COMPACTNESS PHENOMENA IN HOD

GABRIEL GOLDBERG AND ALEJANDRO POVEDA

Abstract. We prove two compactness theorems for HOD. First, if κ is a strong limit singular cardinal with uncountable cofinality and for stationarily many $\delta < \kappa$, $(\delta^+)^{\mathrm{HOD}} = \delta^+$, then $(\kappa^+)^{\mathrm{HOD}} = \kappa^+$. Second, if κ is a singular cardinal with uncountable cofinality and stationarily many $\delta < \kappa$ are singular in HOD, then κ is singular in HOD. We also discuss the optimality of these results and show that the first theorem does not extend from HOD to other ω-club amenable inner models.

1. Introduction

In 1963, Cohen established that Cantor's continuum problem cannot be solved from the accepted ZFC axioms of set theory [Coh63]. This is the problem of determining which among Cantor's transfinite cardinal numbers

$$\aleph_0, \aleph_1, \aleph_2, \ldots \aleph_\omega, \aleph_{\omega+1}, \ldots$$

is the cardinality of the continuum $\mathbb{R}$. More precisely, what Cohen showed is that the axioms cannot rule out that $|\mathbb{R}| = \aleph_2$, while Gödel [Göd39] had already shown that the possibility $|\mathbb{R}| = \aleph_1$ could be ruled out either.

Of course, Cantor himself ruled out that $|\mathbb{R}|$ is $\aleph_0$ by proving that the real numbers form an uncountable set. Later König [Kön05] showed that $|\mathbb{R}|$ is not equal to $\aleph_\omega$, $\aleph_{\omega+\omega}$, or, more generally, $\aleph_\alpha$ for any limit ordinal α of countable cofinality. Soon after Cohen's theorem, Solovay showed that there are no restrictions on the cardinality of the continuum besides the ones established by Cantor and König. For example, it is consistent with the ZFC axioms that $|\mathbb{R}| = \aleph_{19}$ or $|\mathbb{R}| = \aleph_{\omega\cdot\omega+1}$ or $|\mathbb{R}| = \aleph_{\omega_5}$.

The cardinality of the continuum is denoted by $2^{\aleph_0}$, recognizing that $\mathbb{R}$ is equinumerous with the set of functions from $\mathbb{N}$ into a set of size 2. For each cardinal number κ, 2^κ denotes the cardinality of the set of functions from a set of size κ to a set of size 2. The function $\kappa \mapsto 2^\kappa$ is known as the *continuum function.*

After Solovay's result classifying all possible values of $2^{\aleph_0}$, set theorists took up the problem of classifying the possibilities for the continuum function itself. Obviously, we have $2^\kappa \leq 2^\lambda$ whenever $\kappa \leq \lambda$. Also $2^\kappa > \kappa$ by Cantor's theorem, and furthermore $\mathrm{cf}(2^\kappa) > \kappa$ by König's theorem.

Are there any other restrictions on the continuum function, or is the situation analogous to Solovay's theorem for $2^{\aleph_0}$, where no further constraints

2020 *Mathematics Subject Classification.* Primary 03E35; Secondary 03E04.

Key words and phrases. Woodin's HOD conjecture, compactness, Prikry-type forcings.

are possible. In 1966, Easton [Eas70] showed this is precisely the case for regular cardinals, those cardinals κ that are not the limit of fewer than κ smaller cardinals. That is, Easton showed that no restrictions on the behavior of the continuum function on regular cardinals can be established in ZFC except the ones mentioned in the previous paragraphs.

Following Easton's theorem, the outstanding open problem in set theory was to generalize the result to all cardinals, showing without restriction that the continuum function obeys no laws other than those discovered by Cantor and König. This paper is inspired by a theorem of Silver [Sil75], which shows such a generalization of Easton's theorem is not possible: in fact, there are intricate and subtle restrictions on the behavior of the continuum function at singular (i.e., non-regular) cardinals. To this day, the problem of completely classifying the possible behavior of the continuum function at singular cardinals remains open, though the theory of singular cardinal arithmetic has since blossomed into one of the deepest subjects in set theory.

Silver's theorem reveals that the value of $2^{\aleph_{\omega_1}}$ is tied to the values of $2^{\aleph_\alpha}$ for ordinals $\alpha < \omega_1$. More precisely, if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all limit ordinals $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$. More generally, he showed that if κ is a singular cardinal of uncountable cofinality and $2^\lambda = \lambda^+$ for a stationary set of $\lambda < \kappa$, then $2^\kappa = \kappa^+$.

Silver's theorem can be construed as a *compactness property* of the continuum function. The compactness phenomenon describes a general pattern in set theory wherein the properties of a structure are determined by its small substructures. The most familiar compactness phenomena involve infinite structures and their finite substructures: for example, the Compactness Theorem in first-order logic states that the satisfiability of a first-order theory is determined by the satisfiability of its finite fragments. Compactness properties of larger regular cardinals often turn out to be related to large cardinal properties, for instance the tree property and stationary reflection. On the other hand, singular cardinals have been found to have compactness properties that are provable in ZFC; for instance, Shelah's singular compactness theorem in algebra, which led to his solution of Whitehead's problem [She74].

This paper establishes analogs of Silver's theorem in the context of set-theoretic definability, generalizing the phenomenon of singular compactness to this core area of higher set theory. Gödel [Göd46] introduced the concept of *ordinal definability* in an attempt to formalize the intuitive concept of mathematical definability. Roughly speaking, a set is *ordinal definable* if it is definable over the universe of sets using finitely many ordinal numbers as parameters.

