are elementary embeddings $i : V \to N$ and $k : N \to M$ such that $j = k \circ i$ and $\textsc{crt}(k) = \theta$.

A well-known, easily proved fact is that if θ is a generator of j, then letting E be the extender of length θ derived from j and $k : M_E \to M$ be the factor embedding, $\textsc{crt}(k) = \theta$, so θ is witnessed to be a generator by $i = j_E$ and k. We also use the following fact, which essentially reduces to the standard fact that a nonprincipal ultrafilter cannot be supercompact past its space:

Lemma 4.8. *Suppose λ is an uncountable cardinal and $j : V \to M$ is an elementary embedding such that $\textsc{crt}(j) \leq \lambda$ and $j[\lambda] \in M$. Let θ be the least ordinal such that $j[\lambda]$ is definable in M from θ and a point in the range of j. Then θ is a generator of j and $\theta \geq \sup j[\lambda]$.*

Proof. It is clear that θ is a generator. To see that $\theta \geq \sup j[\lambda]$, note that the ultrafilter U derived from j using θ is λ-supercompact. Assume towards a contradiction that $\theta < \sup j[\lambda]$. Let $\delta = \textsc{sp}(U)$. Then $\delta < \lambda$, which contradicts that a nonprincipal ultrafilter cannot be supercompact to a cardinal past its space. (Briefly, note that j_U is continuous at δ^+, but since $\delta^+ \leq \lambda$, $\text{cf}^{M_U}(\sup j_U[\delta^+]) = \delta^+ < j_U(\delta^+)$, with the last inequality following from Kunen's inconsistency theorem [10].) $\square$

Our generalization of Solovay's lemma essentially says that the converse of Theorem 4.8 holds.

Theorem 4.9. *Suppose λ is an uncountable cardinal and $j : V \to M$ is an elementary embedding such that $\text{CRT}(j) \leq \lambda$ and $j[\lambda] \in M$. Let θ be the least generator of j greater than or equal to $\sup j[\lambda]$. Then $j[\lambda]$ is definable in M from θ and a point in the range of j.*

Proof. By Solovay's lemma, we may assume λ is singular. Let ι denote the cofinality of λ and ι_* denote $\sup j[\iota]$. (The case that $\iota < \kappa$ easily reduces to Solovay's lemma, but we do not assume $\kappa \leq \iota$ because we think that Subclaim 2 is of interest even in the case $\iota < \kappa$.)

Let λ_* denote $\sup j[\lambda]$. Let E be the extender of length λ_* derived from j and let M_E be the extender ultrapower. Let $\langle \gamma_\xi : \xi < \iota \rangle$ enumerate a cofinal set of regular cardinals below λ. Let $e : \iota \to \lambda_*$ be the function $e(\xi) = \sup j[\gamma_\xi]$. Note that $e \in M$ since $j[\lambda]$ is in M and $\langle \gamma_\xi : \xi < \iota \rangle$ is in M. Let J be the ideal of bounded subsets of ι. We state the key observation, the proof of which will constitute the bulk of the proof of Theorem 4.9:

Claim 1. *The equivalence class $[e]_J$ of e modulo J is definable in M from $\lambda_*^{+M_E}$ and a point in the range of j.*

For now, we assume Claim 1 and show that the theorem follows.

We first show $j[\lambda]$ is definable in M from $[e]_J$ and a point in the range of j. For $\xi < \iota$, let $\mathcal{T}_\xi$ be a stationary partition of $\text{cof}(\omega) \cap \gamma_\xi$. For any $e' \in [e]_J$, for all sufficiently large $\xi_0 < \iota$, $j[\lambda]$ is the union over $\xi \in [\xi_0, \iota)$ of the sets X_ξ obtained by applying Solovay's lemma to $j(\mathcal{T}_\xi)$ and $e'(\xi)$. In this way, $j[\lambda]$ is definable from $[e]_J$ and $\langle j(\mathcal{T}_\xi) : \xi < \iota \rangle$. Obviously λ_* is definable from $[e]_J$, and hence ι_* is definable from $[e]_J$ by Lemma 4.5. Thus $\langle j(\mathcal{T}_\xi) : \xi < \iota \rangle$ is definable from $[e]_J$ and a point in the range of j by Lemma 4.6. Thus $j[\lambda]$ is definable in M from $[e]_J$ and a point in the range of j. It follows that $j[\lambda]$ is definable in M from $\lambda_*^{+M_E}$ and a point in the range of j.

