Even ordinals and the Kunen inconsistency*

Gabriel Goldberg
Evans Hall
University Drive
Berkeley, CA 94720

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Abstract

This paper contributes to the theory of large cardinals beyond the Kunen inconsistency, or choiceless large cardinal axioms, in the context where the Axiom of Choice is not assumed. The first part of the paper investigates a periodicity phenomenon: assuming choiceless large cardinal axioms, the properties of the cumulative hierarchy turn out to alternate between even and odd ranks. The second part of the paper explores the structure of ultrafilters under choiceless large cardinal axioms, exploiting the fact that these axioms imply a weak form of the author's Ultrapower Axiom [1]. The third and final part of the paper examines the consistency strength of choiceless large cardinals, including a proof that assuming DC, the existence of an elementary embedding $j : V_{\lambda+3} \to V_{\lambda+3}$ implies the consistency of ZFC + I_0. embedding $j : V_{\lambda+3} \to V_{\lambda+3}$ implies that every subset of $V_{\lambda+1}$ has a sharp. We show that the existence of an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$ is equiconsistent with the existence of an elementary embedding from $L(V_{\lambda+2})$ to $L(V_{\lambda+2})$ with critical point below λ. We show that assuming DC, the existence of an elementary embedding $j : V_{\lambda+3} \to V_{\lambda+3}$ implies the consistency of ZFC + I_0. By a recent result of Schlutzenberg [2], an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$ does not suffice.

1 Introduction

Assuming the Axiom of Choice, the large cardinal hierarchy comes to an abrupt halt in the vicinity of an ω-huge cardinal. This is the content of Kunen's Inconsistency Theorem. The anonymous referee of Kunen's 1968 paper [3] raised the question of whether this theorem can be proved without appealing to the Axiom of Choice. This question remains unanswered. If the answer is no, then dropping the Axiom of Choice, a choiceless large cardinal hierarchy extends unimpeded beyond the Kunen

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barrier. The consistency of these large cardinals beyond choice would raise profound philosophical problems, arguably undermining the status of ZFC as a foundation for all of mathematics. (These problems will not be discussed further here.)

Of course, Gödel's Incompleteness Theorem precludes a definitive positive answer to the question of the consistency of any large cardinal axiom, choiceless or not. Instead, one can only hope to develop a large cardinal's theory to the point that it would be unreasonable to doubt its consistency. This has been achieved for some initial segment of the large cardinal hierarchy, although which axioms are secured in this weak sense is not a matter of general agreement. We can all agree, however, that there is scant evidence to date of the consistency of any of the axioms beyond the Kunen inconsistency.

In fact, a number of researchers have tried to *refute* the choiceless large cardinals in ZF. Many partial results towards this appear in Woodin's *Suitable Extender Models II*; for example, [4, Section 7] and [5, Section 5]. In the other direction, the theory of large cardinals just below the Kunen inconsistency has been developed quite extensively: for example, in [5] and [6]. The theory of choiceless large cardinals far beyond the Kunen inconsistency, especially Berkeley cardinals, is developed in [7] and [8]. Following Schlutzenberg [9], we take up the theory of choiceless large cardinals right at the level of the principle that Kunen refuted in ZFC. In particular, we will be concerned with the structure of nontrivial elementary embeddings from $V_{\lambda+n}$ to $V_{\lambda+n}$ where λ is a limit ordinal and n is a natural number. One of the general themes of this work is that, assuming choiceless large cardinal axioms, the structure of $V_{\lambda+2n}$ is very different from that of $V_{\lambda+2n+1}$.

The underlying phenomenon here involves the definability properties of rank-into-rank embeddings, which is the subject of Section 3.2. An ordinal α is said to be *even* if $\alpha = \lambda + 2n$ for some limit ordinal λ and some natural number n; otherwise, α is *odd.*

Theorem 3.3. *Suppose ϵ is an even ordinal.*

(1) No nontrivial elementary embedding from V_ϵ to V_ϵ is definable over V_ϵ.

(2) Every elementary embedding from $V_{\epsilon+1}$ to $V_{\epsilon+1}$ is definable over $V_{\epsilon+1}$.

This theorem was the catalyst for most of this research. It was independently discovered by Schlutzenberg, and is treated in greater detail in the joint paper [10].

We will use the following notation:

Definition 1.1. Suppose M and N are transitive classes and $j : M \to N$ is an elementary embedding. Then the *critical point* of j, denoted $\text{crit}(j)$, is the least ordinal moved by j. The *critical supremum of* j, denoted $\kappa_\omega(j)$, is the least ordinal above $\text{crit}(j)$ that is fixed by j.

Most of the study of rank-to-rank embeddings has focused on embeddings j either from $V_{\kappa_\omega(j)}$ to $V_{\kappa_\omega(j)}$ or from $V_{\kappa_\omega(j)+1}$ to $V_{\kappa_\omega(j)+1}$. The reason, of course, is that assuming the Axiom of Choice, these are the only rank-to-rank embeddings there are. (This well-known fact follows from the proof of Kunen's theorem.) Part of the purpose of this paper is to use Theorem 3.3 to extend this theory to embeddings of V_ϵ and $V_{\epsilon+1}$ where ϵ is an arbitrary even ordinal.

The mysterious analogy between the structure of the inner model $L(\mathbb{R})$ assuming $\text{AD}^{L(\mathbb{R})}$ and that of $L(V_{\lambda+1})$ under the axiom I_0 motivates much of the theory of $L(V_{\lambda+1})$ developed in [5].[1] In Section 4, we attempt to develop a similar analogy between the structure of arbitrary subsets of $V_{\epsilon+1}$ assuming that there is an elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$, and the structure of subsets of $\mathbb{R}$ assuming full AD.

Our main focus in Section 4 is the following sequence of cardinals:

Definition 1.2. We denote by θ_α the supremum of all ordinals that are the surjective image of V_β for some $\beta < \alpha$.

The problem of determining the structure of the cardinals θ_α is a choiceless analog of the (generalized) Continuum Problem. Note that for any limit ordinal λ, θ_λ is a strong limit cardinal[2] and $\theta_{\lambda+1} = (\theta_\lambda)^+$. We conjecture that this phenomenon generalizes periodically:

Conjecture 4.1. *Suppose ϵ is an even ordinal and there is an elementary embedding from $V_{\epsilon+1}$ to $V_{\epsilon+1}$. Then θ_ϵ is a strong limit cardinal and $\theta_{\epsilon+1} = (\theta_\epsilon)^+$.*

Under the Axiom of Determinacy, $\theta_\omega = \omega$ is a strong limit cardinal, $\theta_{\omega+1} = \omega_1$, $\theta_{\omega+2} = \Theta$ is a strong limit cardinal, and $\theta_{\omega+3} = \Theta^+$.

In addition to this numerology, various partial results of Section 4 suggest that Conjecture 4.1 holds, or at least that θ_ϵ is relatively large and $\theta_{\epsilon+1}$ is relatively small. For example:

Theorem 4.3. *Suppose ϵ is an even ordinal. Suppose $j : V_{\epsilon+2} \to V_{\epsilon+2}$. Then there is no surjection from $P((\theta_{\epsilon+1})^{+\lambda})$ onto $\theta_{\epsilon+2}$ where $\lambda = \kappa_\omega(j)$.*

Theorem 4.2. *Suppose ϵ is an even ordinal. Suppose $j : V_{\epsilon+3} \to V_{\epsilon+3}$ is an elementary embedding with critical point κ. Then the interval $(\theta_{\epsilon+2}, \theta_{\epsilon+3})$ contains fewer than κ regular cardinals.*

The attempt to prove Conjecture 4.1 leads to the following principle:

Definition 1.3. We say $V_{\alpha+1}$ *satisfies the Collection Principle* if every binary relation $R \subseteq V_\alpha \times V_{\alpha+1}$ has a subrelation S such that $\text{dom}(S) = \text{dom}(R)$ and $\text{ran}(S)$ is the surjective image of V_α.

From one perspective, the Collection Principle is a weak choice principle. It follows from the Axiom of Choice, because one can take the subrelation S to be a uniformization of R. Another perspective is that the Collection Principle states that $\theta_{\alpha+1}$ is regular in a strong sense. In particular, if $V_{\alpha+1}$ satisfies the Collection Principle, then $\theta_{\alpha+1}$ is a regular cardinal. Under AD, the converse holds at $\omega+2$: if $\theta_{\omega+2}$ is regular, then $V_{\omega+2}$ satisfies the Collection Principle.

[1]The axiom I_0 states that there is an elementary embedding from $L(V_{\lambda+1})$ to $L(V_{\lambda+1})$ with critical point less than λ.

[2]In the context of ZF, a cardinal θ is a *strong limit cardinal* if θ is not the surjective image of $P(\beta)$ for any ordinal $\beta < \theta$.

Theorem 4.12. *Suppose ϵ is an even ordinal. Suppose $j : V_{\epsilon+2} \to V_{\epsilon+2}$ is a nontrivial elementary embedding. Assume $\kappa_\omega(j)$-DC and that $V_{\epsilon+1}$ satisfies the Collection Principle.*[3] *Then $\theta_{\epsilon+2}$ is a strong limit cardinal. Moreover, for all $\beta < \theta_{\epsilon+2}$, $P(\beta)$ is the surjective image of $V_{\epsilon+1}$.*

The proof of this theorem involves generalizing Woodin's Coding Lemma. The theorem yields a new proof of the Kunen inconsistency theorem: assuming the Axiom of Choice, the hypotheses of Theorem 4.12 hold, yet $\theta_{\epsilon+2} = |V_{\epsilon+1}|^+$ is not a strong limit cardinal, and it follows that there is no elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$. (A slightly more detailed proof appears in Corollary 4.13.)

To drive home the contrast between the even and odd levels, we show that the final conclusion of Theorem 4.12 fails at the even levels:

Theorem 4.19. *Suppose ϵ is an even ordinal and there is an elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$. Then for any ordinal γ, there is no surjection from $V_\epsilon \times \gamma$ onto $P(\theta_\epsilon)$.*

Section 5 concerns the theory of ultrafilters assuming choiceless large cardinals. Woodin proved that choiceless large cardinal axioms (combined with "$\kappa_\omega(j)$-DC") imply the existence of measurable successor cardinals. The ultrafilters he produced bear a strong resemblance to the ultrafilters arising in the context of AD. Here we expand upon that theme.

First, we study the ordinal definability of ultrafilters over ordinals:

Theorem 5.17. *Suppose $j : V_{\epsilon+3} \to V_{\epsilon+3}$ is an elementary embedding. Let $\lambda = \kappa_\omega(j)$. Assume λ-DC. Suppose U is a λ^+-complete ultrafilter over an ordinal less than $\theta_{\epsilon+2}$. Then the following hold:*

(1) $U \cap \mathrm{HOD}$ belongs to HOD.

(2) U belongs to an ordinal definable set of cardinality less than λ.

(3) For an OD-cone of $x \in V_\lambda$, the ultrapower embedding j_U is amenable to HOD_x.

This result uses an analog of the Ultrapower Axiom of [1] that is provable from choiceless large cardinals (Theorem 5.12).

Finally, we prove a form of strong compactness for $\kappa_\omega(j)$ where $j : V \to V$ is an elementary embedding:

Theorem 5.24. *Suppose $j : V \to V$ is a nontrivial elementary embedding. Let $\lambda = \kappa_\omega(j)$. Assume λ-DC holds. Then every λ^+-complete filter over an ordinal extends to a λ^+-complete ultrafilter.*

This result is an application of the Ketonen order on filters, a wellfounded partial order on countably complete filters over ordinals that simultaneously generalizes the Jech order on stationary sets and the Mitchell order on normal ultrafilters.

Like many of the arguments of this paper (e.g., Theorem 5.17), the proof of Theorem 5.24 is general enough that it yields a new consequence of I_0:

[3] The choice principle λ-DC is defined in Section 2.2

Theorem 5.25 (ZFC)**.** *Suppose λ is a cardinal and there is an elementary embedding from $L(V_{\lambda+1})$ to $L(V_{\lambda+1})$ with critical point less than λ. Then in $L(V_{\lambda+1})$, every λ^+-complete filter over an ordinal less than $\theta_{\lambda+2}$ extends to a λ^+-complete ultrafilter.*

In the last section of this paper, Section 6, we turn to consistency results. Most of these results predate the groundbreaking theorem of Schlutzenberg [2] that the existence of an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with critical point below λ is equiconsistent with the existence of an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$, but it is useful to keep this theorem in mind to appreciate the statements of our theorems.

We prove the equiconsistency of various choiceless large cardinals associated with the Kunen inconsistency:

Theorem 6.8. *The following statements are equiconsistent over* ZF*:*

(1) For some λ, there is a nontrivial elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$.

(2) For some λ, there is an elementary embedding from $L(V_{\lambda+2})$ to $L(V_{\lambda+2})$ with critical point below λ.

(3) There is an elementary embedding j from V to an inner model M that is closed under $V_{\kappa_\omega(j)+1}$-sequences.

Combined with Schlutzenberg's Theorem, this shows that all of these principles are equiconsistent with the the existence of an elementary embedding from $L(V_{\lambda+1})$ to $L(V_{\lambda+1})$ with critical point below λ.

Our next theorem shows that choiceless large cardinal axioms beyond an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$ are stronger than I_0:

Theorem 6.19. *Suppose λ is an ordinal and there is a Σ_1-elementary embedding $j : V_{\lambda+3} \to V_{\lambda+3}$ with $\lambda = \kappa_\omega(j)$. Assume $\mathrm{DC}_{V_{\lambda+1}}$. Then there is a set generic extension N of V such that $(V_\delta)^N$ satisfies $\mathrm{ZFC} + I_0$ for some $\delta < \lambda$.*

The following result is an immediate corollary:

Corollary. *Over* ZF + DC*, the existence of an elementary embedding from $V_{\lambda+3}$ to $V_{\lambda+3}$ implies the consistency of* $\mathrm{ZFC} + I_0$. □

By Schlutzenberg's Theorem, the hypothesis of Theorem 6.19 cannot be reduced to the existence of an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$, or even the existence of a Σ_0-elementary embedding $j : V_{\lambda+3} \to V_{\lambda+3}$ with $j(V_{\lambda+2}) = V_{\lambda+2}$.

Schlutzenberg [2] poses the problem of calculating the exact consistency strength over ZF of the existence of an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$ in terms of large cardinal axioms compatible with the Axiom of Choice. We sketch how to calculate the consistency strength of this assertion over ZF + DC:

Theorem 6.20. *The following statements are equiconsistent over* ZF + DC*:*

(1) For some ordinal λ, there is an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$.

(2) The Axiom of Choice + I_0.

We defer to the appendix some facts about countably complete filters and ultrafilters that are used in Section 3.3 and Section 5. This is accomplished by considering a version of the Ketonen order studied in [1] that is applicable to countably complete filters on complete Boolean algebras in the context of ZF + DC. This level of generality is overkill, but it makes the proofs slicker.

2 Notation and preliminaries

In this section, we lay out some of the notational conventions we will use in this paper. *Most importantly, we work throughout this paper in* ZF *alone, without assuming the Axiom of Choice, explicitly making note of any other choice principles we use.* Most of the notation discussed here is standard, with the notable exception of Section 2.4, which introduces a class of structures $\langle \mathcal{H}_\alpha \rangle_{\alpha \in \text{Ord}}$ which will be very useful throughout the paper.

2.1 Elementary embeddings

We use the following notation for elementary embeddings:

Definition 2.1. Suppose M and N are structures in the same signature. Then $\mathscr{E}(M,N)$ denotes the set of elementary embeddings from M to N, and $\mathscr{E}(M)$ denotes the set of elementary embeddings from M to itself.

Typically the structures we consider are of the form $(M, \in)$ where M is a transitive set. We will always suppress the membership relation, writing $\mathscr{E}(M)$ when we mean $\mathscr{E}(M, \in)$.

Our notation for the critical sequence of an embedding is pulled from [5]:

Definition 2.2. Suppose M and N are transitive structures and $j \in \mathscr{E}(M,N)$. The *critical point of* j, denoted $\text{crit}(j)$, is the least ordinal moved by j. The *critical sequence of* j is the sequence $\langle \kappa_n(j) \mid n < \omega \rangle$ defined by $\kappa_0(j) = \text{crit}(j)$ and $\kappa_{n+1}(j) = j(\kappa_n(j))$. Finally, the *critical supremum of* j is the ordinal $\kappa_\omega(j) = \sup_{n<\omega} \kappa_n(j)$.

Of course, $\text{crit}(j)$ may not be defined since j may have no critical point. Even if $\text{crit}(j)$ is defined, $\kappa_{n+1}(j)$ may not be for some $n < \omega$, since it is possible that $\kappa_n(j) \notin M$.

2.2 Weak choice principles

We say that $T \subseteq X^{<\lambda}$ is a tree if for all $s \in T$, for all $\alpha < \text{dom}(s)$, $s \restriction \alpha \in T$. A tree $T \subseteq X^{<\lambda}$ is λ-*closed* if for any $s \in X^{<\lambda}$ with $s \restriction \alpha \in T$ for all $\alpha < \text{dom}(s)$, $s \in T$. A *cofinal branch* of a tree $T \subseteq X^{<\lambda}$ is a sequence $s \in X^\lambda$ such that $x \restriction \alpha \in T$ for all $\alpha < \lambda$.

Various weak choice principles will be used throughout the paper. The most important are the following:

Definition 2.3. Suppose λ is a cardinal and X is a set.

- λ-DC_X denotes the principle asserting that every λ-closed tree of sequences $T \subseteq X^{<\lambda}$ with no maximal branches has a cofinal branch.
- λ-DC denotes the principle asserting that λ-DC_Y holds for all sets Y.
- The Axiom of Dependent Choice, or DC, is the principle ω-DC.

In the context of ZF, it may be that there is a surjection from X to Y but no injection from Y to X. We therefore use the following notation:

Definition 2.4. If X and Y are sets, then $X \preceq^* Y$ if there is a partial surjection from Y to X. We let $[X]^Y = \{S \subseteq X \mid S \preceq^* Y\}$.

We use partial surjections because these are what arise naturally in practice, but of course $X \preceq^* Y$ if and only if either $X = \emptyset$ or there is a total surjection from Y to X.

2.3 Filters and ultrafilters

We use the following convention: a filter *over* a set X is a filter *on* the Boolean algebra $P(X)$. Filters on Boolean algebras do not come up until the appendix, so until then, we use the word filter to refer to a filter over some set.

Definition 2.5. Suppose γ is an ordinal. A filter F is *γ-saturated* if there is no sequence $\langle S_\alpha \mid \alpha < \gamma\rangle$ of F-positive sets such that $S_\alpha \cap S_\beta$ is F-null for all $\alpha < \beta < \gamma$; F is *weakly γ-saturated* if there is no sequence $\langle S_\alpha \mid \alpha < \gamma\rangle$ of pairwise disjoint F-positive sets.

If F is γ-complete, then F is γ-saturated if and only if F is weakly γ-saturated.

Definition 2.6. If B is a set, a filter F is *B-complete* if for any $b \in B$ and any $D \subseteq F$ such that $D \preceq^* b$, $\bigcap D \in F$. A filter F is *X-closed* if F is $\{X\}$-complete.

A filter is said to be *countably complete* if it is ω_1-complete.

We will need the standard derived ultrafilter construction:

Definition 2.7. Suppose $h : P(X) \to P(Y)$ is a homomorphism of Boolean algebras and $a \in Y$. The *ultrafilter over X derived from h using a* is the ultrafilter over X defined by the formula $\{A \subseteq X : a \in h(A)\}$.

Our notation for ultrapowers is standard in set theory. If U is an ultrafilter over a set X, then $j_U : V \to M_U$ denotes the associated ultrapower. If $\langle M_x\rangle_{x\in X}$ is a sequence of structures in the same signature, then $\prod_{x\in X} M_x/U$ denotes their ultraproduct.

2.4 The structures $\mathcal{H}_\alpha$

Although the subject of this paper is rank-to-rank embeddings (i.e., elements of $\mathscr{E}(V_\alpha)$ for some ordinal α), it is often convenient to lift these embeddings to act on larger structures. The issue is that many sets are coded in V_α but do not belong to V_α. This is especially annoying when α is a successor ordinal, in which case V_α fails to be closed under Kuratowski pairs. This motivates introducing the following structures:

Definition 2.8. For any set X, let $\mathcal{H}(X)$ denote the union of all transitive sets M such that $M \preceq^* S$ for some $S \in X$. For any ordinal α, let $\mathcal{H}_\alpha = \mathcal{H}(V_\alpha)$.

If κ is a (wellordered) cardinal, then $\mathcal{H}(\kappa)$ is the usual structure $H(\kappa)$. In ZF, however, there may be other structures of the form $\mathcal{H}(X)$. Notice that $\mathcal{H}_{\alpha+1}$ is the collection of sets that are the surjective image of V_α.

Definition 2.9. For any set X, let $\theta(X)$ denote the least ordinal that is not the surjective image of some set $S \in X$. Let $\theta_\alpha = \theta(V_\alpha)$.

Then $\theta(X) = \mathcal{H}(X) \cap \mathrm{Ord}$. The cardinals θ_α are studied in Section 4.

Note that for all α, $V_\alpha \subseteq \mathcal{H}_\alpha$. We claim that every embedding in $\mathscr{E}(V_\alpha)$ extends uniquely to an embedding in $\mathscr{E}(\mathcal{H}_\alpha)$. This is a consequence of a coding of $\mathcal{H}_\alpha$ inside the structure V_α. We proceed to describe one such coding, and then we sketch how this yields the unique extension of embeddings from $\mathscr{E}(V_\alpha)$ to $\mathscr{E}(\mathcal{H}_\alpha)$.

Let $p : V \to V \times V$ denote some *Quine-Rosser pairing function*, which is a bijection that is Σ_0-definable without parameters such that for all infinite ordinals α, $p[V_\alpha] = V_\alpha \times V_\alpha$.

Fix an infinite ordinal α. We will define a partial surjection

$$\Phi_\alpha : V_\alpha \to \mathcal{H}_\alpha$$

For each $x \in V_\alpha$, let $R_x = p[x]$ be the binary relation coded by x, and let D_x denote the field of R_x. Note that $D_x \in V_\alpha$. Let E_α be the set of $x \in V_\alpha$ such that R_x is a wellfounded relation on D_x. For $x \in \mathrm{E}_\alpha$, let $\pi_x : D_x \to M_x$ be the Mostowski collapse. Let

$$\mathrm{C}_\alpha = \{(x, y) \mid x \in \mathrm{E}_\alpha \text{ and } y \in D_x\}$$

For $(x, y) \in \mathrm{C}_\alpha$, let $\Phi_\alpha(x, y) = \pi_x(y)$. Then $\mathrm{ran}(\Phi_\alpha) = \bigcup_{x \in V_\alpha} M_x = \mathcal{H}_\alpha$.

Since Φ_α is definable over $\mathcal{H}_\alpha$, one has that for any $i : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$, $i(\Phi_\alpha(x, y)) = \Phi_\alpha(i(x), i(y))$. Conversely, since the sets C_α and $\Phi_\alpha^{-1}[\in] = \{(u, w) \in \mathrm{C}_\alpha \times \mathrm{C}_\alpha : \Phi_\alpha(u) \in \Phi_\alpha(w)\}$, and $\Phi_\alpha^{-1}[=]$ are definable over V_α, one has that if $j : V_\alpha \to V_\alpha$ is elementary, then setting

$$j^\star(\Phi_\alpha(x, y)) = \Phi_\alpha(j(x), j(y)),$$

the embedding $j^\star : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ is well-defined and elementary.

Definition 2.10. For any $j \in \mathscr{E}(V_\alpha)$, $j^\star$ denotes the unique elementary embedding $k : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ such that $k \restriction V_\alpha = j$.

The rest of the section contains an analysis of $j^\star$ when $j : V_\alpha \to V_\alpha$ is only assumed to be Σ_n-elementary for some $n < \omega$. This is rarely relevant, and the reader can skip it for now.

We claim that if $i : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ is a Σ_0-elementary embedding, then $i(\Phi_\alpha(x,y)) = \Phi_\alpha(i(x), i(y))$ for all $(x,y) \in \mathrm{C}_\alpha$. To see this, fix $(x,y) \in \mathrm{C}_\alpha$. Let $a = \Phi_\alpha(x,y)$. Let M be an admissible set in $\mathcal{H}_\alpha$ such that $x, y \in M$. Then M satisfies the statement "$a = \pi_x(y)$ where π_x is the Mostowski collapse of (D_x, R_x)." Since i is Σ_0-elementary, it follows that $i(M)$ is an admissible set that satisfies "$i(a) = \pi_{i(x)}(i(y))$." This is expressible as a Σ_1 statement, so is upwards absolute to V. Therefore $i(a) = \pi_{i(x)}(i(y))$, or in other words, $i(\Phi_\alpha(x,y)) = \Phi_\alpha(i(x), i(y))$, as desired.

Conversely, suppose $j : V_\alpha \to V_\alpha$ is Σ_1-elementary, and let $j^\star : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ be defined by $j^\star(\Phi_\alpha(x,y)) = \Phi_\alpha(j(x), j(y))$. We claim $j^\star : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ is well-defined and Σ_0-elementary. Note that C_α, $\Phi_\alpha^{-1}[=]$, and $\Phi_\alpha^{-1}[\in]$ are Π_1-definable over V_α. This implies that $j^\star$ is a well-defined $\in$-homomorphism. If $M \in \mathcal{H}_\alpha$ is a transitive set, then taking $x \in \mathrm{E}_\alpha$ such that $M = M_x$, the satisfaction predicate for $(D_x, R_x) \cong M$ is Δ_1-definable over V_α from x, and hence $j \restriction M : M \to j(M)$ is fully elementary. Since every $x \in \mathcal{H}_\alpha$ belongs to some transitive $M \in \mathcal{H}_\alpha$, and such a set M is a Σ_0-elementary substructure of $\mathcal{H}_\alpha$, it follows that j is Σ_0-elementary.

Definition 2.11. Suppose $i : V_\alpha \to V_\alpha$ is a Σ_1-elementary embedding. Then $i^\star$ denotes the unique Σ_0-elementary embedding from $\mathcal{H}_\alpha$ to $\mathcal{H}_\alpha$ extending i.

By a localization of the arguments above, one obtains the following fact about partially elementary $\star$-extensions:

Lemma 2.12. *Suppose $n < \omega$ and $i : V_\alpha \to V_\alpha$ is a Σ_{n+1}-elementary embedding. Then $i^\star : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ is Σ_n-elementary.*

Sketch. For example, take the case $n = 1$. Suppose ψ is a Σ_0-formula, $a \in \mathcal{H}_\alpha$ and $\mathcal{H}_\alpha$ satisfies $\exists v\ \psi(j(a), v)$. Fix $(x_0, y_0) \in \mathrm{C}_\alpha$ with $\Phi_\alpha(x_0, y_0) = a$. Then V_α satisfies that there is some $(x_1, y_1) \in \mathrm{C}_\alpha$ such that (D_{x_1}, R_{x_1}) is an end-extension of $(D_{j(x_0)}, R_{j(x_0)})$ and (D_{x_1}, R_{x_1}) satisfies $\psi(j(y_0), y_1)$. This is Σ_2-expressible in V_α, so by elementarity, there is some $(x_1, y_1) \in \mathrm{C}_\alpha$ such that (D_{x_1}, R_{x_1}) is an end-extension of (D_{x_0}, R_{x_0}) and (D_{x_1}, R_{x_1}) satisfies $\psi(y_0, y_1)$. It follows that M_{x_1} satisfies $\psi(a, \Phi_\alpha(x_1, y_1))$, and hence $\mathcal{H}_\alpha$ satisfies $\exists v\ \psi(a, v)$, as desired. $\square$

3 The definability of rank-to-rank embeddings

3.1 Prior work

The results of this section are inspired by the work of Schlutzenberg [9], which greatly expands upon the following theorem of Suzuki:[4]

[4]Suzuki actually proved the slightly stronger schema that no elementary embedding from V to V is definable from parameters over V.

Theorem 3.1 (Suzuki)**.** *If κ is an inaccessible cardinal, no nontrivial elementary embedding from V_κ to V_κ is definable over V_κ from parameters.*

Schlutzenberg [9] extended this to limit ranks:[5]

Theorem 3.2 (Schlutzenberg)**.** *Suppose λ is a limit ordinal. Then no nontrivial elementary embedding from V_λ to V_λ is definable over V_λ from parameters.*

Schlutzenberg noted that the situation for elementary embeddings from $V_{\lambda+1}$ to $V_{\lambda+1}$, where λ is a limit ordinal, is completely different: *every* elementary embedding from $V_{\lambda+1}$ to $V_{\lambda+1}$ is definable from parameters over $V_{\lambda+1}$. He then asked the corresponding question for elementary embeddings of $V_{\lambda+n}$ for $n > 1$. The answer is given by the following theorem, which is established by the main results of this section:

Theorem 3.3. *Suppose ϵ is an even ordinal.*[6]

(1) No nontrivial elementary embedding from V_ϵ to V_ϵ is definable over V_ϵ from parameters.