The behavior of ordinal definability is highly sensitive to the structure of the universe of sets, and for this reason it is subject to the same independence phenomena that hinder our understanding of the continuum function. The main results of this paper show for the first time that ordinal definability at singular cardinals of uncountable cofinality exhibits patterns of compactness parallel to those that Silver identified for the continuum function.

Our theorems concern two invariants of ordinal definability, which play the role of the continuum function in our analogs of Silver's theorem. First, we define the *ordinal definable cofinality* of an ordinal α, denoted by $\mathrm{cf}^{\mathrm{OD}}(\alpha)$, as the least ordinal δ such that there is an ordinal definable cofinal function from δ to α. Second, we define the *ordinal definable successor of* α, denoted by $\alpha^{+\,\mathrm{OD}}$, as the supremum of all ordinals γ for which there is an ordinal definable surjection from α to γ.[1] With this notation in hand, we can state our compactness theorems for ordinal definability.

Theorem. *Suppose that κ is a singular cardinal with uncountable cofinality and that $\{\delta < \kappa \mid \mathrm{cf}^{\mathrm{OD}}(\delta) < \delta\}$ is stationary. Then $\mathrm{cf}^{\mathrm{OD}}(\kappa) < \kappa$.*

Our theorem on the ordinal definable successor function is significantly harder to prove, and moreover we do not know how to prove it for arbitrary singular cardinals.

Theorem. *Suppose that κ is a singular strong limit cardinal of uncountable cofinality and $\{\delta < \kappa \mid \delta^{+\,\mathrm{OD}} = \delta^+\}$ is stationary. Then $\kappa^{+\,\mathrm{OD}} = \kappa^+$.*

The theorems above are proved by combining the technique of *generic ultrapowers* (see §2.1) with variants of Vopenka's theorem that every set belongs to a forcing extension of HOD. In addition, in §3.1 we employ set-theoretic forcing to show that the hypothesis employed may not be relaxed. Thus, our results are provably optimal.

Finally, we show that the first of our compactness theorems does not extend to arbitrary *ω-club models* (see p.17). This contrasts with the main results of [Gol23] where the first author showed that most known results about HOD — for example, the HOD dichotomy theorem — can actually be proved for an arbitrary inner model that is ω-club amenable.

Theorem. *Assume that every set T belongs to an inner model with a measurable cardinal of Mitchell order 2 above* $\mathrm{rank}(T)$. *Then for every cardinal λ, there is an ω-club inner model M that is correct about cardinals and cofinalities below λ while $(\lambda^+)^M < \lambda^+$.*

The notation of this paper is standard in set theory. In §2 we provide the reader with some preliminaries regarding HOD and the theory of generic ultrapowers. §3 is devoted to prove the above theorems and discuss their optimality. Finally, in §4 we leave some related open questions.

2. Preliminaries and notation

This section collects some set-theoretic tools employed through the paper. The material here is standard and is included just for the benefit of our readers. We also introduce some relevant terminology.

[1] Of course, $\mathrm{cf}^{\mathrm{OD}}(\alpha)$ is just the cofinality of α as computed in the inner model HOD, and $\alpha^{+\,\mathrm{OD}}$ is the least cardinal of HOD that is greater than α.

2.1. **Generic ultrapowers.** Fix a set X. A set $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an *ideal* if $\emptyset \in \mathcal{I}$, $X \notin \mathcal{I}$ and $\mathcal{I}$ is closed under subsets and finite unions. Dually, a set $\mathcal{F} \subseteq \mathcal{P}(X)$ a *filter* if $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$ and $\mathcal{F}$ is closed under supersets and finite intersections. A filter $\mathcal{U}$ is called an *ultrafilter* if it satisfies the following additional property: given $A \in \mathcal{P}(X)$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$; equivalently, $\mathcal{U}$ is a $\subseteq$-maximal filter.

Given an ideal $\mathcal{I} \subseteq \mathcal{P}(X)$ its *dual filter* $\mathcal{I}^*$ is defined as $\{X \setminus A \mid A \in \mathcal{I}\}$. The *sets of $\mathcal{I}$-positive measure* (denoted $\mathcal{I}^+$) is the collection of all $A \in \mathcal{P}(X)$ that do not belong to $\mathcal{I}$. Note that $\mathcal{I}^*$ is a filter, $\mathcal{I}^* \subseteq \mathcal{I}^+$ and $A \in \mathcal{I}^+$ if and only if $A \cap B \neq \emptyset$ for all $B \in \mathcal{I}^*$. These concepts have natural parallels in the setting of filters $\mathcal{F} \subseteq \mathcal{P}(X)$ as well [Jec03, §7].

Given an ideal $\mathcal{I} \subseteq \mathcal{P}(X)$ define an equivalence relation $\sim_{\mathcal{I}}$ on $P(X)$ as follows:

$$A \sim_{\mathcal{I}} B \text{ if and only if } A \triangle B \in \mathcal{I}.$$

This yields a quotient $\mathcal{P}(X)/\mathcal{I}$, which, endowed with the order

$$[X] \leq [Y] \iff X \setminus Y \in \mathcal{I},$$

gives rise to a Boolean algebra. After removing the zero element from $\mathcal{P}(\kappa)/\mathcal{I}$ the partial ordering $\leq$ becomes a separative order.