It is now probably quite clear that $\lambda_*^{+M_E} = \theta$, which implies Theorem 4.9. Still, we include a detailed proof that $\lambda_*^{+M_E} = \theta$. Let $k_E : M_E \to M$ be the factor embedding. Note that $\text{CRT}(k_E)$ is regular in M_E, so by Theorem 4.6, $\text{CRT}(k_E) > \lambda_*$ and hence $\text{CRT}(k_E) \geq \lambda_*^{+M_E}$. In particular for any $\xi < \lambda_*^{+M_E}$, ξ is definable in M from an ordinal less than λ_* and a point in the range of j, so by Theorem 4.8, $j[\lambda]$ is not definable in M from ξ and point in the range of j. Therefore $\lambda_*^{+M_E}$ is the least ordinal θ' such that $j[\lambda]$ is definable in M from θ' and point in the range of j. So by Theorem 4.8, $\lambda_*^{+M_E}$ is a generator of j. So $\lambda_*^{+M_E} = \theta$. Hence $j[\lambda]$ is definable in M from θ.

This proves Theorem 4.9 assuming Claim 1. We turn to the proof of Claim 1.

Proof. We introduce some notation. Denote by $\mathscr{D}^M$ the product $\left(\prod_{\xi<\iota} j(\gamma_\xi)\right)\cap M$. Note that this product is in M since $\langle j(\gamma_\xi) : \xi < \iota\rangle \in M$ by Lemma 4.6. For any extender F derived from j with length in $(\sup j[\iota], \lambda_*]$, we denote the ultrapower by $j_F : V \to M_F$, the factor embedding to M_E by $k_{FE} : M_F \to M_E$, the factor embedding to M by $k_F : M_F \to M$, and the product $\left(\prod_{\xi<\iota} j_F(\gamma_\xi)\right) \cap M_F$ by $\mathscr{D}^{M_F}$. Then $\mathscr{D}^{M_F} \in M_F$ by Lemma 4.6. (Note that Lemma 4.6 applies in this situation since $j[\iota]$ is in the hull that collapses to M_F by Solovay's lemma, and so $j_F[\iota]$ is in M_F. In fact, $j_F[\iota] = j[\iota]$, but this is not really relevant.) In particular $\mathscr{D}^{M_E} = \mathscr{D}^M \cap M_E$ since $j_E(\gamma_\xi) = j(\gamma_\xi)$ for all $\xi < \iota$.

We break the proof into two subclaims.

Subclaim 1. In M, there is a λ_*^{+M}-scale in $\mathscr{D}^M/J$ that is definable from ι_* and a point in the range of j. Moreover, for any such scale $\langle f_\xi : \xi < \lambda_*^{+M}\rangle$, in M_E, $\langle f_\xi : \xi < \lambda_*^{+M_E}\rangle$ is a scale in $\mathscr{D}^{M_E}/J$.

Proof. If in M there is a scale in $\mathscr{D}^M/J$ of length λ_*^{+M}, then there is one definable from ι_* and a point in the range of j: first of all, $\mathscr{D}^M$ is definable from ι_* and a point in the range of j by Lemma 4.6; second, λ_*^{+M} is definable in M from ι_* and a point in the range of j by Lemma 4.5; third, the class of points definable in M from ι_* and a point in the range of j forms an elementary substructure of M by Los's theorem.

Since $\mathscr{D}^M/J$ is $\leq\lambda_*$-directed, the existence of a λ_*^{+M}-scale in $\mathscr{D}^M/J$ in M will follow if we show $|\mathscr{D}^M|^M = \lambda_*^{+M}$. We therefore prove $|\mathscr{D}^M|^M = \lambda_*^{+M}$. This is essentially an application of the local version of Solovay's theorem [5] that SCH holds above a supercompact cardinal. Note that in M, $j(\kappa)$ is $j(\lambda)$-supercompact, since in V, j witnesses that κ is λ-supercompact. (Here we use Kunen's observation [10] that the condition $j(\kappa) > \lambda$ can be omitted in the definition of λ-supercompactness using his inconsistency theorem.)