(2) Every elementary embedding from $V_{\epsilon+1}$ to $V_{\epsilon+1}$ is definable over $V_{\epsilon+1}$ from parameters.

This theorem is the first instance of a periodicity phenomenon in the hierarchy of choiceless large cardinal axioms, leading to a generalization to arbitrary ranks of the basic theory of rank-to-rank embeddings familiar from the ZFC context. (1) is proved as Proposition 3.20 of Section 3.3 and (2) as Theorem 3.13 of Section 3.2.

We note that Schlutzenberg rediscovered Theorem 3.3, and this theorem is the main subject of the joint paper [10].

3.2 Extending embeddings to $V_{\epsilon+1}$

That elementary embeddings of odd ranks are definable (Theorem 3.3 (2)) came as quite a surprise to the author, but with hindsight emerges as a natural generalization a well-known phenomenon from the standard theory of rank-to-rank embeddings.

Definition 3.4. Suppose λ is a limit ordinal and $j : V_\lambda \to V_\lambda$ is an elementary embedding. Then the *canonical extension of* j is the embedding $j^+ : V_{\lambda+1} \to V_{\lambda+1}$ defined by $j^+(X) = \bigcup_{\Gamma \in V_\lambda} j(X \cap \Gamma)$.

While the canonical extension of an embedding in $\mathscr{E}(V_\lambda)$ is not necessarily an elementary embedding, it is true that *if* $i \in \mathscr{E}(V_{\lambda+1})$, then necessarily $i = (i \restriction V_\lambda)^+$. The proof is easy, but since it is relevant below, we give a detailed sketch. Fix $X \in V_{\lambda+1}$. Clearly $i(X \cap \Gamma) \subseteq i(X)$ for all $\Gamma \in V_\lambda$, and this easily implies the inclusion $(i \restriction V_\lambda)^+(X) \subseteq i(X)$. For the reverse inclusion, suppose $a \in i(X)$. Since

[5] Schlutzenberg also proved many other definability results for rank-to-rank embeddings, incorporating, for example, constructibility and ordinal definability.

[6] An ordinal α is said to be *even* if for some limit ordinal λ and some natural number n, $\alpha = \lambda + 2n$; otherwise, α is *odd*.

λ is a limit ordinal, there is some $\Gamma \in V_\lambda$ such that $a \in i(\Gamma)$; for example, one can take $\Gamma = V_{\xi+1}$ where $\xi = \text{rank}(a)$. Now $a \in i(X) \cap i(\Gamma) = i(X \cap \Gamma)$, so $a \in (i \restriction V_\lambda)^+(X)$. The key property of λ that was used in this proof is that any $i : V_\lambda \to V_\lambda$ is a *cofinal embedding* in the sense that for all $a \in V_\lambda$, there is some $\Gamma \in V_\lambda$ with $a \in i(\Gamma)$.

Suppose now that α is an arbitrary infinite ordinal. We want to generalize the canonical extension operation to act on embeddings $j \in \mathscr{E}(V_\alpha)$. It is easy to see that if α is a successor ordinal, the naive generalization (i.e., $j^+(X) = \bigcup_{\Gamma \in V_\alpha} j(X \cap \Gamma)$ for $X \in V_{\alpha+1}$) does not have the desired effect. (For example, adopting this definition, one would have $j^+(\{V_{\alpha-1}\}) = \emptyset$.) Instead, one must make the following tweak:

Definition 3.5. Suppose α is an infinite ordinal and $j : V_\alpha \to V_\alpha$ is an elementary embedding. Then the *canonical extension of* j is the embedding $j^+ : V_{\alpha+1} \to V_{\alpha+1}$ defined by $j^+(X) = \bigcup_{\Gamma \in \mathcal{H}_\alpha} j^\star(X \cap \Gamma)$.

See Section 2.4 for the definition of the structure $\mathcal{H}_\alpha$, and the basic facts about lifting elementary embeddings from V_α to $\mathcal{H}_\alpha$. Thus $j^+(X)$ is the union of all sets of the form $j^\star(X \cap \Gamma)$ where Γ is *coded* in V_α. The following easily verified lemma clarifies the definition:

Proposition 3.6. *Suppose α is an infinite ordinal and $j \in \mathscr{E}(V_\alpha)$. Then*

$$j^+(X) = \begin{cases} \bigcup_{\Gamma \in V_\alpha} j(X \cap \Gamma) & \text{if } \alpha \text{ is a limit ordinal} \\ \bigcup_{\Gamma \in [V_\alpha]^{V_{\alpha-1}}} j^\star(X \cap \Gamma) & \text{if } \alpha \text{ is a successor ordinal} \end{cases} \qquad \square$$

While the definition of the canonical extension operation directly generalizes Definition 3.4, a key new phenomenon arises at successor ranks: it is no longer clear that every elementary embedding $i : V_{\alpha+1} \to V_{\alpha+1}$ satisfies

$$i = (i \restriction V_\alpha)^+$$

It is easy to show that for all $X \in V_{\alpha+1}$, $(i \restriction V_\alpha)^+(X) \subseteq i(X)$, but the reverse inclusion is no longer clear.

In fact, the reverse inclusion is only true for even values of α. This is proved by an induction that simultaneously establishes the *canonical extension property* and the *cofinal embedding property*, which we now define.

Definition 3.7. Suppose ϵ is an ordinal. Then ϵ has the *canonical extension property* if for any $i \in \mathscr{E}(V_{\epsilon+1})$, $i = (i \restriction V_\epsilon)^+$.

The terminology is motivated by equivalence of the canonical extension property with the statement that an elementary embedding in $\mathscr{E}(V_\epsilon)$ extends to at most one embedding in $\mathscr{E}(V_{\epsilon+1})$.

Definition 3.8. Suppose ϵ is an ordinal. Then ϵ has the *cofinal embedding property* if for any $i \in \mathscr{E}(V_\epsilon)$, for any $A \in \mathcal{H}_\epsilon$, there is some $\Gamma \in \mathcal{H}_\epsilon$ with $A \in i^\star(\Gamma)$.

Thus the cofinal embedding property states that every embedding in $\mathscr{E}(V_\epsilon)$ induces a cofinal embedding in $\mathscr{E}(\mathcal{H}_\epsilon)$. The proof that limit ordinals have the canonical extension property generalizes to all ordinals with the cofinal embedding property:

Lemma 3.9. *Suppose ϵ is an ordinal. If ϵ has the cofinal embedding property, then ϵ has the canonical extension property.*

Proof. Fix an elementary embedding $i : V_{\epsilon+1} \to V_{\epsilon+1}$. We must show $i = (i \restriction V_\epsilon)^+$. Take $X \in V_{\epsilon+1}$. The inclusion $(i \restriction V_\epsilon)^+(X) \subseteq i(X)$ is true regardless of parity: if $\Gamma \in \mathcal{H}_\epsilon$, then $i^\star(X \cap \Gamma) = i(X \cap \Gamma) \subseteq i(X)$ by the elementarity of i and the uniqueness of $i^\star$, so $i^+(X) = \bigcup_{\Gamma \in \mathcal{H}_\alpha} i^\star(X \cap \Gamma) \subseteq i(X)$.

To show $i(X) \subseteq (i \restriction V_\epsilon)^+(X)$, suppose $A \in i(X)$. Since ϵ has the cofinal embedding property, there is some $\Gamma \in \mathcal{H}_\epsilon$ such that $A \in i^\star(\Gamma)$. Now

$$A \in i(X) \cap i^\star(\Gamma) = i(X) \cap i(\Gamma) = i(X \cap \Gamma) = i^\star(X \cap \Gamma) \subseteq i^+(X)$$

This completes the proof. □

The periodicity phenomenon is a result of the following lemma:

Lemma 3.10. *Suppose ϵ is an ordinal. If ϵ has the canonical extension property, then $\epsilon + 2$ has the cofinal embedding property.*

Proof. Fix $i \in \mathscr{E}(V_{\epsilon+2})$ and $A \in V_{\epsilon+2}$. Let

$$\Gamma = \{(k^+)^{-1}[A] \mid k \in \mathscr{E}(V_\epsilon)\}$$

Then $\Gamma \preceq^* \mathscr{E}(V_\epsilon) \preceq^* V_{\epsilon+1}$. Since $\Gamma \cup V_{\epsilon+1}$ is transitive, it follows that $\Gamma \in \mathcal{H}_{\epsilon+1}$. □

This allows us to prove the cofinal embedding property and the canonical extension property for even ordinals by induction:

Corollary 3.11. *Every even ordinal has the cofinal embedding property.*

Proof. Suppose λ is a limit ordinal. We show that $\lambda + 2n$ has the cofinal embedding property by induction on $n < \omega$.

We first prove the base case, when $n = 0$. Suppose $A \in \mathcal{H}_\lambda$. Fix $\xi < \lambda$ such that $A \in \mathcal{H}_\xi$. Then we have $\mathcal{H}_\xi \in \mathcal{H}_\lambda$ and $A \in j^\star(\mathcal{H}_\xi)$ since $\mathcal{H}_\xi \subseteq \mathcal{H}_{j(\xi)} = j^\star(\mathcal{H}_\xi)$. This shows that λ has the cofinal embedding property.

For the induction step, assume that $\lambda + 2n$ has the cofinal embedding property. Then by Lemma 3.10, $\lambda + 2n$ has the canonical extension property, and so by Lemma 3.9, $\lambda + 2n + 2$ has the cofinal embedding property. □

Corollary 3.12. *Every even ordinal has the canonical extension property.* □

As an immediate consequence, we have Theorem 3.3 (2):

Theorem 3.13. *Suppose ϵ is an infinite even ordinal and $i : V_{\epsilon+1} \to V_{\epsilon+1}$ is an elementary embedding. Then i is definable over $V_{\epsilon+1}$ from $i[V_\epsilon]$.*

Proof. Clearly i is definable over $\mathcal{H}_{\epsilon+1}$ from $i \restriction V_\epsilon$ since $i = (i \restriction V_\epsilon)^+$ and the canonical extension operation is explicitly defined over $\mathcal{H}_{\epsilon+1}$. Moreover, $i \restriction V_\epsilon$ is definable over $V_{\epsilon+1}$ from $i[V_\epsilon]$ as the inverse of the Mostowski collapse. It follows that i is definable over $\mathcal{H}_{\epsilon+1}$ from $i[V_\epsilon]$. But by coding elements of $\mathcal{H}_{\epsilon+1}$ as elements of $V_{\epsilon+1}$ as in Section 2, one can translate this into a definition of i over $V_{\epsilon+1}$ from $i[V_\epsilon]$. □

Let us put down for safe-keeping the following version of the cofinal embedding property that is often useful:

Definition 3.14. Suppose σ is a set such that $(\sigma, \in)$ is wellfounded and extensional. Then $j_\sigma : M_\sigma \to \sigma$ denotes the inverse of the Mostowski collapse of σ.

Suppose ϵ is an even ordinal. For any $A \in V_{\epsilon+2}$, let $f_A : V_{\epsilon+1} \to V_{\epsilon+2}$ be the partial function defined by $f_A(\sigma) = (j_\sigma^+)^{-1}[A]$.

We leave $f_A(\sigma)$ undefined if one of the following holds:

- $(\sigma, \in)$ is not wellfounded and extensional.
- j_σ is not an elementary embedding from V_ϵ to V_ϵ.

Proposition 3.15. *Suppose $j : V_{\epsilon+2} \to V_{\epsilon+2}$ is an elementary embedding. Then for any $A \in V_{\epsilon+2}$, $A = j^\star(f_A)(j[V_\epsilon])$.*

Proof. Since f_A is definable from A over $\mathcal{H}_{\epsilon+2}$, $j^\star(f_A)(j[V_\epsilon]) = f_{j(A)}(j[V_\epsilon])$. Note that $M_{j[V_\epsilon]} = V_\epsilon$ and $j_{j[V_\epsilon]} = j \restriction V_\epsilon$. Therefore by the elementarity of $j^\star$,

$$f_{j(A)}(j[V_\epsilon]) = ((j \restriction V_\epsilon)^+)^{-1}[j(A)] = (j \restriction V_{\epsilon+1})^{-1}[j(A)] = A$$ □

Our original approach to Theorem 3.13 diverged from the one presented here in that we used a superficially different definition of the canonical extension operation from Definition 3.5. This approach is not as clearly motivated by the canonical extension operation for embeddings from V_λ to V_λ where λ is a limit ordinal (Definition 3.4), but it might illuminate the underlying combinatorics. This construction is described in more detail in [10].

Suppose $j : V_{\epsilon+2} \to V_{\epsilon+2}$ is elementary, and let $\mathcal{U}$ be the ultrafilter derived from j using $j[V_\epsilon]$. Let $j_\mathcal{U} : V \to M_\mathcal{U}$ denote the ultrapower associated to $\mathcal{U}$. It is easy to see that $\text{Ult}(V_{\epsilon+1}, \mathcal{U}) \cong V_{\epsilon+1}$. Assume ϵ has the cofinal embedding property. Then moreover $\text{Ult}(V_{\epsilon+2}, \mathcal{U}) \cong V_{\epsilon+2}$. Therefore we identify $\text{Ult}(V_{\epsilon+2}, \mathcal{U})$ with $V_{\epsilon+2}$. As a consequence, for every $X \in V_{\epsilon+3}$, we can identify $j_\mathcal{U}(X)$ with a subset of $V_{\epsilon+2}$; that is, we identify $j_\mathcal{U}(X)$ with an element of $V_{\epsilon+3}$. Given the cofinal embedding property, it is not hard to show that $j_\mathcal{U} \restriction V_{\epsilon+3}$ is the only possible extension of j to an elementary embedding of $V_{\epsilon+3}$. Therefore one can set $j^+ = j_\mathcal{U}$ instead of using Definition 3.5, and then prove Corollary 3.11 and Corollary 3.12 by very similar arguments to the ones given above.

Obviously the two approaches to the canonical extension operation are very similar, but we just want to highlight that the canonical extension operation, like everything else in set theory, is really an ultrapower construction.

3.3 Undefinability over V_ϵ

In this section, we establish Theorem 3.3 (1). At this point, we have found three proofs, increasing chronologically in complexity and generality.

We begin by giving a sketch of the simplest of these proofs in the successor ordinal case. (Note that the limit case is handled by Schlutzenberg's Theorem 3.2.) Suppose ϵ is an even ordinal and $j : V_{\epsilon+2} \to V_{\epsilon+2}$ is a nontrivial elementary embedding. Let $\mathcal{U}$ be the ultrafilter over $V_{\epsilon+1}$ derived from j using $j[V_\epsilon]$. It turns out that if j is definable over $V_{\epsilon+2}$, then $\mathcal{U}$ belongs to the ultrapower of V by $\mathcal{U}$. A fundamental fact from the ZFC theory of large cardinals, proved for example in [11], is that no countably complete ultrafilter belongs to its own ultrapower. The idea of the proof of Proposition 3.20 is to try to push this through to the current context.

Recall that the *Mitchell order* is defined on countably complete ultrafilters U and W by setting $U \lhd W$ if $U \in \mathrm{Ult}(V, W)$. The "fundamental fact" mentioned above simply states that assuming the Axiom of Choice, the Mitchell order is irreflexive. (In fact, it is wellfounded.) In the context of ZF, especially given the failure of Łoś's Theorem, the Mitchell order is fairly intractable, and in particular, we do not know how to prove its irreflexivity. Instead, we use a variant of the Mitchell order called the internal relation (introduced in the author's thesis [1]) that is more amenable to combinatorial arguments.

Definition 3.16. Suppose U and W are countably complete ultrafilters over sets X and Y. We say U is *internal to* W, and write $U \sqsubset W$, if there is a sequence of countably complete ultrafilters $\langle U_y \mid y \in Y \rangle$ such that for any relation $R \subseteq X \times Y$:[7]

$$\forall^U x\, \forall^W y\ R(x, y) \iff \forall^W y\, \forall^{U_y} x\ R(x, y) \tag{1}$$

Using notation that is standard in ultrafilter theory, (1) states that $U \times W$ is canonically isomorphic to $W\text{-}\sum_{y \in Y} U_y$.

Of course, from this combinatorial definition, it is not clear that the internal relation is related to the Mitchell order at all. Using Łoś's Theorem, however, the following is easy to verify:

Definition 3.17. If U and W are ultrafilters, then the *pushforward of* U *to* M_W is the M_W-ultrafilter $s_W(U) = \{A \in j_W(P(X)) \mid (j_W)^{-1}[A] \in U\}$.

Proposition 3.18 (ZFC)**.** *Suppose U and W are countably complete ultrafilters. Let X be the underlying set of U. Then the following are equivalent:*

(1) $U \sqsubset W$.

(2) $s_W(U) \in M_W$.

(3) $j_U \restriction \mathrm{Ult}(V, W)$ *is definable from parameters over* $\mathrm{Ult}(V, W)$. □

(3) is not the right perspective when Łoś's Theorem does not hold for W. The equivalence of (1) and (2), however, is essentially a consequence of ZF:

[7] For any predicate P, we write "$\forall^U x\ P(x)$" to mean that $\{x \in X : P(x)\} \in U$.

Lemma 3.19. *Suppose U and W are countably complete ultrafilters over X and Y. Then $U \sqsubset W$ if and only if there is a sequence of countably complete ultrafilters $\langle U_y \rangle_{y\in Y}$ such that $[\langle U_y \rangle_{y\in Y}]_W = s_W(U)$.*

Proof. One shows that $[\langle U_y \rangle_{y\in Y}]_W = s_W(U)$ if and only if the equivalence (1) from Definition 3.16 holds.

For the forwards direction, assume $[\langle U_y \rangle_{y\in Y}]_W = s_W(U)$. Fix $R \subseteq X \times Y$. We verify the equivalence (1) from Definition 3.16.

Suppose that $\forall^W y\ \forall^{U_y} x\ R(x,y)$. Then $R^y \in U_y$ for W-almost all $y \in Y$.[8] By assumption, this means $[\langle R^y \rangle_{y\in Y}]_W \in s_W(U)$, so by the definition of $s_W(U)$, $(j_W)^{-1}([\langle R^y \rangle_{y\in Y}]_W) \in U$. In other words, $\{x \in X \mid \forall^W y\ x \in R^y\} \in U$, which means that $\forall^U x\ \forall^W y\ R(x,y)$.

It follows immediately that $\forall^U x\ \forall^W y\ R(x,y)$ implies $\forall^W y\ \forall^{U_y} x\ R(x,y)$: notice that $Z = \{R \subseteq X \times Y : \forall^U x\ \forall^W y\ R(x,y)\}$ and $Z' = \{R \subseteq X \times Y : \forall^W y\ \forall^{U_y} x\ R(x,y)\}$ are both ultrafilters, so since the previous paragraph shows $Z' \subseteq Z$, in fact, $Z = Z'$.

We omit the proof that (1) from Definition 3.16 implies $[\langle U_y \rangle_{y\in Y}]_W = s_W(U)$, since the proof is straightforward and the result is never actually cited. $\square$

The key advantage of the internal relation over the Mitchell order is that its irreflexivity can be proved in ZF by a combinatorial argument that will be given in the appendix:

Theorem 7.15. *Suppose U is a countably complete ultrafilter and $\operatorname{crit}(j_U)$ exists. Then $U \not\sqsubset U$.*

With this in hand, we can prove the undefinability theorem.

Proposition 3.20. *Suppose ϵ is an even ordinal and $j : V_{\epsilon+2} \to V_{\epsilon+2}$ is a nontrivial elementary embedding. Then j is not definable from parameters over $V_{\epsilon+2}$.*

Proof. Assume towards a contradiction that j is definable over $V_{\epsilon+2}$ from parameters. Let $\mathcal{U}$ be the ultrafilter over $V_{\epsilon+1}$ derived from j using $j[V_\epsilon]$. Clearly $\mathcal{U}$ is definable over $V_{\epsilon+2}$ from j, and hence $\mathcal{U}$ is definable over $V_{\epsilon+2}$ from parameters.

Let $k : \operatorname{Ult}(V_{\epsilon+2}, \mathcal{U}) \to V_{\epsilon+2}$ be the canonical factor embedding defined by $k([f]_\mathcal{U}) = j^\star(f)(j[V_\epsilon])$. The elementarity of $j^\star$ implies that k is a well-defined injective homomorphism of structures. Moreover, Proposition 3.15 implies that k is surjective. So $\operatorname{Ult}(V_{\epsilon+2}, \mathcal{U})$ is canonically isomorphic to $V_{\epsilon+2}$, and we will identify the two structures.

Fix a formula $\varphi(v, w)$ and a parameter $B \in V_{\epsilon+2}$ such that

$$\mathcal{U} = \{A \in V_{\epsilon+2} \mid V_{\epsilon+2} \vDash \varphi(A, B)\}$$

Let $\mathcal{U}_\sigma = \{A \in V_{\epsilon+2} \mid V_{\epsilon+2} \vDash \varphi(f_A(\sigma), f_B(\sigma))\}$ where f_A and f_B are as defined in Definition 3.14.

[8] For $y \in Y$, R^y denotes the set $\{x \in X \mid (x, y) \in R\}$.

We claim that $[\langle \mathcal{U}_\sigma \mid \sigma \in V_{\epsilon+1}\rangle]_\mathcal{U} = s_\mathcal{U}(\mathcal{U})$. This follows from the elementarity of j, which implies

$$\begin{aligned}[\langle \mathcal{U}_\sigma \mid \sigma \in V_{\epsilon+1}\rangle]_\mathcal{U} &= \{A \in V_{\epsilon+2} \mid V_{\epsilon+2} \vDash \varphi(f_A(j[V_\lambda]), B)\}\\ &= \{A \subseteq V_{\epsilon+1} \mid j^{-1}[A] \in \mathcal{U}\}\\ &= s_\mathcal{U}(\mathcal{U})\end{aligned}$$

It is also easy to show that $\mathcal{U}_\sigma$ is a countably complete ultrafilter for $\mathcal{U}$-almost all $\sigma \in V_{\epsilon+1}$. By Lemma 3.19, $\langle \mathcal{U}_\sigma \mid \sigma \in V_{\epsilon+1}\rangle$ witnesses $\mathcal{U} \sqsubset \mathcal{U}$. Since $j_\mathcal{U} \restriction \epsilon = j \restriction \epsilon$, $j_\mathcal{U}$ has a critical point. This contradicts Corollary 7.15. $\square$

3.4 Supercompactness properties of ultrapowers

The proof of Proposition 3.20 raises a number of questions that also arise naturally in the study of choiceless cardinals. The one we will focus on concerns the supercompactness properties of ultrapowers. Suppose ϵ is an even ordinal, $j : V_{\epsilon+2} \to V_{\epsilon+2}$ is an elementary embedding, and $\mathcal{U}$ is the ultrafilter over $V_{\epsilon+1}$ derived from j using $j[V_\epsilon]$. Motivated by the proof of Corollary 3.12, one might ask when $j_\mathcal{U}[S] \in M_\mathcal{U}$ for various sets S. This is related to the definability of elementary embeddings because if $S \in V_\beta$ is transitive and j extends to an elementary embedding $i : V_\beta \to N$, then $j_\mathcal{U}[S] \in M_\mathcal{U}$ if and only if $i \restriction S$ is definable over N from parameters in $i[V_\beta] \cup V_{\epsilon+2}$.

Of course, by Corollary 3.12, $j_\mathcal{U}[V_{\epsilon+1}]$ belongs to $M_\mathcal{U}$. It follows easily that $j_\mathcal{U}[S] \in M_\mathcal{U}$ for all $S \preceq^* V_{\epsilon+1}$. (See Definition 2.4 for this notation.) The converse remains open: if $j_\mathcal{U}[S] \in M_\mathcal{U}$, must $S \preceq^* V_{\epsilon+1}$? We still do not know the answer to this question, even in the following special case: with $\mathcal{U}$ as above, it is not hard to show that $j_\mathcal{U}[P(\epsilon^+)]$ belongs to $M_\mathcal{U}$, yet it is far from clear whether one can prove $P(\epsilon^+) \preceq^* V_{\epsilon+1}$ without making further assumptions. Of course, assuming the Axiom of Choice, if U is an ultrafilter over X, S is a set, and $j_U[S] \in M_U$, then $|S| \leq |X|$. (See [1, Proposition 4.2.31].) We are simply asking whether a very special case of this fact can be proved in ZF.

We begin by proving a general theorem that subsumes the ZFC result mentioned above:

Theorem 3.21. *Suppose X is a set such that $X \times X \preceq^* X$. Suppose U is an ultrafilter over X, $\kappa = \text{crit}(j_U)$, and $\alpha \geq \kappa$ is an ordinal. If $j_U[\alpha] \in M_U$, then $\alpha \preceq^* X$.*

Note that we implicitly assume that U has a critical point.

Definition 3.22. Suppose U is an ultrafilter over a set X. A sequence $\langle S_x \mid x \in X\rangle$ is an *A-supercompactness sequence for U* if it has the following properties:

- For all $x \in X$, $S_x \subseteq A$.
- Every $a \in A$ belongs to S_x for U-almost all $x \in X$.

- For any X-indexed sequence $\langle a_x \mid x \in X\rangle$ with $a_x \in S_x$ for U-almost all $x \in X$, there is some $a \in A$ with $a_x = a$ for U-almost all $x \in X$.[9]

Supercompact ultrafilters are the combinatorial manifestation of supercompact embeddings:

Lemma 3.23. *$\langle S_x \mid x \in X\rangle$ is an A-supercompactness sequence for U if and only if $[\langle S_x \mid x \in X\rangle]_U = j_U[A]$.* □

The first step of Theorem 3.21 is the following well-known fact:

Lemma 3.24. *Suppose U is an ultrafilter over a set X and $\kappa = \text{crit}(j_U)$. Then there is a surjection from X to κ.*

Proof. Since U is κ-complete but not κ^+-complete, there is a strictly decreasing sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$ of sets $A_\alpha \in U$ such that $\bigcap_{\alpha<\kappa} A_\alpha = \emptyset$. Let $f : X \to \kappa$ be defined by

$$f(x) = \min\{\alpha \mid x \notin A_{\alpha+1}\}$$

Since the sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$ is strictly decreasing, f is a surjection. □

The second step of Theorem 3.21 is more involved:

Lemma 3.25. *Suppose U is an ultrafilter over a set X, $\kappa = \text{crit}(j_U)$, $\gamma \geq \kappa$ is an ordinal, and $\langle S_x \mid x \in X\rangle$ is a γ-supercompactness sequence for U. Then for any surjection p from X to κ, there is a cofinal function from X to γ that is ordinal definable from $\langle S_x \mid x \in X\rangle$ and p.*

In fact, the function produced by Lemma 3.25 will be Σ_1-definable from $\langle S_x\rangle_{x\in X}$ and p, but ordinal definability will suffice for our applications.

Proof. There are two cases.

Case 1. For U-almost all $x \in X$, $\sup S_x < \gamma$.

It does no harm to assume that $\sup S_x < \gamma$ for all $x \in X$. This is because the sequence obtained from $\langle S_x \mid x \in X\rangle$ by replacing S_x with the empty set whenever $\sup S_x \geq \gamma$ is a supercompactness sequence (and is ordinal definable from $\langle S_x \mid x \in X\rangle$).

Let $f : X \to \gamma$ be the function defined by

$$f(x) = \sup(S_x \cap \gamma)$$

Then f is cofinal in γ, which proves Lemma 3.25 in Case 1. To see that f is cofinal, fix an ordinal $\alpha < \gamma$. The definition of a supercompactness sequence implies that the set $B_\alpha = \{x \in X \mid \alpha \in S_x\}$ belongs to U, so we may fix an $x \in X$ such that $\alpha \in S_x$. Then $f(x) = \sup S_x > \alpha$. Thus f is cofinal in γ.

[9] In the case that A is not wellorderable, the right concept seems to be that of a *normal* A-supercompactness sequence, which has the stronger property that for any X-indexed sequence $\langle B_x \mid x \in X\rangle$ with $\emptyset \neq B_x \subseteq S_x$ for U-almost all $x \in X$, there is some $a \in A$ with $a \in B_x$ for U-almost all $x \in X$.

Case 2. For U-almost all $x \in X$, $\sup S_x = \gamma$.

As in Case 1, it does no harm to assume $\sup S_x = \gamma$ for all $x \in X$.

