THE KETONEN ORDER

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Abstract. We study a partial order on countably complete ultrafilters introduced by Ketonen [2] as a generalization of the Mitchell order. The following are our main results: the order is wellfounded; its linearity is equivalent to the Ultrapower Axiom, a principle introduced in the author's dissertation [1]; finally, assuming the Ultrapower Axiom, the Ketonen order coincides with Lipschitz reducibility in the sense of generalized descriptive set theory.

§1. Introduction. The subject of this paper is a partial order on the class of countably complete ultrafilters originally introduced by Jussi Ketonen [2] in the early 1970s. This order, which we call the *Ketonen order*, is a generalization of the Mitchell order on normal ultrafilters to a consistently linear order on (essentially) all countably complete ultrafilters.

The Mitchell order is the partial order on the class of normal ultrafilters defined by setting $U \lhd W$ if U belongs to the ultrapower of the universe of sets by W. The Mitchell order turns out to be wellfounded, and the rank of a normal ultrafilter in the Mitchell order has come to be viewed as a measure of its complexity. All the known canonical inner models (for example, the Mitchell-Steel models [4]) satisfy the statement that the Mitchell order is linear. In other words, the normal ultrafilters of these models are linearly ordered by their complexity as measured by their rank in the Mitchell order.

The best-studied generalization of the Mitchell order is obtained by directly extending the original definition to more general objects than normal ultrafilters. For example, one can define the Mitchell order on arbitrary countably complete ultrafilters U and W by again setting $U \lhd W$ if $U \in M_W$.

The Ketonen order, on the other hand, is obtained by incorporating in addition a more subtle variation, demanding not that U belongs M_W but instead that $j_W[U]$ is contained in a sufficiently small ultrafilter of M_W. (See Theorem 3.1 below for the details.)

Restricting to the class of normal ultrafilters, Theorem 3.3 shows that the Ketonen order and the Mitchell order coincide. The point of this paper is to begin to extend the theory of normal ultrafilters under the Mitchell order to a theory of all countably complete ultrafilters under the Ketonen order. For example, the first result of this paper generalizes the main structural features of the Mitchell order to the Ketonen order:

Key words and phrases. Ultrafilter, Lipschitz order, Ultrapower Axiom.

THEOREM 1.1. *The Ketonen order is a wellfounded partial order of the class of all countably complete ultrafilters on ordinals.*

What sets the Ketonen order apart from the usual generalizations of the Mitchell order is that it is consistent with very large cardinals that the Ketonen order is linear. In fact, the main result of this paper shows that the Ketonen order is linear in all known canonical inner models.

We will actually prove something much more general, which arguably shows that the Ketonen order is linear not only in the known canonical inner models, but also in all canonical inner models that will ever be constructed: we establish that the linearity of the Ketonen order follows from the *Ultrapower Axiom*, a combinatorial principle with far-reaching consequences for the structure of large cardinals that itself arguably should hold in any model that is anything like the canonical inner models constructed to date.

Moreover, we show that the Ultrapower Axiom is *equivalent* to the linearity of the Ketonen order:

THEOREM 1.2. *The following are equivalent:*

(1) The Ketonen order is linear.
(2) The Ultrapower Axiom holds.

The Ultrapower Axiom has many consequences in the theory of countably complete ultrafilters, so in light of Theorem 1.2, the linearity of the Ketonen order is itself a surprisingly powerful structural principle.

We conclude this paper by considering the relationship between the Ketonen order and a fundamental measure of complexity from descriptive set theory, the Lipschitz order. We define a direct generalization of the Lipschitz order on subsets of Cantor space and prove the following theorem:

THEOREM 1.3 (UA). *For any ordinal δ, the Lipschitz order coincides with the Ketonen order on countably complete ultrafilters on δ.*

We infer from this that the Ultrapower Axiom implies the determinacy of certain long games involving ultrafilters. We conclude the paper with a number of questions about the generalized Lipschitz order.

§2. Preliminaries.

2.1. Ultrafilters and ultrapowers. Throughout this paper, we will often need to discuss objects U which are not ultrafilters but do appear to be ultrafilters inside of some transitive model. For this we use the following self-explanatory terminology:

DEFINITION 2.1. Suppose N is a transitive model of ZFC. A set $U \in N$ is an *ultrafilter of N* if N satisfies that U is an ultrafilter; U is a *countably complete ultrafilter of N* if in addition N satisfies that U is countably complete.

Relativizing Scott's construction of the ultrapower of the universe, we can form the ultrapower of a transitive model N by an ultrafilter of N. For this, we use the following notation:

Definition 2.2. Suppose N is a transitive model of ZFC and U is an ultrafilter of N. We write $(M_U)^N$ to denote the ultrapower of N by U using functions in N. We write $(j_U)^N$ to denote the ultrapower embedding from N to $(M_U)^N$ associated to U. For any function $f \in N$, we write $[f]^N_U$ to denote the point represented by f in $(M_U)^N$.

We will often omit the parentheses in this notation, writing j^N_U and M^N_U. In the case $N = V$, we omit the superscript altogether, writing $j_U : V \to M_U$ for the ultrapower of V by an ultrafilter U.

In all our applications, U will be a countably complete ultrafilter of N, and therefore M^N_U will be wellfounded and so we identify it with the transitive inner model to which it is isomorphic.

Definition 2.3. Suppose U is an ultrafilter on a set X. We say that U *concentrates* on a set A if $A \cap X$ belongs to U. If $a \in X$, the *principal ultrafilter concentrated at* a is the set of all $A \subseteq X$ such that $a \in A$.

In other words, U concentrates on A if $[\mathrm{id}]_U \in j_U(A)$. If U is an ultrafilter on an ordinal δ, then U concentrates on an ordinal α if and only if either $\alpha \in U$ or $\delta \leq \alpha$.

2.2. The Ultrapower Axiom and the seed order. The Ultrapower Axiom is a combinatorial principle motivated by the Comparison Lemma of inner model theory. To express the axiom succinctly, we introduce a coarse notion of comparison.

Definition 2.4. Suppose N and P are transitive models of ZFC and $j : N \to P$ is an elementary embedding. We say j is an *internal ultrapower embedding* of N if there is an ultrafilter U of N such that $P = M^N_U$ and $j = j^N_U$.

We will refer to internal ultrapower embeddings of the universe V simply as *ultrapower embeddings*.

Definition 2.5. Suppose M_0, M_1, and N are transitive models of set theory. We write $(i_0, i_1) : (M_0, M_1) \to N$ to denote that $i_0 : M_0 \to N$ and $i_1 : M_1 \to N$ are elementary embeddings.

Definition 2.6. Suppose $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are ultrapower embeddings. A *comparison* of (j_0, j_1) is a pair $(i_0, i_1) : (M_0, M_1) \to N$ of internal ultrapower embeddings such that $i_0 \circ j_0 = i_1 \circ j_0$.

Ultrapower Axiom. Every pair of ultrapower embeddings of the universe of sets admits a comparison.

The Ultrapower Axiom is not provable in ZFC. Indeed, the Ultrapower Axiom implies that the Mitchell order is linear on normal ultrafilters (as a consequence of Theorem 3.3 and Theorem 4.5 below), but Kunen-Paris [3] exhibit a forcing extension in which the least measurable cardinal carries many normal ultrafilters, any two of are necessarily incomparable in the Mitchell order. The same argument shows that the Ultrapower Axiom is not a consequence of any large cardinal axiom.

On the other hand, if there are no measurable cardinals then the Ultrapower Axiom holds vacuously. Thus ZFC + UA is trivially equiconsistent with ZFC. A less trivial fact is that ZFC + a measurable cardinal is equiconsistent with ZFC + UA + a measurable cardinal. This is true because the inner model $L[U]$ is a model of UA. Similarly, ZFC + a Woodin cardinal is equiconsistent with ZFC + UA + a Woodin cardinal; the proof proceeds by constructing the Mitchell-Steel inner model with one Woodin cardinal.