It follows that $j(\kappa)$ is λ_*^{+M}-supercompact in M: if $\iota < \kappa$, then we have that $j(\kappa)$ is $j(\lambda)^+$ supercompact in M, and if $\kappa \leq \iota$ then $\lambda_* < j(\lambda)$. Therefore

$$|\mathscr{D}^M|^M = (\lambda_*^\iota)^M = \lambda_*^{+M} \cdot (2^\iota)^M$$

by the local version of Solovay's theorem that SCH holds above a supercompact applied in M.

In fact $(2^\iota)^M < \lambda_*$: we claim that in M there is a strongly inaccessible cardinal between λ and λ_*. To see this, let $\langle \kappa_n : n < \omega\rangle$ denote the critical sequence of j, and let $n < \omega$ be least such that $\lambda < \kappa_{n+1}$. Then $\kappa_n < \lambda$ since we assumed $\kappa_0 < \lambda$. Thus $\kappa_{n+1} < \lambda_*$. Moreover since $P(\lambda) \subseteq M$, κ_n is inaccessible, and hence κ_{n+1} is inaccessible in M. (That $P(\lambda) \subseteq M$ follows from $j[\lambda] \in M$ since for any $A \subseteq \lambda$, $A = \{\alpha < \lambda : j(\alpha) \in j(A)\}$.) □

Subclaim 2. The function e is an exact upper bound of $\mathscr{D}^{M_E}/J$.

Proof. We show first that $\mathscr{D}^{M_E}$ is cofinal in e. This follows from Lemma 4.6: by Lemma 4.6, $j_E \circ f \in M_E$ for all $f : \iota \to V$. The collection of all $j_E \circ f$ for $f \in \prod_{\xi<\iota} \gamma_\xi$ is clearly cofinal in e, recalling that $j_E \circ f = j \circ f$ for such f.

Now we show that e is an upper bound of $\mathscr{D}^{M_E}/J$. Suppose $f \in \mathscr{D}^{M_E}$. For an extender F derived from j with length in $(\sup j[\iota], \lambda_*)$ and some $\bar{f} \in M_F$, $f = k_{FE}(\bar{f})$ since M_E is the direct limit of such M_F. By the elementarity of k_{FE}, $\bar{f} \in \mathscr{D}^{M_F}$. Since the length of F is strictly below λ_*, the space of F is strictly below λ. Thus for some $\xi_0 < \iota$, j_F is continuous at all regular cardinals $\delta \geq \gamma_{\xi_0}$. Since $\bar{f} \in \mathscr{D}^{M_F}$, $\bar{f}(\xi) < j_F(\gamma_\xi)$ for $\xi < \iota$. For $\xi \in [\xi_0, \iota)$, we may therefore choose $\alpha_\xi < \gamma_\xi$ such that $\bar{f}(\xi) < j_F(\alpha_\xi)$. For $\xi < \xi_0$, set $\alpha_\xi = 0$. Let $h = \langle \alpha_\xi : \xi < \iota \rangle$. Then $\bar{f} <_J j_F(h)$, and so $f = k_{FE}(\bar{f}) <_J j_E(h) < e$, as desired. □

Using Subclaim 1 and Subclaim 2, we prove Claim 1. Fix by Subclaim 1 a scale $\langle f_\xi : \xi < \lambda_*^+ \rangle$ in $\mathscr{D}^M$ definable from ι_* and a point in the range of j. Then $\langle f_\xi : \xi < \lambda_*^{+M_E} \rangle$ is definable from $\lambda_*^{+M_E}$ and a point in the range of j by Lemma 4.5. By Subclaim 1, $\langle f_\xi : \xi < \lambda_*^{+M_E} \rangle$ is a scale in $\mathscr{D}^{M_E}$. Thus $[e]_J$ is definable in M from $\langle f_\xi : \xi < \lambda_*^{+M_E} \rangle$ as the equivalence class of any exact upper bound of $\langle f_\xi : \xi < \lambda_*^{+M_E} \rangle$. Claim 1 follows from this and the definability of $\langle f_\xi : \xi < \lambda_*^{+M_E} \rangle$.