The first step is to show that the sets S_x cannot have a common limit point of uniform cofinality κ in the following sense:

Claim. *Suppose $\nu < \gamma$. There is no sequence of sets $\langle E_x \mid x \in X\rangle$ such that for all $x \in X$, $E_x \subseteq S_x \cap \nu$, E_x has ordertype κ, and E_x is cofinal in ν.*

Proof. Fix one last cofinal set $E \subseteq \nu$ of ordertype κ. Let

$$M_x = L[S_x, E_x, E]$$

Since there is a definable sequence $\langle <_x \mid x \in X\rangle$ such that $<_x$ is a wellorder of M_x, Łoś's Theorem holds for the ultraproduct $M = \prod_{x\in X} M_x/U$. Thus M is a proper class model of ZFC, although it may be that M is illfounded. That being said, $\gamma+1$ is contained in the wellfounded part of M. (As usual, the wellfounded part of M is taken to be transitive.) Indeed, $[\langle S_x \mid x \in X\rangle]_U = j_U[\gamma]$ by Lemma 3.23, so M contains a wellorder of ordertype γ and hence is wellfounded up to $\gamma + 1$.

The purpose of including the set E in each M_x is to ensure that M correctly computes the cofinality of $\sup j_U[\nu]$. The argument is standard, at least in the context of the Axiom of Choice. Since $E \in M_x$ for every $x \in X$, $j_U[E] = j_U(E) \cap j_U[\gamma]$ belongs to M. Since $\text{ot}(E) = \kappa$, $\text{ot}(j_U[E]) = \kappa$. Therefore M satisfies that $\sup j_U[E]$ has cofinality κ. Since E is a cofinal subset of ν, $\sup j_U[E] = \sup j_U[\nu]$. Thus:

$$\text{cf}^M(\sup j_U[\nu]) = \kappa \tag{2}$$

Let $E_* = [\langle E_x \mid x \in X\rangle]_U$. Then Łoś's Theorem implies that the following hold in M:

- $E_* \subseteq j_U[\nu]$.
- E_* has ordertype $j_U(\kappa)$.
- E_* is cofinal in $j_U(\nu)$.

The first bullet point uses that $j_U[\gamma] \cap j_U(\nu) = j_U[\nu]$.

Since $E_* \subseteq j_U[\nu]$, $j_U[\nu]$ is cofinal in $j_U(\nu)$, and hence

$$\sup j_U[\nu] = j_U(\nu)$$

Combining this with (2),

$$\text{cf}^M(j_U(\nu)) = \kappa$$

But $\text{cf}(\nu) = \kappa$ in M_x for every $x \in X$, so by Łoś's Theorem

$$\text{cf}^M(j_U(\nu)) = j_U(\kappa)$$

Thus $\kappa = j_U(\kappa)$. This contradicts that κ is the critical point of j_U. $\square$

We now sketch the last idea of the proof. Let C_x denote the set of limit points of S_x. Suppose towards a contradiction that there is no cofinal function from X to γ. Then the intersection $\bigcap_{x \in X} C_x$ is a closed unbounded subset of γ, and hence should contain a point ν of cofinality κ. This almost contradicts the claim. Therefore to finish, we carefully go through the standard proof that $\bigcap_{x \in X} C_x$ is closed unbounded, checking that it either produces a cofinal function from X to γ that is (ordinal) definable from $\langle S_x \rangle_{x \in X}$ and p or else produces a genuine counterexample to the claim.

Now, the details. We define various objects by a transfinite recursion with stages indexed by ordinals α. The first α stages of the construction produce ordinals

$$\langle \delta_x^\beta \mid (x, \beta) \in X \times \alpha \rangle$$

such that for each $x \in X$, $\langle \delta_x^\beta \mid \beta < \alpha \rangle$ is an increasing sequence of elements of S_x. It remains to define δ_x^α for each $x \in X$. There are two possibilities:

Subcase 1. The set $\{\delta_x^\beta + 1 \mid x \in X, \beta < \alpha\}$ is bounded below γ.

In this case, set

$$\delta^\alpha = \sup\{\delta_x^\beta + 1 : x \in X, \beta < \alpha\}$$

and for each $x \in X$:

$$\delta_x^\alpha = \min(S_x \setminus \delta^\alpha)$$

Subcase 2. The set $\{\delta_x^\beta + 1 \mid x \in X, \beta < \alpha\}$ is cofinal in γ.

If this case arises, the construction terminates.

The construction must halt at some stage $\alpha_* \leq \kappa$. To see this, assume towards a contradiction that it does not. Let $E_x = \{\delta_x^\beta \mid \beta < \kappa\}$. Since the construction did not halt at stage κ, E_x is bounded strictly below γ. The construction ensures that if $\alpha < \beta < \kappa$, then $\delta_x^\alpha < \delta_x^\beta$ for any $x, y \in X$. It follows that all the sets E_x have the same supremum, say ν. But for every $x \in X$, $E_x \subseteq S_x \cap \nu$, E_x has ordertype κ, and E_x is cofinal in ν. This contradicts the claim.

Suppose first that α_* is a limit ordinal. Then for each $\alpha < \alpha_*$, let $\delta_\alpha = \sup_{\beta < \alpha} \delta_x^\beta + 1$. Since α_* is the first stage at which the construction halts, $\delta_\alpha < \gamma$ for all $\alpha < \alpha_*$. Let

$$f(x) = \delta_{p(x)}$$

for those x such that $p(x) < \alpha_*$. Since p is a surjection from X to κ, the range of f is equal to $\{\delta_\beta \mid \beta < \alpha_*\}$, which is cofinal in γ.

Otherwise $\alpha = \beta + 1$ for some ordinal β. Then of course the function

$$f(x) = \delta_x^\beta$$

must be cofinal in γ. □

Theorem 3.21 is a consequence of Lemma 3.25 and the following elementary fact:

Lemma 3.26. *Suppose X is a set, δ is an ordinal, and for each $\gamma \leq \delta$, f_γ is a cofinal function from X to γ. Suppose $d : X \to X \times X$ is a surjection. Then there is a surjection g from X to δ that is ordinal definable from d and $\langle f_\gamma \mid \gamma \leq \delta \rangle$.*

Proof. We will define a sequence $\langle g_\gamma \mid \gamma \leq \delta\rangle$ by recursion and set $g = g_\delta$. For $\alpha \leq \delta$, suppose $\langle g_\gamma \mid \gamma < \alpha\rangle$ is given. Define $h : X \times X \to \delta$ by setting $h(x, y) = g_\gamma(y)$ where $\gamma = f_\alpha(x)$. Then let $g_\alpha = h \circ d$. □

Proof of Theorem 3.21. By Lemma 3.24, there is a surjection $p : X \to \kappa$. For each $\gamma \leq \delta$, $\langle S_x \cap \gamma \mid x \in X\rangle$ is a γ-supercompactness sequence. Applying Lemma 3.25, let $f_\gamma : X \to \gamma$ be the least cofinal function ordinal definable from p and $\langle S_x \cap \gamma \mid x \in X\rangle$. The hypotheses of Lemma 3.26 are now satisfied by taking $d : X \times X \to X$ to be any surjection. As a consequence, $\delta \preceq^* X$, which proves the theorem. □

3.5 Ordinal definability and ultrapowers

We finally turn to a generalization of Theorem 3.21 that has no ZFC analog:

Theorem 3.27. *Suppose X is a set, γ is an ordinal, U is an ultrafilter over $X \times \gamma$, κ is the critical point of j_U, and $\theta \geq \kappa$ is an ordinal. If $j_U[\theta] \in M_U$ and $j_U(\theta) = \theta$, then $\theta \preceq^* X$.*

Very roughly, this theorem says that supercompactness up to a fixed point θ cannot be the result of the wellorderable part of an ultrafilter. The proof of Theorem 3.27 uses the following version of Vopěnka's Theorem, which for reasons of citation we reduce to Bukovsky's Theorem:

Theorem 3.28 (Vopěnka)**.** *Suppose X and T are sets such that X is ordinal definable from T. Then for any $x \in X$, $\text{HOD}_{T,x}$ is an ι-cc generic extension of HOD_T where ι is the least regular cardinal of $\text{HOD}_{T,x}$ greater than or equal to $\theta\{X\}$.*

Proof. Recall that $\theta\{X\}$ is the least ordinal not the surjective image of X.

By Bukovsky's Theorem ([12], Fact 3.9), it suffices to verify that HOD_T has the uniform ι-covering property in $\text{HOD}_{T,x}$. This amounts to the following task. Suppose α and β are ordinals and $f : \alpha \to \beta$ is a function in $\text{HOD}_{T,x}$. We must find a function $F : \alpha \to P(\beta)$ in HOD_T such that for all $\xi < \alpha$, $F(\xi)$ is a set of cardinality less than ι containing $f(\xi)$ as an element.

Since f is $\text{OD}_{T,x}$, there is an OD_T function $g : \alpha \times X \to \beta$ such that $g(\xi, x) = f(\xi)$ for all $\xi < \alpha$. Let $F(\xi) = \{g(\xi, u) \mid u \in X\}$. Clearly F is OD_T, so $F \in \text{HOD}_T$. Fix $\xi < \alpha$. By definition, $f(\xi) = g(\xi, x) \in F(\xi)$. Finally, since $F(\xi) \preceq^* X$, $\iota \not\preceq^* X$, and $F(\xi)$ is wellorderable, $|F(\xi)| < \iota$. □

Proof of Theorem 3.27. Let $j = j_U$.

Fix a function $S : X \times \gamma \to P(\theta)$ such that $[S]_U = j[\theta]$. (That is, S is a θ-supercompactness sequence for U.) For any set T from which X is ordinal definable, let

$$M_T = \prod_{(x,\xi)\in X\times\gamma} \text{HOD}_{T,x}/U$$

Notice that $j[\theta] \in M_S$ since $S(x, \xi) \in \text{HOD}_{S,x}$ for all $(x, \xi) \in X \times \gamma$. It follows that for any set T,

$$P(\theta) \cap \text{HOD}_{S,T} \subseteq M_{S,T}$$

This is a basic fact about supercompactness, recast in our context. The point is that if $A \in P(\theta) \cap \mathrm{HOD}_{S,T}$, then $j(A)$ and $j \restriction \theta$ both belong to $M_{S,T}$, so $A \in M_{S,T}$ since $A = (j \restriction \theta)^{-1}[j(A)]$.

The key idea of the proof is to construct a sequence $T = \langle T(\nu) \mid \nu < \beta_* \rangle$ of subsets of θ such that

$$\{T(\nu) \mid \nu < \beta_*\} = P(\theta) \cap M_{S,T} \tag{3}$$

The construction proceeds by recursion. Suppose $T \restriction \beta = \langle T(\nu) \mid \nu < \beta \rangle$ has been defined. Assume that $\{T(\nu) \mid \nu < \beta\} \subsetneq P(\theta) \cap M_{S,T\restriction\beta}$, and let $T_\beta \subseteq \theta$ be the least set in the canonical wellorder of $P(\theta) \cap M_{S,T\restriction\beta}$ that does not belong to $\{T(\nu) \mid \nu < \beta\}$.

Eventually, one must reach an ordinal β such that $\{T(\nu) \mid \nu < \beta\} = P(\theta) \cap M_{S,T\restriction\beta}$: otherwise one obtains a sequence $\langle T(\nu) \mid \nu \in \mathrm{Ord}\rangle$ of distinct subsets of θ, violating the Replacement and Powerset Axioms. At the least such ordinal β, the construction terminates, and one sets $\beta_* = \beta$ and $T = \langle T(\nu) \mid \nu < \beta_* \rangle$, securing (3).

Let δ be the least ordinal such that $(2^\delta)^{\mathrm{HOD}_{S,T}} > \theta$. We claim that

$$j[P(\delta) \cap \mathrm{HOD}_{S,T}] \in M_{S,T} \tag{4}$$

Let $P_{\mathrm{bd}}(\delta)$ denote the set of bounded subsets of δ. Since $(2^{<\delta})^{\mathrm{HOD}_{S,T}} \leq \theta$ and $j[\theta] \in M_{S,T}$, $j[P_{\mathrm{bd}}(\delta) \cap \mathrm{HOD}_{S,T}] \in M_{S,T}$ by a standard argument: letting $f : \theta \to P_{\mathrm{bd}}(\delta) \cap \mathrm{HOD}_{S,T}$ be a surjection with $f \in \mathrm{HOD}_{S,T}$, $j[P_{\mathrm{bd}}(\delta) \cap \mathrm{HOD}_{S,T}] = j(f)[j[\theta]] \in M_{S,T}$.

We claim that $j(\delta) = \sup j[\delta]$. This will imply (4), since then $j[P(\delta) \cap \mathrm{HOD}_{S,T}]$ is equal to the set of $A \in P(j(\delta)) \cap M_{S,T}$ such that $A \cap \alpha \in j[P_{\mathrm{bd}}(\delta) \cap \mathrm{HOD}_{S,T}]$ for all $\alpha < j(\delta)$. Here we make essential use of the equality (3).

Let $\delta_* = \sup j[\delta]$. Let

$$P = \mathrm{Ult}(\mathrm{HOD}_{S,T}, U)$$

so j restricts to an elementary embedding from $\mathrm{HOD}_{S,T}$ to P. To prove that $j(\delta) = \delta_*$, it suffices by the minimality of δ and the elementarity of j to show that $(2^{\delta_*})^P > j(\theta)$, or, since $j(\theta) = \theta$, that $(2^{\delta_*})^P > \theta$. Suppose towards a contradiction that this is false, so $(2^{\delta_*})^P \leq \theta$.

Since $j \restriction P_{\mathrm{bd}}(\delta) \cap \mathrm{HOD}_{S,T} \in M_{S,T}$, $j \restriction P_{\mathrm{bd}}(\delta) \cap \mathrm{HOD}_{S,T} \in \mathrm{HOD}_{S,T}$. Therefore the one-to-one function $h : P(\delta) \cap \mathrm{HOD}_{S,T} \to P(\delta_*) \cap P$ defined by $h(A) = j(A) \cap \delta_*$ belongs to $\mathrm{HOD}_{S,T}$. Since $(2^{\delta_*})^P \leq \theta$, there is an injective function from $\mathrm{ran}(h)$ to θ in P, hence in $M_{S,T}$, and hence in $\mathrm{HOD}_{S,T}$ by (3). Therefore in $\mathrm{HOD}_{S,T}$,

$$P(\delta) \cap \mathrm{HOD}_{S,T} \preceq \mathrm{ran}(h) \preceq^* \theta$$

Since $\mathrm{HOD}_{S,T}$ satisfies the Axiom of Choice, it follows that $(2^\delta)^{\mathrm{HOD}_{S,T}} \leq \theta$, which contradicts the definition of δ. This contradiction establishes that $\delta_* = \sup j[\delta]$, finishing the proof of (4).

Let $\iota = \theta^{+\mathrm{HOD}_{S,T}}$. Then

$$\iota \leq j(\iota) = \theta^{+P} \leq \theta^{+M_{S,T}} \leq \iota$$

The first equality uses that $j(\theta) = \theta$, and the final inequality follows from (3). Hence $j(\iota) = \iota$. Since $\iota \leq (2^\delta)^{\mathrm{HOD}_{S,T}}$, $j[\iota] \in M_{S,T}$ as an immediate consequence of (4). We omit the proof, which is similar to the proof above that $j[P_{\mathrm{bd}}(\delta) \cap \mathrm{HOD}_{S,T}] \in M_{S,T}$.

We finally show that $\theta \preceq^* X$. Assume towards a contradiction that this fails. Then by Theorem 3.28, for every $x \in X$, $\mathrm{HOD}_{S,T,x}$ is an ι-cc generic extension of $\mathrm{HOD}_{S,T}$. By elementarity, it follows that $M_{S,T}$ is an ι-cc generic extension of P. In particular, P is stationary correct in $M_{S,T}$ at ι. This allows us to run Woodin's proof [13] of the Kunen Inconsistency Theorem to reach our final contradiction.

Let $B = \{\xi < \iota \mid \mathrm{cf}(\xi) = \omega\}$. Recall that κ denotes the critical point of j. Since $\mathrm{HOD}_{S,T}$ satisfies the Axiom of Choice, the Solovay Splitting Theorem [14] applied in $\mathrm{HOD}_{S,T}$ yields a partition $\langle B_\nu \mid \nu < \kappa \rangle$ of $(B)^{\mathrm{HOD}_{S,T}}$ into $\mathrm{HOD}_{S,T}$-stationary sets. Let $\langle B'_\nu \mid \nu < j(\kappa)\rangle = j(\langle B_\nu \mid \nu < \kappa\rangle)$. Then B'_κ is P-stationary in ι. Therefore B'_κ is $M_{S,T}$-stationary in ι since P is stationary correct in $M_{S,T}$ at ι. Since $j[\iota] \in M_{S,T}$ is an ω-closed unbounded set in $M_{S,T}$ and $M_{S,T}$ satisfies that B'_κ is a stationary set of ordinals of cofinality ω, the intersection $j[\iota] \cap B'_\kappa$ is nonempty. Fix $\xi < \iota$ such that $j(\xi) \in B'_\kappa$. Clearly $\xi \in B$, so since $\langle B_\nu \mid \nu < \kappa\rangle$ partitions B, there is some $\nu < \iota$ such that $\xi \in B_\nu$. Now $j(\xi) \in j(B_\nu) = B'_{j(\nu)}$. Therefore the intersection $B'_{j(\nu)} \cap B'_\kappa$ is nonempty, and so since $\langle B'_\nu \mid \nu < j(\kappa)\rangle$ is a partition, $j(\nu) = \kappa$. This contradicts that κ is the critical point of j. $\square$

As a corollary of Theorem 3.27, we answer the following question of Schlutzenberg. Suppose $j : V \to V$ is an elementary embedding. Is every set ordinal definable from parameters in the range of j? The question is motivated by the well-known ZFC fact that if $j : V \to M$ is an elementary embedding, then every element in M is ordinal definable in M from parameters in the range of j.

The answer to Schlutzenberg's question, however, is no.

Theorem 3.29. *Suppose $\epsilon \leq \eta \leq \eta'$ are ordinals, ϵ is even, and $j : V_\eta \to V_{\eta'}$ is a cofinal elementary embedding such that $j(\epsilon) = \epsilon$. Then $j^\star \restriction \theta_\epsilon$ is not definable over $\mathcal{H}_{\eta'}$ from parameters in $j^\star[\mathcal{H}_\eta] \cup V_\epsilon \cup \theta_{\eta'}$.*

To put this theorem in a more familiar context, let us state a special case.

Corollary 3.30. *Suppose $j : V \to V$ is an elementary embedding. Let $\lambda = \kappa_\omega(j)$. Then $j[\lambda]$ is not ordinal definable from parameters in $j[V] \cup V_\lambda$.* $\square$

Proof of Theorem 3.29. Let $\theta = \theta_\epsilon$. Since $j(\epsilon) = \epsilon$, $j^\star(\theta) = \theta$. Suppose towards a contradiction that the theorem fails. Then there is a set $p \in V_\eta$, a set $a \in V_\epsilon$, an ordinal $\alpha < \eta'$, and a formula φ such that $j^\star[\theta]$ is the unique set $k \in \mathcal{H}_{\eta'}$ such that $\mathcal{H}_{\eta'}$ satisfies $\varphi(k, j(p), a, \alpha)$. By Proposition 3.15 (or trivially if ϵ is a limit ordinal), there is an ordinal ξ such that $\xi + 2 \leq \epsilon$, a set $x \in V_{\xi+1}$, and a function $f : V_{\xi+1} \to V_\epsilon$ such that $j(f)(x) = a$.

Let γ be an ordinal such that $\alpha < j(\gamma)$. Define $g : V_{\xi+1} \times \gamma \to P(\theta)$ so that $j^\star(g)(x, \alpha) = j^\star[\theta]$: let $g(u, \beta)$ be the unique set $k \in \mathcal{H}_\eta$ such that $\mathcal{H}_\eta$ satisfies $\varphi(k, p, f(u), \beta)$. Let U be the ultrafilter over $V_{\xi+1} \times \gamma$ derived from j using (x, α). It is easy to check that $[g]_U = j_U[\theta]$: $[h]_U \in [g]_U$ if and only if $j(h)(x, \alpha) \in j(g)(x, \alpha)$ if and only if $j(h)(x, \alpha) = j(\nu)$ for some $\nu < \theta$ if and only if there is some $\nu < \theta$

such that $h(u,\beta) = \nu$ for U-almost all (u,β). By Theorem 3.27, there is a surjection from $V_{\xi+1}$ to θ, and since $V_{\xi+1} \in V_\epsilon$, this contradicts the definition of θ. $\square$

We include a final result $\mathrm{AD}_{\mathbb{R}}$-like result about ordinal definability assuming an elementary embedding from $V_{\epsilon+3}$ to $V_{\epsilon+3}$:

Theorem 3.31. *Suppose ϵ is an even ordinal and there is a Σ_1-elementary embedding from $V_{\epsilon+3}$ to $V_{\epsilon+3}$. Then there is no sequence of functions $\langle f_\alpha \mid \alpha < \theta_{\epsilon+2}\rangle$ such that for all $\alpha < \theta_{\epsilon+2}$, f_α is a surjection from $V_{\epsilon+1}$ to α.*

This result cannot be proved from the existence of an elementary embedding $j : V_{\epsilon+2} \to V_{\epsilon+2}$ (if this hypothesis is consistent): the inner model $L(V_{\epsilon+1})[j]$ satisfies that there is an elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$, but using that $L(V_{\epsilon+1})[j]$ satisfies that $V = \mathrm{HOD}_{V_{\epsilon+1},i}$ where $i = j \restriction L(V_{\epsilon+1})[j]$, one can easily show that in $L(V_{\epsilon+1})[j]$, there is a sequence $\langle f_\alpha \mid \alpha < \theta_{\epsilon+2}\rangle$ such that for all $\alpha < \theta_{\epsilon+2}$, f_α is a surjection from $V_{\epsilon+2}$ to α.

Proof of Theorem 3.31. Suppose towards a contradiction that $\langle f_\alpha \mid \alpha < \theta_{\epsilon+2}\rangle$ is such a sequence. As a consequence of this assumption and Lemma 3.26, $\theta_{\epsilon+2}$ is regular.

Note that $\langle f_\alpha \mid \alpha < \theta_{\epsilon+2}\rangle \in \mathcal{H}_{\epsilon+3}$. Using the notation from Section 2, fix $u \in \mathrm{C}_{\epsilon+3}$ such that $\Phi_{\epsilon+3}(u) = \langle f_\alpha \mid \alpha < \theta_{\epsilon+2}\rangle$. (Recall that $\mathrm{C}_{\epsilon+3}$ is the set of codes in $V_{\epsilon+3}$ for elements of $\mathcal{H}_{\epsilon+3}$.)

For $\ell = 0, 1$, suppose

$$j_\ell : (V_{\epsilon+2}, u_\ell) \to (V_{\epsilon+2}, u)$$

is an elementary embedding. Necessarily, $u_\ell = j_\ell^{-1}[u]$. Assume $j_0 \restriction V_\epsilon = j_1 \restriction V_\epsilon$. We claim that $j_0^\star[\theta_{\epsilon+2}] = j_1^\star[\theta_{\epsilon+2}]$. First, let $F \subseteq \theta_{\epsilon+2}$ be the set of common fixed points of $j_0 \restriction \theta_{\epsilon+2}$ and $j_1 \restriction \theta_{\epsilon+2}$. Since $\theta_{\epsilon+2}$ is regular, F is ω-closed unbounded. For each $\alpha \in F$, there is some $g^\ell_\alpha \in \mathcal{H}_{\epsilon+2}$ such that $j_\ell^\star(g^\ell_\alpha) = f_\alpha$. Indeed, one can set $g^\ell_\alpha = \Phi(u_\ell)_\alpha$. Now $j_0^\star[\alpha] = j_0^\star[g^\ell_\alpha[V_{\epsilon+1}]] = f_\alpha[j_0[V_{\epsilon+1}]]$. Similarly $j_1^\star[\alpha] = f_\alpha[j_1[V_{\epsilon+1}]]$. Since $j_0 \restriction V_\epsilon = j_1 \restriction V_\epsilon$, $j_0 \restriction V_{\epsilon+1} = j_1 \restriction V_{\epsilon+1}$ by Corollary 3.11. It follows that $j_0^\star[\alpha] = j_1^\star[\alpha]$. Since F is unbounded in $\theta_{\epsilon+2}$, this implies $j_0^\star[\theta_{\epsilon+2}] = j_1^\star[\theta_{\epsilon+2}]$.

Let

$$\begin{aligned}
\mathcal{E}_0 &= \{k \mid k \in \mathscr{E}((V_{\epsilon+2}, k^{-1}[u]), (V_{\epsilon+2}, u))\}\\
\mathcal{E}_1 &= \{k \restriction V_\epsilon \mid k \in \mathcal{E}_1\}
\end{aligned}$$

Let $C = \bigcap_{k\in\mathcal{E}_0} k^\star[\theta_{\epsilon+2}]$. For any $i \in \mathcal{E}_1$, let $A_i \subseteq \theta_{\epsilon+2}$ be equal to $k^\star[\theta_{\epsilon+2}]$ for any $k \in \mathcal{E}_1$ extending i. This is well-defined by the previous paragraph. Clearly $C = \bigcap\{A_i \mid i \in \mathcal{E}_0\}$. Since $\theta_{\epsilon+2}$ is regular and $\mathcal{E}_0 \preceq^* V_{\epsilon+1}$, it follows that C is ω-closed unbounded in $\theta_{\epsilon+2}$.

Now suppose $j : V_{\epsilon+3} \to V_{\epsilon+3}$ is Σ_1-elementary. Then

$$j^\star(\mathcal{E}_0) = \{k \mid k \in \mathscr{E}((V_{\epsilon+2}, k^{-1}[u]), (V_{\epsilon+2}, u))\} \tag{5}$$

$$j^\star(C) = \bigcap_{k\in j^\star(\mathcal{E}_0)} k^\star[\theta_{\epsilon+2}] \tag{6}$$

Verifying these equalities is a bit tricky since we only know that $j^\star : \mathcal{H}_{\epsilon+3} \to \mathcal{H}_{\epsilon+3}$ is Σ_0-elementary. (See Lemma 2.12.) (5) is proved by writing $\mathcal{E}_0 = \bigcap_{n<\omega} \mathcal{E}_0^n$ where

$$\mathcal{E}_0^n = \{k \mid k : (V_{\epsilon+2}, k^{-1}[u]) \to_{\Sigma_n} (V_{\epsilon+2}, u)\}$$

Notice that $\mathcal{E}_0^n$ is Σ_0-definable over $\mathcal{H}_{\epsilon+3}$ from u and $V_{\epsilon+2}$, and $j^\star(\mathcal{E}_0) = \bigcap_{n<\omega} j^\star(\mathcal{E}_0^n)$. This easily yields (5). (6) is proved by checking that the $\star$-operation on $\mathscr{E}(V_{\epsilon+2})$ is Σ_0-definable over $\mathcal{H}_{\epsilon+3}$ from the parameter $\mathcal{H}_{\epsilon+2}$.

It follows that $j \restriction V_{\epsilon+2} \in j^\star(\mathcal{E}_0)$, and hence $j^\star(C) \subseteq j^\star[\theta_{\epsilon+2}]$. The only way this is possible is if $|C| < \text{crit}(j)$. But C is unbounded in $\theta_{\epsilon+2}$. This contradicts that $\theta_{\epsilon+2}$ is regular. $\square$

The Axiom of Choice implies the existence of a sequence $\langle f_\alpha \mid \alpha < \theta_{\epsilon+2}\rangle$ such that for all $\alpha < \theta_{\epsilon+2}$, f_α is a surjection from $V_{\epsilon+1}$ to α, so Theorem 3.31 yields a new proof of the Kunen Inconsistency Theorem.

4 The θ_α sequence

4.1 The main conjecture

In this section we study the sequence of cardinals θ_α (Definition 2.9). The results we will prove suggest that if ϵ is an even ordinal, then assuming choiceless large cardinal axioms, θ_ϵ should be relatively large and $\theta_{\epsilon+1}$ should be relatively small.

Conjecture 4.1. *Suppose ϵ is an even ordinal and there is an elementary embedding from $V_{\epsilon+1}$ to $V_{\epsilon+1}$.*

- *θ_ϵ is a strong limit cardinal.*[10]
- *$\theta_{\epsilon+1} = (\theta_\epsilon)^+$.*

We note that if ϵ is a limit ordinal, then one can prove in ZF that θ_ϵ is a strong limit cardinal and $\theta_{\epsilon+1} = (\theta_\epsilon)^+$.

4.2 ZF theorems

Our first two theorems towards Conjecture 4.1 are the following:

Theorem 4.2. *Suppose ϵ is an even ordinal. Suppose $j : V_{\epsilon+3} \to V_{\epsilon+3}$ is an elementary embedding with critical point κ. Then the interval $(\theta_{\epsilon+2}, \theta_{\epsilon+3})$ contains fewer than κ regular cardinals.*

This theorem shows that $\theta_{\epsilon+3}$ is not too much larger than $\theta_{\epsilon+2}$. Theorem 4.14 shows that under further assumptions, $\theta_{\epsilon+2}$ is an inaccessible limit of regular cardinals, so Theorem 4.2 captures a genuine difference between the even and odd levels.