There is another presentation of the poset $(\mathcal{P}(X)/\mathcal{I} \setminus \{[\emptyset]\}, \leq)$ as $\mathbb{P} := (\mathcal{I}^+, \subset)$. The two posets are forcing equivalent in the sense that they give rise to the same generic extensions. Indeed, the former poset is the separative quotient of the latter. A V-generic filter $G \subseteq \mathbb{P}$ yields a V-ultrafilter on X extending the dual filter $\mathcal{I}^*$. If $\mathcal{I}$ is κ-complete then G is also V-κ-complete.[2] For details concerning these facts see [Jec03, §22].

Suppose that $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ is an ideal containing all singletons and that it is κ-complete (i.e., $\bigcup_{\alpha<\lambda} A_\alpha \in \mathcal{I}$ provided $\langle A_\alpha \mid \alpha < \lambda\rangle \subseteq \mathcal{I}$ with $\lambda < \kappa$).

Let $G \subseteq \mathbb{P}$ be a V-generic filter. Working in $V[G]$, we can define the *generic ultrapower of V by G*. Namely, in $V[G]$ one defines the structure

$$\mathrm{Ult}(V, G) := \langle ({}^X V) \cap V)/{=_G}, \in_G\rangle$$

where for each two functions $f, g\colon X \to V$ (in V)

$$f =_G g \text{ if and only if } \{x \in X \mid f(x) = g(x)\} \in G$$

and

$$[f]_G \in_G [g]_G \text{ if and only if } \{x \in X \mid f(x) \in g(x)\} \in G.$$

Here and in the future we will denote by $[f]_G$ the $=_G$ equivalence class of f, omitting the subscript when there is no chance of confusion.

It turns out that $\mathrm{Ult}(V, G)$ is a model of ZFC, yet not necessarily well-founded, even if G is V-κ-complete for an uncountable cardinal κ. As usual, an appropriate version of Łoś's theorem holds. Namely,

$$\mathrm{Ult}(V, G) \models \varphi([f_1], \ldots, [f_n]) \iff \{x \in X \mid V \models \varphi(f_1(x), \ldots, f_n(x))\} \in G,$$

[2] A filter G is V-κ-complete if given $\lambda < \kappa$ and a sequence $\langle A_\alpha \mid \alpha < \lambda\rangle$ in V of sets in G then $\bigcap_{\alpha<\lambda} A_\alpha \in G$.

where $\varphi(v_1, \dots, v_n)$ is a first order formula in the language $\{=, \in\}$.

This ensures that the map $j_G \colon \langle V, \in\rangle \to \mathrm{Ult}(V, G)$ given by $a \mapsto [c_a]_G$ is an elementary embedding, where $c_a \colon X \to V$ is the constant function with value a.

The combinatorial properties of the ideal $\mathcal{I}$ (in V) are related to the properties of the embedding $j_G : V \to \mathrm{Ult}(V, G)$ in the generic extension $V[G]$. For example, $\mathcal{I}$ is κ-complete if and only if the maximal condition $[X]$ forces that the critical point of j_G is at least κ.

The generic ultrapower construction will play a prominent role in the forthcoming §3. We refer the reader to [Jec03, §22] or Foreman's excellent handbook chapter [For09] for any notion not considered in this account.

Remark 2.1. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a filter and $\mathcal{I} := \{X \setminus F \mid F \in \mathcal{F}\}$ be its dual ideal. Since the $\mathcal{F}$-positive sets are exactly the $\mathcal{I}$-positive sets, all the previous comments remain valid starting with a filter $\mathcal{F}$ and taking $\mathbb{P} := (\mathcal{F}^+, \subseteq)$. This will be the approach we take through §3. We decided to phrase the discussion here in the language of ideals just because this is the approach pursued in reference text, such as [Jec03, For09].

Definition 2.2. Given a filter $\mathcal{F} \subseteq \mathcal{P}(X)$ and functions $f, g \colon X \to V$, denote
$$f <_{\mathcal{F}} g \iff \{x \in X \mid f(x) < g(x)\} \in \mathcal{F}.$$
Similarly, if $\mathcal{I} \subseteq \mathcal{P}(X)$ is an ideal,
$$f <_{\mathcal{I}} g \iff \{x \in X \mid f(x) \geq g(x)\} \in \mathcal{I}.$$

2.2. **Ordinal definability and forcing.** A set X is called *ordinal definable* if it is definable by a formula of the language of set theory using ordinals as parameters. More formally, there is $\varphi(x, \vec{y})$ and $\langle \alpha_*, \alpha_0, \dots \alpha_n\rangle \subseteq \mathrm{Ord}$ such that
$$x \in X \Leftrightarrow V_{\alpha^*} \models \varphi(x, \alpha_0, \dots, \alpha_n).$$
The class of ordinal definable sets is denoted by OD. This contains the class of ordinals Ord and satisfies all the ZFC axioms except for extensionality. Another caveat is that OD is not transitive. To mitigate this pathology one looks at a special subclass of OD – the *Hereditarily Ordinal Definable* sets, HOD. A set X is Hereditarily Ordinal Definable (or, simply, in HOD) if $X \in \mathrm{OD}$ and the transitive closure of $\{X\}$ is contained in OD. It turns that HOD is an *inner model*; namely, it is a transitive class containing the ordinals and satisfying all the ZFC axioms.