The Ultrapower Axiom might therefore be viewed as a principle that clarifies the large cardinal structure of the universe of sets without "adding consistency strength." It is open, however, whether this pattern persists to large cardinals beyond the current inner models. For example, it is open whether the Ultrapower Axiom is consistent with a supercompact cardinal, and this seems like a good test question for inner model theory at that level.

Comparisons in this coarse sense induce a natural relation on the class of countably complete ultrafilters:

Definition 2.7. The *seed order* is defined on countably complete ultrafilters on ordinals as follows. Suppose U_0 and U_1 are countably complete ultrafilters on ordinals.

- $U_0 <_S U_1$ if there is a comparison $(i_0, i_1) : (M_{U_0}, M_{U_1}) \to N$ of (j_{U_0}, j_{U_1}) such that $i_0([\mathrm{id}]_{U_0}) < i_1([\mathrm{id}]_{U_1})$.
- $U_0 \leq_S U_1$ if there is a comparison $(i_0, i_1) : (M_{U_0}, M_{U_1}) \to N$ of (j_{U_0}, j_{U_1}) such that $i_0([\mathrm{id}]_{U_0}) \leq i_1([\mathrm{id}]_{U_1})$.

The totality of the seed order is trivially equivalent to the Ultrapower Axiom:

Proposition 2.8. *The following are equivalent:*

- *The Ultrapower Axiom.*
- *If U_0 and U_1 are countably complete ultrafilters on ordinals, then either $U_0 <_S U_1$ or $U_1 \leq_S U_0$.* ⊣

We will show later on that the seed order is wellfounded and in particular that one cannot have both $U_0 <_S U_1$ and $U_1 \leq_S U_0$ simultaneously. While one can show the wellfoundedness of the seed order, one cannot prove its transitivity in ZFC:

Proposition 2.9. *Assume the seed order is transitive. Then the Ultrapower Axiom holds.*

The proposition is a consequence of the following easily proved lemma:

Lemma 2.10. *Suppose U is a countably complete ultrafilter on an ordinal δ. Suppose α is an ordinal, and let p_α denote the principal ultrafilter on $\alpha + 1$ concentrated at α.*

- *If U concentrates on α, then $U <_S p_\alpha$.*
- *If U does not concentrate on α, then $p_\alpha \leq_S U$.* ⊣

To prove Theorem 2.10, one uses that (j_U, id) is a comparison of (j_{p_α}, j_U).

Proof of Theorem 2.9. Suppose U and W are countably complete ultrafilters on ordinals. We will show that the pair (j_U, j_W) admits a comparison.

Let α be any ordinal on which U concentrates. It is easy to construct a countably complete ultrafilter W' on an ordinal such that $j_{W'} = j_W$ but W'

does not concentrate on α. (For example, one can let W' be the ultrafilter derived from j_W using $\langle[\mathrm{id}]_W, j_W(\alpha)\rangle$ where $\langle -, -\rangle : \mathrm{Ord} \times \mathrm{Ord} \to \mathrm{Ord}$ denotes a definable pairing function.)

By Theorem 2.10,

$$U <_S p_\alpha \leq_S W'$$

where p_α denotes the principal ultrafilter on $\alpha + 1$ concentrated at α. By the transitivity of the seed order, $U <_S W'$. In particular, there is a comparison of $(j_U, j_{W'})$. But $j_{W'} = j_W$, so there is a comparison of (j_U, j_W), as desired. ⊣

§3. The Ketonen order. The conclusion one should draw from Theorem 2.9 is that in the context of ZFC alone, the seed order is not really the right order to study. This rather pedestrian observation was the author's motivation for considering the Ketonen order:

Definition 3.1. Suppose U and W are countably complete ultrafilters on ordinals. The *Ketonen order* is defined by setting $U <_{\Bbbk} W$ if $j_W[U]$ is contained in a countably complete ultrafilter of M_W that concentrates on $[\mathrm{id}]_W$.

We start by proving that the restriction of the Ketonen order to normal ultrafilters is the Mitchell order.

Definition 3.2. Suppose κ is a cardinal. An ultrafilter U on κ is *normal* if U is κ-complete and $[\mathrm{id}]_U = \kappa$.

Suppose U and W are normal ultrafilters. The *Mitchell order* is defined by setting $U \lhd W$ if $U \in M_W$.

Proposition 3.3. *If U and W are normal ultrafilters on a cardinal κ. Then $U \lhd W$ if and only if $U <_{\Bbbk} W$.*

Proof. First assume $U \lhd W$, and we will show $U <_{\Bbbk} W$. Define

$$U_* = \{A \subseteq j_W(\kappa) : A \in M_W \text{ and } A \cap \kappa \in U\}$$

Since $U \in M_W$, the set U_* is definable from parameters in M_W, and hence U_* belongs to M_W. It is easy to check that U_* is a countably complete ultrafilter of M_W. Moreover, by definition U_* concentrates on κ. Finally U_* contains $j_W[U]$: if $A \in U$, then $j_W(A) \cap \kappa = A \in U$, so $j_W(A) \in U_*$ by definition. It follows that $U <_{\Bbbk} W$.

Conversely, assume $U <_{\Bbbk} W$, and we will show $U \lhd W$. Fix a countably complete ultrafilter U_* of M_W containing $j_W[U]$ and concentrating on κ. We claim $U = U_* \cap P(\kappa)$. For any set $A \subseteq \kappa$,

$$\begin{aligned} A \in U &\iff j_W(A) \in U_* \\ &\iff j_W(A) \cap \kappa \in U_* \qquad (1)\\ &\iff A \in U_* \qquad (2)\end{aligned}$$

(1) follows from the fact that U_* concentrates on κ, while (2) follows from the fact that κ is the critical point of j_W. Thus $U = U_* \cap P(\kappa) \in M_W$, so $U \lhd W$. ⊣

In fact, the Mitchell order, the Ketonen order, the seed order, and the Lipschitz order (see Section 5) all coincide on normal ultrafilters. The point of this section is to show that the basic structural properties of the Mitchell order on normal ultrafilters generalize to the Ketonen order. We begin by reformulating the definition of the Ketonen order in two simple ways. This requires two definitions. The first comes from the classical theory of ultrafilters:

DEFINITION 3.4. Suppose U is an ultrafilter, I is a set in U, and $\langle W_i : i \in I\rangle$ is a sequence of ultrafilters on a set Y. Then $U\text{-}\lim_{i\in I} W_i$ is the ultrafilter on Y consisting of all subsets A of Y such that $\{i \in I : A \in W_i\}$ belongs to U.

We also need a generalization of the notion of a comparison:

DEFINITION 3.5. Suppose $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are ultrapower embeddings. A pair of elementary embeddings $(i_0, i_1) : (M_0, M_1) \to N$ is a *semicomparison* of (j_0, j_1) if i_1 is an internal ultrapower embedding of M_1 and $i_0 \circ j_0 = i_1 \circ j_1$.

We warn that the notion of a semicomparison of (j_0, j_1) is not symmetric in j_0 and j_1.

PROPOSITION 3.6. *If U and W are countably complete ultrafilters on ordinals ϵ and δ, then the following are equivalent:*

(1) $U <_{\Bbbk} W$.
(2) $U = W\text{-}\lim_{\alpha\in I} U_\alpha$ where $I \in W$ and for all $\alpha \in I$, U_α is a countably complete ultrafilter on ϵ that concentrates on α.
(3) There is a semicomparison $(k, h) : (M_U, M_W) \to N$ of (j_U, j_W) such that $k([\mathrm{id}]_U) < h([\mathrm{id}]_W)$.

PROOF. We first show that (1) implies (2). Since $U <_{\Bbbk} W$, we can fix a countably complete ultrafilter U_* of M_W containing $j_W[U]$ and concentrating on $[\mathrm{id}]_W$. By replacing U_* with $U_* \cap j_W(P(\epsilon))$, we may assume without loss of generality that the underlying set of U_* is $j_W(\epsilon)$. Fix a function $f : \delta \to V$ such that $[f]_W = U_*$. By Los's Theorem, there is a set $I \in W$ such that for all $\alpha \in I$, $f(\alpha)$ is a countably complete ultrafilter on ϵ that concentrates on α. For $\alpha \in I$, let $U_\alpha = f(\alpha)$. For any $A \subseteq \epsilon$, $A \in U$ if and only if $j_W(A) \in U_*$ if and only if $\{\alpha \in I : A \in U_\alpha\} \in W$. It follows that $U = W\text{-}\lim_{\alpha\in I} U_\alpha$, as desired.