[10] Recall that in the context of ZF, a cardinal is defined to be a *strong limit cardinal* if it is not the surjective image of the powerset of any smaller cardinal.

Theorem 4.3. *Suppose ϵ is an even ordinal. Suppose $j : V_{\epsilon+2} \to V_{\epsilon+2}$. Then for any $\alpha < \kappa_\omega(j)$, there is no surjection from $P(\theta_{\epsilon+1}^{+\alpha})$ onto $\theta_{\epsilon+2}$.*

While we cannot show that $\theta_{\epsilon+2}$ is a strong limit, this theorem shows that it has some strong limit-like properties. Theorem 4.12 below proves the stronger fact that $V_{\epsilon+1}$ surjects onto $P(\alpha)$ for all $\alpha < \theta_{\epsilon+2}$, but this theorem requires weak choice assumptions.

The following lemma, which is a key aspect of the proof of both Theorem 4.2 and Theorem 4.3, roughly states that rank-to-rank embeddings have no generators in the interval $(\theta_{\epsilon+2}, \theta_{\epsilon+3})$.

Lemma 4.4. *Suppose ϵ is an even ordinal and $j : V_{\epsilon+1} \to V_{\epsilon+1}$ is an elementary embedding. Then for every ordinal $\nu < \theta_{\epsilon+1}$, there are ordinals $\alpha, \beta < \theta_\epsilon$ and a function $g : \alpha \to \theta_{\epsilon+1}$ such that $\nu = j^\star(g)(\beta)$.*

Proof. Let R be a prewellorder of V_ϵ of length $\nu+1$. Then $j(R)$ is a prewellorder of V_ϵ of length at least $\nu+1$. Fix $a \in V_\epsilon$ with $\operatorname{rank}_{j(R)}(a) = \nu$ in R. By Corollary 3.12, find an ordinal ξ such that $\xi + 2 \leq \epsilon$ and $a = j^\star(f)(x)$ for some $f : V_{\xi+1} \to V_\epsilon$ and $x \subseteq V_\xi$. Define $d : V_{\xi+1} \to \theta_{\epsilon+1}$ by setting $d(u) = \operatorname{rank}_R(f(u))$. Then $j^\star(d)(x) = \nu$, so $\nu \in \operatorname{ran}(j^\star(d))$. Let $g : \alpha \to \theta_{\epsilon+1}$ be the order-preserving enumeration of $\operatorname{ran}(d)$, and note that $\alpha < \theta_\epsilon$ since d witnesses $\operatorname{ran}(d) \preceq^* V_{\xi+1}$. By elementarity, $j^\star(g)$ enumerates $\operatorname{ran}(j^\star(d))$, and hence there is some $\beta < \theta_\epsilon$ such that $j^\star(g)(\beta) = \nu$. □

Proof of Theorem 4.2. Let η be the ordertype of the set of regular cardinals in the interval $(\theta_{\epsilon+2}, \theta_{\epsilon+3})$. Then η is fixed by $j^\star$, so $\eta \neq \kappa$. Suppose towards a contradiction that $\eta > \kappa$. Let δ be the κ-th regular cardinal in $(\theta_{\epsilon+2}, \theta_{\epsilon+3})$. Then $j^\star(\delta)$ is a regular cardinal strictly above δ, so $j^\star(\delta)$ is not equal to $\sup j^\star[\delta]$, which has cofinality δ. Therefore $\sup j^\star[\delta] < j^\star(\delta)$. By Lemma 4.4, there are ordinals $\alpha, \beta < \theta_{\epsilon+2}$ and a function $g : \alpha \to \delta$ such that $j^\star(g)(\beta) = \sup j^\star[\delta]$. Since δ is regular, there is some ordinal $\rho < \delta$ such that $\operatorname{ran}(g) \subseteq \rho$. Therefore $j^\star(g)(\beta) < j^\star(\rho) < \sup j^\star[\delta]$, which is a contradiction. □

We now turn to the size of $\theta_{\epsilon+2}$ for ϵ even.

Proof of Theorem 4.3. Let $E = \langle D(j,a) \mid a \in [\theta_\epsilon]^{<\omega}\rangle$ be the extender of length θ_ϵ derived from j. Notice that E is definable over $\mathcal{H}_{\epsilon+2}$ from $j^\star \restriction P_{\text{bd}}(\theta_\epsilon)$, hence from $j \restriction V_{\epsilon+1}$, and hence from $j \restriction V_\epsilon$ by Corollary 3.11. In fact, there is a partial sequence $\langle F(\sigma) \mid \sigma \in V_{\epsilon+1}\rangle$ definable without parameters over $\mathcal{H}_{\epsilon+2}$ such that $E = F(j[V_\epsilon])$: to be explicit, $F(\sigma)$ is the extender of length θ_ϵ derived from k where $k = ((\pi_\sigma)^+)^\star$ for $\pi_\sigma : \sigma \to M$ the Mostowski collapse of σ.

Let $j_E : \mathcal{H}_{\epsilon+2} \to N$ be the associated ultrapower embedding, and let $k : N \to \mathcal{H}_{\epsilon+2}$ be the associated factor embedding, defined by $k([f,a]_E) = j^\star(f)(a)$. Let $\nu = \operatorname{crit}(k)$. Note that ν must exist or else $j_E \restriction \theta_{\epsilon+2} = j^\star \restriction \theta_{\epsilon+2}$, contrary to the undefinability of $j^\star \restriction \theta_{\epsilon+2}$ over $\mathcal{H}_{\epsilon+3}$ from parameters, like E, that lie in $V_{\epsilon+2}$ (Lemma 3.23).

Note that ν is a generator of $j^\star$, in the sense that for any ordinals $\alpha, \beta < \nu$ and any function $g : \alpha \to \theta_{\epsilon+2}$, $\nu \neq j^\star(g)(\beta)$: otherwise $\nu \in \operatorname{ran}(k)$ by definition. As an immediate consequence of Lemma 4.4, it follows that $\nu \geq \theta_{\epsilon+1}$. Since $j^\star$ fixes every cardinal in the interval $(\theta_{\epsilon+1}, \theta_{\epsilon+1}^{+\kappa})$, it follows that $\nu \geq \theta_{\epsilon+1}^{+\kappa}$.

Notice that for any $\eta < \theta_{\epsilon+1}^{+\kappa}$ and any set $A \subseteq \eta$, $j_E(A) = j^\star(A)$. Indeed, $j_E(A) \cap \nu = j^\star(A) \cap \nu$ for any set of ordinals A. Suppose towards a contradiction that $p : P(\eta) \to \theta_{\epsilon+2}$ is a surjection. Define $g : V_{\epsilon+1} \to P(\theta_{\epsilon+2})$ by setting $g(\sigma) = p \circ j_{F(\sigma)}[P(\eta)]$.

Let $\mathcal{U}$ be the ultrafilter over $V_{\epsilon+1}$ derived from j using $j[V_\epsilon]$. It is easy to check that $j_{\mathcal{U}} \restriction \mathcal{H}_{\epsilon+2} = j^\star \restriction \mathcal{H}_{\epsilon+2}$. Therefore $[g]_{\mathcal{U}} = j_{\mathcal{U}}(g)(j[V_\epsilon]) = j_{\mathcal{U}}(p) \circ j_E[P(\eta)] = j_{\mathcal{U}}(p) \circ j_{\mathcal{U}}[P(\eta)] = j_{\mathcal{U}} \circ p[P(\eta)] = j_{\mathcal{U}}[\theta_{\epsilon+2}]$. This shows that $j_{\mathcal{U}}[\theta_{\epsilon+2}] \in M_{\mathcal{U}}$, so by Theorem 3.21, it follows that $\theta_{\epsilon+2} \preceq^* V_{\epsilon+1}$, which is a contradiction.

This shows that for any $\eta < \theta_{\epsilon+1}^{+\kappa}$, $P(\eta)$ does not surject onto $\theta_{\epsilon+2}$. The same argument applied to the finite iterates of j shows that for any $n < \omega$, $\eta < \theta_{\epsilon+1}^{+\kappa_n(j)}$, $P(\eta)$ does not surject onto $\theta_{\epsilon+2}$. This proves the theorem. $\square$

As a corollary of the proof of Theorem 4.3, we have the following fact, which exhibits a difference between the even and odd levels with regard to Lemma 4.4:

Proposition 4.5. *Suppose ϵ is an even ordinal and $j : V_{\epsilon+2} \to V_{\epsilon+2}$ is an elementary embedding. Then j has a generator in the interval $(\theta_{\epsilon+1}, \theta_{\epsilon+2})$.* $\square$

4.3 The Coding Lemma

One of the central theorems in the analysis of $L(V_{\lambda+1})$ assuming the axiom I_0 is Woodin's generalization of the Moschovakis Coding Lemma. Here we prove a new Coding Lemma. This Coding Lemma lifts Woodin's to structures of the form $L(V_{\epsilon+1})$ where ϵ is even and the appropriate generalization of I_0 holds. But moreover, the proof adds a new twist to Woodin's and as a consequence it applies to a host of models beyond $L(V_{\epsilon+1})$. For example, the Coding Lemma holds in $\text{HOD}(V_{\epsilon+1})$, and more interestingly, the Coding Lemma holds in V itself under what seem to be reasonable assumptions.

Definition 4.6. Suppose ϵ and η are ordinals, $\varphi : V_{\epsilon+1} \to \eta$ is a surjection, and R is a binary relation on $V_{\epsilon+1}$.

- A relation $\bar{R} \subseteq R$ is a *φ-total subrelation of R* if $\varphi[\text{dom}(\bar{R})] = \varphi[\text{dom}(R)]$.
- A set of binary relations Γ on $V_{\epsilon+1}$ is a *code-class for η* if for any surjection $\psi : V_{\epsilon+1} \to \eta$, every binary relation on $V_{\epsilon+1}$ has a ψ-total subrelation in Γ.
- The *Coding Lemma holds at ϵ* if every ordinal $\eta < \theta_{\epsilon+2}$ has a code-class Γ such that $\Gamma \preceq^* V_{\epsilon+1}$.

The Coding Lemma has a number of important consequences. For example:

Proposition 4.7. *Suppose ϵ is an ordinal at which the Coding Lemma holds. Then $\theta_{\epsilon+2}$ is a strong limit cardinal. In fact, for any $\eta < \theta_{\epsilon+2}$, $P(\eta) \preceq^* V_{\epsilon+1}$.*

Proof. Fix a code-class Γ for η with $\Gamma \preceq^* V_{\epsilon+1}$. Fix a surjection $\varphi : V_{\epsilon+1} \to \eta$. It is immediate that $P(\eta) = \{\varphi[\mathrm{dom}(R)] \mid R \in \Gamma\}$. Therefore since $\Gamma \preceq^* V_{\epsilon+1}$, $P(\eta) \preceq^* V_{\epsilon+1}$. $\square$

Definition 4.8. Then the *Collection Principle* states that every class binary relation R whose domain is a set has a set-sized subrelation $\bar{R}$ such that $\mathrm{dom}(\bar{R}) = \mathrm{dom}(R)$.

It seems that one needs a local form of the Collection Principle to prove the Coding Lemma:

Theorem 4.9. *Suppose ϵ is an even ordinal and M is an inner model containing $V_{\epsilon+1}$. Suppose there is an embedding $j \in \mathscr{E}(V_{\epsilon+2} \cap M)$ with $\mathrm{crit}(j) = \kappa$. Assume $(\mathcal{H}_{\epsilon+2})^M$ satisfies κ-DC and the Collection Principle. Then M satisfies the Coding Lemma at ϵ.*

Note that we only require the *first-order* Collection Principle to hold in $(\mathcal{H}_{\epsilon+2})^M$.

We begin by proving a Weak Coding Lemma, which requires some more definitions.

Definition 4.10. Suppose ϵ and η are ordinals, $\varphi : V_{\epsilon+1} \to \eta$ is a surjection, and R is a binary relation on $V_{\epsilon+1}$.

- A relation $\bar{R} \subseteq R$ is a *φ-cofinal subrelation of R* if either $\varphi[\mathrm{dom}(R)]$ is not cofinal in η or $\varphi[\mathrm{dom}(\bar{R})]$ is cofinal in $\varphi[\mathrm{dom}(R)]$.
- A set of binary relations Γ on $V_{\epsilon+1}$ is a *weak code-class for η* if for any surjection $\psi : V_{\epsilon+1} \to \eta$, every binary relation on $V_{\epsilon+1}$ has a ψ-cofinal subrelation in Γ.
- The *Weak Coding Lemma holds at ϵ* if every ordinal $\eta < \theta_{\epsilon+2}$ has a weak code-class Γ such that $\Gamma \preceq^* V_{\epsilon+1}$.

Lemma 4.11. *Suppose ϵ is an even ordinal and M is an inner model containing $V_{\epsilon+1}$. Suppose there is an elementary embedding from $V_{\epsilon+2} \cap M$ to $V_{\epsilon+2} \cap M$ with critical point κ. Assume $(\mathcal{H}_{\epsilon+2})^M$ satisfies κ-DC. Then the Weak Coding Lemma holds at ϵ in M.*

Proof. Assume towards a contradiction that M does not satisfy the Weak Coding Lemma at ϵ. Let η be the least ordinal for which there is no weak code-class Γ with $\Gamma \preceq^* V_{\epsilon+1}$. Notice that η is definable in $(\mathcal{H}_{\epsilon+2})^M$, and hence is fixed by any embedding in $\mathscr{E}((\mathcal{H}_{\epsilon+2})^M)$.

We now use that $(\mathcal{H}_{\epsilon+2})^M$ satisfies κ-DC to construct a sequence

$$\langle (A_\alpha, \varphi_\alpha) \mid \alpha < \kappa \rangle$$

by recursion. Each A_α will be a binary relation on $V_{\epsilon+1}$, and each φ_α will be a surjection from $V_{\epsilon+1}$ to η. Suppose $\langle (A_\alpha, \varphi_\alpha) \mid \alpha < \beta \rangle$ has been defined. Let Γ be the collection of binary relations on $V_{\epsilon+1}$ definable (from parameters) over

$(V_{\epsilon+1}, A_\alpha)$ for some $\alpha < \beta$. Obviously, $\Gamma \preceq^* V_{\epsilon+1}$, so by choice of η, Γ is not a weak code-class for η. We can therefore choose a binary relation A_β on $V_{\epsilon+1}$ and a surjection φ_β from $V_{\epsilon+1}$ to η such that A_β has no φ_β-cofinal subrelation in Γ. This completes the construction.

Fix an embedding $j \in \mathscr{E}(V_{\epsilon+2} \cap M)$ with critical point κ. Then j extends uniquely to $j^\star : (\mathcal{H}_{\epsilon+2})^M \to (\mathcal{H}_{\epsilon+2})^M$. Apply $j^\star$ twice to $\langle (A_\alpha, \varphi_\alpha) \mid \alpha < \kappa\rangle$:

$$\langle (A^1_\alpha, \varphi^1_\alpha) : \alpha < \kappa_1\rangle = j^\star(\langle (A_\alpha, \varphi_\alpha) \mid \alpha < \kappa\rangle)$$
$$\langle (A^2_\alpha, \varphi^2_\alpha) \mid \alpha < \kappa_2\rangle = j^\star(\langle (A^1_\alpha, \varphi^1_\alpha) \mid \alpha < \kappa_1\rangle)$$

Notice the following equality:

$$j^\star(A^1_\kappa, \varphi^1_\kappa) = (A^2_{\kappa_1}, \varphi^2_{\kappa_1}) \tag{7}$$

(7) implies:

$$j[A^1_\kappa] \text{ is a } \varphi^2_{\kappa_1}\text{-cofinal subrelation of } A^2_{\kappa_1}.$$

The point here is that $j^\star(\eta) = \eta$, so $j^\star[\eta]$ is cofinal in η. Hence

$$\varphi^2_{\kappa_1}[\mathrm{dom}(j[A^1_\kappa])] = j^\star[\varphi^1_\kappa[\mathrm{dom}(A^1_\kappa)]]$$

is cofinal in $\mathrm{dom}(A^2_{\kappa_1})$. (Note that for every $\beta < \kappa$, the set $\mathrm{dom}(A_\beta)$ is cofinal in η: otherwise for all $\alpha < \beta$, A_α is vacuously a φ_β-cofinal subrelation of A_β.)

Recall, however, that our construction ensured that for all $\beta < \kappa$, A_β has no φ_β-cofinal subrelation that is definable over $(V_{\epsilon+1}, A_\alpha)$ for some $\alpha < \beta$. By elementarity, $A^2_{\kappa_1}$ has no $\varphi^2_{\kappa_1}$-cofinal subrelation that is boldface definable over $(V_{\epsilon+1}, A_\alpha)$ for some $\alpha < \kappa_1$. We reach a contradiction by showing that $j^\star[A^1_\kappa]$ is definable over $(V_{\epsilon+1}, A^1_\kappa)$.

Combinatorially, the following equation is the new ingredient in this proof:

$$j(j)(A^1_\kappa) = A^2_\kappa \tag{8}$$

(More formally $(j^\star(j \restriction V_\epsilon))^+(A^1_\kappa) = A^2_\kappa$; this notation is just too unwieldy.) (8) implies that $A^1_\kappa = j(j)^{-1}[A^2_\kappa]$. By Theorem 3.13, $j(j) \restriction V_{\epsilon+1}$ is definable over $V_{\epsilon+1}$ from its restriction to V_ϵ, and therefore A^1_κ is definable from parameters over $(V_{\epsilon+1}, A^2_\kappa)$. Similarly, $j[A^1_\kappa]$ is definable over $(V_{\epsilon+1}, A^1_\kappa)$. It follows that $j[A^1_\kappa]$ is definable over $(V_{\epsilon+1}, A^2_\kappa)$. □

The proof that the Weak Coding Lemma implies the Coding Lemma is a direct generalization of Woodin's:

Proof of Theorem 4.9. Assume towards a contradiction that the Coding Lemma fails in M at ϵ. Let η be the least ordinal for which there is no code-class Γ such that $\Gamma \preceq^* V_{\epsilon+1}$.

Let $\psi : V_{\epsilon+1} \to \eta$ be an arbitrary surjection. We begin by using the Collection Principle to construct a set $\Lambda \preceq^* V_{\epsilon+1}$ that is a code-class for every $\alpha < \eta$. Consider the relation $S \subseteq V_{\epsilon+1} \times \mathcal{H}_{\epsilon+2}$ defined by

$$S(a, \Gamma) \iff \Gamma \text{ is a code-class for } \psi(a) \text{ with } \Gamma \preceq^* V_{\epsilon+1}$$

By the minimality of η, S is a total relation. Since $\mathcal{H}_{\epsilon+2}$ satisfies the Collection Principle, there is a total relation $\bar{S} \subseteq S$ with $\bar{S} \in \mathcal{H}_{\epsilon+2}$. Let

$$\Lambda = \bigcup_{\Gamma \in \operatorname{ran}(\bar{S})} \Gamma$$

Clearly Λ is a code-class for every $\alpha < \eta$. Since $\bar{S} \in \mathcal{H}_{\epsilon+2}$, $\Lambda \preceq^* V_{\epsilon+1}$, so there is a surjection $\pi : V_{\epsilon+1} \to \Lambda$.

Applying the Weak Coding Lemma, fix a weak code-class Σ for η with $\Sigma \preceq^* V_{\epsilon+1}$. Let Γ be the collection of binary relations on $V_{\epsilon+1}$ definable over $(\mathcal{H}_{\epsilon+2}, \pi)$ using parameters in Σ. Clearly $\Gamma \preceq^* V_{\epsilon+1}$. We finish by showing that Γ is a code-class for η.

Fix a surjection $\varphi : V_{\epsilon+1} \to \eta$ and a binary relation R on $V_{\epsilon+1}$. We must find a φ-total subrelation $\bar{R}$ of R that belongs to Γ. First consider the relation

$$S(a, u) \iff \pi(u) \text{ is a } \varphi\text{-total subrelation of } R \restriction_\varphi \psi(a)$$

(Here $R \restriction_\varphi \beta = R \restriction \{b \mid \varphi(b) < \beta\}$.) Notice that S is a total relation due to the construction of Λ. Let $\bar{S} \in \Sigma$ be a ψ-cofinal subrelation of S. Since $\bar{S} \subseteq S$, for all $u \in \operatorname{ran}(\bar{S})$, $\pi(u) \subseteq R$. Moreover since $\bar{S}$ is a ψ-cofinal subrelation of S, for cofinally many $\beta < \eta$, there is some $u \in \operatorname{ran}(\bar{S})$ such that $\pi(u)$ is a φ-total subrelation of $R \restriction \beta$. Let

$$\bar{R} = \bigcup_{u \in \operatorname{ran}(\bar{S})} \pi(u)$$

Then $\bar{R}$ is a φ-total subrelation of R. Moreover $\bar{R} \in \Gamma$ since $\bar{R}$ is definable over $(\mathcal{H}_{\epsilon+2}, \pi)$ using the parameter $\bar{S} \in \Sigma$. $\square$

Theorem 4.12. *Suppose ϵ is an even ordinal. Suppose there is an elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$ with critical point κ. Assume $\mathcal{H}_{\epsilon+2}$ satisfies the Collection Principle and κ-DC. Then $\theta_{\epsilon+2}$ is a strong limit cardinal.* $\square$

As a corollary of Theorem 4.12, we have a proof of the Kunen Inconsistency Theorem that seems new:

Corollary 4.13 (ZFC)**.** *There is no elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$.*

Proof. Assume towards a contradiction that there is an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$. The Axiom of Choice implies that all the hypotheses of Theorem 4.9 are satisfied when $M = V$. Therefore by Theorem 4.12, $\theta_{\lambda+2}$ is a strong limit cardinal. On the other hand, the Axiom of Choice implies $\theta_{\lambda+2} = |V_{\lambda+1}|^+$, which is not a strong limit cardinal. $\square$

Another consequence of the Coding Lemma beyond Theorem 4.12 is the following theorem:

Theorem 4.14. *Suppose ϵ is an even ordinal. Suppose there is an elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$ with critical point κ. Assume $\mathcal{H}_{\epsilon+2}$ satisfies the Collection Principle and κ-DC. Then $\theta_{\epsilon+2}$ is a limit of regular cardinals.*

The theorem also generalizes easily to inner models M as in Theorem 4.9, but here it seems one must require that j is a proper embedding in the sense of [5], and since we do not want to introduce the notion of a proper embedding, we omit the proof.

The proof uses the following lemma, which is a direct generalization of a construction due to Woodin:

Lemma 4.15 ([5], Lemma 6)**.** *Suppose ϵ is an even ordinal and $j : V_{\epsilon+2} \to V_{\epsilon+2}$ is an elementary embedding. Then for any set $A \subseteq V_{\epsilon+2}$, there is some set $B \in V_{\epsilon+2}$ such that $j(B) = B$ and $A \in L(V_{\epsilon+1}, B)$.* $\square$

We warn that (2.2) in the proof of Lemma 6 of [5] contains a typo. We also need a routine generalization of another theorem of Woodin:

Theorem 4.16 ([5], Lemma 22)**.** *Suppose ϵ is an even ordinal, B is a subset of $V_{\epsilon+1}$, and $j : L(V_{\epsilon+1}, B) \to L(V_{\epsilon+1}, B)$ is an elementary embedding such that $j(B) = B$. Let θ be $\theta_{\epsilon+2}$ as computed in $L(V_{\epsilon+1}, B)$. Then θ is a limit of regular cardinals in $L(V_{\epsilon+1}, B)$.* $\square$

We note that although Woodin's proof seems to use λ-DC, this is not really necessary by the proof of Theorem 4.18 below.

Proof of Theorem 4.14. Fix a cardinal $\eta < \theta_{\epsilon+2}$. We will show that there is a regular cardinal in the interval $(\eta, \theta_{\epsilon+2})$. By the Coding Lemma, there is a code-class Γ for η such that $\Gamma \preceq^* V_{\epsilon+1}$. Let $\varphi : V_{\epsilon+1} \to \eta$ be a surjection in M and let $A \subseteq V_{\epsilon+1}$ be a set such that $\Gamma \subseteq L(V_{\epsilon+1}, A)$ and such that $L(V_{\epsilon+1}, A)$ satisfies that $\eta < \theta_{\epsilon+2}$. Let B be a set such that $j(B) = B$ and $A \in L(V_{\epsilon+1}, B)$.

Let θ be $\theta_{\epsilon+2}$ as computed in $L(V_{\epsilon+1}, B)$. We claim that for every ordinal $\gamma < \theta$, $\gamma^\eta \subseteq L(V_{\epsilon+1}, B)$. To see this, fix $s : \eta \to \gamma$, and we will show that $s \in L(V_{\epsilon+1}, B)$. Let $\psi : V_{\epsilon+1} \to \gamma$ be a surjection. Let $R = \{(x, y) \mid s(\varphi(x)) = \psi(y)\}$. Since Γ is a code-class for η, there is a subrelation $\bar{R}$ of R in Γ such that $\text{dom}(\bar{R}) = \text{dom}(R)$. But $\bar{R} \in L(V_{\epsilon+1}, B)$ and s is clearly coded by $\bar{R}$. Therefore $s \in L(V_{\epsilon+1}, B)$, as desired.

By Theorem 4.16, θ is a limit of regular cardinals in $L(V_{\epsilon+1}, B)$. Therefore let $\iota \in (\eta, \theta)$ be a regular cardinal of $L(V_{\epsilon+1}, B)$. Since $\iota^\eta \subseteq L(V_{\epsilon+1}, B)$, $\text{cf}(\iota) \in (\eta, \theta)$. In particular, there is a regular cardinal in the interval $(\eta, \theta_{\epsilon+2})$. $\square$

Question 4.17. Suppose ϵ is an even ordinal. Suppose there is an elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$ with critical point κ. Assume $\mathcal{H}_{\epsilon+2}$ satisfies the Collection Principle and κ-DC. Must $\theta_{\epsilon+2}$ be a limit of measurable cardinals?

Let us now show that for certain inner models, one can avoid the extra assumptions in Theorem 4.9.

Theorem 4.18. *Suppose N is an inner model of* ZFC*, ϵ is an even ordinal, $A \subseteq V_{\epsilon+1}$, and W is a set. Let $M = N(V_{\epsilon+1}, A)[W]$. Suppose there is an elementary embedding from $M \cap V_{\epsilon+2}$ to $M \cap V_{\epsilon+2}$. Then the Coding Lemma holds in M at ϵ.*

Sketch. We just describe how to modify the proofs above to avoid assuming Collection and Dependent Choice.

The first point is that one can prove $(\mathcal{H}_{\epsilon+2})^M$ satisfies the Collection Principle. To see this, note that by the definition of M, for any $X \in M$, there is an ordinal γ such that $X \preceq^* V_{\epsilon+1} \times \gamma$ in M. Therefore fix a surjection

$$f : V_{\epsilon+1} \times \gamma \to (\mathcal{H}_{\epsilon+2})^M$$

with $f \in M$. For each $x \in V_{\epsilon+1}$, let $\Gamma_x = f[\{x\} \times \gamma]$ and let $<_x$ be the wellorder of H_x induced by f. Suppose $R \subseteq (\mathcal{H}_{\epsilon+2})^M$ is a definable relation whose domain belongs to $(\mathcal{H}_{\epsilon+2})^M$. Then $R \in M$. For each $x \in V_{\epsilon+1}$, let $r_x(a)$ be the $<_x$-least b in H_x such that $(a, b) \in R$. The sequence $\langle r_x \mid x \in V_{\epsilon+1}\rangle$ belongs to M, and indeed it belongs to $(\mathcal{H}_{\epsilon+2})^M$. This is because one can define a partial surjection from $\text{dom}(R)$ to r_x uniformly in x. Therefore $\bigcup_{x\in V_{\epsilon+1}} r_x \in (\mathcal{H}_{\epsilon+2})^M$, and letting $\bar{R} = \bigcup_{x\in V_{\epsilon+1}} r_x$, we have $\text{dom}(\bar{R}) = \text{dom}(R)$. This verifies that $(\mathcal{H}_{\epsilon+2})^M$ satisfies the Collection Principle.

To finish the sketch, we describe how in this special case one can avoid the use of Dependent Choice in the proof of the Weak Coding Lemma (Lemma 4.11). One need only modify the construction of the sequence $\langle (A_\alpha, \varphi_\alpha) \mid \alpha < \kappa\rangle$. One instead constructs a sequence $\langle (A_{x,\alpha}, \varphi_{x,\alpha}) \mid (x, \alpha) \in V_{\epsilon+1} \times \kappa\rangle$. The construction proceeds as follows. Suppose that $\langle (A_{x,\alpha}, \varphi_{x,\alpha}) \mid (x, \alpha) \in V_{\epsilon+1} \times \beta\rangle$ has been defined. By the failure of the Weak Coding Lemma, there is some (A, φ) such that A has no φ-cofinal subrelation that is definable over $(V_{\epsilon+1}, A_{x,\alpha})$ for some $(x, \alpha) \in V_{\epsilon+1} \times \beta$. For each $x \in V_{\epsilon+1}$, let $(A_{x,\beta}, \varphi_{x,\beta})$ be the $<_x$-least such $(A, \varphi) \in H_x$, if one exists. This completes the construction.