At many places in this paper we shall be preoccupied with the following issue. Suppose that $\mathbb{P} \in \mathrm{OD}$ is a forcing poset and $G \subseteq \mathbb{P}$ is V-generic – how does $\mathrm{HOD}^{V[G]}$ compare to HOD^V? Here HOD^V (resp. $\mathrm{HOD}^{V[G]}$) stands for the class HOD as computed in V (resp. in $V[G]$). In special circumstances this comparison can be done; e.g., if $\mathbb{P}$ is *cone/weakly homogeneous.*

Definition 2.3. A poset $\mathbb{P}$ is *weakly homogeneous* if for for all $p, q \in \mathbb{P}$ there is an automorphism $\varphi \colon \mathbb{P} \to \mathbb{P}$ making $\varphi(p)$ and q compatible.

Similarly, $\mathbb{P}$ is *cone-homogeneous* if for all $p, q \in \mathbb{P}$ there are $p^* \leq p$ and $q^* \leq q$ together with an isomorphism $\varphi \colon \mathbb{P}/p^* \to \mathbb{P}/q^*$.

We used $\mathbb{P}/p$ to denote the subposet of $\mathbb{P}$ with universe $\{q \in \mathbb{P} \mid q \leq p\}$. It is clear that every weakly homogeneous forcing is cone-homogeneous.

Lemma 2.4 (Folklore)**.** *If $\mathbb{P} \in \mathrm{OD}$ is a cone-homogeneous forcing poset then $\mathrm{HOD}^{V[G]} \subseteq \mathrm{HOD}^V$ for all V-generic $G \subseteq \mathbb{P}$.*

Sometimes we will need to assume (see e.g. §3.1) that HOD encompasses large cardinals which exist in V. This can be done by forcing "$V = \mathrm{HOD}$" with *McAloon iteration* coding V into the continuum function. The said iteration preserves large cardinals (see [FHR15, BP23]) and produces a model V such that $V \subseteq \mathrm{HOD}^{V^{\mathbb{Q}}}$ for any set-sized forcing $\mathbb{Q}$.

3. Two compactness theorems for HOD

In this section we prove our compactness theorems for HOD (Theorems 3.4 and 3.5). Our results are very much in the spirit of Silver's classical theorem that the generalized continuum hypothesis cannot first fail at a singular cardinal of uncountable cofinality [Sil75]. The overall idea is to extract some information about ordinal definability from Silver's argument, which heavily uses the technique of generic ultrapowers from §2.1.

Let us begin with the following key result:

Theorem 3.1 (Casey–Goldberg)**.** *For any strong limit cardinal λ,*

$$\mathrm{cf}(\lambda^{+\mathrm{HOD}}) \in \{\omega, \mathrm{cf}(\lambda), \lambda^+\}$$

Proof. Let $\kappa := \lambda^{+\mathrm{HOD}}$ and let $\delta := \mathrm{cf}(\kappa)$. Let us assume that $\omega < \delta < \lambda^+$.

We must show that $\delta = \mathrm{cf}(\lambda)$. Let $\mathcal{F}$ denote the restriction of the closed unbounded filter on κ to HOD; i.e., $\mathcal{F} := \mathrm{Cub}_\kappa \cap \mathrm{HOD}$. Since Cub_κ is ordinal definable it is easy to check that $\mathcal{F} \in \mathrm{HOD}$ and that it is a filter in HOD. We will denote by $\mathcal{F}^+$ the $\mathcal{F}$-positive sets, as computed in HOD; namely, the collection of all $A \in \mathcal{P}(\kappa)^{\mathrm{HOD}}$ intersecting all members of $\mathcal{F}$.

Claim 3.1.1. *In HOD, $\mathcal{F}$ is weakly normal in the sense that if $S \in \mathcal{F}^+$ and $f \colon S \to \kappa$ is a regressive function in HOD, there is $\beta < \kappa$ such that*

$$\{\alpha \in S \mid f(\alpha) \leq \beta\} \in \mathcal{F}^+.^3$$

Moreover, if $\gamma \in \mathrm{Ord}$, $\mathrm{cf}^V(\gamma) \neq \delta$, $S \in \mathcal{F}^+$, and $f \colon S \to \gamma$ is any function in HOD, then there is some $\beta < \gamma$ such that

$$\{\alpha \in S \mid f(\alpha) \leq \beta\} \in \mathcal{F}^+.$$

Finally, $(\mathcal{F}^+, \subset)$ is forcing equivalent in HOD to a poset of size less than $(2^\delta)^{+V}$.

[3] This notion of weak normality is weaker than the one considered in [Kan76].

Proof of claim. The bounding properties of $\mathcal{F}$ in HOD follow from the corresponding properties of Cub_κ in V. That is, if $S \in \mathcal{F}^+$ and $f : S \to \kappa$ is a regressive function in V, then f is bounded on a stationary set, and if γ is an ordinal whose cofinality is not δ and $f : S \to \gamma$ is any function in V, then f is bounded on a stationary set. We leave these as exercises with the following hints. First, by restricting to a club in κ of ordertype δ, one can reduce to the more familiar case that $\kappa = \delta$. Second, to prove the statement about functions into γ, one can split into cases based on whether the cofinality of γ is less than δ or greater than δ; in the former case, one appeals to the δ-completeness of the club filter, and in the latter case, one uses that functions from δ to γ are bounded *everywhere*.