Next we show that (2) implies (3). Let U_* be the ultrafilter of M_W represented by the function $\alpha \mapsto U_\alpha$. Let $h : M_W \to N$ be the ultrapower of M_W by U_*. Define $k : M_U \to N$ by setting $k([f]_U) = [j_W(f)]^{M_W}_{U_*}$. It is easy to check that k is a well-defined elementary embedding and moreover that (k, h) is a semicomparison of (j_U, j_W). Finally $k([\mathrm{id}]_U) = [\mathrm{id}]^{M_W}_{U'} < h([\mathrm{id}]_W)$. The final inequality follows by applying Los's Theorem to the ultrapower of V by W, using the fact that for all $\alpha \in I$, $[\mathrm{id}]_{U_\alpha} < j_{U_\alpha}(\alpha)$.

We finally show that (3) implies (1). For this, let U_* be the ultrafilter of M_W on $j_W(\epsilon)$ derived from h using the ordinal $k([\mathrm{id}]_U)$.

We first show that U_* contains $j_W[U]$. Suppose $A \in U$, and we will show $j_W(A) \in U_*$. Note that $[\mathrm{id}]_U \in j_U(A)$, so $k([\mathrm{id}]_U) \in k(j_U(A))$. By the definition of a semicomparison, $k(j_U(A)) = h(j_W(A))$, and therefore $k([\mathrm{id}]_U) \in h(j_W(A))$. Since $j_W(A) \subseteq j_W(\epsilon)$ and $k([\mathrm{id}]_U) \in h(j_W(A))$, $j_W(A)$ belongs to the ultrafilter of M_W on $j_W(\epsilon)$ derived from h using $k([\mathrm{id}]_U)$. In other words, $j_W(A) \in U_*$.

We finally show that U_* concentrates on $[\mathrm{id}]_W$. Since $k([\mathrm{id}]_U) < h([\mathrm{id}]_W)$, formally $k([\mathrm{id}]_U) \in h([\mathrm{id}]_W)$. It follows that $[\mathrm{id}]_{U_*}^{M_W} \in j_{U_*}^{M_W}([\mathrm{id}]_W)$. In other words, U_* concentrates on $[\mathrm{id}]_W$. ⊣

Theorem 3.6 (3) is obviously reminiscent of the definition of the seed order. Indeed, since every comparison is a semicomparison, Theorem 3.6 (3) yields:

COROLLARY 3.7. *The Ketonen order $<_{\Bbbk}$ extends the seed order $<_S$.* ⊣

It is convenient to introduce a non-strict version of the Ketonen order:

DEFINITION 3.8. Suppose U and W are countably complete ultrafilters on ordinals. The *non-strict Ketonen order* is defined by setting $U \leq_{\Bbbk} W$ if $j_W[U]$ is contained in a countably complete ultrafilter of M_W that concentrates on $[\mathrm{id}]_W + 1$.

The equivalences of Theorem 3.6 also go through for the non-strict Ketonen order:

PROPOSITION 3.9. *If U and W are countably complete ultrafilters on ordinals ϵ and δ, then the following are equivalent:*

(1) $U \leq_{\Bbbk} W$.
(2) $U = W\text{-}\lim_{\alpha\in I} U_\alpha$ where $I \in W$ and for all $\alpha \in I$, U_α is a countably complete ultrafilter on ϵ that concentrates on $\alpha + 1$.
(3) There is a semicomparison $(k,h) : (M_U, M_W) \to N$ of (j_U, j_W) such that $k([\mathrm{id}]_U) \leq h([\mathrm{id}]_W)$.

COROLLARY 3.10. *The non-strict Ketonen order $\leq_{\Bbbk}$ extends the non-strict seed order $\leq_S$.* ⊣

The relationship between the Ketonen order and the non-strict Ketonen order is quite straightforward. The following equivalence relation ultimately turns out to be the equivalence relation given by the non-strict Ketonen order (Theorem 3.21):

DEFINITION 3.11. Suppose U and W are ultrafilters. The *change-of-space equivalence relation* is defined by setting $U \equiv_{\Bbbk} W$ if there is a set $A \in U \cap W$ such that $U \cap P(A) = W \cap P(A)$.

Change-of-space equivalence can be reformulated in a couple of ways:

LEMMA 3.12. *Suppose U and W are countably complete ultrafilters. The following are equivalent:*

(1) $U \equiv_{\Bbbk} W$.
(2) $j_U = j_W$ and $[\mathrm{id}]_U = [\mathrm{id}]_W$.
(3) There exist elementary embeddings $(k,h) : (M_U, M_W) \to N$ such that $k \circ j_U = h \circ j_W$ and $k([\mathrm{id}]_U) = h([\mathrm{id}]_W)$.
(4) U and W concentrate on the same sets.

PROOF. We only include the proof that (3) implies (4). Suppose U concentrates on the set A. Then $[\mathrm{id}]_U \in j_U(A)$. Applying k, $k([\mathrm{id}]_U) \in k(j_U(A))$. Replacing like terms, $h([\mathrm{id}]_W) \in h(j_W(A))$. Pulling back by h, $[\mathrm{id}]_W \in j_W(A)$. Therefore W concentrates on the set A. Similarly if W concentrates on A, then U concentrates on A. Thus U and W concentrate on the same sets. ⊣

LEMMA 3.13. *The Ketonen order is invariant under change-of-space equivalence. In other words, if $U \equiv_{\Bbbk} U'$ and $W \equiv_{\Bbbk} W'$ are countably complete ultrafilters on ordinals, then $U <_{\Bbbk} W$ if and only if $U' <_{\Bbbk} W'$.*

The most straightforward relationship between the Ketonen order and the non-strict Ketonen order involves change-of-space equivalence:

LEMMA 3.14. *Suppose U and W are countably complete ultrafilters on ordinals. The following are equivalent:*
(1) $U \leq_{\Bbbk} W$.
(2) Either $U <_{\Bbbk} W$ or $U \equiv_{\Bbbk} W$.

PROOF. We prove (1) implies (2). Suppose $U \leq_{\Bbbk} W$. By Theorem 3.9 (2), there is a semicomparison $(k, h) : (M_U, M_W) \to N$ such that $k([\mathrm{id}]_U) \leq h([\mathrm{id}]_W)$. If strict inequality holds, then by Theorem 3.6 (2), $U <_{\Bbbk} W$. Otherwise $k([\mathrm{id}]_U) = h([\mathrm{id}]_W)$, so $U \equiv_{\Bbbk} W$ by Theorem 3.12 (3). ⊣

We now prove the various structural properties of the Ketonen order. We start with a useful lemma which is analogous to Theorem 2.10:

LEMMA 3.15. *Suppose $U <_{\Bbbk} W$ are countably complete ultrafilters on ordinals. If W concentrates on the ordinal $\beta + 1$, then U concentrates on β.*

PROOF. Fix a sequence $\langle U_\alpha : \alpha \in I\rangle$ of countably complete ultrafilters defined on a set $I \in W$ such that $U = W\text{-}\lim_{\alpha\in I} U_\alpha$ and for all $\alpha \in I$, U_α concentrates on α. Note that $I \cap (\beta+1) \in W$ and for all $\alpha \in I \cap (\beta + 1)$, U_α concentrates on β. Since $U = W\text{-}\lim_{\alpha\in I} U_\alpha$, this implies that U concentrates on β. ⊣

Another variant is the following:

LEMMA 3.16. *Suppose U and W are countably complete ultrafilters on ordinals. Suppose U concentrates on the ordinal β but W does not. Then $U <_{\Bbbk} W$.*

SKETCH. Let δ be the underlying set of W. Let $I = \delta \setminus \beta$ and for $\alpha \in I$, let $U_\beta = U$. The sequence $\langle U_\alpha : \alpha \in I\rangle$ witnesses $U <_{\Bbbk} W$. ⊣

For each ordinal α, let $\mathscr{U}_\alpha$ be the class of countably complete ultrafilters U that concentrate on α but on no smaller ordinal. Theorem 3.16 tells us that if $\alpha < \beta$, then all the ultrafilters in $\mathscr{U}_\alpha$ are below all the ultrafilters in $\mathscr{U}_\beta$ in the Ketonen order. Thus the Ketonen order is the sum of the orders $\langle \mathscr{U}_\alpha : \alpha \in \mathrm{Ord}\rangle$.