This completes the proof of Theorem 4.9.

Remark 4.10. The proof of Solovay's lemma for regular cardinals specializes in the case that $M = V$ to Woodin's proof of Kunen's inconsistency theorem (which appears in [11]). Our proof of Theorem 4.9 specializes in that case to Zapletal's proof of Kunen's inconsistency theorem (which also appears in [11]).

5. Generalized normal ultrafilters

Definition 5.1. Suppose λ is an uncountable cardinal and $\mathcal{U}$ is a normal fine ultrafilter on $P(\lambda)$. Then $U_\mathcal{U}$ denotes the uniform ultrafilter derived from $\mathcal{U}$ using θ where θ is the least generator of $\mathcal{U}$ greater than or equal to $\sup j_\mathcal{U}[\lambda]$.

The following is immediate from Theorem 4.9.

Theorem 5.2. *Suppose λ is an uncountable cardinal and $\mathcal{U}$ is a normal fine ultrafilter on $P(\lambda)$. Then $\mathcal{U} \equiv_{\text{RK}} U_\mathcal{U}$.*

The normal fine ultrafilters on $P(\lambda)$ break into two types.

Definition 5.3. A normal fine ultrafilter on $P(\lambda)$ is TYPE 0 if $\textsc{crt}(\mathcal{U}) \leq \text{cf}(\lambda)$ and TYPE 1 if $\text{cf}(\lambda) < \textsc{crt}(\mathcal{U})$.

We will show that a normal fine ultrafilter on $P(\lambda)$ is TYPE 1 if and only if it is Rudin-Keisler equivalent to a normal fine ultrafilter on $P(\lambda^+)$.

Proposition 5.4. *Suppose $\mathcal{U}$ is a normal fine ultrafilter on $P(\lambda)$. Then*

$$\textsc{sp}(U_\mathcal{U}) = \begin{cases} \lambda & \text{if } \mathcal{U} \text{ is TYPE 0} \\ \lambda^+ & \text{if } \mathcal{U} \text{ is TYPE 1} \end{cases}$$

Proof. We may assume by Corollary 4.2 that λ is singular. By Theorem 4.9, $U_\mathcal{U}$ is derived from $\mathcal{U}$ using $\lambda_*^{+M_E}$ where $\lambda_* = \sup j_\mathcal{U}[\lambda]$ and E is the extender of length λ_* derived from $\mathcal{U}$.

Suppose first that $\textsc{crt}(\mathcal{U}) \leq \text{cf}(\lambda)$. Clearly $\textsc{sp}(U_\mathcal{U}) \geq \lambda$ since $\lambda_* < \lambda_*^{+M_E}$. But since $\textsc{crt}(\mathcal{U}) \leq \text{cf}(\lambda)$, $j_\mathcal{U}$ is discontinuous at λ and so $\lambda_* < j(\lambda)$. Since $j_\mathcal{U}(\lambda)$ is a limit cardinal of $M_\mathcal{U}$, follows that $\lambda_*^{+M_E} < j_\mathcal{U}(\lambda)$, and so $\textsc{sp}(U_\mathcal{U}) \leq \lambda$. Thus $\textsc{sp}(U_\mathcal{U}) = \lambda$.

Suppose instead that $\text{cf}(\lambda) < \textsc{crt}(\mathcal{U})$. Then $j_\mathcal{U}(\lambda) = \lambda_*$. Since $\lambda_*^{+M_E}$ is a generator of $\mathcal{U}$, $\lambda_*^{+M_E} \neq j_\mathcal{U}(\lambda^+)$. Thus $j_\mathcal{U}(\lambda^+) > \lambda_*^{+M_E}$, so $\textsc{sp}(U_\mathcal{U}) \leq \lambda^+$. Moreover, if $\xi < \lambda^+$, then $j_E(\xi) < \lambda^{+M_E}$ and hence $j_E(\xi) = j_\mathcal{U}(\xi)$, since λ^{+M_E} is the critical point of the factor map from M_E to $M_\mathcal{U}$. Thus $j_\mathcal{U}(\xi) < \lambda^{+M_E}$. Since $\xi < \lambda^+$ was arbitrary, it follows that $\textsc{sp}(U_\mathcal{U}) = \lambda^+$. $\square$

Proposition 5.4 has a counterintuitive corollary.