Now one considers:

$$\langle (A^1_{x,\alpha}, \varphi^1_{x,\alpha}) \mid (x, \alpha) \in V_{\epsilon+1} \times \kappa_1\rangle = j(\langle (A_{x,\alpha}, \varphi_{x,\alpha}) \mid (x, \alpha) \in V_{\epsilon+1} \times \kappa\rangle)$$

Fix any $x \in V_{\epsilon+1}$ such that $(A^1_{x,\kappa}, \varphi^1_{x,\kappa})$ is defined. One then uses $(A^1_{x,\kappa}, \varphi^1_{x,\kappa})$ in place of $(A^1_\kappa, \varphi^1_\kappa)$. The rest of the proof is unchanged. □

We conclude this section by showing that the Coding Lemma fails at odd ordinals in a strong sense. For example, we show that if ϵ is even and there is an elementary embedding from $V_{\epsilon+3}$ to $V_{\epsilon+3}$, then $V_{\epsilon+2}$ does not surject onto $P(\theta_{\epsilon+2})$. By Proposition 4.7, this implies that the Coding Lemma does not hold at $\epsilon + 1$.

Theorem 4.19. *Suppose ϵ is an even ordinal and there is an elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$. Then for any ordinal γ, there is no surjection from $V_\epsilon \times \gamma$ onto $P(\theta_\epsilon)$.*

Proof. Suppose towards a contradiction that the theorem fails. Let $\theta = \theta_\epsilon$. Let γ be the least ordinal such that $P(\theta) \preceq^* V_\epsilon \times \gamma$. We first show

$$\gamma \preceq^* V_\epsilon \times P(\theta) \preceq^* V_{\epsilon+1} \tag{9}$$

For the first inequality, fix a surjection $f : V_\epsilon \times \gamma \to P(\theta)$. Define

$$g : V_\epsilon \times P(\theta) \to \gamma$$

by setting $g(A, S) = \min\{\xi \mid f(A, \xi) = S\}$. Let T be the range of g. Then $f[V_\epsilon \times T] = P(\theta)$, so by the minimality of γ, it must be that $|T| = \gamma$. Thus $\gamma \preceq^* V_\epsilon \times P(\theta)$.

For the second inequality, note that $\theta \preceq^* V_\epsilon$ so $P(\theta) \preceq^* P(V_\epsilon) = V_{\epsilon+1}$. It follows easily that $V_\epsilon \times P(\theta) \preceq^* V_{\epsilon+1}$.

By our large cardinal hypothesis, there is an elementary embedding $j : \mathcal{H}_{\epsilon+2} \to \mathcal{H}_{\epsilon+2}$. Note that $f \in \mathcal{H}_{\epsilon+2}$ by (9). By the elementarity of j, $j(f)$ is a surjection from $V_\epsilon \times j(\gamma)$ to $P(\theta)$; therefore, for some $(a, \alpha) \in V_\epsilon \times j(\gamma)$, $j(f)(a, \alpha) = j[\theta]$. By the cofinal embedding property (Proposition 3.15), or trivially if ϵ is a limit ordinal, there is an ordinal ξ such that $\xi + 2 \leq \epsilon$, a set $x \subseteq V_\xi$, and a function $g : V_{\xi+1} \to V_\epsilon$ such that $j(g)(x) = a$. Let U be the ultrafilter over $V_{\xi+1} \times \gamma$ derived from j using (x, α). Let $h : V_{\xi+1} \times \gamma \to P(\theta)$ be defined by $h(u, \beta) = f(g(u), \beta)$. Then it is easy to see that $[h]_U = j_U[\theta]$. This contradicts Theorem 3.27. $\square$

It is a bit strange that for example we require an embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$ to show this structural property of $P(\theta_\epsilon)$. The theorem implies that $P(\theta_\epsilon)$ cannot be wellordered, so for example in the case "$\epsilon = \kappa_\omega(j)$," one cannot reduce the large cardinal hypothesis to an embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ assuming the consistency of ZFC plus I_1. (Similar results hold for $j : V_{\epsilon+2} \to V_{\epsilon+2}$, considering the model $L(V_{\epsilon+1})[j]$.) Inspecting the proof, however, one obtains the following result:

Theorem 4.20. *Suppose ϵ is an even ordinal and there is an elementary embedding from $V_{\epsilon+1}$ to $V_{\epsilon+1}$. Then there is no surjection from V_ϵ onto $P(\theta_\epsilon)$.* $\square$

The following question is related to Theorem 4.19 and might be more tractable than the question of whether $\theta_{\epsilon+1} = (\theta_\epsilon)^+$:

Question 4.21. Suppose ϵ is an even ordinal and there is an elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$. Is there a surjection from $P(\theta_\epsilon)$ to $\theta_{\epsilon+1}$?

5 Ultrafilters and the Ketonen order

5.1 Measurable cardinals and the Ulam argument

With the goal of refuting strong choiceless large cardinal axioms in mind, Woodin [4] showed that various consequences of the Axiom of Choice follow from the existence of large cardinals at the level of supercompact and extendible cardinals. While developing set theoretic geology in the choiceless context, Usuba realized that the apparently much weaker notion of a Löwenheim-Skolem cardinal does just as well as a supercompact.

Definition 5.1. A cardinal κ is a *Löwenheim-Skolem cardinal* if for all ordinals $\alpha < \kappa \leq \gamma$, for any $a \in V_\gamma$, there is an elementary substructure $X \prec V_{\gamma+1}$ such that $[X]^{V_\alpha} \cap V_\gamma \subseteq X$, $a \in X$, and for some $\beta < \kappa$, $X \preceq^* V_\beta$.

Here we give a proof of Ulam's theorem on the atomicity of saturated filters in ZF assuming the existence of two strategically placed Löwenheim-Skolem cardinals.

Our arguments are inspired by the ones in [8], and our result generalizes some of the theorems of that paper while simultaneously reducing their large cardinal hypotheses.

Recall that the usual proof of Ulam's theorem uses a splitting argument that seems to make heavy use of a strong form of the Axiom of Dependent Choice. Here it is shown that this can be avoided if one is allowed to take two elementary substructures.

Theorem 5.2. *Suppose γ is a cardinal, $\kappa_0 < \kappa_1$ are Löwenheim-Skolem cardinals above γ, and δ is an ordinal. Suppose F is a filter over δ that is V_{κ_1}-complete and weakly γ-saturated. Then for some cardinal $\eta < \gamma$, there is a partition $\langle S_\alpha \mid \alpha < \eta\rangle$ of δ such that $F \restriction S_\alpha$ is an ultrafilter for all $\alpha < \eta$.*

Proof. Since κ_1 is a Löwenheim-Skolem cardinal, we can fix an elementary substructure $X \prec V_{\delta+\omega+1}$ with the following properties:

- $X \preceq^* V_\beta$ for some $\beta < \kappa_1$.
- $\gamma, \kappa_0, \kappa_1, \delta$, and F belong to X.
- $[X]^{V_\alpha} \cap V_{\delta+\omega} \subseteq X$ for every $\alpha < \kappa_0$.

Let $\pi : H_X \to V_{\delta+\omega+1}$ be the inverse of the Mostowski collapse of X, and let $\bar{\gamma}, \bar{\kappa}_0, \bar{\kappa}_1, \bar{\delta}$, and $\bar{F}$ be the preimages under π of $\gamma, \kappa_0, \kappa_1, \delta$, and F respectively.

For each ordinal $\xi < \delta$, let U_ξ denote the H_X-ultrafilter over $\bar{\delta}$ derived from π using ξ. Since π has critical point above κ_0 and H_X is closed under V_α-sequences for every $\alpha < \kappa_0$, for all $\xi < \delta$, U_ξ is V_{κ_0}-complete. (More precisely, U_ξ generates a V_{κ_0}-complete filter.) Since $\{U_\xi \mid \xi < \delta\} \subseteq P(H_X)$ and $H_X \preceq^* V_\beta$, $\{U_\xi \mid \xi < \delta\} \preceq^* V_{\beta+1}$.

For each $\xi < \delta$, let $B_\xi = \{\xi' \mid U_{\xi'} = U_\xi\}$. We make the obvious observation that the map sending B_ξ to U_ξ is a (well-defined) one-to-one correspondence. It follows that $\{B_\xi \mid \xi < \delta\} \preceq^* V_{\beta+1}$. Let $T \subseteq \delta$ be the set of $\xi < \delta$ such that B_ξ is F-positive. Since F is V_{κ_1}-complete and $\kappa_1 > \beta + 1$, $T \in F$. (This is a standard argument: $\{B_\xi \mid \xi \in \delta \setminus T\}$ is a collection of F-null sets with $\{B_\xi \mid \xi \in \delta \setminus T\} \preceq^* V_{\beta+1}$, and hence $\bigcap_{\xi\in\delta\setminus T} B_\xi$ is F-null by the V_{κ_1}-completeness of F. Therefore the complement of $\bigcap_{\xi\in\delta\setminus T} B_\xi$ belongs to F. Note that $\xi \in T$ if and only if $B_\xi \subseteq T$, and hence $T = \bigcup_{\xi\in T} B_\xi$. It follows that T is the complement of $\bigcap_{\xi\in\delta\setminus T} B_\xi$, so $T \in F$ as desired.) Since F is weakly γ-saturated and $\{B_\xi \mid \xi \in T\}$ is a partition of δ into positive sets, $|\{B_\xi \mid \xi \in T\}| < \gamma$. Given the one-to-one correspondence described above, it follows that $|\{U_\xi \mid \xi \in T\}| < \gamma$.

Now since κ_0 is a Löwenheim-Skolem cardinal and $\gamma < \kappa_0$, we can fix an elementary substructure $Y \prec V_{\delta+\omega+2}$ with $\{U_\xi \mid \xi \in T\} \in Y$, $X \in Y$, $\gamma \subseteq Y$, and $Y \preceq^* V_{\beta'}$ for some $\beta' < \kappa_0$. Since $|\{U_\xi \mid \xi \in T\}| < \gamma$, it follows that $U_\xi \in Y$ for all $\xi \in T$. Notice, however, that $U_\xi \cap Y \in X$ since $[X]^{V_{\beta'}} \cap V_{\delta+\omega} \subseteq X$. For $\xi \in T$, let

$$A_\xi = \bigcap\{A \in U_\xi \mid A \in Y\}$$

Notice that $A_\xi \in U_\xi$ for all $\xi \in T$ since U_ξ is V_{κ_0}-complete.

We claim that $A_{\xi_0} \cap A_{\xi_1} = \emptyset$ whenever $U_{\xi_0} \neq U_{\xi_1}$. (Obviously if $U_{\xi_0} = U_{\xi_1}$, then $A_{\xi_0} = A_{\xi_1}$.) To see this, note that since $U_{\xi_0} \neq U_{\xi_1}$ and $Y \prec V_{\delta+\omega+2}$, there is some $A \in H_X \cap Y$ with $A \in U_{\xi_0}$ and $\bar{\delta} \setminus A \in U_{\xi_1}$. It follows that $A_{\xi_0} \subseteq A$ and $A_{\xi_1} \subseteq \bar{\delta} \setminus A$, and hence $A_{\xi_0} \cap A_{\xi_1} = \emptyset$, as desired.

Let $S = \{\xi \in T \mid \bar{F} \subseteq U_\xi\}$. Thus $S = T \cap \bigcap\{A \in F \mid A \in X\}$, so since T and $\bigcap\{A \in F \mid A \in X\}$ belong to F, $S \in F$. We claim that for all $\xi \in S$, A_ξ is an atom of $\bar{F}$ in H_X. Fix $\xi \in S$, and suppose towards a contradiction that E_0 and E_1 are disjoint $\bar{F}$-positive subsets of A_ξ that belong to H_X. Since $\pi(E_0)$ is F-positive, $\pi(E_0) \cap S \neq \emptyset$, so fix $\xi_0 \in \pi(E_0) \cap S$. Note that $E_0 \in U_{\xi_0}$ since U_{ξ_0} is the ultrafilter derived from π using ξ_0. Similarly fix $\xi_1 \in \pi(E_1) \cap S$, and note that $E_1 \in U_{\xi_1}$. Since E_0 and E_1 are disjoint, it follows that $U_{\xi_0} \neq U_{\xi_1}$. In particular, one of them is not equal to U_ξ. Assume without loss of generality that $U_{\xi_0} \neq U_\xi$. Since $E_0 \in U_{\xi_0}$, we have $E_0 \cap A_{\xi_0} \neq \emptyset$. It follows that $A_\xi \cap A_{\xi_0} \neq \emptyset$, and this contradicts that $A_{\xi_0} \cap A_{\xi_1} = \emptyset$ whenever $U_{\xi_0} \neq U_{\xi_1}$.

Therefore $\{A_\xi \mid \xi \in S\}$ is a set of atoms for $\bar{F}$. We claim $\bigcup_{\xi \in S} A_\xi \in F$. Suppose not, towards a contradiction. In other words, the set

$$E = \bar{\delta} \setminus \bigcup_{\xi \in S} A_\xi$$

is F-positive. For every $\xi \in S$, since $A_\xi \in U_\xi$ and $\bar{F} \subseteq U_\xi$ for all $\xi \in S$, necessarily $\bar{F} \restriction A_\xi = U_\xi$. Note that E belongs to U_ξ for some $\xi \in S$: indeed, since $\pi(E)$ is F-positive and $S \in F$, $\pi(E) \cap S \neq \emptyset$; now for any $\xi \in \pi(E) \cap S$, $E \in U_\xi$. But then $E \cap A_\xi \neq \emptyset$, which contradicts that $E = \bar{\delta} \setminus \bigcup_{\xi \in S} A_\xi$.

Finally, note that $\{A_\xi \mid \xi \in T\} \in H_X$ since $[H_X]^\gamma \cap V_{\bar{\delta}+\omega} \subseteq H_X$. Hence H_X satisfies that there is a partition of an $\bar{F}$-large set into fewer than γ-many atoms. By the elementarity of π, $V_{\delta+\omega+1}$ satisfies that there is a partition of an F-large set into fewer than γ-many atoms. Obviously this is absolute to V, which completes the proof. $\square$

Corollary 5.3. *Suppose η is a limit of Löwenheim-Skolem cardinals and there is an elementary embedding from $V_{\eta+2}$ to $V_{\eta+2}$. If η is regular, then η is measurable, and if η is singular, then η^+ is measurable.* $\square$

5.2 Preliminaries on the Ketonen order

Recall the notion of an ultrafilter comparison (Definition 7.6) that played a role in Proposition 3.20. One obtains an order on ultrafilters over ordinals by setting $U <_{\Bbbk} W$ if there is an ultrafilter comparison from (U, id) to (W, id). Let us give a more concrete definition of this order.

Definition 5.4. Suppose F is a filter over X and $\langle G_x \mid x \in X\rangle$ is a sequence of filters over Y. Then the *F-limit of* $\langle G_x \mid x \in X\rangle$ is the filter

$$F\text{-}\lim_{x \in X} G_x = \{A \subseteq Y \mid \{x \in X \mid A \in G_x\} \in F\}$$

Definition 5.5. Suppose δ is an ordinal. The *Ketonen order* is defined on countably complete ultrafilters over δ by setting $U \leq_{\Bbbk} W$ if U is of the form $W\text{-}\lim_{\alpha<\delta} Z_\alpha$ where Z_α is a countably complete ultrafilter over δ concentrating on $\alpha + 1$.

The following fact is easy to verify:

Lemma 5.6. *The Ketonen order is transitive and anti-symmetric.* $\square$

Definition 5.7. Set $U <_{\Bbbk} W$ if $U \leq_{\Bbbk} W$ and $W \not\leq_{\Bbbk} U$.

Equivalently, $U <_{\Bbbk} W$ if $U \leq_{\Bbbk} W$ and $U \neq W$. Also $U <_{\Bbbk} W$ if and only if U is of the form $W\text{-}\lim_{\alpha<\delta} Z_\alpha$ where Z_α is a countably complete ultrafilter over δ concentrating on α. Also, as we mentioned above, $U <_{\Bbbk} W$ if there is an ultrafilter comparison from (U, id) to (W, id). In the appendix, we give a proof of the following theorem:

Theorem 7.13 (DC)**.** *The Ketonen order is wellfounded.* $\square$

Definition 5.8 (DC)**.** The *Ketonen rank of* U, denoted $\sigma(U)$, is the rank of U in the Ketonen order.

A straightforward alternate characterization of the Ketonen order turns out to be important here.

Definition 5.9. Suppose δ is an ordinal. A function $h : P(\delta) \to P(\delta)$ is *Lipschitz* if for all $\alpha < \delta$ and all $A, B \subseteq \delta$ with $A \cap \alpha = B \cap \alpha$, $h(A) \cap \alpha = h(B) \cap \alpha$, and h is *strongly Lipschitz* if for all $\alpha < \delta$ and all $A, B \subseteq \delta$ with $A \cap \alpha = B \cap \alpha$, $h(A) \cap (\alpha + 1) = h(B) \cap (\alpha + 1)$.

Lemma 5.10. *Suppose U and W are countably complete ultrafilters over δ.*

- *$U \leq_{\Bbbk} W$ if and only if there is a countably complete Lipschitz homomorphism $h : P(\delta) \to P(\delta)$ such that $h^{-1}[W] = U$.*
- *$U <_{\Bbbk} W$ if and only if there is a countably complete strongly Lipschitz homomorphism $h : P(\delta) \to P(\delta)$ such that $h^{-1}[W] = U$.* $\square$

Notice that an elementary embedding from $P(\delta)$ to $P(\delta)$ is a countably complete Lipschitz homomorphism.

5.3 Ordinal definability and the Ultrapower Axiom

The Ultrapower Axiom (UA) is an inner model principle introduced by the author in [1] to develop the general theory of countably complete ultrafilters and in particular the theory of strongly compact and supercompact cardinals. Implicit in the statement of UA is the assumption of the Axiom of Choice, and dropping that assumption, there are a number of inequivalent reformulations of the principle. In ZFC, however, the Ultrapower Axiom is equivalent to the linearity of the Ketonen order. The following theorem therefore shows that in one sense, the existence of a Reinhardt cardinal almost implies the Ultrapower Axiom.

Theorem 5.11. *Suppose $j : V \to V$ is an elementary embedding and $\kappa_\omega(j)$-DC holds. Then for any ordinals δ and ξ, the set of countably complete ultrafilters over δ of Ketonen rank ξ has cardinality strictly less than $\kappa_\omega(j)$.* $\square$

We will prove a more technical theorem that also applies to $L(V_{\lambda+1})$ and other small models.

Theorem 5.12. *Suppose ϵ is an ordinal, M is an inner model containing $V_{\epsilon+1}$, and $\delta < \theta^M_{\epsilon+2}$ is an ordinal. Suppose there is a nontrivial embedding $j \in \mathscr{E}(V_{\epsilon+3} \cap M)$ such that $j \restriction P^M(\delta)$ belongs to M. Assume M satisfies λ-DC where $\lambda = \kappa_\omega(j)$. Then in M, for any ordinal $\xi < \theta_{\epsilon+3}$, the set of countably complete ultrafilters over δ of Ketonen rank ξ is wellorderable and has cardinality strictly less than λ.*

Why $V_{\epsilon+3}$? The point of this large cardinal hypothesis is that working in M, every countably complete ultrafilter over an ordinal less than $\theta_{\epsilon+2}$ belongs to $\mathcal{H}_{\epsilon+3}$, and moreover the Ketonen order and its rank function are definable over $\mathcal{H}_{\epsilon+3}$. Therefore by the remarks following Definition 2.8, an embedding $j \in \mathscr{E}(V_{\epsilon+3} \cap M)$ lifts to an embedding $j^\star \in \mathscr{E}((\mathcal{H}_{\epsilon+3})^M)$ that is Ketonen order preserving and in addition respects Ketonen ranks in the sense that $\sigma(j^\star(U)) = j^\star(\sigma(U))$ for any countably complete ultrafilter U over an ordinal less than $\theta_{\epsilon+2}$. This is what is needed for the proof of Theorem 5.12.

Proof of Theorem 5.12. By considering the least counterexample, we may assume without loss of generality that j fixes δ and ξ. Since M satisfies λ-DC, it suffices to show that in M, there is no λ-sequence of distinct countably complete ultrafilters over δ of Ketonen rank ξ. Suppose towards a contradiction that $\langle U_\alpha \mid \alpha < \lambda\rangle$ is such a sequence. Note that $\langle U_\alpha \mid \alpha < \lambda\rangle$ is coded by an element of $V_{\epsilon+3} \cap M$, so we can apply j to it. This yields:

$$\langle U^1_\alpha \mid \alpha < \lambda\rangle = j(\langle U_\alpha \mid \alpha < \lambda\rangle)$$
$$\langle U^2_\alpha \mid \alpha < \lambda\rangle = j(\langle U^1_\alpha \mid \alpha < \lambda\rangle)$$

Let κ be the critical point of j. We claim that the following hold for $A \in P^M(\delta)$:

$$A \in U^1_\kappa \iff j(A) \in U^2_{j(\kappa)} \tag{10}$$
$$A \in U^1_\kappa \iff j(i)(A) \in U^2_\kappa \tag{11}$$

where $i = j \restriction P^M(\delta)$. (10) is trivial since $j(U^1_\kappa) = U^2_{j(\kappa)}$. (11) is slightly more subtle because we are not assuming $j \restriction P(P(\delta))$ belongs to M, and therefore $j(j)(U^1_\kappa)$ is not obviously well-defined. Note however that for all $\alpha < \kappa$, $A \in U_\alpha$ if and only if $i(A) \in U^1_\alpha$. By the elementarity of j, for all $\alpha < j(\kappa)$, $A \in U^1_\alpha$ if and only if $j(i)(A) \in U^2_\alpha$. In particular, $A \in U^1_\kappa$ if and only if $j(i)(A) \in U^2_\kappa$, proving (11).

Now notice that in M, i and $j(i)$ are countably complete Lipschitz homomorphisms from $P(\delta)$ to $P(\delta)$. Therefore by the characterization of the Ketonen order in terms of Lipschitz homomorphisms (Lemma 5.10), (10) and (11) imply:

$$U^1_\kappa \leq_{\Bbbk} U^2_{j(\kappa)}$$
$$U^1_\kappa \leq_{\Bbbk} U^2_\kappa$$

Now we use that j fixes ξ, which implies that the three ultrafilters U^1_κ, U^2_κ, and $U^2_{j(\kappa)}$ have Ketonen rank ξ. Since the Ketonen order is wellfounded, it follows that the inequalities above cannot be strict, and so since the Ketonen order is antisymmetric, $U^2_{j(\kappa)} = U^1_\kappa = U^2_\kappa$. This contradicts that $\langle U^2_\alpha \mid \alpha < \lambda\rangle$ is a sequence of distinct ultrafilters. $\square$

Although Theorem 5.12 shows that the Ketonen order is *almost* linear under choiceless large cardinal assumptions, true linearity is incompatible with $\kappa_\omega(j)$-DC:

Proposition 5.13. *Suppose $j : V_{\lambda+2} \to V_{\lambda+2}$ is an elementary embedding with $\lambda = \kappa_\omega(j)$. Suppose λ is a limit of Löwenheim-Skolem cardinals. Assume the restriction of the Ketonen order to λ^+-complete ultrafilters over λ^+ is linear. Then* ω_1-DC *is false.*

Proof. We begin by outlining the proof. Assume towards a contradiction that ω_1-DC is true. We first prove that the Ketonen least ultrafilter U on λ^+, which exists by the linearity of the Ketonen order, extends the ω-closed unbounded filter. Next, we show that, assuming ω_1-DC, there is a normal ultrafilter W on λ^+ extending the ω_1-closed unbounded filter.

Let us show that the existence of the ultrafilters U and W actually implies that ω_1 is measurable, contradicting ω_1-DC. Clearly $U <_\Bbbk W$. As a consequence $U = W\text{-}\lim_{\alpha<\lambda^+} U_\alpha$ where U_α is a countably complete ultrafilter such that $\alpha \in U_\alpha$ for W-almost all $\alpha < \lambda^+$. For each $\alpha < \lambda^+$, let δ_α be the least ordinal such that $\delta_\alpha \in U_\alpha$. Then $\delta_\alpha = \alpha$ for W-almost all $\alpha < \lambda^+$: otherwise, since W is normal, there is a $\delta < \lambda^+$ such that $\delta_\alpha = \delta$ for W-almost all $\alpha < \lambda^+$, which implies $\delta \in W\text{-}\lim_{\alpha<\lambda^+} U_\alpha = U$, contradicting that U is a uniform ultrafilter over λ^+. Fix an ordinal $\alpha < \lambda^+$ of cofinality ω_1 such that $\delta_\alpha = \alpha$. Then U_α is a fine ultrafilter over α; that is, every set in U is cofinal in α. This implies that $\text{cf}(\alpha)$ carries a uniform countably complete ultrafilter D: let $f : \alpha \to \omega_1$ be any monotone function and let $D = f_*(U_\alpha)$. This means that ω_1 is measurable, which is a contradiction.

The first step is to show that DC implies that there is a normal filter over λ^+ extending the ω-closed unbounded filter. For this, we show that the *weak club filter* is normal. This is the filter F generated by sets of the form

$$\{\sup(\sigma \cap \lambda^+) \mid \sigma \prec M\}$$

where M is a structure in a countable language containing λ^+. The normality of this filter, given the Löwenheim-Skolem hypothesis, is proved by Usuba as [15, Proposition 3.5]. By DC, the set

$$S = \{\alpha < \lambda^+ \mid \text{cf}(\alpha) = \omega\}$$

is F-positive. Hence $F \restriction S$ is a normal filter extending the ω-closed unbounded filter. By the Woodin argument and Theorem 5.2, $F \restriction S$ is atomic. Therefore there is some $T \subseteq S$ such that $F \restriction T$ is an ultrafilter. Of course $F \restriction T$ is normal since F is, and hence we have obtained a normal ultrafilter U extending the ω-closed unbounded filter.

We claim that no uniform ultrafilter over λ^+ lies below U in the Ketonen order. To see this, suppose $Z <_{\Bbbk} U$, and we will show that there is some $\delta < \lambda^+$ such that $\delta \in Z$. Suppose $Z = U\text{-}\lim_{\xi<\lambda^+} D_\xi$ where D_ξ is a countably complete ultrafilter over λ^+ with $\xi \in D_\xi$ for U-almost all $\xi < \lambda^+$. For U-almost all $\xi < \lambda^+$, ξ has cofinality ω and D_ξ is a countably complete ultrafilter with $\xi \in D_\xi$, so there is some $\delta_\xi < \xi$ with $\delta_\xi \in D_\xi$. Since U is normal, there is a fixed ordinal $\delta < \lambda^+$ such that $\delta_\xi = \delta$ for U-almost all $\xi < \lambda^+$. Hence $\delta \in D_\xi$ for U-almost all $\xi < \lambda^+$, so since $Z = U\text{-}\lim_{\xi<\lambda^+} D_\xi$, $\delta \in Z$.

Thus U is the Ketonen least ultrafilter over λ^+.

Using ω_1-DC, one can show that $\{\alpha < \lambda^+ \mid \text{cf}(\alpha) = \omega_1\}$ is positive with respect to the weak club filter. It follows as above that the ω_1-closed unbounded filter extends to a normal ultrafilter. As explained in the first two paragraphs, this leads to the conclusion that ω_1 is measurable, which contradicts our assumption that ω_1-DC holds. $\square$

It would not be that surprising if it turned out to be possible to refute the linearity of the Ketonen order outright from choiceless large cardinals. If the linearity of the Ketonen order is consistent with choiceless large cardinals, however, then perhaps there is an interesting theory of choiceless large cardinals in which choice fails low down. We will not pursue this idea further here since it leads to highly speculative territory. We do note that one can make do with a weaker choice assumption in the proof of Theorem 5.12:

Theorem 5.14. *Suppose ϵ is an ordinal, M is an inner model of* DC *containing $V_{\epsilon+1}$, and $\delta < \theta^M_{\epsilon+2}$ is an ordinal. Suppose there is a nontrivial $j \in \mathscr{E}(V_{\epsilon+3} \cap M)$ such that $j \restriction P^M(\delta)$ belongs to M. Assume $\lambda = \kappa_\omega(j)$ is a limit of Löwenheim-Skolem cardinals in M. Then the following hold in M:*

(1) For any ordinal $\xi < \theta_{\epsilon+3}$, the set of countably complete ultrafilters over δ of Ketonen rank ξ is the surjective image of V_α for some $\alpha < \lambda$.