Finally, $(\mathcal{F}^+, \subset)$ is equivalent in HOD to a forcing of size less than $(2^\delta)^{+V}$ because its separative quotient $\mathbb{Q}$ has cardinality less than $(2^\delta)^{+V}$: note that the underlying set of $\mathbb{Q}$ is precisely the set of equivalence classes of $\mathcal{F}^+$ modulo the non-stationary ideal on κ. In V, choose a set $\mathcal{T} \subseteq \mathcal{F}^+$ such that for each $S \in \mathcal{F}^+$, there is exactly one $S' \in \mathcal{T}$ such that $S \triangle S'$ is non-stationary (in V). Then $|\mathcal{T}|^{\mathrm{HOD}} = |\mathbb{Q}|^{\mathrm{HOD}}$, and $V \models$ "$|\mathcal{T}| \leq 2^\delta$". To see this last inequality, fix a closed unbounded set $C \subseteq \kappa$ of ordertype δ. Then $\langle S \cap C \mid S \in \mathcal{T}\rangle$ is a sequence of distinct subsets of C from which we deduce that $V \models |\mathcal{T}| \leq |\mathcal{P}(C)| = 2^\delta$. □

By forcing over HOD with $(\mathcal{F}^+, \subset)$, we extend $\mathcal{F}$ to a HOD-weakly normal HOD-ultrafilter G on κ with the property that if γ is an ordinal with $\mathrm{cf}^V(\gamma) \neq \delta$ then every $f\colon \kappa \to \gamma$ in HOD is bounded on a set in G. (This is by the moreover part of the claim together with a density argument.) In particular, the generic ultrapower map $i\colon \mathrm{HOD} \to N := \mathrm{Ult}(\mathrm{HOD}, G)$ is continuous at ordinals γ of V-cofinality distinct from δ. Note that N may not be well-founded so N-ordinals may fail to be ordinals.

Since G is HOD-weakly normal (because so is $\mathcal{F}$, by the previous claim) it follows that

$$N \models \text{“}[\mathrm{id}]_G = \sup i[\kappa] < i(\kappa)\text{”}.^4$$

Assume towards a contradiction that

$$\mathrm{cf}(\lambda) \neq \delta.$$

By this assumption and our previous comments, $N \models$ "$i(\lambda) = \sup i[\lambda]$" (i.e., *$i$ is continuous at λ*) as every $f\colon \kappa \to \lambda$ is bounded on a set in G.

Since $\kappa := \lambda^{+\mathrm{HOD}}$, $\{\xi < \kappa \mid \mathrm{cf}^{\mathrm{HOD}}(\xi) \leq \lambda\} \in G$, and so

$$N \models \text{“}\,\mathrm{cf}(\sup i[\kappa]) \leq \sup i[\lambda]\text{”}.$$

Let us next argue that this inequality is strict.

[4] Since N is not-well founded, statements of the form "$\sup(i[\kappa]) < i(\kappa)$" may not have the intended meaning when interpreted in V. We are grateful to the anonymous for urging us to clarify this point.

Note that $\mathrm{HOD}[G] \models$ "$\mathrm{cf}(\sup i[\kappa]) = \kappa$": Indeed, κ remains regular in $\mathrm{HOD}[G]$ because G comes from a forcing equivalent to another of cardinality less than $(2^\delta)^{+V} < \lambda$.

Because $\mathrm{HOD}[G] \models$ "$\mathrm{cf}(\sup i[\lambda]) = \mathrm{cf}(\lambda) \leq \lambda < \kappa$" it follows that

$$\mathrm{HOD}[G] \models \text{“}\mathrm{cf}(\sup i[\kappa]) \neq \mathrm{cf}(\sup i[\lambda])\text{”}.$$

Therefore N satisfies the same; namely,

$$N \models \text{“}\mathrm{cf}(\sup i[\kappa]) \neq \mathrm{cf}(\sup i[\lambda])\text{”}.$$

In particular, we must have

$$N \models \text{“}\mathrm{cf}(\sup i[\kappa]) < \sup i[\lambda]\text{”}.$$

Let $C \in N$ be such that

$$N \models \text{“}C \text{ is a closed cofinal subset of } \sup i[\kappa] \text{ of order-type } \mathrm{cf}(\sup i[\kappa])\text{”}.$$

Recall that i is continuous at ordinals whose V-cofinality is not equal to δ. In particular, i is continuous at ordinals whose $V[G]$-cofinality γ lies between $(2^\delta)^+$ and λ: by preservation of regular cardinals, such an ordinal has the same cofinality in V. Thus for any such γ, a familiar argument shows that $i[\kappa] \cap C$ is γ-closed cofinal in $\sup i[\kappa]$. Hence $i^{-1}[C]$ is cofinal in κ.

Let $B := i^{-1}[C]$, and note that there is some $A \in \mathrm{HOD}$ unbounded in κ contained in B because G is generic for a partial order of size less than $(2^\delta)^{+V} < \lambda \leq \kappa$. Since $i[A] \subseteq C$, letting $f\colon A \to \kappa$ be the transitive collapse, $i[\kappa] \subseteq \bar{C}$ where $\bar{C} := i(f)[C]$.

Note that

$$N \models \text{“}\mathrm{otp}(\bar{C}) = \mathrm{otp}(C)\text{”}.$$

Fix a $V[G]$-regular cardinal $\gamma \in (\delta, \lambda)$ such that

$$N \models \text{“}\mathrm{otp}(C) < i(\gamma)\text{”}.$$

Then $i[\gamma] \subseteq \bar{C} \cap i(\gamma)$, and so

$$N \models \text{“}i[\gamma] \text{ is bounded above by } \sup(\bar{C} \cap i(\gamma)) < i(\gamma)\text{”}.$$

In particular, i is discontinuous at γ. However i must be continuous at γ because $\mathrm{cf}(\gamma) \neq \delta$. This yields a contradiction showing that our original assumption that $\mathrm{cf}(\lambda) \neq \delta$ was false. □

The proof of Theorem 3.4 requires another technical result.