To prove the transitivity of the Ketonen order, we need a simple lemma that allows us to copy the structure of the Ketonen order into an ultrapower:

LEMMA 3.17. *Suppose $U <_{\Bbbk} W$ are countably complete ultrafilters on ordinals, and Z is a countably complete ultrafilter. Suppose $j_Z[W]$ is contained in a countably complete ultrafilter W_* of M_Z. Then $j_Z[U]$ is contained in some $U_* <_{\Bbbk}^{M_Z} W_*$.*

PROOF. Fix a sequence of countably complete ultrafilters $\langle U_\alpha : \alpha \in I\rangle$ witnessing $U <_{\Bbbk} W$. Let $\langle U^*_\alpha : \alpha \in I^*\rangle = j_Z(\langle U_\alpha : \alpha \in I\rangle)$. Working in M_Z, let

$$U_* = W_*\text{-}\lim_{\alpha\in I^*} U^*_\alpha$$

which is defined since $I^* = j_Z(I) \in W_*$. The elementarity of j_Z easily implies that $\langle U^*_\alpha : \alpha \in I^*\rangle$ witnesses $U_* <_{\Bbbk} W_*$ in M_Z. We finish by showing $j_Z[U] \subseteq U_*$.

Suppose $A \in U$. Then $\{\alpha \in I : A \in U_\alpha\} \in W$, so $j_Z(\{\alpha \in I : A \in U_\alpha\}) \in W_*$ since $j_Z[W] \subseteq W_*$. In other words, $\{\alpha \in I^* : j_Z(A) \in U^*_\alpha\} \in W_*$. It follows that $j_Z(A) \in U_*$, as desired. ⊣

Theorem 3.17 easily implies the transitivity of the Ketonen order:

Corollary 3.18. *Suppose $U <_\Bbbk W \leq_\Bbbk Z$ are countably complete ultrafilters on ordinals. Then $U <_\Bbbk Z$.*

Proof. Let W_* be a countably complete ultrafilter of M_Z containing $j_Z[W]$ and concentrating on the ordinal $[\mathrm{id}]_Z + 1$. By Theorem 3.17, $j_Z[U]$ is contained in some $U_* <_\Bbbk^{M_Z} W_*$. Applying Theorem 3.15 in M_Z, U_* concentrates on $[\mathrm{id}]_Z$. Therefore U_* witnesses that $U <_\Bbbk Z$. ⊣

The proof that the Ketonen order is wellfounded is somewhat subtle, and apparently it was not known to Ketonen (who proved it only in the special case of weakly normal ultrafilters where a different argument can be used). Given Theorem 3.17, however, the wellfoundedness proof closely follows the proof that the Mitchell order is wellfounded on normal ultrafilters:

Theorem 3.19. *The Ketonen order is wellfounded.*

Proof. Assume towards a contradiction that δ is the least ordinal α such that the Ketonen order is illfounded on ultrafilters concentrating on α. Fix a descending sequence $U_0 >_\Bbbk U_1 >_\Bbbk \cdots$ of countably complete ultrafilters concentrating on δ.

Let $N = M_{U_0}$. We define by recursion a sequence $U^*_1 >^N_\Bbbk U^*_2 >^N_\Bbbk \cdots$ of countably complete ultrafilters of N concentrating on $[\mathrm{id}]_{U_0}$. We will also maintain that for all integers $n \geq 1$, $j_{U_0}[U_n]$ is contained in U^*_n.

For the base case, the definition of the Ketonen order yields a countably complete ultrafilter U^*_1 of N containing $j_{U_0}[U_1]$ and concentrating on $[\mathrm{id}]_{U_0}$.

Suppose U^*_n has been defined, and we will define U^*_{n+1}. Applying Theorem 3.17 with $Z = U_0$, $U = U_{n+1}$, $W = U_n$, and $W_* = U^*_n$ yields an ultrafilter U_* of N containing $j_{U_0}[U_{n+1}]$ such that $U_* <^N_\Bbbk U^*_n$. Set $U^*_{n+1} = U_*$. Theorem 3.15 implies U^*_{n+1} concentrates on $[\mathrm{id}]_{U_0}$ since $U^*_{n+1} <^N_\Bbbk U^*_n$ and U^*_n concentrates on $[\mathrm{id}]_{U_0}$.

Thus we have obtained a Ketonen descending sequence of ultrafilters of N concentrating on $[\mathrm{id}]_{U_0}$. Note that $[\mathrm{id}]_{U_0} < j_{U_0}(\delta)$. Since N is closed under countable sequences, N satisfies that there is an ordinal $\alpha < j_{U_0}(\delta)$ such that the Ketonen order is illfounded on countably complete ultrafilters concentrating on α. By the elementarity of j_{U_0}, there is an ordinal $\alpha < \delta$ such that the Ketonen order is illfounded on countably complete ultrafilters concentrating on α. This contradicts the minimality of δ. ⊣

The wellfoundedness of the Ketonen order immediately implies the strictness of the Ketonen order. (This can also be proved by the simpler and more general argument of Theorem 5.9, which for example shows the strictness of the natural extension of the Ketonen order to countably incomplete ultrafilters.)

Corollary 3.20. *If U is a countably complete ultrafilter, then $U \not<_\Bbbk U$.* ⊣

As another corollary, we can analyze the equivalence relation given by the non-strict Ketonen order:

Corollary 3.21. *Suppose* U *and* W *are countably complete ultrafilters. Then the following are equivalent:*

(1) $U \leq_{\Bbbk} W$ *and* $W \leq_{\Bbbk} U$.
(2) $U \equiv_{\Bbbk} W$.

Proof. We show (1) implies (2). Suppose $U \leq_{\Bbbk} W$ and $W \leq_{\Bbbk} U$. To see that $U \equiv_{\Bbbk} W$, it suffices by Theorem 3.14 to show that $U \not<_{\Bbbk} W$. Assume towards a contradiction that $U <_{\Bbbk} W$. Since $U <_{\Bbbk} W \leq_{\Bbbk} U$, Theorem 3.18 implies $U <_{\Bbbk} U$. This contradicts Theorem 3.20. ⊣

We can use Theorem 3.7 and Theorem 3.19 to establish the wellfoundedness of the seed order:

Proposition 3.22. *The seed order is wellfounded on countably complete ultrafilters on ordinals.*

Proof. The seed order wellfounded because it is contained in the Ketonen order, which is wellfounded by Theorem 3.19. ⊣

§4. The linearity of the Ketonen order. In this section, we prove the main theorem of this paper: the linearity of the Ketonen order implies the Ultrapower Axiom. Before we state this, we list a number of equivalent formulations of the linearity of the Ketonen order.