(2) The set of V_λ-complete ultrafilters of Ketonen rank ξ is wellorderable and has cardinality strictly less than λ.

The proof uses the following lemma, whose analog in the context of AC is well known and does not require the supercompactness assumption that we make below:

Lemma 5.15. *Assume M is a model of set theory and $j : M \to N$ is an elementary embedding with critical point κ. Assume there is a Löwenheim-Skolem cardinal η in M such that $\kappa < \eta < j(\kappa)$ and $j[V_\xi \cap M] \in N$ for all $\xi < \eta$. Then for any set $S \in M$ such that $j(S) = j[S]$, there is some $\alpha < \kappa$ such that M satisfies $S \preceq^* V_\alpha$.*

Proof. We first observe that if there is a surjection $f : S \to S'$ in M, then

$$j(S') = j(f)[j(S)] = j(f)[j[S]] = j[S']$$

Work in M, and assume towards a contradiction that $S \not\preceq^* V_\alpha$ for any $\alpha < \kappa$. Fix $\gamma > \beta$ and an elementary substructure $X \prec V_\gamma$ with $S \in X$, $[X]^{V_\alpha} \subseteq X$, and for some $\nu < \eta$, $X \preceq^* V_\nu$. Let $S' = X \cap S$. Notice that there is no surjection from V_α to S' for any $\alpha < \kappa$: if there is, then $S' \in X$ since X is closed under V_α-sequences, and hence $S' = S$ because $S \in X$ and $(S \setminus S') \cap X = \emptyset$; but then $S \preceq^* V_\alpha$, which is a contradiction. Let ξ be the least rank of a set a that is in bijection with S'. Then $\kappa \leq \xi < \eta$.

We now leave M. On the one hand, $j(\xi) > j(\kappa) > \eta > \xi$. On the other hand, $j(S') = j[S']$ by our first observation. Let $f : a \to S'$ be a bijection in M, and notice that $a \in N$ and $j(f) \circ j \restriction a \in N$ is a bijection between $j[S']$ and a that belongs to N. Therefore in N, $|j(S')| = |a|$. It follows that in N, ξ is the least rank of a set in bijection with $j(S')$. This contradicts that $j(\xi) > \xi$. $\square$

Proof of Theorem 5.14. Suppose towards a contradiction that the theorem fails.

We work in M for the time being. Let $\xi < \theta_{\epsilon+3}$ be the least ordinal such that the set of countably complete ultrafilters over δ of Ketonen rank ξ is not the surjective image of V_β for any $\beta < \lambda$. Let S be any set of countably complete ultrafilters over δ of Ketonen rank ξ. Leaving M, a generalization of the proof of Theorem 5.12 will show that $j(S) = j[S]$. Assume otherwise, fix $U \in j(S) \setminus j[S]$ and consider $j(U)$ and $j(j)(U)$. On the one hand, these ultrafilters must be distinct: by elementarity $j(U) \notin j(j[S]) = j(j)[j(S)]$, whereas evidently $j(j)(U) \in j(j)[j(S)]$. On the other hand, $j \restriction P(\delta)$ and $j(j) \restriction P(\delta)$ witness that $U \leq_{\Bbbk} j(U), j(j)(U)$ in M, and therefore $U = j(U) = j(j)(U)$ since $\sigma(U) = \sigma(j(U)) = \sigma(j(j)(U)) = \xi$ since of course j and $j(j)$ fix the definable ordinal ξ. This is a contradiction, so in fact $j(S) = j[S]$.

By Lemma 5.15, M satisfies that $S \preceq^* V_\alpha$ for some $\alpha < \text{crit}(j)$. This contradiction proves (1).

Now consider the set S of V_λ-complete ultrafilters over δ of Ketonen rank ξ. By an argument similar to that of Theorem 5.2, one can use the Löwenheim-Skolem assumption to find a "discretizing family" for S, or in other words a function $f : S \to P(\delta)$ such that $f(U) \in U \setminus W$ for all $W \in S$ except for U. Then the function $g(U) = \min f(U)$ is an injection from S into δ, so S is wellorderable. Since $\theta_\alpha < \lambda$ for all $\alpha < \lambda$, it follows that $|S| < \lambda$, proving (2). $\square$

We remark that an argument similar to the proof of Theorem 5.14 can be used to establish the Coding Lemma (Theorem 4.9) from a Löwenheim-Skolem hypothesis rather than dependent choice.

The semi-linearity of the Ketonen order given by Theorem 5.12 implies that V is in a sense "close to HOD." (No such closeness result is known to be provable from large cardinal axioms consistent with the Axiom of Choice, so this perhaps complicates the intuition that choiceless large cardinal axioms imply that HOD is a small model.) We first state the theorem in two special cases:

Theorem 5.16. *Suppose λ is a cardinal such that λ-DC holds. Assume $M = L(V_{\lambda+1})$ or $M = V$. Suppose there is an elementary embedding $j : M \to M$ with $\kappa_\omega(j) = \lambda$. Then M satisfies the following statements:*

(1) *Every countably complete ultrafilter over an ordinal belongs to an ordinal definable set of size less than λ.*

(2) *Every λ^+-complete ultrafilter over an ordinal δ is ordinal definable from a subset of δ.*

(3) *For any set of ordinals S, every λ^+-complete ultrafilter is amenable to* HOD_S.

(4) *For any λ^+-complete ultrafilter U over an ordinal, the ultrapower embedding j_U is amenable to* HOD_x *for a cone of $x \in V_\lambda$.* $\square$

We now prove a more technical result that immediately implies the previous theorem.

Theorem 5.17. *Suppose ϵ is an even ordinal, M is an inner model of* DC *containing $V_{\epsilon+1}$. Assume there is an elementary embedding j from $V_{\epsilon+3} \cap M$ to $V_{\epsilon+3} \cap M$ such that $j \restriction P(\delta) \in M$ for all $\delta < \theta^M_{\epsilon+2}$. Assume $\lambda = \kappa_\omega(j)$ is a limit of Löwenheim-Skolem cardinals in M. Then the following hold in M:*

(1) *Every countably complete ultrafilter over an ordinal below $\theta_{\epsilon+2}$ belongs to an ordinal definable set of size less than λ.*

(2) *Every λ^+-complete ultrafilter over an ordinal $\delta < \theta_{\epsilon+2}$ is ordinal definable from a subset of δ.*

(3) *For any set of ordinals S, every λ^+-complete ultrafilter over an ordinal $\delta < \theta_{\epsilon+2}$ is amenable to* HOD_S.

(4) *For any λ^+-complete ultrafilter U over an ordinal less than $\theta_{\epsilon+2}$, the ultrapower embedding j_U is amenable to* HOD_x *for a cone of $x \in V_\lambda$.*

Proof. We work entirely in M, using only the conclusion of Theorem 5.14.

(1) is clear from Theorem 5.12.

For (2), suppose U is a countably complete ultrafilter over δ. Let ξ be the Ketonen rank of U, and let $\langle U_\alpha : \alpha < \eta\rangle$ enumerate the λ^+-complete ultrafilters of Ketonen rank ξ. Choose a set $A \subseteq \delta$ such that $A \in U$ and $A \notin U_\alpha$ for any $\alpha < \eta$; this is possible because $\eta < \lambda$ and the ultrafilters in question are λ^+-complete. Since U is the unique λ^+-complete ultrafilter over δ of Ketonen rank ξ such that $A \in U$, U is ordinal definable from A.

We now prove (3). Let $\bar{U} = U \cap \text{HOD}_S$. We must show that $\bar{U} \in \text{HOD}_S$. Fix an OD set P of cardinality less than λ such that $U \in P$. Note that $F = \bigcap P$ is ordinal definable. Let $\bar{F} = F \cap \text{HOD}_S$. Then $\bar{F} \in \text{HOD}_S$. Using λ^+-completeness, it is obvious that F is λ-saturated, and it follows that $\bar{F}$ is λ-saturated in HOD_S. Applying the Ulam splitting theorem inside HOD_S, there is some $\eta < \lambda$ and a partition $\langle A_\alpha \mid \alpha < \eta\rangle \in \text{HOD}_S$ of δ into atoms of $\bar{F}$. Since $\bigcup A_\alpha = \delta$, there is some $\alpha < \eta$ such that $A_\alpha \in U$. It follows that $\bar{F} \restriction A_\alpha \subseteq U \cap \text{HOD}_S = \bar{U}$, and since $\bar{F} \restriction A_\alpha$ is a HOD_S-ultrafilter, this implies that $\bar{F} \restriction A_\alpha = \bar{U}$. Clearly $\bar{F} \restriction A_\alpha \in \text{HOD}$, and therefore so is $\bar{U}$.

We only sketch the proof of (4), which requires knowledge of the proof of Vopěnka's Theorem. (This is the theorem stating that every set of ordinals is set-generic over HOD; see [13].) We first show that for any cardinal γ, there is some

$x \in V_\lambda$ such that $j_U \restriction P(\gamma)$ is amenable to HOD_x. Let E be the HOD-extender of length $j_U(\gamma)$ derived from j_U. Notice that $E \subseteq \text{HOD}$ by (3), and moreover E belongs to an ordinal definable set X of size less than λ since U does.

The set X is (essentially) a condition in the Vopěnka forcing to add E to HOD, and below this condition, the Vopěnka algebra has cardinality less than λ, since it is isomorphic to $P(X) \cap \text{OD}$. It follows that E belongs to HOD_x where $x \in V_\lambda$ is the generic for this Vopěnka forcing below the condition given by X. Each ultrafilter of E lifts uniquely to an ultrafilter of HOD_x by the Lévy-Solovay Theorem [16]: these ultrafilters are λ^+-complete and x is HOD-generic for a forcing of size less than λ. It follows that the HOD_x-extender of j_U of length $j_U(\gamma)$ can be computed from E inside HOD_x, simply by lifting all the measures of E to HOD_x. But from this extender, one can decode $j_U \restriction P(\gamma) \cap \text{HOD}_x$. This shows that there is some $x \in V_\lambda$ such that $j_U \restriction P(\gamma)$ is amenable to HOD_x.

If follows from the pigeonhole principle that there is some $x_0 \in V_\lambda$ such that j_U is amenable to HOD_{x_0}. Now for any $x \geq_{\text{OD}} x_0$, j_U is amenable to HOD_x by exactly the same argument we used above to show that E extends from HOD to HOD_x. This proves (4). $\square$

The fixed point filter associated to a set of elementary embeddings plays a key role in the theory developed in [5]:

Definition 5.18. Suppose j is a function and X is a set. Then

$$\text{Fix}(j, X) = \{x \in X \mid j(x) = x\}$$

Suppose σ is a set of functions. Then $\text{Fix}(\sigma, X) = \bigcap_{j\in\sigma} \text{Fix}(j, X)$.

Suppose $\mathcal{E}$ is a set of elementary embeddings whose domains contain the set X. Suppose B is a set. Then the *fixed point filter B-generated by $\mathcal{E}$ on X*, denoted $\mathcal{F}^B(\mathcal{E}, X)$, is the filter over X generated by sets of the form $\text{Fix}(\sigma, X)$ where $\sigma \subseteq \mathcal{E}$ and $\sigma \preceq^* b$ for some $b \in B$.

The sort of techniques we have been using yield the following representation theorem for ultrafilters over ordinals, which says that in the land of choiceless cardinals, every ultrafilter over an ordinal is one set away from a fixed point filter:

Theorem 5.19 (DC)**.** *Suppose ϵ is an ordinal and $j : V_{\epsilon+3} \to V_{\epsilon+3}$ is an elementary embedding. Assume $\lambda = \kappa_\omega(j)$ is a limit of Löwenheim-Skolem cardinals. Suppose $\delta < \theta_{\epsilon+2}$ is an ordinal and U is a V_λ-complete ultrafilter over δ. Then there is an ordinal definable set of elementary embeddings $\mathcal{E}$ and a set $A \subseteq \delta$ such that $U = \mathcal{F}^{V_\lambda}(\mathcal{E}, \delta) \restriction A$.*

Proof. The proof is by contradiction. Suppose ξ is least possible Ketonen rank of a V_λ-complete ultrafilter over an ordinal for which the theorem fails. Obviously $\xi < \theta_{\epsilon+3}$. Let $j : V_{\epsilon+3} \to V_{\epsilon+3}$ be an elementary embedding. Note that ξ is definable in $\mathcal{H}_{\epsilon+3}$, and therefore $j(\xi) = \xi$. It follows that $j(U) = U$ for any V_λ-complete ultrafilter over an ordinal $\delta < \theta_{\epsilon+2}$ of Ketonen rank ξ: as in Theorem 5.12, $j \restriction P(\delta)$ is a countably complete Lipschitz homomorphism witnessing $U \leq_{\Bbbk} j(U)$, while $j(U)$ has rank ξ since $j(\xi) = \xi$. Let $\mathcal{E}$ be the set of Σ_5-elementary embeddings $k : V_{\epsilon+3} \to V_{\epsilon+3}$ such that $k(\xi) = \xi$. The argument we have just given shows that $k(U) = U$ for any $k \in \mathcal{E}$.

Let $F = \mathcal{F}^{V_\lambda}(\mathcal{E}, \delta)$. Clearly, F is ordinal definable: in fact, F is definable over $\mathcal{H}_{\epsilon+3}$ from an ordinal parameter. We claim F is κ-saturated where $\kappa = \text{crit}(j)$. This follows from Woodin's proof of the Kunen inconsistency theorem. Suppose F is not κ-saturated, so there is a partition $\langle S_\alpha \mid \alpha < \kappa\rangle$ of δ into pairwise disjoint F-positive sets. Let $\langle T_\alpha \mid \alpha < j(\kappa)\rangle = j(\langle S_\alpha \mid \alpha < \kappa\rangle)$. Since F is first-order definable over $\mathcal{H}_{\epsilon+3}$, T_α is F-positive for all α. In particular, T_κ is F-positive, or in other words, T_κ has nonempty intersection with every set in F. The set of fixed points of j below δ belongs to F, so T_κ contains an ordinal η that is fixed by j. Now $\eta \in S_\alpha$ for some $\alpha < \kappa$, and therefore $\eta = j(\eta) \in j(S_\alpha) = T_{j(\alpha)}$. It follows that $T_{j(\alpha)} \cap T_\kappa \neq \emptyset$, so since the sets $\langle T_\alpha \mid \alpha < \kappa\rangle$ are pairwise disjoint, $j(\alpha) = \kappa$. This contradicts that κ is the critical point of j.

Using the Löwenheim-Skolem cardinals, it is easy to show that F is V_λ-complete. Therefore by Theorem 5.2, F is atomic.

We now show that $F \subseteq U$. Suppose $k \in \mathcal{E}$. As we noted in the first paragraph, $k(U) = U$. Therefore k is a countably complete Lipschitz homomorphism with $k^{-1}[U] = U$. If U contains the set of ordinals that are not fixed by k, then k witnesses that U is strictly below U in the Ketonen order, which is impossible. Since U is an ultrafilter, U must instead concentrate on fixed points of k. Since U is V_λ-complete, it follows that U contains the basis generating F as in Definition 5.18, so $F \subseteq U$.

Since F is atomic, there is some atom A of F such that $U = F \restriction A$, and this completes the proof. $\square$

A number of interesting questions remain. We state them in the context of I_0, which is arguably the simplest special case, but obviously the same questions are relevant in the choiceless large cardinal context.

Question 5.20. Assume I_0. In $L(V_{\lambda+1})$, is there a surjection from $V_{\lambda+1}$ onto the set of λ^+-complete ultrafilters over λ^+?

The question is at least somewhat subtle, since one can show that in $L(V_{\lambda+1})$, there is a $\delta < \theta_{\lambda+2}$ such that there is no surjection from $V_{\lambda+1}$ onto the set of λ^+-complete ultrafilters over δ. In fact, one can take $\delta = (\delta^2_1)^{L(V_{\lambda+1})}$. This is exactly parallel to the situation in $L(\mathbb{R})$. An even more basic question is whether the ultrapower of λ^+ by the unique normal ultrafilter over λ^+ concentrating on ordinals of cofinality ω is smaller than $\lambda^{+\lambda}$.

Another question, directly related to Theorem 5.12, concerns the size of antichains in the Ketonen order:

Question 5.21. Assume I_0. In $L(V_{\lambda+1})$, if $\langle U_\alpha \mid \alpha < \lambda\rangle$ is a sequence of λ^+-complete ultrafilters over ordinals, must there be $\alpha \leq \beta < \lambda$ such that $U_\alpha \leq_{\Bbbk} U_\beta$?

A positive answer would bring us even closer to a "proof of the Ultrapower Axiom" from choiceless cardinals. Actually one can prove a weak version of this for λ^+-sequences of ultrafilters, whose statement and proof are omitted.

5.4 The filter extension property

We now turn to a different application of the Ketonen order: extending filters to ultrafilters. For this, we need the Ketonen order on countably complete filters, which was introduced in [1]:

Definition 5.22. Suppose δ is an ordinal. The *Ketonen order on filters* is defined on countably complete filters F and G on δ as follows:

- $F <_{\Bbbk} G$ if $F \subseteq G\text{-}\lim_{\alpha<\delta} F_\alpha$ where for G-almost all $\alpha < \delta$, F_α is a countably complete filter over δ with $\alpha \in F_\alpha$.
- $F \leq_{\Bbbk} G$ if $F \subseteq G\text{-}\lim_{\alpha<\delta} F_\alpha$ where for G-almost all $\alpha < \delta$, F_α is a countably complete filter over δ with $\alpha + 1 \in F_\alpha$.

The notation is a bit unfortunate since the Ketonen order on ultrafilters (Definition 5.5) need not be equal to the restriction of the Ketonen order on filters to the class of ultrafilters (although the latter order is an extension of the former one). For example, assuming I_0, in $L(V_{\lambda+1})$, any ω-club ultrafilter over λ^+ lies below any ω_1-club ultrafilter over λ^+ in the Ketonen order on filters, but not in the Ketonen order on ultrafilters. (We do not know whether the two Ketonen orders can diverge assuming ZFC, though it seems very likely that this can be forced.) The Ultrapower Axiom obviously implies that the Ketonen order on filters coincides with the Ketonen order on ultrafilters. We will not make substantial use of the Ketonen order on ultrafilters for the rest of the paper, so this ambiguity causes no real problem.

A distinctive feature of the Ketonen order on filters is that $\leq_{\Bbbk}$ is not antisymmetric; similarly $F \leq_{\Bbbk} G$ but $G \not\leq_{\Bbbk} F$ does not imply $F <_{\Bbbk} G$. This makes it hard to generalize arguments like Theorem 5.12 from countably complete ultrafilters to countably complete filters. Still, many of the key combinatorial properties of the Ketonen order do generalize. For example, it is easy to see that the Ketonen order on filters is transitive. Most importantly, we show in the appendix that this order is wellfounded:

Theorem 7.12 (DC)**.** *The Ketonen order on countably complete filters is wellfounded.* □

In the choiceless context, we say a cardinal κ is *strongly compact* if for every set X, there is a κ-complete fine ultrafilter over $P_\kappa(X)$. Suppose $j : V \to V$ is an elementary embedding and λ-DC holds where $\lambda = \kappa_\omega(j)$. It seems possible that λ^+ is then strongly compact. While we do not know how to prove this, and expect it is not provable, we can establish a consequence of strong compactness that is equivalent to strong compactness in ZFC. The consequence we are referring to is the *filter extension property*, which is said to hold at κ if every κ-complete filter over an ordinal extends to a κ-complete ultrafilter. If κ is strongly compact, then a standard argument, which does not require the Axiom of Choice, shows that the filter extension property holds at κ. (On the other hand, the proof that every κ-complete filter extends to a κ-complete ultrafilter does use the Axiom of Choice, and in fact any cardinal with this stronger form of the filter extension property must be inaccessible.)

Theorem 5.23. *Suppose $j : V \to V$ is an elementary embedding. Assume λ-DC holds where $\lambda = \kappa_\omega(j)$. Then every λ^+-complete filter over an ordinal extends to a λ^+-complete ultrafilter.* $\square$

This is an immediate consequence of the following more local theorem:

Theorem 5.24. *Suppose ϵ is an even ordinal and $\nu \leq \epsilon$ is a limit of Löwenheim-Skolem cardinals. Suppose there is an elementary embedding $j : V_{\epsilon+3} \to V_{\epsilon+3}$ with $\kappa_\omega(j) \leq \nu$. Then every V_ν-complete filter over an ordinal less than $\theta_{\epsilon+2}$ extends to a V_ν-complete ultrafilter.*

Proof. For any elementary embedding $k : V_{\epsilon+2} \to V_{\epsilon+2}$, let $k' : \mathcal{H}_{\epsilon+3} \to \mathcal{H}_{\epsilon+3}$ be defined by $k' = (k^+)^\star$, assuming that $k^+ : V_{\epsilon+3} \to V_{\epsilon+3}$ is Σ_1-elementary, so that $(k^+)^\star$ is well-defined.

Let $\lambda = \kappa_\omega(j)$, where j is as in the statement of the theorem. We begin with a basic observation, whose proof is lifted from a claim in [7]: for any $n < \omega$ and any $\xi \leq \epsilon$, there is a Σ_n-elementary embedding $i : \mathcal{H}_{\epsilon+3} \to \mathcal{H}_{\epsilon+3}$ such that $\kappa_\omega(i) = \lambda$ and $i(\xi) = \xi$. To see this, suppose the claim fails for some n. Consider the least ξ for which there is no such embedding. Then ξ is first-order definable from λ over $\mathcal{H}_{\epsilon+3}$, so $j'(\xi) = \xi$. But then j' itself witnesses that ξ is not a counterexample to our basic observation, and this is a contradiction.

Suppose towards a contradiction that the theorem fails. Fix an ordinal $\eta < \theta_{\epsilon+2}$ and a filter F that is minimal in the Ketonen order among all V_ν-complete filters over η that do not extend to V_ν-complete ultrafilters.

Fix natural numbers $n_0 < n_1 < n_2$ that are sufficiently far apart for the following proof to work. For concreteness, one can take $n_0 = 10$, $n_1 = 15$, and $n_2 = 20$.

Let $\mathcal{E}$ be the set containing every elementary embedding $k : V_{\epsilon+2} \to V_{\epsilon+2}$ whose extension $k' : \mathcal{H}_{\epsilon+3} \to \mathcal{H}_{\epsilon+3}$ is well-defined and Σ_{n_0}-elementary, fixes ν and η, and has F in its range. Note that $\mathcal{E}$ is Σ_{n_1}-definable over $V_{\epsilon+3}$. Since ν and η can be coded by a single ordinal $\xi \leq \epsilon$, we can fix a Σ_{n_2}-elementary embedding $i : \mathcal{H}_{\epsilon+3} \to \mathcal{H}_{\epsilon+3}$ with $\kappa_\omega(i) = \lambda$, $i(\nu) = \nu$, and $i(\eta) = \eta$.

Let $G = \mathcal{F}^{V_\nu}(\mathcal{D}, \eta)$ where $\mathcal{D}$ is the set of embeddings $k' : \mathcal{H}_{\epsilon+3} \to \mathcal{H}_{\epsilon+3}$ induced by embeddings $k \in \mathcal{E}$ as in the previous paragraph. The filter G is λ-saturated, as a consequence of Woodin's proof of the Kunen inconsistency theorem. Suppose towards a contradiction that there is a partition $\langle S_\alpha : \alpha < \lambda\rangle$ of η into G-positive sets. Let

$$\langle T_\alpha : \alpha < \lambda\rangle = i(\langle S_\alpha : \alpha < \lambda\rangle)$$

Since $\mathcal{E}$ is Σ_{n_1}-definable over $V_{\epsilon+3}$ and i is Σ_{n_2}-elementary on $V_{\epsilon+3}$, $i(G) = \mathcal{F}^{V_\nu}(\mathcal{D}, \eta)$, where $\mathcal{D}$ is the set of embeddings $k' : \mathcal{H}_{\epsilon+2} \to \mathcal{H}_{\epsilon+2}$ induced by embeddings $k \in i(\mathcal{E})$. Notice that $i \restriction V_{\epsilon+2} \in i(\mathcal{E})$: this follows easily by our choice of i and the fact that $i(F) \in \text{ran}(i)$. Let κ be the critical point of i. Since T_κ is $i(G)$-positive, it follows that $\{\xi \mid i(\xi) = \xi\} \cap T_\kappa$ is nonempty. Fix ξ such that $i(\xi) = \xi$ and $\xi \in T_\kappa$. Note that $\xi \in S_\alpha$ for some α since $\langle S_\alpha \mid \alpha < \lambda\rangle$ is a partition of η. Therefore since $i(\xi) = \xi$, $\xi \in i(S_\alpha) = T_{i(\alpha)}$. Since κ is the critical point of i, $i(\alpha) \neq \kappa$. But $\xi \in T_\kappa \cap T_{i(\alpha)}$, and this contradicts that $\langle T_\alpha \mid \alpha < \lambda\rangle$ is a partition.

Since ν is a limit of Löwenheim-Skolem cardinals, G is V_ν-complete, and so Theorem 5.2 implies that there is some $\rho < \lambda$ and a partition $\langle A_\alpha \mid \alpha < \rho\rangle$ of η into G-positive sets such that $G \restriction A_\alpha$ is an ultrafilter for all $\alpha < \rho$.

The main claim is that $G \cup F$ generates a proper filter. Granting the claim, the proof is completed as follows. Let H be the filter generated by $G \cup F$. Since G and F are V_ν-complete filters, given that H is proper, in fact H is V_ν-complete. In particular, for some $\alpha < \rho$, A_α is H-positive. Let $U = G \restriction A_\alpha$, which is an ultrafilter by definition. Since $H \restriction A_\alpha$ is a proper filter and the ultrafilter $U = G \restriction A_\alpha$ is contained in $H \restriction A_\alpha$, in fact, $H \restriction A_\alpha = U$. Since $F \subseteq H \subseteq U$, U is a V_ν-complete extension of F. This contradicts our choice of F, and completes the proof modulo the claim.

We finish by showing that $G \cup F$ generates a proper filter. Suppose it does not, so there is a set in G whose complement is in F. Since $G = \mathcal{F}^{V_\nu}(\mathcal{E}, \eta)$, this means that there is some $\beta < \nu$ and a sequence $\langle i_x \mid x \in V_\beta\rangle \subseteq \mathcal{E}$ such that the set

$$T = \bigcup\{\alpha < \eta \mid i'_x(\alpha) > \alpha\}$$

belongs to F. Let $j_x = i'_x$.

Fix $x \in V_\beta$ for the rest of the paragraph. Since $i_x \in \mathcal{E}$, there is a filter F_x such that $j_x(F_x) = F$. Moreover, $j_x : \mathcal{H}_{\epsilon+3} \to \mathcal{H}_{\epsilon+3}$ is Σ_{n_0}-elementary, $j_x(\nu) = \nu$, and $j_x(\eta) = \eta$. It follows that F_x does not extend to a V_ν-complete ultrafilter: this is because j_x is Σ_{n_0}-elementary and it is a Σ_{n_0}-expressible fact in $\mathcal{H}_{\epsilon+3}$ that $j_x(F_x) = F$ does not extend to a V_ν-complete ultrafilter.

Let D^x_α denote the ultrafilter over η derived from j_x using α. For $\alpha \in T$, let

$$D_\alpha = \bigcap\{D^x_\alpha \mid j_x(\alpha) > \alpha\}$$

Thus for all $\alpha \in T$, D_α is a countably complete filter and $\alpha \in D_\alpha$.

Notice that

$$\bigcap_{x \in V_\beta} F_x \subseteq F\text{-}\lim_{\alpha \in T} D_\alpha \tag{12}$$

The proof is a matter of unwinding the definitions. Fix $A \in \bigcap_{x\in V_\beta} F_x$. For each $x \in V_\beta$, let $S_x = \{\alpha < \eta \mid A \in D^x_\alpha\}$. In other words, $S_x = j_x(A)$, and so since $A \in F_x$, $S_x \in j_x(F_x) = F$. Let $S = \bigcap_{x\in V_\beta} S_x$. Since F is V_ν-complete, $S \in F$. By definition, for $\alpha \in S \cap T$, $A \in \bigcap D^x_\alpha \subseteq D_\alpha$. Since $S \cap T \in F$, this means that for F-almost all α, $A \in D_\alpha$. In other words, $A \in F\text{-}\lim_{\alpha\in T} D_\alpha$, as desired.

Since F_x is V_ν-complete for every $x \in V_\beta$, $\bigcap_{x\in V_\beta} F_x$ is a V_ν-complete filter. Since $\alpha \in D_\alpha$ for all $\alpha \in T$, (12) implies that $\bigcap_{x\in V_\beta} F_x <_{\Bbbk} F$. Since F is a minimal counterexample to the theorem, it follows that there is a V_ν-complete ultrafilter W that extends $\bigcap_{x\in V_\beta} F_x$.