Definition 3.2. Let $V \subseteq W$ be two transitive models of ZFC and $\kappa \in V$ be such that $V \models$ "κ is a regular cardinal". We say that the pair (V, W) has the *κ-uniform cover property* if for every function $f \in W$ with $\mathrm{dom}(f) \in V$ and $\mathrm{ran}(f) \subseteq V$ there is yet another function $F \in V$ with $\mathrm{dom}(F) = \mathrm{dom}(f)$, and for all $i \in \mathrm{dom}(f)$, $f(i) \in F(i)$ and $V \models |F(i)| < \kappa$.

If $\mathbb{P} \in V$ is a κ-cc forcing poset then standard arguments show that $(V, V[G])$ has the κ-uniform cover property. Conversely, a remarkable theorem by Bukovský [Buk73] says that if (V, W) has the κ-uniform cover property then there is a poset $\mathbb{P} \in V$ that has the κ-cc in V, $W \models$ "$|\mathbb{P}| \leq 2^\kappa$" and W is a generic extension of V by $\mathbb{P}$ (see [Sch20, Theorem 3.11])

Lemma 3.3. *Suppose $\mathbb{P} \in \mathrm{HOD}$ is a κ-cc forcing and $G \subseteq \mathbb{P}$ is a V-generic. Let N be the class of sets that are hereditarily definable in the structure $\langle V[G], V, G, \in\rangle$ from ordinal parameters. Then the pair (HOD, N) has the κ-uniform cover property.*

In particular, N is a forcing extension of HOD by a forcing $\mathbb{Q} \in \mathrm{HOD}$ such that $\mathrm{HOD} \models$ "$\mathbb{Q}$ is κ-cc" and $N \models$ "$|\mathbb{Q}| \leq 2^\kappa$".

Proof. Clearly, $\mathrm{HOD} \subseteq N$. We verify that (HOD, N) has the κ-uniform cover property. Fix an ordinal λ and a function $f : \lambda \to \lambda$ that is definable in the structure $\langle V[G], V, G, \in\rangle$ from ordinal parameters. Let $\varphi(x_0, x_1, x_2)$ be a formula in the language of $\langle V[G], V, G, \in\rangle$ such that for some ordinal β, $f(\xi) = \zeta$ if and only if $\langle V[G], V, G, \in\rangle$ satisfies $\varphi(\xi, \zeta, \beta)$. Then let

$$F(\xi) = \{\zeta < \lambda \mid \exists p \in \mathbb{P}\, (p \Vdash_{\mathbb{P}} \varphi(\xi, \zeta, \beta))\}.$$

Note that F is ordinal definable (because so is $\mathbb{P}$) and that $f(\xi) \in F(\xi)$. Since $\mathbb{P}$ is κ-cc it also follows that $\mathrm{HOD} \models |F(\xi)| < \kappa$. $\square$

We are now in a position to prove our first main result:

Theorem 3.4. *If κ is a strong limit singular cardinal of uncountable cofinality and $\{\delta < \kappa \mid (\delta^+)^{\mathrm{HOD}} = \delta^+\}$ is stationary then $(\kappa^+)^{\mathrm{HOD}} = \kappa^+$.*

Proof. The first attempt at a proof, on which the correct proof will elaborate, proceeds as follows. Let $\iota = \mathrm{cf}(\kappa)$ and fix $f \colon \iota \to \kappa$ an increasing continuous cofinal function. Let $\mathcal{F}$ be the club filter on ι. Then, by assumption,

$$S := \{\xi < \iota \mid f(\xi)^{+\,\mathrm{HOD}} = f(\xi)^+\} \in \mathcal{F}^+.$$

By forcing with $\mathcal{F}^+$ below S one produces a generic filter $G \subseteq \mathcal{F}^+$ extending the filter $\mathcal{F}$, which is ι-complete and normal in V. In particular,

$$(\dagger)\ \ \{\xi < \iota \mid f(\xi)^{+\,\mathrm{HOD}} = f(\xi)^+\} \in G.$$

Then we take the generic ultrapower $j_G : V \to M_G$, using only functions $f \colon \iota \to V$ in the ground model V (see §2.1). By V-normality of G,

$$(\dagger\dagger)\ \ X \in G \iff \iota \in j_G(X).$$

The ultrapower M_G has its own version of κ, the unique ordinal κ_* of M_G that is "κ-like" in the sense that each of its predecessors has cardinality less than κ, whereas the set of predecessors of κ_* has cardinality exactly κ. Indeed, $\kappa_* = j_G(f)(\iota)$, where as above $f \colon \iota \to \kappa$ is any cofinal continuous function in V. Note that if M_G is well-founded, then $\kappa_* = \kappa$, but we must deal with the possibility that M_G is ill-founded.

Let us begin with an easy (yet useful) observation.