The simplest way in which the Ketonen order is linear assuming the Ultrapower Axiom is the following:

Proposition 4.1 (UA). *Suppose* U *and* W *are countably complete ultrafilters on ordinals. Either* $U <_{\Bbbk} W$ *or* $W \leq_{\Bbbk} U$. *In fact, the Ketonen order and the seed order coincide.*

Proof. By the totality of the seed order (Theorem 2.8), it suffices to show that the Ketonen order and the seed order coincide. We will show that if U and W are countably complete ultrafilters on ordinals, then $U <_{\Bbbk} W$ if and only if $U <_S W$. It suffices by Theorem 3.7 to show that $U <_{\Bbbk} W$ implies $U <_S W$. Assume $U <_{\Bbbk} W$ and, towards a contradiction, that $U \not<_S W$. Since the seed order is total, $W \leq_S U$. Since the Ketonen order extends the seed order, $W \leq_{\Bbbk} U$. Thus $W \leq_{\Bbbk} U <_{\Bbbk} W$, so by the transitivity of the Ketonen order (Theorem 3.18), $W <_{\Bbbk} W$. This contradicts the irreflexivity of the Ketonen order. ⊣

The Ketonen order is not strictly speaking a linear order of the class of all countably complete ultrafilters on ordinals. Indeed, any two $\equiv_{\Bbbk}$-equivalent ultrafilters are incomparable in the (strict) Ketonen order. One way to get around this is to restrict our attention to fine ultrafilters:

Definition 4.2. An ultrafilter U on an ordinal δ is *fine* if δ is the least ordinal that belongs to U.

Equivalently, U is fine if and only if U extends the tail filter. Fine ultrafilters select a unique element from each $\equiv_{\Bbbk}$-equivalence class:

Lemma 4.3. *If* U *is an ultrafilter on an ordinal, there is a unique fine ultrafilter* U' *such that* $U \equiv_{\Bbbk} U'$. *In particular, if* U *and* W *are fine ultrafilters on ordinals, then* $U \equiv_{\Bbbk} W$ *if and only if* $U = W$. ⊣

Corollary 4.4. *The non-strict Ketonen order is antisymmetric on fine countably complete ultrafilters on ordinals. Moreover, if U and W are countably complete fine ultrafilters, then $U \leq_{\Bbbk} W$ if and only if $U <_{\Bbbk} W$ or $U = W$.* ⊣

Along with Theorem 4.1, this easily implies the following theorem:

Theorem 4.5 (UA). *The Ketonen order is a linear order on the class of countably complete fine ultrafilters on ordinals.* ⊣

One final way of stating the linearity of the Ketonen order will be convenient going forward.

Definition 4.6. Suppose $(X, \prec)$ is a wellorder and U and W are countably complete ultrafilters on X. The *Ketonen order associated to* $(X, \prec)$ is defined by setting $U \prec^{\Bbbk} W$ if $U = W\text{-}\lim_{i\in I} U_i$ where $I \in W$ and for all $i \in I$, U_i is a countably complete ultrafilter on X concentrating on $\{x \in X : x \prec i\}$.

The association of Ketonen orders is well-defined on ordertypes:

Lemma 4.7. *If $(X_0, \prec_0)$ and $(X_1, \prec_1)$ are isomorphic wellorders, then their associated Ketonen orders are isomorphic.* ⊣

This implies in particular that all the characterizations of the Ketonen order from Section 3 generalize to arbitrary wellorders. We will only need the characterization in terms of semicomparisons:

Lemma 4.8. *Suppose $(X, \prec)$ is a wellorder and U and W are countably complete ultrafilters on X. Then the following are equivalent:*

(1) $U \prec^{\Bbbk} W$.
(2) There is a semicomparison $(k, h) : (M_U, M_W) \to N$ of (j_U, j_W) such that $k([\mathrm{id}]_U) \prec^* h([\mathrm{id}]_W)$ where $\prec^* = k \circ j_U(\prec)$. ⊣

The following lemma is evident:

Lemma 4.9. *The following are equivalent:*

(1) For all countably complete ultrafilters U and W on ordinals, either $U <_{\Bbbk} W$ or $W \leq_{\Bbbk} U$.
(2) The relation $<_{\Bbbk}$ restricts to a linear order on the class of countably complete fine ultrafilters on ordinals.
(3) For any ordinal α, the relation $<_{\Bbbk}$ restricts to a linear order on the set of countably complete ultrafilters on α.
(4) The Ketonen order associated to any wellorder is a linear order. ⊣

We abbreviate the various equivalent statements from Theorem 4.9 by saying "the Ketonen order is linear."

Theorem 4.10. *The following are equivalent:*

(1) The Ketonen order is linear.
(2) The Ultrapower Axiom holds. ⊣

Theorem 4.10 is an immediate consequence of our next theorem, which shows how to define a comparison of a pair of ultrafilters given the linearity of the Ketonen order.

We will use the following terminology which is slightly imprecise but always clear from context:

Definition 4.11. If U is a countably complete ultrafilter on X and W is another countably complete ultrafilter, a *cover* of $j_W[U]$ is a countably complete ultrafilter of M_W on $j_W(X)$ that contains $j_W[U]$.

Theorem 4.12. *Assume the Ketonen order is linear. Suppose U and W are countably complete ultrafilters on ϵ and δ respectively. Suppose U_* is a countably complete ultrafilter of M_W on $j_W(\epsilon)$ and W_* is a countably complete ultrafilter of M_U on $j_U(\delta)$ such that the following hold:*

(1) U_* is $<_{\Bbbk}^{M_W}$-minimal among all covers of $j_W[U]$.
(2) W_* is $<_{\Bbbk}^{M_U}$-minimal among all covers of $j_U[W]$.

Then $(j^{M_U}_{W_*}, j^{M_W}_{U_*})$ is a comparison of (j_U, j_W).

Let us make a remark regarding Theorem 4.12 (1) that applies equally well to Theorem 4.12 (2). Theorem 4.12 (1) is formulated externally to M_W: indeed, $j_W[U]$ need not belong to M_W. Still, $<_{\Bbbk}^{M_W}$ is a (truly) wellfounded order on the class of countably complete ultrafilters of M_W on ordinals. Since there is a countably complete ultrafilter of M_W containing $j_W[U]$, namely $j_W(U)$, there must be a $<_{\Bbbk}^{M_W}$-minimal cover of $j_W[U]$. The linearity of the Ketonen order in fact implies that this cover is unique, but this will not be used in the proof of Theorem 4.12. In fact, the heart of the proof is contained in a lemma that does not require the linearity of the Ketonen order at all:

Lemma 4.13. *Suppose U and W are countably complete ultrafilters on ϵ and δ respectively. Suppose U_* is a countably complete ultrafilter of M_W on $j_W(\epsilon)$ and W_* is a countably complete ultrafilter of M_U on $j_U(\delta)$ such that the following hold:*

- *U_* is $<_{\Bbbk}^{M_W}$-minimal among all covers of $j_W[U]$.*
- *W_* is a cover of $j_U[W]$.*

For any semicomparison $(k, h) : (M^{M_U}_{W_*}, M^{M_W}_{U_*}) \to P$ of $(j^{M_U}_{W_*} \circ j_U, j^{M_W}_{U_*} \circ j_W)$, the following hold:

$$h(j^{M_W}_{U_*}([\mathrm{id}]_W)) \leq k([\mathrm{id}]_{W_*}) \tag{3}$$

$$h([\mathrm{id}]_{U_*}) \leq k(j^{M_U}_{W_*}([\mathrm{id}]_U)) \tag{4}$$

For the proof we need a simple lemma that follows from the irreflexivity of the Ketonen order:

Lemma 4.14. *Suppose W is a countably complete ultrafilter on an ordinal and $k : M_W \to N$ and $h : M_W \to N$ are elementary embeddings from M_W to a common inner model N such that $k \circ j_W = h \circ j_W$. If h is an internal ultrapower embedding of M_W, then $h([\mathrm{id}]_W) \leq k([\mathrm{id}]_W)$.*

Proof. Note that (k, h) is a semicomparison of (j_W, j_W). Assume towards a contradiction that $k([\mathrm{id}]_W) < h([\mathrm{id}]_W)$. Then by Theorem 3.6, (k, h) witnesses that $W <_{\Bbbk} W$, which contradicts Theorem 3.20 (or Theorem 5.9 below). $\dashv$

A significant generalization of Theorem 4.14 is proved in the author's thesis [1, Theorem 3.5.10]:

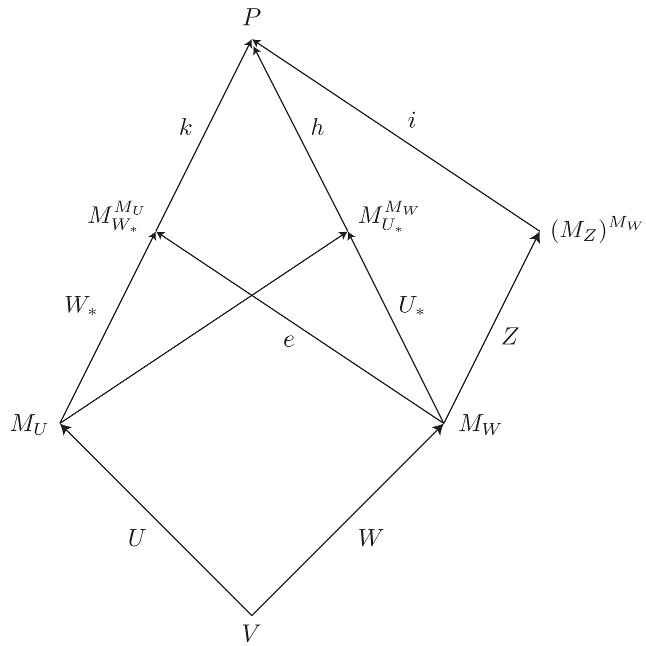

FIGURE 1. The proof of Theorem 4.13.

THEOREM 4.15. *Suppose M and N are inner models, $h : M \to N$ and $k : M \to N$ are elementary embeddings, and h is an internal extender ultrapower of M. Then $h(\alpha) \leq k(\alpha)$ for all $\alpha \in \mathrm{Ord}$.* $\dashv$

In fact, even the assumption that h is an extender ultrapower can be relaxed: by [1, Theorem 3.5.11], it suffices to assume that h is definable over M from parameters. For our purposes here, however, Theorem 4.14 is already general enough.

PROOF OF THEOREM 4.13. We first prove (3). There is a unique elementary embedding

$$e : M_W \to M_{W_*}^{M_U}$$

such that $e \circ j_W = j_{W_*}^{M_U} \circ j_U$ and $e([\mathrm{id}]_W) = [\mathrm{id}]_{W_*}$. This is defined by setting $e([f]_W) = [j_U(f)]_{W_*}^{M_U}$. We now apply the minimality of internal ultrapower embeddings of M_W (Theorem 4.14). Note that $k \circ e$ and $h \circ j_{U_*}^{M_W}$ are both elementary embeddings from M_W to P, but $h \circ j_{U_*}^{M_W}$ is an internal ultrapower embedding. Moreover,

$$k \circ e \circ j_W = k \circ j_{W_*}^{M_U} \circ j_U = h \circ j_{U_*}^{M_W} \circ j_W$$

The final equality comes from the fact that (k, h) is a semicomparison of $(j_{W_*}^{M_U} \circ j_U, j_{U_*}^{M_W} \circ j_W)$. We can therefore apply Theorem 4.14 to conclude that $h(j_{U_*}^{M_W}([\mathrm{id}]_W)) \leq k(e([\mathrm{id}]_W)) = k([\mathrm{id}]_{W_*})$, proving (3).

We now prove (4). To simplify notation, we define the following ordinal:

$$\alpha = j^{M_U}_{W_*}([\mathrm{id}]_U)$$

Let Z be the ultrafilter of M_W on $j_W(\epsilon)$ derived from $h \circ j^{M_W}_{U_*}$ using $k(\alpha)$. Since $h \circ j^{M_W}_{U_*}$ is an internal ultrapower embedding of M_W, Z is a countably complete ultrafilter of M_W on $j_W(\epsilon)$. Moreover, Z contains $j_W[U]$: for any set $A \subseteq \epsilon$,

$$\begin{aligned} j_W(A) \in Z &\iff k(\alpha) \in h \circ j^{M_W}_{U_*}(j_W(A)) \\ &\iff k(\alpha) \in k \circ j^{M_U}_{W_*}(j_U(A)) \\ &\iff \alpha \in j^{M_U}_{W_*}(j_U(A)) \\ &\iff j^{M_U}_{W_*}([\mathrm{id}]_U) \in j^{M_U}_{W_*}(j_U(A)) \\ &\iff [\mathrm{id}]_U \in j_U(A) \\ &\iff A \in U \end{aligned}$$

Since U_* is $<^{M_W}_{\Bbbk}$-minimal among all covers of $j_W[U]$, $Z \not<_{\Bbbk} U_*$ in M_W.

Since Z is derived from $h \circ j^{M_W}_{U_*}$ using $k(\alpha)$, there is a factor embedding $i : (M_Z)^{M_W} \to P$ specified by the following properties:

$$i \circ j^{M_W}_Z = h \circ j^{M_W}_{U_*} \tag{5}$$

$$i([\mathrm{id}]_Z) = k(\alpha) \tag{6}$$

Note that i is a definable class of M_W: it is defined from parameters over M_W by (5) and (6). Therefore by (5), (i, h) is a semicomparison of $(j^{M_W}_Z, j^{M_W}_{U_*})$ in M_W. The fact that $Z \not<_{\Bbbk} U_*$ in M_W implies

$$h([\mathrm{id}]_{U_*}) \leq i([\mathrm{id}]_Z) = k(\alpha) = k(j^{M_U}_{W_*}([\mathrm{id}]_U))$$

proving (4). ⊣

Theorem 4.13 can be read as asserting that the natural ultrafilter representing the embedding $j^{M_W}_{U_*} \circ j_W$ does not exceed the one representing $j^{M_U}_{W_*} \circ j_U$ in the Ketonen order. To make this precise, we must specify what this natural ultrafilter is as well as what we mean by the Ketonen order in this context.

Definition 4.16. Suppose U is an ultrafilter on X, and W_* is an ultrafilter of M_U on $j_U(Y)$. Then $U\text{-}\sum W_*$ denotes the ultrafilter on $X \times Y$ derived from $j^{M_U}_{W_*} \circ j_U$ using $(j^{M_U}_{W_*}([\mathrm{id}]_U), [\mathrm{id}]_{W_*})$.

Lemma 4.17. *Suppose U is an ultrafilter and W_* is an ultrafilter of M_U. Then $j_{U\text{-}\sum W_*} = j^{M_U}_{W_*} \circ j_U$, and $[\mathrm{id}]_{U\text{-}\sum W_*} = (j^{M_U}_{W_*}([\mathrm{id}]_U), [\mathrm{id}]_{W_*})$.* ⊣

In the context of Theorem 4.12, we would like to use Theorem 4.13 to conclude (in ZFC alone) that the ultrafilters $U\text{-}\sum W_*$ and $W\text{-}\sum U_*$ are either the same or else incomparable in the Ketonen order. We will then apply the linearity of the Ketonen order to conclude that $U\text{-}\sum W_* = W\text{-}\sum U_*$. More accurately, we will show that $U\text{-}\sum W_*$ is the *flip* of $W\text{-}\sum U_*$ (and hence their ultrapowers coincide):

Definition 4.18. If U is an ultrafilter on $X \times Y$, $\mathrm{flip}(U)$ denotes the ultrafilter on $Y \times X$ consisting of all $A \subseteq Y \times X$ such that $\{(x, y) \in X \times Y : (y, x) \in A\} \in U$.

Note that $U\text{-}\sum W_*$ and $W\text{-}\sum U_*$ are not ultrafilters on ordinals but rather on the products $\epsilon \times \delta$ and $\delta \times \epsilon$ respectively. Thus instead of using the usual Ketonen order from Section 3, we will use the Ketonen order associated to the lexicographic wellorder of pairs of ordinals:

DEFINITION 4.19. Let $<_{\text{lex}}$ denote the lexicographic order on pairs of ordinals.