Recall that for every $x \in V_\beta$, F_x does not extend to a V_ν-complete ultrafilter. It follows that there is a set in W whose complement belongs to F_x. Since ν is a limit of Löwenheim-Skolem cardinals, for some ordinal $\gamma > \epsilon$, there is an elementary substructure $X \prec V_\gamma$ with $V_\beta \subseteq X$, $\langle F_x \mid x \in V_\beta\rangle \in X$, $W \in X$, and $X \preceq^* V_\zeta$ for some $\zeta < \nu$. Let S be the intersection of all W-large sets that belong to X. Since

$X \preceq^* V_\zeta$, $\zeta < \nu$, and W is V_ν-complete, $S \in W$. We claim that the complement of S belongs to $\bigcap_{x \in V_\beta} F_x$. To see this, fix an $x \in V_\beta$. There is a W-large set $A \in X$ whose complement belongs to F_x since X is an elementary substructure of V_γ that contains F_x and W. Since S is the intersection of all W-large sets in X, $S \subseteq A$. Hence the complement of S contains the complement of A, and it follows that the complement of S belongs to F_x.

The existence of a set $S \in W$ whose complement is in $\bigcap_{x \in V_\beta} F_x$ contradicts that W extends $\bigcap_{x \in V_\beta} F_x$. This contradiction proves the claim that $G \cup F$ generates a proper filter, and thereby proves the theorem as explained above. $\square$

By a similar argument, we also have the following consequence of I_0:

Theorem 5.25 (ZFC). *Suppose there is an elementary embedding from $L(V_{\lambda+1})$ to $L(V_{\lambda+1})$ with critical point below λ. Then in $L(V_{\lambda+1})$, every λ^+-complete filter over an ordinal less than $\theta_{\lambda+2}$ extends to a λ^+-complete ultrafilter.* $\square$

6 Consistency results

6.1 Introduction

In a groundbreaking recent development, Schlutzenberg [2] has proved the consistency of the existence of an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$ relative to ZF + I_0:

Theorem 6.1 (Schlutzenberg). *Assume λ is an even ordinal and*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

is an elementary embedding with $\text{crit}(j) < \lambda$. *Let $M = L(V_{\lambda+1})[j \restriction V_{\lambda+2}]$. Then $V_{\lambda+2} \cap M = V_{\lambda+2} \cap L(V_{\lambda+1})$. Hence M satisfies that there is an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$.*

It follows that the existence of an elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$ is equiconsistent with I_0. Moreover, neither hypothesis implies that $V_{\epsilon+1}^\#$ exists.

If ϵ is even and $\mathcal{H}_{\epsilon+2}$ satisfies the Collection Principle, every elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$ extends to a Σ_0-elementary embedding from $V_{\epsilon+3}$ to $V_{\epsilon+3}$. Therefore the existence of a Σ_1-elementary embedding from $V_{\epsilon+3}$ to $V_{\epsilon+3}$ is in some sense the first rank-to-rank axiom beyond an elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$. (Also see Theorem 6.8.) We can prove the existence of sharps from this principle:

Theorem 6.2. *Suppose ϵ is an even ordinal and there is a Σ_1-elementary embedding from $V_{\epsilon+3}$ to $V_{\epsilon+3}$. Then $A^\#$ exists for every $A \subseteq V_{\epsilon+1}$.*

Recall the following theorem:

Proposition 6.3. *Suppose λ is a cardinal and there is an elementary embedding $j : L(V_\lambda) \to L(V_\lambda)$ such that $\kappa_\omega(j) = \lambda$. Then $V_\lambda^\#$ exists and for some $\alpha < \lambda$, there is an elementary embedding from V_α to V_α.*

We will prove the following somewhat unexpected equiconsistency at the $V_{\lambda+2}$ to $V_{\lambda+2}$ level, which shows that Proposition 6.3 does not generalize to the other even levels:

Theorem 6.8. *The following statements are equiconsistent over* ZF*:*

(1) For some λ, there is a nontrivial elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$.

(2) For some λ, there is an elementary embedding from $L(V_{\lambda+2})$ to $L(V_{\lambda+2})$ with critical point below λ.

(3) There is an elementary embedding j from V to an inner model M that is closed under $V_{\kappa_\omega(j)+1}$-sequences.

Combined with Schlutzenberg's theorem, all these principles are equiconsistent with the existence of an elementary embedding from $L(V_{\lambda+1})$ to $L(V_{\lambda+1})$ with critical point below λ. In particular, *the existence of an elementary embedding from $L(V_{\lambda+1})$ to $L(V_{\lambda+1})$ with critical point below λ is equiconsistent with the existence of an elementary embedding from $L(V_{\lambda+2})$ to $L(V_{\lambda+2})$ with critical point below λ.*

We then turn to some long-unpublished work of the author. The following is technically an open question:

Question 6.4. Does the existence of a nontrivial elementary embedding from V to V imply the consistency of ZFC + I_0?

Combining a forcing technique due to Woodin [4] and the Laver-Cramer theory of inverse limits [6], we provide the following partial answer:

Theorem 6.19. *Suppose λ is an ordinal and there is a Σ_1-elementary embedding $j : V_{\lambda+3} \to V_{\lambda+3}$ with $\lambda = \kappa_\omega(j)$. Assume $\mathrm{DC}_{V_{\lambda+1}}$. Then there is a set generic extension N of V such that $(V_\lambda)^N$ satisfies $\mathrm{ZFC} + I_0$.*

In particular, in the presence of DC, the existence of a Σ_1-elementary embedding $j : V_{\lambda+3} \to V_{\lambda+3}$ with $\lambda = \kappa_\omega(j)$ implies the consistency of $\mathrm{ZFC} + I_0$.

We also briefly outline a proof of the following theorem:

Theorem 6.20. *The following statements are equiconsistent over* ZF + DC*:*

(1) For some λ, $\mathscr{E}(V_{\lambda+2}) \neq \{\mathrm{id}\}$.

(2) For some λ, λ-DC holds and $\mathscr{E}(V_{\lambda+2}) \neq \{\mathrm{id}\}$.

(3) The Axiom of Choice $+ I_0$.

The equivalence of (2) and (3) is Schlutzenberg's Theorem.

6.2 Equiconsistencies and sharps

We begin with the equiconsistencies for embeddings of the even levels. Here we need some basic observations about ultrapowers assuming weak choice principles, which we will later apply to inner models of the form $L(V_{\epsilon+1})[C]$, which satisfy these principles.

Lemma 6.5. *Suppose ϵ is an even ordinal and M is an inner model containing $V_{\epsilon+1}$. Suppose $j : V^M_{\epsilon+2} \to V^M_{\epsilon+2}$ is a Σ_1-elementary embedding. Assume that for all relations $R \subseteq V_{\epsilon+1} \times M$ in M, there is some $S \subseteq R$ in M such that $\mathrm{dom}(S) = \mathrm{dom}(R)$ and, in M, $\mathrm{ran}(S) \preceq^* V_{\epsilon+1}$. Let $\mathcal{U}$ be the M-ultrafilter derived from j using $j[V_\epsilon]$. Then the ultrapower of M by $\mathcal{U}$ satisfies Łoś's Theorem. Moreover, if $\mathcal{U} \in M$, then in M, $\mathrm{Ult}(M,\mathcal{U})$ is closed under $V_{\epsilon+1}$-sequences.*

Proof. To establish Łoś's Theorem, it suffices to show that if $R \subseteq V_{\epsilon+1} \times M$ belongs to M and $\mathrm{dom}(R) \in \mathcal{U}$, has a *$\mathcal{U}$-uniformization* in M, which is just a $f \subseteq R$ in M such that $\mathrm{dom}(f) \in \mathcal{U}$. We can reduce to the case of relations on $V_{\epsilon+1} \times V_{\epsilon+1}$. Given $R \subseteq V_{\epsilon+1} \times M$, take $S \subseteq R$ with $\mathrm{dom}(S) = \mathrm{dom}(R)$ and $\mathrm{ran}(S) \preceq^* V_{\epsilon+1}$. Fix a surjection $p : V_{\epsilon+1} \to \mathrm{ran}(S)$. Let

$$R' = \{(x, y) \mid (x, p(y)) \in S\}$$

If g is a $\mathcal{U}$-uniformization of T, then $p \circ g$ is a $\mathcal{U}$-uniformization of S, and therefore $p \circ g$ is a $\mathcal{U}$-uniformization of R.

Therefore fix $R \subseteq V_{\epsilon+1} \times V_{\epsilon+1}$ in M with $\mathrm{dom}(R) \in \mathcal{U}$. We have that $j[V_\epsilon] \in j(\mathrm{dom}(R))$ by the definition of a derived ultrafilter. Note that $\mathrm{dom}(j(R)) = j(\mathrm{dom}(R))$ by the Σ_1-elementarity of j. (Here we extend j to act on R, which is essentially an element of $V_{\epsilon+2}$.) Therefore $j[V_\epsilon] \in \mathrm{dom}(j(R))$.

Fix $y \in V_{\epsilon+1}$ such that $(j[V_\epsilon], y) \in R$. Then the function f_y given by Definition 3.14 has the property that $y = j(f)(j[V_\epsilon])$ (by the proof of Corollary 3.12), but also $f_y \in M$ since f_y is definable over $V_{\epsilon+1}$ from y. Let $g = f \cap R$, so $g \subseteq R$. Note that $j(f)(j[V_\epsilon]) = y$ has the property that $(j[V_\epsilon], y) \in j(R)$, and hence $j(g)(j[V_\epsilon])$ is defined and is equal to y. In other words, $j[V_\epsilon] \in j(\{x \in V_{\epsilon+1} \mid (x, g(x)) \in R\})$, again using the Σ_1-elementarity of j on $V_{\epsilon+2}$. This means that $\mathrm{dom}(g) \in \mathcal{U}$, so g is a $\mathcal{U}$-uniformization of R that belongs to M.

We finally show that if $\mathcal{U} \in M$, then $\mathrm{Ult}(M,\mathcal{U})$ is closed under $V_{\epsilon+1}$-sequences in M. We might as well assume $V = M$, since what we are trying to prove is first-order over M. Let $N = \mathrm{Ult}(V,\mathcal{U})$. We cannot assume N is transitive, but we will abuse notation by identifying certain points in N with their extensions.

We first show that every set $X \in [N]^{V_{\epsilon+1}}$ is covered by a set $Y \in N$ such that $Y \preceq^* V_{\epsilon+1}$ in N. Let $p : V_{\epsilon+1} \to X$ be a surjection. Let $R \subseteq V_{\epsilon+1} \times V$ be the relation defined by $R(x, f)$ if $p(x) = [f]_\mathcal{U}$. Take $S \subseteq R$ such that $\mathrm{dom}(S) = \mathrm{dom}(R)$ and $\mathrm{ran}(S) \preceq^* V_{\epsilon+1}$. Then

$$X \subseteq \{j(f)([\mathrm{id}]_\mathcal{U}) \mid f \in \mathrm{ran}(S)\} \subseteq \{g([\mathrm{id}]_\mathcal{U}) \mid f \in j(S)\}$$

Let $Y = \{g([\mathrm{id}]_\mathcal{U}) \mid f \in j(S)\}$. Then $X \subseteq Y$, $Y \in N$, and $Y \preceq^* V_{\epsilon+1}$ in N.

Now we show that N is closed under $V_{\epsilon+1}$-sequences. It suffices to show that $[N]^{V_{\epsilon+1}} \subseteq N$. Fix $X \in [N]^{V_{\epsilon+1}}$. Take $Y \in N$ with $X \subseteq Y$ and $Y \preceq^* V_{\epsilon+1}$ in N. Let $q : V_{\epsilon+1} \to Y$ be a surjection that belongs to N. Consider the set

$$A = \{x \in V_{\epsilon+1} \mid q(x) \in X\}$$

By Corollary 3.12, $A \in N$. Hence $q[A] = X$ belongs to N. This finishes the proof. $\square$

To prove the wellfoundedness of the ultrapower seems to require a stronger hypothesis which is related to Schlutzenberg's results on ultrapowers using Löwenheim-Skolem cardinals.

Lemma 6.6. *Suppose ϵ is an even ordinal and $j : V_{\epsilon+2} \to V_{\epsilon+2}$ is a Σ_1-elementary embedding. Assume that every transitive set N containing $V_{\epsilon+1}$ has an elementary substructure H containing $V_{\epsilon+1}$ such that $H \preceq^* V_{\epsilon+1}$. Let $\mathcal{U}$ be the ultrafilter over $V_{\epsilon+1}$ derived from j using $j[V_\epsilon]$. Then $\text{Ult}(V,\mathcal{U})$ is wellfounded.*

Proof. Assume towards a contradiction that the lemma fails. Let α be an ordinal greater than ϵ such that V_α is a Σ_4-elementary substructure of V. Then $\text{Ult}(V_\alpha,\mathcal{U})$ is illfounded. Let H be an elementary substructure of V_α containing $V_{\epsilon+1}$ and $\mathcal{U}$ such that $H \preceq^* V_{\epsilon+1}$. (Take $H = H' \cap V_\alpha$ where H' is an elementary substructure of a $N = V_\alpha \cup \{V_\alpha \times \mathcal{U}\}$.)

Let P be the Mostowski collapse of H. Let $\mathcal{W} = \mathcal{U} \cap P$. Since $V_{\epsilon+1} \subseteq H$, $\mathcal{W}$ is the image of $\mathcal{U}$ under the Mostowski collapse map. Therefore by elementarity, $\text{Ult}(P,\mathcal{W})$ is illfounded. Note that there is a Σ_2-elementary embedding $k : \text{Ult}(P,\mathcal{W}) \to j_{\mathcal{U}}(P)$ defined by $k([f]_{\mathcal{W}}) = [f]_{\mathcal{U}}$. Therefore $j_{\mathcal{U}}(P)$ is illfounded. Let $E \subseteq V_{\epsilon+1} \times V_{\epsilon+1}$ be a wellfounded relation whose Mostowski collapse is P. Then in $\text{Ult}(V,\mathcal{U})$, $j_{\mathcal{U}}(E)$ has Mostowski collapse $j_{\mathcal{U}}(P)$ since Łoś's Theorem holds by Lemma 6.5. (Note that the Löwenheim-Skolem hypothesis of this lemma is stronger than the collection hypothesis from Lemma 6.5.) It follows that $j_{\mathcal{U}}(E)$ is illfounded. Since $E \subseteq V_{\epsilon+1}$, $j_{\mathcal{U}}(E) \cong j(E)$. Therefore $j(E)$ is illfounded. (Here we must extend j slightly to act on binary relations.) This contradicts that j is a Σ_1-elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$, since such an embedding preserves wellfoundedness. $\square$

The following lemma gives an example of a structure satisfying the hypotheses of Lemma 6.5 and Lemma 6.6:

Lemma 6.7. *Suppose $j : V_{\epsilon+2} \to V_{\epsilon+2}$ is a Σ_1-elementary embedding. Let $\mathcal{U}$ be the ultrafilter over $V_{\epsilon+1}$ derived from j using $j[V_\epsilon]$. Then for any class C, the ultrapower of $L(V_{\epsilon+1})[C]$ by $\mathcal{U}$ using functions in $L(V_{\epsilon+1})[C]$ is wellfounded and satisfies Łoś's Theorem.*

Proof. Let $M = L(V_{\epsilon+1})[C]$. The Löwenheim-Skolem hypothesis of Lemma 6.6 holds inside M as an immediate consequence of the fact that M satisfies that every set is ordinal definable from parameters in $V_{\epsilon+1} \cup \{C \cap M\}$. This yields Skolem functions $\langle f_x \mid x \in V_{\epsilon+1}\rangle$ for any transitive structure N: if $\varphi(v_0, v_1)$ is a formula, $f_x(\varphi, p)$ is the least $a \in \text{OD}_{C\cap M,x}$ in the canonical wellorder of $\text{OD}_{C\cap M,x}$ such that $N \vDash \varphi(a, p)$. (Obviously this would work for any structure N in a countable language.) If $V_{\epsilon+1} \subseteq N$, then closing under these Skolem functions, one obtains an elementary substructure $H \prec N$ containing $V_{\epsilon+1}$ such that $H \preceq^* V_{\epsilon+1}$. Since this hypothesis implies the collection hypothesis from Lemma 6.5, Łoś's Theorem holds for the ultrapower in question.

For the proof of wellfoundedness, we would like to apply Lemma 6.6 inside M, but the problem arises that $\mathcal{U} \cap M$ may not belong to M. Note, however, that it suffices to show that $\text{Ult}(M', \mathcal{U} \cap M')$ is wellfounded where $M' = L(V_{\epsilon+1})[C, \mathcal{U}]$: if the ultrapower $i : M' \to \text{Ult}(N, \mathcal{U} \cap N)$ is wellfounded, then since $\text{Ult}(M, \mathcal{U} \cap M)$ elementarily embeds into $i(M)$ via the canonical factor map, $\text{Ult}(M, \mathcal{U} \cap M)$ is wellfounded as well. Since M' is of the form $L(V_{\epsilon+1})[C']$ for some class C' coding C and $\mathcal{U}$, the previous paragraph yields that the hypothesis of Lemma 6.6 holds inside M'. Therefore Lemma 6.6 yields the wellfoundedness of $\text{Ult}(M', \mathcal{U} \cap M')$, which completes the proof. $\square$

Theorem 6.8. *The following theories are equiconsistent:*

(1) For some λ, there is a Σ_1-elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$.

(2) For some λ, there is an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$.

(3) For some λ, there is an elementary embedding from $L(V_{\lambda+2})$ to $L(V_{\lambda+2})$ with critical point below λ.

(4) There is an elementary embedding $j : V \to M$ where M is an inner model that is closed under under $V_{\lambda+1}$-sequences for $\lambda = \kappa_\omega(j)$.

Proof. Clearly each statement is implied by the next (except for (4)!), so it suffices to show that (1) implies that (4) holds in an inner model. Assume (1). Let λ be the least ordinal such that there is a Σ_1-elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$. Then λ is a limit ordinal. (1) still holds in $L(V_{\lambda+2})[\mathcal{U}]$, and so applying the proof of Lemma 6.7 and Lemma 6.5, we obtain that $L(V_{\lambda+2})[\mathcal{U}]$ satisfies (4). This proves the theorem. $\square$

The following is a proof, without requiring λ-DC or any choice principles, of [5, Lemma 28]:

Theorem 6.9. *Suppose ϵ is an even ordinal, $A, B \subseteq V_{\epsilon+1}$, $A \in L(V_{\epsilon+1}, B)$, and $j : L(V_{\epsilon+1}, B) \to L(V_{\epsilon+1}, B)$ is an elementary embedding that fixes B. Assume*

$$(\theta_{\epsilon+2})^{L(V_{\epsilon+1}, A)} < (\theta_{\epsilon+2})^{L(V_{\epsilon+1}, B)}$$

Then $A^\#$ exists, and $A^\# \in L(V_{\epsilon+1}, B)$.

Proof. For $D \subseteq V_{\epsilon+1}$, let $M_D = L(V_{\epsilon+1}, D)$ and let $\theta_D = (\theta_{\epsilon+2})^{L(V_{\epsilon+1}, D)}$. If $D \in M_B$, let $\mathcal{U}_D$ be the M_D-ultrafilter over $V_{\epsilon+1}$ derived from j using $j[V_\epsilon]$. Let $j_D : M_D \to \text{Ult}(M_D, \mathcal{U}_D)$ be the ultrapower embedding.

Note that $M_A \cap V_{\epsilon+2} \preceq^* V_{\epsilon+1}$ in M_B. Therefore

$$j \restriction M_A \cap V_{\epsilon+2} \in M_B$$

This yields that $\mathcal{U}_A \in M_B$. Thus within M_B, one can compute the ultrapower

$$j_A : M_A \to \mathrm{Ult}(M_A, \mathcal{U}_A)$$

In particular, $j_A \restriction \theta_B \in M_B$. By a standard argument, $j_B \restriction \theta_B \notin M_B$. (*Sketch:* Assume not. Inside M_B, compute first $j_B \restriction L_{\theta_B}(V_{\epsilon+1})$, then $\mathcal{U}_B$, and finally $j_B : M_B \to M_B$. Now in M_B, there is a definable embedding from V to V, contradicting Theorem 3.1.)

Since $j_A \restriction \theta_B \in M_B$ and $j_B \restriction \theta_B \notin M_B$, it must be that $j_A \restriction \theta_B \neq j_B \restriction \theta_B$. Let $k : \mathrm{Ult}(M_A, \mathcal{U}_A) \to j_B(M_A)$ be the factor embedding. Note that $k \restriction V_{\epsilon+1}$ is the identity. Therefore $j_A(A) = j_B(A) = j(A)$, and so by elementarity and wellfoundedness, $\mathrm{Ult}(M_A, \mathcal{U}_A) = j_B(M_A) = M_{j(A)}$. Since $j_A \restriction \theta_B \neq j_B \restriction \theta_B$, k has a critical point, and $\mathrm{crit}(k) < \theta_B$. Clearly $\mathrm{crit}(k) > \epsilon$ since $k \restriction V_{\epsilon+1}$ is the identity. Thus we have produced an elementary embedding

$$k : L(V_{\epsilon+1}, j(A)) \to L(V_{\epsilon+1}, j(A))$$

with critical point between ϵ and θ_B. This implies that $j(A)^\#$ exists. Moreover, since $j_B \restriction Z \in M_B$ for all transitive sets such that $Z \preceq^* V_{\epsilon+1}$ in M_B, the same holds true of k. In particular, the normal $M_{j(A)}$-ultrafilter W on $\mathrm{crit}(k)$ derived from k belongs to M_B. Here we use the Coding Lemma (Theorem 4.18) to see that

$$P(\mathrm{crit}(k)) \cap M_A \subseteq P(\mathrm{crit}(k)) \cap M_B \preceq^* V_{\epsilon+1}$$

in M_B.

Since W is $V_{\epsilon+1}$-closed (Definition 2.5), it is easy to check that the ultrapower of $M_{j(A)}$ by W satisfies Łoś's Theorem. This ultrapower is wellfounded since it admits a factor embedding into k. The elementary embedding

$$j_W : M_{j(A)} \to M_{j(A)}$$

is therefore definable over M_B. Thus M_B satisfies that $j(A)^\#$ exists. By elementarity, M_B satisfies that $A^\#$ exists. By absoluteness, $A^\#$ exists, and $A^\# \in M_B$. $\square$

Corollary 6.10. *Suppose ϵ is an even ordinal and there is a Σ_1-elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$. Then $A^\#$ exists for every $A \subseteq V_{\epsilon+1}$ such that*

$$(\theta_{\epsilon+2})^{L(V_{\epsilon+1}, A)} < \theta_{\epsilon+1}$$

Proof. Fix $A \subseteq V_{\epsilon+1}$. By Lemma 4.15, there is a set $B \subseteq V_{\epsilon+1}$ such that $A \in L(V_{\epsilon+1}, B)$, $j(B) = B$, and

$$(\theta_{\epsilon+2})^{L(V_{\epsilon+1}, A)} < (\theta_{\epsilon+2})^{L(V_{\epsilon+1}, B)}$$

Taking the ultrapower of $L(V_{\epsilon+1}, B)$ by the ultrafilter derived from j, Lemma 6.7 shows that one obtains an elementary embedding $i : L(V_{\epsilon+1}, B) \to L(V_{\epsilon+1}, B)$ such that $i(B) = B$. By Theorem 6.9, this implies that $A^\#$ exists. $\square$

Corollary 6.11. *Suppose ϵ is an even ordinal and there is a Σ_1-elementary embedding from $V_{\epsilon+3}$ to $V_{\epsilon+3}$. Then $A^\#$ exists for every $A \subseteq V_{\epsilon+1}$.*

Proof. We claim that for all $A \subseteq V_{\epsilon+1}$, $(\theta_{\epsilon+2})^{L(V_{\epsilon+1},A)} < \theta_{\epsilon+2}$. The corollary then follows by applying Corollary 6.10. To prove the claim, note that $L(V_{\epsilon+1}, A)$ satisfies that there is a sequence $\langle f_\alpha \mid \alpha < \theta_{\epsilon+2}\rangle$ such that for all $\alpha < \theta_{\epsilon+2}$, $f_\alpha : V_{\epsilon+1} \to \alpha$ is a surjection; this is immediate from the fact that $L(V_{\epsilon+1})$ satisfies that every set is ordinal definable from parameters in $V_{\epsilon+1} \cup \{A\}$. By Theorem 3.31, this does not hold in V, and therefore $(\theta_{\epsilon+2})^{L(V_{\epsilon+1},A)} < \theta_{\epsilon+2}$. □

By a similar proof, we obtain the following consistency strength separation:

Theorem 6.12. *The existence of a Σ_1-elementary embedding from $V_{\lambda+3}$ to $V_{\lambda+3}$ implies the consistency of* ZF *plus the existence of an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$.*

This follows immediately from a more semantic fact:

Proposition 6.13. *Suppose ϵ is an even ordinal and there is a Σ_1-elementary embedding from $V_{\epsilon+3}$ to $V_{\epsilon+3}$. Then there is a set $E \subseteq V_{\epsilon+1}$ and an inner model $M \subseteq L(V_{\epsilon+1}, E)$ containing $V_{\epsilon+1}$ such that M satisfies that there is an elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$.*

Proof. Fix an elementary embedding $j : V_{\epsilon+2} \to V_{\epsilon+2}$. Then the model $M = L(V_{\epsilon+1})[j]$ satisfies that there is an elementary embedding from $V^M_{\epsilon+2}$ to $V^M_{\epsilon+2}$, namely $j \restriction V^M_{\epsilon+2}$. This model also satisfies that there is a sequence $\langle f_\alpha \mid \alpha < \theta_{\epsilon+2}\rangle$ such that for all $\alpha < \theta_{\epsilon+2}$, $f_\alpha : V_{\epsilon+1} \to \alpha$ is a surjection. Thus $(\theta_{\epsilon+2})^M < \theta_{\epsilon+2}$ by the proof of Theorem 3.31. By condensation, this implies that $V_{\epsilon+2} \cap M \preceq^* V_{\epsilon+1}$. Therefore

$$V^M_{\epsilon+2} \cup \{j \restriction V^M_{\epsilon+2}\} \in \mathcal{H}_{\epsilon+2}$$

It follows that there is a wellfounded relation E on $V_{\epsilon+1}$ whose Mostowski collapse is $V^M_{\epsilon+2} \cup \{j \restriction V^M_{\epsilon+2}\}$. Hence $M \subseteq L(V_{\epsilon+1}, E)$, as desired. □

Proof of Theorem 6.12. By minimizing, we may assume λ is a limit ordinal. By Proposition 6.13, there is a set $E \subseteq V_{\lambda+1}$ and an inner model $M \subseteq L(V_{\lambda+1}, E)$ containing $V_{\lambda+1}$ such that M satisfies that there is an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$. By Corollary 6.11, $E^\#$ exists. Therefore $L(V_{\lambda+1}, E)$ has a proper class of inaccessible cardinals. Fix an inaccessible δ of $L(V_{\lambda+1}, E)$ such that $\delta > \lambda$. Then $M \cap V_\delta$ is a model satisfying ZF plus the existence of an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$. □

6.3 Forcing choice

It is natural to wonder whether choiceless large cardinal axioms really are stronger than the traditional large cardinals in terms of the consistency hierarchy. Perhaps the situation is analogous to the status of the full Axiom of Determinacy in that traditional large cardinal axioms imply the existence of an inner model of ZF containing choiceless large cardinals. Could fairly weak traditional large cardinal axioms imply the consistency of the axioms we have been considering in this paper? Using the techniques of inner model theory, one can show that the choiceless cardinals imply the existence of inner models with many Woodin cardinals. But what about large cardinal axioms currently out of reach of inner model theory?

In fact, Woodin showed that one can prove that certain very large cardinals are equiconsistent with their choiceless analogs. For example:

Theorem 6.14 (Woodin)**.** *The following theories are equiconsistent:*

- ZF + there is a proper class of supercompact cardinals*.*
- ZFC + there is a proper class of supercompact cardinals*.*

In this context, we are using the following definition of a supercompact cardinal:

Definition 6.15. A cardinal κ is *supercompact* if for all $\alpha \geq \kappa$, for some $\beta \geq \alpha$ and some transitive set N with $[N]^{V_\alpha} \subseteq N$, there is an elementary embedding $j : V_\beta \to N$ such that $\text{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

The proof shows that if there is a proper class of supercompact cardinals, there is a class forcing extension preserving all supercompact cardinals in which the Axiom of Choice holds. (Not every countable model of ZF is an inner model of a model of ZFC, since for example every inner model of a model of ZFC has a proper class of regular cardinals. More recently, Usuba showed that the existence of a proper class of Löwenheim-Skolem cardinals suffices to carry out Woodin's forcing construction.)