Corollary 5.5. *Suppose λ is an uncountable cardinal and $\mathcal{U}$ is a normal fine ultrafilter on $P(\lambda)$. Then there is a set $X \in \mathcal{U}$ such that $|X| = \lambda^{<\textsc{crt}(\mathcal{U})}$.*

One might expect that every set in $\mathcal{U}$ has cardinality $\lambda^{<\delta}$ where δ is the least ordinal such that $j_\mathcal{U}(\delta) > \lambda$, since this is typically the space on which one takes a huge measure to lie. By Corollary 5.5, this fails whenever $\textsc{crt}(\mathcal{U}) \leq \text{cf}(\lambda) < \delta \leq \lambda$. (This only occurs past a huge cardinal.)

Lemma 5.6. *Suppose λ is an uncountable cardinal and $\mathcal{U}$ is a normal fine ultrafilter on $P(\lambda)$. Then $M_\mathcal{U}$ is closed under $\textsc{sp}(U_\mathcal{U})$-sequences.*

Proof. Note that $M_\mathcal{U}$ is closed under $P_\kappa(\lambda)$-sequences where $\kappa = \textsc{crt}(\mathcal{U})$, since $j[\lambda] \in M_\mathcal{U}$ by normality and $j[P_\kappa(\lambda)]$ is easily computed from $j[\lambda]$. But by Proposition 5.4, $\textsc{sp}(U_\mathcal{U}) = |P_\kappa(\lambda)|$. $\square$

The next proof is really a trivial corollary of what we have already done except for the small amount of extender theory that is needed, and which we spell out in detail.

Theorem 5.7. *Suppose $2^{<\lambda} = \lambda$ and $\mathcal{U}$ is a normal fine ultrafilter on $P(\lambda)$. Then $U_\mathcal{U}$ is Dodd solid.*

Proof. Let $\theta = [\text{id}]_{U_\mathcal{U}}$, which by definition is the least generator of $\mathcal{U}$ above $\lambda_* = \sup j_\mathcal{U}[\lambda]$. Let $E = U_\mathcal{U}|\theta$, the extender of $U_\mathcal{U}$ below θ, which is of course the same as the extender of $\mathcal{U}$ below θ. We must show that $E \in M_{U_\mathcal{U}}$, or in other words that $E \in M_\mathcal{U}$.

First of all, $E|\lambda_* \in M_\mathcal{U}$ since $E|\lambda_*$ is easily computed from the restriction of $j_\mathcal{U}$ to $\bigcup_{\alpha<\lambda} P(\alpha)$, which is in $M_\mathcal{U}$ since $M_\mathcal{U}$ is closed under λ-sequences and $\lambda = 2^{<\lambda}$.

This implies $E \in M_\mathcal{U}$ by the following argument. Note that E is the extender of length θ derived from $j_{E|\lambda_*}$ since θ is the least generator of $\mathcal{U}$ above λ_*. But

$j_{E|\lambda_*} \restriction M_{\mathcal{U}}$ can be defined over $M_{\mathcal{U}}$ by taking the ultrapower by $E|\lambda_* \in M_{\mathcal{U}}$ which is correctly computed in $M_{\mathcal{U}}$ by closure under λ-sequences. Now E is easily computed from $j_{E|\lambda_*} \restriction P(\text{SP}(E))$, and $P(\text{SP}(E)) \subseteq P(\text{SP}(U_{\mathcal{U}})) \subseteq M_{\mathcal{U}}$ by Lemma 5.6. □

We modify the Mitchell order slightly in order to state the general theorem for normal fine ultrafilters in a simple way.

Definition 5.8. Suppose U_0, U_1 are countably complete ultrafilters. The internal relation $\sqsubset$ is defined by $U_0 \sqsubset U_1$ if $j_{U_0} \restriction M_{U_1}$ is an internal ultrapower embedding of M_{U_1}.