Given this notation, we can now prove Theorem 4.12:

PROOF OF THEOREM 4.12. To prove the theorem, it suffices to show that $U\text{-}\sum W_* = \text{flip}(W\text{-}\sum U_*)$: then we have $j_{U\text{-}\sum W_*} = j_{\text{flip}(W\text{-}\sum U_*)} = j_{W\text{-}\sum U_*}$, which by Theorem 4.17 yields $j^{M_U}_{W_*} \circ j_U = j^{M_W}_{U_*} \circ j_W$, or in other words, $(j^{M_U}_{W_*}, j^{M_W}_{U_*})$ is a comparison of (j_U, j_W).

We first show:

$$U\text{-}\textstyle\sum W_* \not<^{\Bbbk}_{\text{lex}} \text{flip}(W\text{-}\sum U_*) \tag{7}$$

Assume towards a contradiction that $U\text{-}\sum W_* <^{\Bbbk}_{\text{lex}} \text{flip}(W\text{-}\sum U_*)$. The following identities are easily verified using Theorem 4.17:

$$\begin{aligned} j_{U\text{-}\sum W_*} &= j^{M_U}_{W_*} \circ j_U & j_{\text{flip}(W\text{-}\sum U_*)} &= j^{M_W}_{U_*} \circ j_W \\ [\text{id}]_{U\text{-}\sum W_*} &= (j^{M_U}_{W_*}([\text{id}]_U), [\text{id}]_{W_*}) & [\text{id}]_{\text{flip}(W\text{-}\sum U_*)} &= ([\text{id}]_{U_*}, j^{M_W}_{U_*}([\text{id}]_W)) \end{aligned}$$

By Theorem 4.8 (and an application of Theorem 4.17), $U\text{-}\sum W_* <^{\Bbbk}_{\text{lex}} \text{flip}(W\text{-}\sum U_*)$ is equivalent to the existence of a semicomparison

$$(k, h) : (M^{M_U}_{W_*}, M^{M_W}_{U_*}) \to N$$

of $(j^{M_U}_{W_*} \circ j_U, j^{M_W}_{U_*} \circ j_W)$ such that

$$k(j^{M_U}_{W_*}([\text{id}]_U), [\text{id}]_{W_*}) <_{\text{lex}} h([\text{id}]_{U_*}, j^{M_W}_{U_*}([\text{id}]_W))$$

Therefore either $k(j^{M_U}_{W_*}([\text{id}]_U)) < h([\text{id}]_{U_*})$ or $k([\text{id}]_{W_*}) < h(j^{M_W}_{U_*}([\text{id}]_W))$. Either way, this contradicts Theorem 4.13.

A similar argument shows:

$$U\text{-}\textstyle\sum W_* \not<^{\Bbbk}_{\text{lex}} \text{flip}(W\text{-}\sum U_*) \tag{8}$$

By the linearity of the Ketonen order, $<^{\Bbbk}_{\text{lex}}$ linearly orders the set of countably complete ultrafilters on $\epsilon \times \delta$ (Theorem 4.9). (7) and (8) therefore imply $U\text{-}\sum W_* = \text{flip}(W\text{-}\sum U_*)$. ⊣

§5. The Lipschitz order. In this last section, we define a generalization of the Lipschitz order and raise the question of whether the Ultrapower Axiom is equivalent to a long determinacy principle.

5.1. The Lipschitz order on subsets of 2^ω. We begin by recalling the definition of Lipschitz reducibility on the Cantor space 2^ω.

DEFINITION 5.1. A function $f : 2^\omega \to 2^\omega$ is *Lipschitz* if for all $x \in 2^\omega$, for any number $n < \omega$, $f(x) \restriction n$ only depends on $x \restriction n$.

To be absolutely clear, in the definition of a Lipschitz function, when we say $f(x) \restriction n$ depends only on $x \restriction n$, we mean that if $x' \in 2^\omega$ is such that $x' \restriction n = x \restriction n$, then $f(x') \restriction n = f(x) \restriction n$.

It will be convenient to define a general concept of a (many-one) reduction:

Definition 5.2. If $G, H \subseteq X$, a function $f : X \to X$ *reduces G to H* if for all $x \in X$, we have $x \in G$ if and only if $f(x) \in H$.

Of course, this is just a longwinded way of saying $f^{-1}[H] = G$, but the terminology suggests that the problem of determining membership in G is reduced via f to the problem of determining membership in H.

Definition 5.3. If $G, H \subseteq 2^\omega$, we say G is *Lipschitz reducible* to H, and write $G \leq_L H$, if there is a Lipschitz function $f : 2^\omega \to 2^\omega$ that reduces G to H.

The following fundamental theorem explains the empirical fact that definable sets of reals are wellordered by their descriptive set theoretic complexity:

Theorem 5.4 (Wadge, Martin-Monk, Martin). *The Borel subsets of 2^ω form a semi-linear, wellfounded hierarchy under the Lipschitz order:*

- *(Semi-linearity) For any Borel sets $G, H \subseteq 2^\omega$, either $G \leq_L H$ or $H \leq_L 2^\omega \setminus G$.*
- *(Wellfoundedness) Any collection of Borel sets has a $\leq_L$-minimal element.*

Under large cardinal hypotheses, this theorem can be extended to more general pointclasses of definable sets. For example, if there is a proper class of Woodin cardinals, then the theorem holds with "Borel" replaced by "universally Baire." This explains the ubiquity of wellfounded semilinear hierarchies of definability in descriptive set theory.

Unsurprisingly, Theorem 5.4 does not extend to the more pathological sets of reals associated with the Axiom of Choice. With regard to semi-linearity, an easy recursive argument using AC produces $G, H \subseteq 2^\omega$ such that $G \not\leq_L H$ and $H \not\leq_L 2^\omega \setminus G$. Given the failure of semi-linearity, it is reasonable in the context of undefinable sets to make the following definition:

Definition 5.5. A function $f : 2^\omega \to 2^\omega$ is a *contraction* if for all $x \in 2^\omega$, for all numbers $n < \omega$, $f(x)(n)$ depends only on $x \restriction n$.

Again, to be clear, that $f(x)(n)$ depends only on $x \restriction n$ means that if $x' \in 2^\omega$ is such that $x' \restriction n = x \restriction n$, then $f(x')(n) = f(x)(n)$.

A contraction is literally a contraction mapping of the metric space 2^ω with Lipschitz constant $\frac{1}{2}$. It is tempting to try to define a strict Lipschitz order by setting $G <_L H$ if G is reducible to H by a contraction, but in fact there are many $G \subseteq 2^\omega$ such that $G <_L G$. The true constraint is given by the following theorem:

Proposition 5.6. *No contraction of 2^ω reduces a set to its complement.*

Proof. Suppose $f : 2^\omega \to 2^\omega$ is a contraction and $G \subseteq 2^\omega$. By the Contraction Mapping Theorem, f has a unique fixed point x. (Explicitly, x is defined by setting $x(n) = f(x \restriction n)$.) Now f cannot reduce G to $2^\omega \setminus G$ or else we have $x \in G$ if and only if $f(x) \in 2^\omega \setminus G$, or in other words, $x \in G$ if and only if $x \notin G$. ⊣

This motivates the definition of the strict Lipschitz order:

Definition 5.7. For $G, H \subseteq 2^\omega$, $G <_L H$ if G is reducible to both H and $2^\omega \setminus H$ via contractions.

Clearly $G <_L H$ implies $G \leq_L H$ and $H \not\leq_L G$ (using Theorem 5.6 and the easily verified fact that $G \leq_L H <_L I$ implies $G <_L I$). If the conclusion of Theorem 5.4 holds for G and H, the converse is also true. Moreover if H is self-dual, then $G <_L H$ if and only if G is reducible to H by a contraction.

5.2. The Lipschitz order on subsets of 2^δ. Fix an ordinal δ. We define Lipschitz functions on 2^δ and the Lipschitz order on subsets of 2^δ as a direct generalization of the concepts from the previous section.

Definition 5.8. A function $f : 2^\delta \to 2^\delta$ is:

- *Lipschitz* if for all $x \in 2^\delta$, for all $\alpha < \delta$, $f(x) \restriction \alpha$ depends only on $x \restriction \alpha$.
- a *contraction* if for all $x \in 2^\delta$, $f(x)(\alpha)$ depends only on $x \restriction \alpha$.