In particular, this theorem implies that the existence of an elementary embedding from V_λ to V_λ, in ZF alone, implies the consistency of the existence of a proper class of supercompact cardinals in ZFC. Indeed, the same ideas produce models of ZFC with many n-huge cardinals from the same hypothesis. But the question arises whether the weakest of the choiceless large cardinal axioms in fact implies the consistency (with ZFC) of all the traditional large cardinal axioms.

In this section, we combine Woodin's method of forcing choice and a reflection theorem due to Scott Cramer to prove the following theorem:

Theorem 6.16. *Over* ZF + DC*, the existence of a* Σ_1*-elementary embedding from* $V_{\lambda+3}$ *to* $V_{\lambda+3}$ *implies* $\text{Con}(\text{ZFC} + I_0)$*.* $\square$

This will follow as an immediate consequence of Theorem 6.19 below.

We appeal to the following result due to Scott Cramer:

Theorem 6.17 (Cramer, [6])**.** *Suppose* λ *is a cardinal,* $V^{\#}_{\lambda+1}$ *exists, and there is a* Σ_1*-elementary embedding from* $(V_{\lambda+1}, V^{\#}_{\lambda+1})$ *to* $(V_{\lambda+1}, V^{\#}_{\lambda+1})$*. Assume* $\text{DC}_{V_{\lambda+1}}$*. Then there is a cardinal* $\bar{\lambda} < \lambda$ *such that* $V_{\bar{\lambda}} \prec V_\lambda$ *and there is an elementary embedding from* $L(V_{\bar{\lambda}+1})$ *to* $L(V_{\bar{\lambda}+1})$ *with critical point less than* $\bar{\lambda}$*.* $\square$

This uses the method of *inverse limit reflection*, which is the technique used to prove reflection results at the level of I_0. For smaller large cardinals, reflection results are typically not very deep, and tend to require no use of the Axiom of Choice. It is not clear, however, whether inverse limit reflection can be carried out without the use of DC. This is the underlying reason that DC is required as a hypothesis in Theorem 6.19.

We also appeal to the following theorem of Woodin:

Theorem 6.18 (Woodin, [4, Theorem 226])**.** *Suppose δ is supercompact, $\bar{\lambda} < \delta$ is such that $V_{\bar{\lambda}} \prec V_\delta$, and $j : V_{\bar{\lambda}+1} \to V_{\bar{\lambda}+1}$ is an elementary embedding. Then there is a weakly homogeneous partial order $\mathbb{P} \subseteq V_{\bar{\lambda}+1}$ definable over $V_{\bar{\lambda}+1}$ without parameters, a condition $p \in \mathbb{P}$, and a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ such that for any V-generic filter $G \subseteq \mathbb{P}$ with $p \in G$, the following hold:*

- *$V_{\bar{\lambda}+1}[G] = V[G]_{\bar{\lambda}+1}$ and $j[G] \subseteq G$.*
- *$V[G]$ satisfies $\bar{\lambda}$-DC.*
- *$\mathbb{Q} = (\dot{\mathbb{Q}})_G$ is a $\bar{\lambda}^+$-closed partial order in $V[G]_\delta$.*
- *For any $V[G]$-generic filter $H \subseteq \mathbb{Q}$, $V[G][H]_\delta$ satisfies ZFC.* □

Combining these two theorems, we show:

Theorem 6.19. *Suppose λ is an ordinal and there is a Σ_1-elementary embedding $j : V_{\lambda+3} \to V_{\lambda+3}$ with $\lambda = \kappa_\omega(j)$. Assume $\mathrm{DC}_{V_{\lambda+1}}$. Then there is a set generic extension N such that for some $\delta < \lambda$, $(V_\delta)^N$ satisfies ZFC $+ I_0$.*

Proof. By Corollary 6.11, $V^{\#}_{\lambda+1}$ exists. Since $V^{\#}_{\lambda+1}$ is definable without parameters in $V_{\lambda+2}$, any elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$ restricts to an elementary embedding from $(V_{\lambda+1}, V^{\#}_{\lambda+1})$ to $(V_{\lambda+1}, V^{\#}_{\lambda+1})$. Therefore the hypotheses of Theorem 6.17 are satisfied. It follows that there is a cardinal $\bar{\lambda} < \lambda$ such that $V_{\bar{\lambda}} \prec V_\lambda$ and there is an elementary embedding from $j : L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1})$ with critical point less than $\bar{\lambda}$. We can assume (by the ultrapower analysis) that j is definable over V_λ from parameters in $V_{\bar{\lambda}+2}$.

Now let $\delta < \lambda$ be a supercompact cardinal of V_λ such that $\delta > \bar{\lambda}$ and $V_\delta \prec V_\lambda$. (If $k : V_\lambda \to V_\lambda$ is elementary, then any point above $\bar{\lambda}$ on the critical sequence of k will do.) The embedding $j : L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1})$ is definable from parameters in $V_{\bar{\lambda}+2}$, so since $V_\delta \prec V_\lambda$, it then follows that j restricts to an elementary embedding from $L_\delta(V_{\bar{\lambda}+1})$ to $L_\delta(V_{\bar{\lambda}+1})$ that is definable over $L_\delta(V_{\bar{\lambda}+1})$.

The hypotheses of Theorem 6.18 hold in V_λ (taking j equal to $j \restriction V_{\bar{\lambda}+1}$). Let $\mathbb{P}$, p, and $\dot{\mathbb{Q}}$ be as in Theorem 6.18 applied in V_λ. Let $G \subseteq \mathbb{P}$ be V-generic with $p \in G$ and $H \subseteq (\dot{\mathbb{Q}})_G$ be $V[G]$-generic. We claim that $V[G][H]_\delta$ satisfies ZFC $+ I_0$. The fact that $V[G][H]_\delta$ satisfies ZFC is immediate from Theorem 6.18 applied in V_λ. (Here we use that $V[G][H]_\delta = V_\lambda[G][H]_\delta$, which follows from the fact that $\mathbb{P} * \dot{\mathbb{Q}} \in V_\lambda$.) Moreover $j[G] \subseteq G$, $\mathbb{P} \in L_\delta(V_{\bar{\lambda}+1})$, and $j(\mathbb{P}) = \mathbb{P}$, so by standard forcing theory, j extends to an elementary embedding from $L_\delta(V_{\bar{\lambda}+1})[G]$ to $L_\delta(V_{\bar{\lambda}+1})[G]$. Since $V_{\bar{\lambda}+1}[G] = V[G]_{\bar{\lambda}+1} = V[G][H]_{\bar{\lambda}+1}$, it follows that j extends to an elementary embedding from $L_\delta(V[G][H]_{\bar{\lambda}+1})$ to $L_\delta(V[G][H]_{\bar{\lambda}+1})$. Therefore $V[G][H]_\delta$ is a model of ZFC $+ I_0$, completing the proof. □

We finish by very briefly sketching the following equiconsistency:

Theorem 6.20. *The following statements are equiconsistent over* ZF + DC*:*

(1) For some λ, $\mathscr{E}(V_{\lambda+2}) \neq \{\mathrm{id}\}$.

*(2) For some λ, λ-*DC *holds and $\mathscr{E}(V_{\lambda+2}) \neq \{\mathrm{id}\}$.*

(3) The Axiom of Choice + I_0.

The equiconsistency of (2) and (3) is due to Schlutzenberg.

The equiconsistency uses Schlutzenberg's Theorem (Theorem 6.1) to reduce to the situation where inverse limit reflection [6] can be applied.

Theorem 6.21. *Assume there is an embedding $j \in \mathscr{E}(L(V_{\lambda+1}))$ with $\lambda = \kappa_\omega(j)$. Assume* DC *holds in $L(V_{\lambda+1})$. Then for any infinite cardinal $\gamma < \lambda$, if γ-*DC *holds in V_λ, then γ-*DC *holds in $L(V_{\lambda+1})$.*

Proof. Assume γ-DC holds in V_λ. By a standard argument, it suffices to show that $\gamma\text{-DC}_{V_{\lambda+1}}$ holds in $L(V_{\lambda+1})$. Suppose T is a γ-closed tree on $V_{\lambda+1}$ with no maximal branches. We must find a cofinal branch of T. Fix $\alpha < (\theta_{\lambda+2})^{L(V_{\lambda+1}}$ such that $T \in L_\alpha(V_{\lambda+1})$. By inverse limit reflection [6], there exist $\gamma < \bar{\lambda} < \bar{\alpha} < \lambda$ and an elementary embedding $J : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1})$ with $T \in \operatorname{ran}(J)$. Let $\bar{T} = J^{-1}(T)$. Working in V_λ, γ-DC yields a cofinal branch $\bar{b} \subseteq \bar{T}$. Since $\bar{b}$ is a γ-sequence of elements of $V_{\bar{\lambda}+1}$, $\bar{b} \in L_1(V_{\bar{\lambda}+1})$. Therefore $\bar{\in} L_{\bar{\alpha}}(V_{\bar{\lambda}+1})$. (We may assume without loss of generality that $\bar{\alpha} \geq 1$.) Now $J(\bar{b})$ is a cofinal branch of T, as desired. $\square$

Corollary 6.22. *Assume there is an embedding $j \in \mathscr{E}(L(V_{\lambda+1}))$ with $\lambda = \kappa_\omega(j)$. Assume* DC *holds in $L(V_{\lambda+1})$. Then for any infinite cardinal $\gamma < \lambda$, if γ-*DC *holds in V_λ, then γ-*DC *holds in $L(V_{\lambda+1})[j \restriction V_{\lambda+2}]$.*

Proof. Let $M = L(V_{\lambda+1})[j \restriction V_{\lambda+2}]$. Again, it suffices to show $\gamma\text{-DC}_{V_{\lambda+1}}$ holds in M. But by Schlutzenberg's Theorem, $V_{\lambda+2} \cap M = V_{\lambda+2} \cap L(V_{\lambda+1})$, so M satisfies $\gamma\text{-DC}_{V_{\lambda+1}}$ if and only if $L(V_{\lambda+1})$ does. Applying Theorem 6.21 then yields the corollary. $\square$

Proof of Theorem 6.20. Assume (1). We may assume $V = L(V_{\lambda+1})[j]$ for a nontrivial embedding $j \in \mathscr{E}(V_{\lambda+2})$ with $\kappa_\omega(j) = \lambda$. We build a forcing extension satisfying (2). Let $\langle \mathbb{Q}_\alpha \mid \alpha < \lambda \rangle$ be Woodin's class Easton iteration for forcing AC, as computed in V_λ. (See [4, Theorem 226].) Let $\mathbb{P}$ be the inverse limit of the sequence $\langle \mathbb{Q}_\alpha \rangle_{\alpha<\lambda}$, and let $\dot{\mathbb{P}}_{\alpha,\lambda}$ be the factor forcing, so $\mathbb{Q}_\alpha * \dot{\mathbb{P}}_{\alpha,\lambda} \cong \mathbb{P}$. By construction, there is an increasing sequence $\langle \kappa_\alpha \rangle_{\alpha<\lambda}$ such that $\mathbb{Q}_\alpha$ forces κ_α-DC over V_λ and $\mathbb{P}_{\alpha,\lambda}$ is κ_α^+-closed in V. But by Corollary 6.22, $\mathbb{Q}_\alpha$ forces κ_α-DC over V. Therefore using that $\mathbb{P}_{\alpha,\lambda}$ is κ_α^+-closed in V, if $G \subseteq \mathbb{P}$ is V-generic, $V[G \cap \mathbb{Q}_\alpha]$ is closed under κ_α-sequences in $V[G]$ and $V[G]$ satisfies κ_α-DC. Since the cardinals κ_α increase to λ, this shows that $V[G]$ satisfies λ-DC. Moreover the standard master condition argument for I_1-embeddings, given for example in [17, Lemma 5.2], shows that G can be chosen so that the embedding j lifts to an elementary embedding $j^* : V[G]_{\lambda+2} \to V[G]_{\lambda+2}$.

The equiconsistency of (2) and (3) is Schlutzenberg's Theorem [2]. $\square$

7 Appendix

In this appendix, we collect together the wellfoundedness proofs for the various Ketonen orders we have used throughout the paper. We take a more general approach by considering a Ketonen order on countably complete filters on complete Boolean algebras. The orders we have considered so far belong to the special case where the Boolean algebras involved are atomic. In our view, the more abstract approach significantly clarifies the wellfoundedness proofs. For a more concrete approach, see the treatment in the author's thesis [1].

Definition 7.1. Suppose $\mathbb{B}_0$ and $\mathbb{B}_1$ are complete Boolean algebras. A *σ-map* from $\mathbb{B}_0$ to $\mathbb{B}_1$ is a function that preserves 0, 1, and countable meets.

We work with σ-maps rather than countably complete homomorphisms so that our results apply to the Ketonen order on filters in addition to the Ketonen order on ultrafilters: if $\langle F_y \mid y \in Y\rangle$ is a sequence of countably complete filters over X, then the function $h : P(X) \to P(Y)$ defined by $h(A) = \{y \in Y \mid A \in F_y\}$ is a σ-map, but is a countably complete homomorphism only if F_y is an ultrafilter for all $y \in Y$.

Given a σ-map from $\mathbb{B}_0$ to $\mathbb{B}_1$, one can define a $\mathbb{B}_1$-valued relation on names for ordinals in $V^{\mathbb{B}_0}$ and $V^{\mathbb{B}_1}$:

Definition 7.2. Suppose $\mathbb{B}_0$ and $\mathbb{B}_1$ are complete Boolean algebras, $\dot{\alpha}_0 \in V^{\mathbb{B}_0}$ and $\dot{\alpha}_1 \in V^{\mathbb{B}_1}$ are names for ordinals, and $h : \mathbb{B}_0 \to \mathbb{B}_1$ is a σ-map. Then

$$[\![\dot{\alpha}_0 < \dot{\alpha}_1]\!]_h = \bigvee_{\beta \in \mathrm{Ord}} h\left([\![\dot{\alpha}_0 < \beta]\!]_{\mathbb{B}_0}\right) \cdot [\![\dot{\alpha}_1 = \beta]\!]_{\mathbb{B}_1}$$

Note that "$\dot{\alpha}_0 < \dot{\alpha}_1$" is not a formula in the forcing language associated to either $\mathbb{B}_0$ or $\mathbb{B}_1$. The notation should be regarded as purely formal.

This notation is motivated by the following considerations. Suppose $h : \mathbb{B}_0 \to \mathbb{B}_1$ is a complete homomorphism. Then there is an embedding $i : V^{\mathbb{B}_0} \to V^{\mathbb{B}_1}$ defined by $i(\dot{x}) = h \circ \dot{x}$. In this case, $[\![\dot{\alpha}_0 < \dot{\alpha}_1]\!]_h = [\![i(\dot{\alpha}_0) < \dot{\alpha}_1]\!]_{\mathbb{B}_1}$. More generally,

$$[\![\dot{\alpha}_0 < \dot{\alpha}_1]\!]_h = [\![h([\![\dot{\alpha}_0 < \dot{\alpha}_1]\!]_{\mathbb{B}_0}) \in \dot{G}_{\mathbb{B}_1}]\!]_{\mathbb{B}_1}$$

Recall that $\dot{G}_{\mathbb{B}}$ denotes the canonical name for a generic ultrafilter in the forcing language associated with the complete Boolean algebra $\mathbb{B}$. Given our assertion above that "$\dot{\alpha}_0 < \dot{\alpha}_1$" is not a formula in the forcing language associated to $\mathbb{B}_0$, the reader may want to take some time interpreting the right-hand side of the formula above.

The following lemma asserts a form of wellfoundedness for the relation given by Definition 7.2:

Lemma 7.3. *Suppose $\langle \mathbb{B}_n, h_{n,m} : \mathbb{B}_n \to \mathbb{B}_m \mid m \leq n < \omega \rangle$ is an inverse system of complete Boolean algebras and σ-maps. Suppose for each $n < \omega$, $\dot{\alpha}_n$ is a $\mathbb{B}_n$-name for an ordinal. Then $\bigwedge_{n<\omega} h_{n,0}([\![\dot{\alpha}_{n+1} < \dot{\alpha}_n]\!]_{h_{n+1,n}}) = 0$.*

Proof. Assume towards a contradiction that the lemma is false. Let β be the least ordinal such that for some $\langle \mathbb{B}_n, h_{n,m}, \dot{\alpha}_n : m \leq n < \omega \rangle$ witnessing the failure of the lemma, $[\![\dot{\alpha}_0 = \beta]\!]_{\mathbb{B}_0} \cdot \bigwedge_{n<\omega} h_{n,0}([\![\dot{\alpha}_{n+1} < \dot{\alpha}_n]\!]_{h_{n+1,n}}) \neq 0$.

The definition of $[\![\dot{\alpha}_1 < \dot{\alpha}_0]\!]_{h_{1,0}}$ yields:

$$\begin{aligned}[\![\dot{\alpha}_0 = \beta]\!]_{\mathbb{B}_0} \cdot [\![\dot{\alpha}_1 < \dot{\alpha}_0]\!]_{h_{1,0}} &= [\![\dot{\alpha}_0 = \beta]\!]_{\mathbb{B}_0} \cdot h_{1,0}([\![\dot{\alpha}_1 < \beta]\!]_{\mathbb{B}_1}) \\ &\leq h_{1,0}([\![\dot{\alpha}_1 < \beta]\!]_{\mathbb{B}_1}) \qquad (13)\end{aligned}$$

For each $m < \omega$, let $a_m = \bigwedge_{m \leq n < \omega} h_{n,m}([\![\dot{\alpha}_{n+1} < \dot{\alpha}_n]\!]_{h_{n+1,n}})$. Since $h_{1,0}$ is a σ-map,

$$a_0 = [\![\dot{\alpha}_1 < \dot{\alpha}_0]\!]_{h_{1,0}} \cdot h_{1,0}(a_1)$$

As a consequence of this and (13):

$$\begin{aligned}[\![\dot{\alpha}_0 = \beta]\!]_{\mathbb{B}_0} \cdot a_0 &= [\![\dot{\alpha}_0 = \beta]\!]_{\mathbb{B}_0} \cdot [\![\dot{\alpha}_1 < \dot{\alpha}_0]\!]_{h_{1,0}} \cdot h_{1,0}(a_1) \\ &\leq h_{1,0}([\![\dot{\alpha}_1 < \beta]\!]_{\mathbb{B}_1}) \cdot h_{1,0}(a_1) \\ &= h_{1,0}([\![\dot{\alpha}_1 < \beta]\!]_{\mathbb{B}_1} \cdot a_1)\end{aligned}$$

By our choice of β, $[\![\dot{\alpha}_0 = \beta]\!]_{\mathbb{B}_0} \cdot a_0 \neq 0$, so we can conclude that $[\![\dot{\alpha}_1 < \beta]\!]_{\mathbb{B}_1} \cdot a_1 \neq 0$. Therefore there is some $\xi < \beta$ such that $[\![\dot{\alpha}_1 = \xi]\!]_{\mathbb{B}_1} \cdot a_1 \neq 0$. This contradicts the minimality of β. $\square$

Definition 7.4. Suppose F_0 and F_1 are countably complete filters on the complete Boolean algebras $\mathbb{B}_0$ and $\mathbb{B}_1$. A *σ-reduction* $h : F_0 \to F_1$ is a σ-map $h : \mathbb{B}_0 \to \mathbb{B}_1$ such that $F_0 \subseteq h^{-1}[F_1]$. Suppose $\dot{\alpha}_0 \in V^{\mathbb{B}_0}$ and $\dot{\alpha}_1 \in V^{\mathbb{B}_1}$ are names for ordinals. A *σ-comparison* $h : (F_0, \dot{\alpha}_0) \to (F_1, \dot{\alpha}_1)$ is a σ-reduction $h : F_0 \to F_1$ such that $[\![\dot{\alpha}_0 < \dot{\alpha}_1]\!]_h \in F_1$.

Theorem 7.5. *There is no infinite sequence of σ-comparisons and countably complete filters of the form $\cdots \xrightarrow{h_{3,2}} (F_2, \dot{\alpha}_2) \xrightarrow{h_{2,1}} (F_1, \dot{\alpha}_1) \xrightarrow{h_{1,0}} (F_0, \dot{\alpha}_0)$.*

Proof. Assume towards a contradiction that there is such a sequence. Fix $n < \omega$. Since $h_{n+1,n} : (F_{n+1}, \dot{\alpha}_{n+1}) \to (F_n, \dot{\alpha}_n)$ is a σ-comparison,

$$[\![\dot{\alpha}_{n+1} < \dot{\alpha}_n]\!]_{h_{n+1,n}} \in F_n$$

Let $h_{n,0} = h_{1,0} \circ \cdots \circ h_{n,n-1}$. Clearly $h_{n,0} : F_n \to F_0$ is a σ-reduction, and therefore

$$h_{n,0}([\![\dot{\alpha}_{n+1} < \dot{\alpha}_n]\!]_{h_{n+1,n}}) \in F_0$$

Since F_0 is a countably complete filter, $\bigwedge_{m<\omega} h_{m,0}([\![\dot{\alpha}_{m+1} < \dot{\alpha}_m]\!]_{h_{m+1,m}}) \in F_0$. In particular, this infinite meet is not 0, contrary to Lemma 7.3. $\square$

We now use this to prove the wellfoundedness of the Ketonen order and the irreflexivity of the internal relation. This is a matter of specializing the theorems we have proved to the case of atomic Boolean algebras.

Definition 7.6. A *pointed filter* on a set X is a pair (F, f) where F is a countably complete filter over X and $f : X \to \mathrm{Ord}$ is a function.

Every function $f : X \to \mathrm{Ord}$ can be associated to the $P(X)$-name τ_f for the ordinal $f(x_G)$ where $x_G \in X$ is the point in X selected by the generic (i.e., principal) ultrafilter $G \subseteq P(X)$. More concretely, the name τ_f is defined by setting $\mathrm{dom}(\tau_f) = \bigcup_{x \in X} f(x)$ and

$$\tau_f(\alpha) = \{x \in X \mid \alpha < f(x)\}$$

for all $\alpha \in \mathrm{dom}(\tau_f)$. Identifying (F, f) and (F, τ_f), Definition 7.4 is transformed as follows:

Definition 7.7. Suppose (F, f) and (G, g) are pointed filters over X and Y and $Z = \langle Z_y \mid y \in Y\rangle$ is a sequence of countably complete filters over X.

- Z is a *filter reduction from F to G* if $F = G\text{-}\lim_{y \in Y} Z_y$.
- Z is a *filter comparison from (F, f) to (G, g)* if Z is a filter reduction from F to G and for G-almost all $y \in Y$, for Z_y-almost all $x \in X$, $f(x) < g(y)$.

We write $Z : F \to G$ to indicate that Z is a filter reduction from F to G. We write $Z : (F, f) \to (G, g)$ to indicate that Z is a filter comparison from (F, f) to (G, g).

Clearly, σ-reductions and σ-comparisons generalize filter reductions and filter comparisons. Let us state this more precisely.

Definition 7.8. Let Φ be the function sending a σ-map $h : P(X) \to P(Y)$ to the sequence $\langle Z_y \mid y \in Y\rangle$ where $Z_y = \{A \subseteq X \mid y \in h(A)\}$ is the filter over X derived from h using y.

Lemma 7.9. *Suppose (F, f) and (G, g) are pointed filters over X and Y. Suppose $h : P(X) \to P(Y)$ is a σ-map. Then $\Phi(h)$ is a filter reduction from F to G if and only if h is a σ-reduction from F to G. Moreover $\Phi(h)$ is a filter comparison from (F, f) to (G, g) if and only if h is a σ-comparison from (F, τ_f) to (G, τ_g).* □

As an immediate corollary of these lemmas and Theorem 7.5, we have the following theorems:

Theorem 7.10. *There is no descending sequence of pointed filters and filter comparisons of the form $\cdots \xrightarrow{Z_3} (F_2, f_2) \xrightarrow{Z_2} (F_1, f_1) \xrightarrow{Z_1} (F_0, f_0)$.* □

Of course, the Ketonen order on filters can be characterized in terms of the notion of a filter reduction:

Lemma 7.11. *Suppose F_0 and F_1 are countably complete filters over ordinals. Then $F_0 <_{\Bbbk} F_1$ in the Ketonen order on filters if and only if there is a σ-comparison from (F_0, id) to (F_1, id).* □

Theorem 7.12 (DC)**.** *The Ketonen order on filters is wellfounded.* □

Whenever $U <_{\Bbbk} W$ in the Ketonen order on ultrafilters, $U <_{\Bbbk} W$ in the Ketonen order on filters. Therefore Theorem 7.12 implies:

Theorem 7.13 (DC)**.** *The Ketonen order on ultrafilters is wellfounded.* □

We finally prove the irreflexivity of the internal relation.

Lemma 7.14. *Suppose U and W are countably complete ultrafilters over sets X and Y. Suppose $\langle Z_y \mid y \in Y\rangle$ is a sequence of countably complete ultrafilters witnessing $U \sqsubset W$. Suppose κ is an ordinal and $g : Y \to \kappa$ is a function such that for any $\alpha < \kappa$, $g(y) > \alpha$ for W-almost all $y \in Y$. Then for any function $f : X \to \kappa$, $Z : (U, f) \to (W, g)$.*

Proof. Let $\langle U_y \mid y \in Y\rangle$ witness that $U \sqsubset W$. Then easily

$$U = W\text{-}\lim_{y \in Y} Z_y$$

So $\langle Z_y \mid y \in Y\rangle$ is an ultrafilter reduction from U to W. Fix a function $f : X \to \kappa$.

We must now verify that for W-almost all $y \in Y$, for Z_y-almost all $x \in X$, $f(x) < g(y)$. Since $\langle Z_y \mid y \in Y\rangle$ witnesses $U \sqsubset W$, it suffices to show that for U-almost all $x \in X$, for W-almost all $y \in Y$, $f(x) < g(y)$. But this is a trivial consequence of our assumption on g and W. □

Corollary 7.15. *Suppose U is a countably complete ultrafilter such that j_U has a critical point. Then $U \not\sqsubset U$.*

Proof. Let X be the underlying set of U. The fact that j_U has a critical point κ implies that there is a function $g : X \to \kappa$ such that for any $\alpha < \kappa$, $g(x) > \alpha$ for U-almost all $x \in X$. Assume $U \sqsubset U$. Then by Lemma 7.14, there is an ultrafilter comparison from (U, g) to (U, g). This contradicts Theorem 7.10. □

References

[1] Gabriel Goldberg. *The Ultrapower Axiom.* PhD thesis, Harvard University, 2019.

[2] Farmer Schlutzenberg. On the consistency of ZF with an elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$. *arXiv preprint,* arXiv:2006.01077, 2020.

[3] Kenneth Kunen. Elementary embeddings and infinitary combinatorics. *J. Symbolic Logic*, 36:407–413, 1971.

[4] W. Hugh Woodin. Suitable extender models I. *J. Math. Log.*, 10(1-2):101–339, 2010.

[5] W. Hugh Woodin. Suitable extender models II: beyond ω-huge. *J. Math. Log.*, 11(2):115–436, 2011.

[6] Scott S. Cramer. Inverse limit reflection and the structure of $L(V_{\lambda+1})$. *J. Math. Log.*, 15(1):1550001, 38, 2015.

[7] Joan Bagaria, Peter Koellner, and W. Hugh Woodin. Large cardinals beyond choice. *Bull. Symb. Log.*, 25(3):283–318, 2019.

[8] Raffaella Cutolo. Berkeley cardinals and the structure of $L(V_{\delta+1})$. *J. Symb. Log.*, 83(4):1457–1476, 2018.

[9] Farmer Schlutzenberg. Reinhardt cardinals and non-definability. *arXiv preprint arXiv:2002.01215*, 2020.

[10] Gabriel Goldberg and Farmer Schlutzenberg. Periodicity in the cumulative hierarchy. *arXiv preprint,* arXiv:2006.01103, 2020.

[11] Robert M. Solovay, William N. Reinhardt, and Akihiro Kanamori. Strong axioms of infinity and elementary embeddings. *Ann. Math. Logic*, 13(1):73–116, 1978.

[12] Toshimichi Usuba. The downward directed grounds hypothesis and very large cardinals. *J. Math. Log.*, 17(2):1750009, 24, 2017.

[13] W. Hugh Woodin. In search of Ultimate-L: the 19th Midrasha Mathematicae Lectures. *Bull. Symb. Log.*, 23(1):1–109, 2017.

[14] Robert M. Solovay. Real-valued measurable cardinals. In *Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967)*, pages 397–428, 1971.

[15] Toshimichi Usuba. A note on Löwenheim-Skolem cardinals. *arXiv preprint,* arXiv:2004.01515, 2020.

[16] A. Lévy and R. M. Solovay. Measurable cardinals and the continuum hypothesis. *Israel J. Math.*, 5:234–248, 1967.

[17] Joel Hamkins. Fragile measurability. *J. Symbolic Logic*, 59(1):262–282, 1994.