Claim 3.4.1. *In $V[G]$, $|(\kappa_*^+)^{M_G}| \geq \kappa^+$.*

Proof of claim. In V, fix a sequence of functions $\langle h_\alpha \rangle_{\alpha<\kappa^+} \subseteq \prod_{\xi<\iota} f(\xi)^+$ that is increasing in the order of domination modulo the bounded ideal on ι; namely, for each $\alpha < \beta < \kappa^+$, $\{\xi < \iota \mid h_\alpha(\xi) \geq h_\beta(\xi)\}$ is bounded in ι. Such a sequence exists because this reduced product is κ^+-directed. Note that in $V[G]$, $\langle j_G(h_\alpha)(\iota)\rangle_{\alpha<\kappa^+}$ is an increasing sequence of length κ^+ consisting of predecessors of $(\kappa_*^+)^{M_G}$. Thus $|(\kappa_*^+)^{M_G}| \geq \kappa^{+V}$. But $\kappa^{+V} = \kappa^{+V[G]}$ since G is added by $(\mathcal{F}^+, \subseteq)$, which is a forcing of size $2^\iota < \kappa$. $\square$

Let $H \coloneqq \mathrm{HOD}^{M_G}$. By (†) and (††) above, $(\kappa_*^+)^H = (\kappa_*^+)^{M_G}$.

Let N denote the inner model of $V[G]$ consisting of all sets hereditarily ordinal definable in the structure $\langle V[G], V, G, \in\rangle$. The model N is a κ-cc forcing extension of HOD by Lemma 3.3, and so $(\kappa^+)^{\mathrm{HOD}} = (\kappa^+)^N$.[5] If the structure H were a subclass of N, then we could finally conclude that

$$(\kappa^+)^{\mathrm{HOD}} = (\kappa^+)^N \geq |(\kappa_*^+)^H| = |(\kappa_*^+)^{M_G}| \geq \kappa^+.$$

The intuition that H should be a subclass of N comes from our experience with well-founded ultrapowers. The structure M_G is definable over the structure $\langle V[G], V, G, \in\rangle$, and so if M_G were well-founded, then any element of H, being ordinal definable in M_G, would be ordinal definable in $\langle V[G], V, G, \in\rangle$; this would yield $H \subseteq N$. If H is ill-founded, however, then ordinals of M_G are not really ordinals, so it is not clear that ordinal definable elements of M_G are ordinal definable in $\langle V[G], V, G, \in\rangle$. To handle the possibility that H is not well-founded, we take a different approach.

Instead, we consider the V-ultrafilter $\mathcal{U}$ on κ given by $\mathcal{U} \coloneqq f_*(G)$ where

$$f_*(G) \coloneqq \{A \in \mathcal{P}(\kappa)^V \mid f^{-1}[A] \in G\},$$

and the ultrapower

$$H_0 \coloneqq \mathrm{Ult}(\mathrm{HOD}, \mathcal{U} \cap \mathrm{HOD})$$

of HOD by $\mathcal{U} \cap \mathrm{HOD}$, using only functions in HOD.

An important observation is that $\mathcal{U}$ is ordinal definable in the structure $\langle V[G], V, G, \in\rangle$. This is because $f'_*(G) = \mathcal{U}$ for any increasing continuous cofinal map $f' : \iota \to \kappa$. Therefore $H_0 \subseteq N$: The point is that the structure H_0 has for its universe the class of ordinal definable functions from κ into HOD, which is a subclass of HOD and hence of N; the (possibly nonstandard) membership and equality predicates of H_0 are ordinal definable over $\langle V[G], V, G, \in\rangle$ as they are definable from $\mathcal{U} \cap \mathrm{HOD}$, which belongs to N.

Let $\kappa_0 \coloneqq [\mathrm{id}]_{\mathcal{U}}$. Then κ_0 is the unique κ-like ordinal of H_0, in the same sense that κ_* is the unique κ-like ordinal of M_G. The argument from above shows that $|(\kappa_0^+)^{H_0}| = |(\kappa^+)^{\mathrm{HOD}}|$ in $V[G]$. We also obtain the following:

Claim 3.4.2. $N \models \kappa^+ = |(\kappa_0^+)^{H_0}|$.

[5]Here, and hereafter, HOD is in the sense of V.

Proof of claim. By Lemma 3.3, $(\kappa^+)^N = (\kappa^+)^{\mathrm{HOD}}$. Thus, as $H_0, \mathrm{HOD} \subseteq N$ and $|(\kappa_0^+)^{H_0}| = |(\kappa^+)^{\mathrm{HOD}}|$ in $V[G]$, $N \models |(\kappa_0^+)^{H_0}| = |(\kappa^+)^{\mathrm{HOD}}| = \kappa^+$. □

Next we work towards showing that the previous claim is incompatible with "$(\kappa^+)^{\mathrm{HOD}} < \kappa^+$". This will yield the desired contradiction and as a result will lead to the proof of the theorem.

Let us begin with an auxiliary claim:

Claim 3.4.3. *There is a $<_{\mathcal{U}}$-increasing sequence $\langle g_\alpha \mid \alpha < (\kappa^+)^{\mathrm{HOD}}\rangle \subseteq \prod_{\delta<\kappa}(\delta^+)^V$ in* HOD*, such that letting $\gamma_\alpha := [g_\alpha]_{\mathcal{U}}$, $\langle \gamma_\alpha \mid \alpha < (\kappa^+)^{\mathrm{HOD}}\rangle$ is an increasing cofinal sequence in $(\kappa_0^+)^{H_0}$.*

Proof of claim. We note first that there is such a sequence in N. This is simply because N satisfies that $|(\kappa_0^+)^{H_0}| = (\kappa^+)^{\mathrm{HOD}} = \kappa^+$, and moreover by the proof of this fact, N satisfies that $\mathrm{cf}((\kappa_0^+)^{H_0}) = \kappa^+$, so in N one can choose representatives for an increasing cofinal sequence in $(\kappa_0^+)^{H_0}$, which is simply a $<_{\mathcal{U}}$-increasing sequence $\langle g_\alpha \mid \alpha < (\kappa^+)^{\mathrm{HOD}}\rangle \subseteq \prod_{\delta<\kappa}(\delta^+)^V$ such that letting $\gamma_\alpha := [g_\alpha]_{\mathcal{U}}$, $\langle \gamma_\alpha \mid \alpha < (\kappa^+)^{\mathrm{HOD}}\rangle$ is cofinal in $(\kappa_0^+)^{H_0}$.