To be absolutely clear, in the definition of a Lipschitz function, when we say $f(x) \restriction \alpha$ depends only on $x \restriction \alpha$, we mean that if $x' \in 2^\delta$ is such that $x' \restriction \alpha = x \restriction \alpha$, then $f(x') \restriction \alpha = f(x) \restriction \alpha$. The meaning is similar in the definition of a contraction.

Note that every contraction is Lipschitz. Every contraction has a unique fixed point, defined by recursion setting $x(\alpha) = f(x \restriction \alpha)$. We therefore obtain the analog of Theorem 5.6:

Theorem 5.9. *No contraction of 2^δ reduces a set to its complement.* ⊣

We can now define the Lipschitz order and the strict Lipschitz order:

Definition 5.10. Suppose $G, H \subseteq 2^\delta$.

- $G \leq_L H$ if G is reducible to H by a Lipschitz function.
- $G <_L H$ if G is reducible to both H and $2^\delta \setminus H$ by contractions.

The following is an immediate consequence of Theorem 5.9:

Lemma 5.11. $<_L$ *is irreflexive.* ⊣

We now consider the Lipschitz order on ultrafilters. Note that 2^δ is isomorphic to $P(\delta)$, so any subset of $P(\delta)$ can be identified with a subset of 2^δ. To avoid confusion, we make this identification completely explicit:

Definition 5.12. For each $S \subseteq \delta$, let $\chi_S \in 2^\delta$ denote the characteristic function of S. For any set $G \subseteq P(\delta)$, let $\tilde{G} = \{\chi_S : S \in G\}$.

Under UA, the relationship between the two orders is quite simple:

Theorem 5.13 (UA). *If U and W are countably complete ultrafilters on δ, then $U \leq_{\Bbbk} W$ if and only if $\tilde{U} \leq_L \tilde{W}$.*

To prove this, we will characterize the Ketonen order as a refinement of the Lipschitz order.

Definition 5.14. A function $f : 2^\delta \to 2^\delta$ is *countably complete* if it is a countably complete endomorphism of the Boolean algebra 2^δ.

In our view, the following theorem offers a very different perspective on the Ketonen order, even though it is ultimately a rather simple reformulation of the definition:

Theorem 5.15. *If U and W are countably complete ultrafilters on δ, then $U \leq_{\Bbbk} W$ if and only if $\tilde{U}$ is reducible to $\tilde{W}$ by a countably complete Lipschitz function.*

Proof. Suppose $f : 2^\delta \to 2^\delta$ is a function. Let $U^f = \langle U^f_\alpha : \alpha < \delta\rangle$ be defined by $U^f_\alpha = \{S \subseteq \delta : \alpha \in f(\chi_S)\}$. Then the following hold:

- The function $f \mapsto U^f$ is a bijection.
- f is countably complete if and only if each U_α is a countably complete ultrafilter.
- f is Lipschitz if and only if each U_α concentrates on $\alpha + 1$.
- f reduces $\tilde{U}$ to $\tilde{W}$ if and only if $U = W\text{-}\lim_{\alpha<\delta} U^f_\alpha$.

The theorem follows. ⊣

The analog of Theorem 5.15 for the strict Ketonen order is not as pretty. The issue is that no function from 2^δ to itself is both a contraction and a Boolean homomorphism; there is no ultrafilter that concentrates on 0. We do not even formulate the analog (although it is not that difficult to do), but we do record the following fact, which follows from the proof of Theorem 5.15:

Lemma 5.16. *If U and W are countably complete ultrafilters on δ, then $U <_{\Bbbk} W$ implies $\tilde{U} <_L \tilde{W}$.* ⊣

Using Theorem 5.15 and Theorem 5.16, we can prove Theorem 5.13.

Proof of Theorem 5.13. By Theorem 5.15, the Lipschitz order extends the Ketonen order. To finish, it suffices to show the converse. Fix countably complete ultrafilters U and W on δ. Assume $\tilde{U} \leq_L \tilde{W}$, and we will show $U \leq_{\Bbbk} W$. Suppose not. Then $W <_{\Bbbk} U$ by the linearity of the Ketonen order. Now Theorem 5.16 yields $\tilde{W} <_L \tilde{U}$. It follows that $\tilde{W} <_L \tilde{U} \leq_L \tilde{W}$, so $\tilde{W} <_L \tilde{W}$, contradicting that $<_L$ is irreflexive (Theorem 5.11). ⊣

Notice that for $G, H \subseteq 2^\delta$, $G \leq_L H$ if and only if Player II has a winning strategy in the *long Lipschitz game*:

Definition 5.17. Suppose δ is an ordinal and $G, H \subseteq 2^\delta$. The *long Lipschitz game* $G_L(G, H)$ is a two-player game of length $2 \cdot \delta$ on $\{0, 1\}$. In a play of $G_L(G, H)$, I and II alternate playing 0s and 1s with I playing at all even stages (including limits). In this way, I and II produce elements $x_{\mathrm{I}}, x_{\mathrm{II}} \in 2^\delta$, and II is declared the winner if either of the following conditions is met:

- $x_{\mathrm{I}} \in G$ and $x_{\mathrm{II}} \in H$.
- $x_{\mathrm{I}} \notin G$ and $x_{\mathrm{II}} \notin H$.

The linearity of the Lipschitz order can be recast as a long determinacy principle which follows from UA.

Definition 5.18. *Lipschitz Determinacy for Ultrafilters on δ* (LDU_δ) is the statement that for all countably complete ultrafilters U and W on δ, $G_L(\tilde{U}, \tilde{W})$ is determined. *Lipschitz Determinacy for Ultrafilters* (LDU) states that LDU_δ holds for all ordinals δ.

Theorem 5.19 (UA). *Lipschitz Determinacy for Ultrafilters holds.* ⊣

Our main open question is whether the converse holds:

QUESTION 5.20. Does LDU imply the Ultrapower Axiom?

There are some interesting consequences of Ultrafilter Determinacy. For example, the proof of Theorem 3.3 generalizes to show the following:

PROPOSITION 5.21. *The Lipschitz order and the Mitchell order coincide on normal ultrafilters. Thus* LDU *implies that the Mitchell order is linear on normal ultrafilters.* $\dashv$

It is not clear whether one can even prove the wellfoundedness of the Lipschitz order on countably complete ultrafilters, even assuming LDU. On the other hand, using the Martin-Monk proof of the wellfoundedness of the Wadge order, one can establish the following curious fact:

PROPOSITION 5.22 (ZF+ DC). *If every set of reals has the Baire Property, then for all ordinals δ, the Lipschitz order is wellfounded on subsets of 2^δ.* $\dashv$

One can actually use this observation along with the characterization of the Ketonen order in Theorem 5.15 to give a very different proof of the wellfoundedness of the Ketonen order.

QUESTION 5.23 (ZF + DC). Does the Axiom of Determinacy imply LDU? Does the Axiom of Determinacy imply the linearity of the Mitchell order on normal ultrafilters?

In fact, the Axiom of Determinacy does imply LDU_δ if $\delta < \omega_3$. It seems plausible that one can show LDU_δ holds for all $\delta < \aleph_{\epsilon_0}$ using Jackson's analysis of ultrafilters on projective ordinals.

The possible structure of the Lipschitz order on arbitrary subsets of large ordinals is itself far from clear:

DEFINITION 5.24. If δ is an ordinal, then *Lipschitz Determinacy holds at δ* (LD_δ) if for all $G, H \subseteq 2^\delta$, $G_L(G,H)$ is determined.

The principle $\mathrm{AD}_\mathbb{R}$ implies LD_α for all countable ordinals α, since it implies determinacy for all games of countable length. It is not clear whether LD_α for countable α follows from AD alone. A more interesting question is whether Lipschitz Determinacy consistently extends to uncountable ordinals:

QUESTION 5.25 (ZF + DC). Is LD_{ω_1} consistent? Can LD_δ hold for all $\delta < \Theta$? What about LD_δ for all ordinals δ?

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