As an immediate corollary of Theorem 3.9, we have the following:

Proposition 5.9. *Suppose U_0 and U_1 are Dodd solid ultrafilters. Then $U_0 <_S U_1$ implies $U_0 \sqsubset U_1$.*

Proof. If $U_0 <_S U_1$ then $U_0 <_M U_1$ and $\text{SP}(U_0) \leq \text{SP}(U_1)$. Since U_1 is Dodd solid, M_{U_1} is closed under $\text{SP}(U_1)$-sequences. It follows that $j_{U_0}^{M_{U_1}} = j_{U_0} \restriction M_{U_1}$ so $U_0 \sqsubset U_1$. □

The converse is not actually true: in fact if $\kappa_0 < \kappa_1$ and U_0 and U_1 are normal ultrafilters on κ_0 and κ_1 respectively, then $U_1 \sqsubset U_0$ (since $j_{U_0}(j_{U_1}) = j_{U_1} \restriction M_{U_0}$ by Kunen's commuting ultrapowers lemma) though of course $U_1 \not<_S U_0$. The next two lemmas say that all counterexamples to the converse resemble this one. We use these to deal with the TYPE 1 case in Theorem 5.12.

Lemma 5.10. *Suppose δ is an ordinal. Restricted to uniform countably complete ultrafilters on δ, the seed order extends the internal relation.*

Proof. Suppose U_0 and U_1 are uniform countably complete ultrafilters on δ and $U_0 \sqsubset U_1$. We show that $U_0 <_S U_1$. The pair $\langle j_{U_0}(j_{U_1}), j_{U_0} \restriction M_{U_1} \rangle$ are the embeddings associated to a comparison of $\langle U_0, U_1 \rangle$ by internal ultrafilters, since $j_{U_0} \restriction M_{U_1}$ is an internal ultrapower embedding of M_{U_1} by the definition of the internal relation. So to see that $U_0 <_S U_1$, it suffices to show that $j_{U_0}(j_{U_1})([\text{id}]_{U_0}) < j_{U_0}([\text{id}]_{U_1})$. By Los's theorem, this is equivalent to the statement that

$$\{\alpha < \delta : j_{U_1}(\alpha) < [\text{id}]_{U_1}\} \in U_0$$

But since $\delta = \text{SP}(U_1)$, for all $\alpha < \delta$, $j_{U_1}(\alpha) < [\text{id}]_{U_1}$. □

Lemma 5.11 (UA). *Suppose $U_0 \sqsubset U_1$ and $U_1 \sqsubset U_0$. Then $j_{U_0}(j_{U_1}) = j_{U_1} \restriction M_{U_0}$ and $j_{U_1}(j_{U_0}) = j_{U_0} \restriction M_{U_1}$.*

Proof. We just sketch the proof. The key idea here is that of a canonical comparison, which we do not define here. The following are the key properties of canonical comparisons used in the proof:

(1) The comparison of $\langle U_0, U_1\rangle$ inducing $\langle j_{U_0}(j_{U_1}), j_{U_0} \restriction M_{U_1}\rangle$ is canonical.
(2) The comparison of $\langle U_0, U_1\rangle$ inducing $\langle j_{U_1} \restriction M_{U_0}, j_{U_1}(j_{U_0})\rangle$ is canonical.
(3) (Ultrapower Axiom) A pair of ultrafilters admits a unique canonical comparison.

It follows that $j_{U_0}(j_{U_1}) = j_{U_1} \restriction M_{U_0}$ and $j_{U_1}(j_{U_0}) = j_{U_0} \restriction M_{U_1}$. □

More detailed proofs will appear in a separate paper.

Theorem 5.12 (UA). *Suppose* $2^{<\lambda} = \lambda$. *Then the internal relation wellorders the set of normal fine ultrafilters on* $P(\lambda)$.

Proof. Suppose $\mathcal{U}_0$ and $\mathcal{U}_1$ are normal fine ultrafilters on $P(\lambda)$. We show that $\mathcal{U}_0 \sqsubset \mathcal{U}_1$ if and only if $U_{\mathcal{U}_0} <_S U_{\mathcal{U}_1}$. Clearly this proves the theorem. Set $U_0 = U_{\mathcal{U}_0}$ and $U_1 = U_{\mathcal{U}_1}$. Since $\mathcal{U}_0 \equiv_{\text{RK}} U_0$ and $\mathcal{U}_1 \equiv_{\text{RK}} U_1$, it suffices to show that $U_0 \sqsubset U_1$ if and only if $U_0 <_S U_1$.

That $U_0 <_S U_1$ implies $U_0 \sqsubset U_1$ is immediate from Proposition 5.9 since U_0 and U_1 are Dodd solid.

Now suppose $U_0 \not<_S U_1$, or in other words $U_1 \leq_S U_0$. We must show $U_0 \not\sqsubset U_1$. If $U_0 = U_1$, this is immediate, so assume instead that $U_1 <_S U_0$. Then $U_1 \sqsubset U_0$ by Proposition 5.9. We assume toward a contradiction that $U_0 \sqsubset U_1$ as well. In this case by Lemma 5.10 and the wellfoundedness of the seed order, we must have that $\text{SP}(U_0) \neq \text{SP}(U_1)$. Therefore λ is a singular cardinal and $\mathcal{U}_0, \mathcal{U}_1$ are of different types. Since $U_1 <_S U_1$, we therefore have $\text{SP}(U_1) = \lambda$ and $\text{SP}(U_0) = \lambda^+$.

Let $\kappa_0 = \text{CRT}(U_0)$ and $\kappa_1 = \text{CRT}(U_1)$. By Lemma 5.11, $j_{U_0}(j_{U_1}) = j_{U_1} \restriction M_{U_0}$ and $j_{U_1}(j_{U_0}) = j_{U_0} \restriction M_{U_1}$. This implies that $j_{U_0}(\kappa_1) = \kappa_1$ and $j_{U_1}(\kappa_0) = \kappa_0$. Since j_{U_0} fixes no ordinals in $[\kappa_0, \lambda^+]$ and j_{U_1} fixes no ordinals in $[\kappa_1, \lambda]$ (by the Kunen inconsistency theorem [10]), the intervals $[\kappa_0, \lambda^+]$ and $[\kappa_1, \lambda]$ are disjoint. Therefore either $\kappa_0 > \lambda$ or $\kappa_1 > \lambda^+$. The former implies $\kappa_0 = \lambda^+$, contradicting that successor cardinals cannot be measurable. The latter is obviously impossible since $\text{SP}(U_1) = \lambda$. This is a contradiction, so $U_0 \not\sqsubset U_1$. □

We close with a trivial variant of Theorem 5.12.

Definition 5.13. Suppose U is an ultrafilter and $X \in U$. Then $U \restriction X$ denotes the ultrafilter $\{A \subseteq X : A \in U\}$.

If $\mathcal{U}_0$ and $\mathcal{U}_1$ are normal fine ultrafilters on $P(\lambda)$, then the *modified Mitchell order* $<^*_M$ on $P(\lambda)$ is defined by $\mathcal{U}_0 <^*_M \mathcal{U}_1$ if there is a set $X \in \mathcal{U}_0$ such that $\mathcal{U}_0 \restriction X \in M_{\mathcal{U}_1}$.

The proof of Theorem 5.12 shows that assuming $2^{<\lambda} = \lambda$, no TYPE 0 ultrafilter on $P(\lambda)$ lies above a TYPE 1 ultrafilter on $P(\lambda)$ in the internal relation. Thus if $\mathcal{U}_0$ and $\mathcal{U}_1$ are normal fine ultrafilters on $P(\lambda)$ with $\mathcal{U}_0 \sqsubset \mathcal{U}_1$, $\mathcal{U}_0$ concentrates on a set X of size at most $\lambda^{<\text{CRT}(\mathcal{U}_1)}$. Since $M_{\mathcal{U}_1}$ is closed under $\lambda^{<\text{CRT}(\mathcal{U}_1)}$-sequences, $P(X) \in M_{\mathcal{U}_1}$, and so $\mathcal{U}_0 \restriction X$ belongs to $M_{\mathcal{U}_1}$ by Theorem 5.12. Another version of Theorem 5.12 is therefore the following:

Theorem 5.14 (UA). *Suppose $2^{<\lambda} = \lambda$. Then the modified Mitchell order wellorders the set of normal fine ultrafilters on $P(\lambda)$.*

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