Now we pull the sequence down to HOD. Let $\mathbb{P} := (\mathcal{F}^+, \subseteq)$ denote our poset. Since κ is a strong limit cardinal (in V) and $|\mathbb{P}|^V < \kappa$ there is some V-regular $\gamma < \kappa$ such that $\mathbb{P}$ is γ-cc. By Lemma 3.3, the pair (HOD, N) has the γ-uniform cover property, so there is $\mathbb{Q} \in \mathrm{HOD}$ with $N \models$ "$|\mathbb{Q}| = 2^\gamma$" and $N = \mathrm{HOD}[F]$ where F is a HOD-generic filter for $\mathbb{Q}$. Note that

$$(2^\gamma)^N \leq (2^\gamma)^{V[G]} < \kappa,$$

because κ remains a strong limit cardinal in $V[G]$.

Let $\langle \dot{g}_\alpha \mid \alpha < (\kappa^+)^{\mathrm{HOD}}\rangle \in \mathrm{HOD}$ be a sequence of $\mathbb{Q}$-names for functions in HOD such that $(\dot{g}_\alpha)_F = g_\alpha$. Since $|\mathbb{Q}| < \kappa$, there is a condition $p \in F$ deciding the value of $\dot{g}_\alpha$ for unboundedly many $\alpha < \kappa^{+\mathrm{HOD}}$; that is, for an unbounded set $S \subseteq (\kappa^+)^{\mathrm{HOD}}$ in HOD, for each $\alpha \in S$, $p \Vdash_{\mathbb{P}} \dot{g}_\alpha = \check{g}_\alpha$. Now $\langle g_\alpha \mid \alpha \in S\rangle \in \mathrm{HOD}$ is as desired.[6] □

Assume towards a contradiction that $(\kappa^+)^{\mathrm{HOD}} < \kappa^+$. Let us define a factor embedding $k : H_0 \to H$ by

$$k([g]_{\mathcal{U}}) := j_G(g)(\kappa_*).$$

This equals $j_G(g \circ f)(\iota)$ and is a well-defined elementary embedding.

The next claim ensures that the ultrafilter $\mathcal{D}$ defined below is ordinal definable in the structure $\langle V[G], V, G, \in\rangle$:

Claim 3.4.4. *$k[\kappa_0^{+H_0}]$ has a least upper bound $\nu < (\kappa_*^+)^H$ in H.*

Proof of claim. By Theorem 3.1, $\mathrm{cf}(\kappa^{+\mathrm{HOD}}) \leq \iota$. Let $\rho := \mathrm{cf}(\kappa^{+\mathrm{HOD}})$ and $A \subseteq \kappa^{+\mathrm{HOD}}$ be a cofinal set of ordertype ρ. Then $\langle \gamma_\alpha\rangle_{\alpha\in A}$ is cofinal in

[6] We thank the second referee for pointing that the proof of this claim that appeared in the first draft of this paper was garbled to the point of incorrectness.

$(\kappa_0^+)^{H_0}$, and hence $\langle k(\gamma_\alpha)\rangle_{\alpha\in A}$ is cofinal in $k[\kappa_0^{+H_0}]$. But $\langle k(\gamma_\alpha)\rangle_{\alpha\in A}\in M_G$: Letting $\langle g^*_\alpha\rangle_{\alpha<j_G(\kappa^{+\mathrm{HOD}})} = j_G(\langle g_\alpha\rangle_{\alpha<\kappa^{+\mathrm{HOD}}})$,

$$\langle k(\gamma_\alpha)\rangle_{\alpha\in A} = \langle g^*_\alpha(\kappa_*)\rangle_{\alpha\in j_G[A]}$$

with $j_G[A]\in M_G$. (As $\mathrm{crit}(j_G)=\iota$, $A\in V$, and $|A|\leq\iota$.) Since $\langle k(\gamma_\alpha)\rangle_{\alpha\in A}$ is a set of ordinals in M_G, it has a least upper bound ν, and since $\langle k(\gamma_\alpha)\rangle_{\alpha\in A}$ is cofinal in $k[\kappa_0^{+H_0}]$, ν is the least upper bound of $k[\kappa_0^{+H_0}]$.

Note that $\nu<(\kappa_*^+)^H$: First, by our comments after Claim 3.4.1, $(\kappa_*^+)^H = (\kappa_*^+)^{M_G}$, so $(\kappa_*^+)^H$ is regular in M_G. Second, the previous argument shows that $\mathrm{cf}^{M_G}(\nu)\leq\iota$, which is less than κ. Therefore, $\nu<(\kappa_*^+)^H$. □

Let $\mathcal{D}$ be the H_0-ultrafilter on $(\kappa_0^+)^{H_0}$ derived from k using ν; namely,

$$\mathcal{D}=\{S\in\mathcal{P}^{H_0}((\kappa_0^+)^{H_0})\mid \nu\in k(S)\}.$$

Let $H_1 \coloneqq \mathrm{Ult}(H_0,\mathcal{D})$, again using only functions in H_0. Let $i: H_0\to H_1$ be the ultrapower embedding, and let $\bar\nu\coloneqq[\mathrm{id}]_{\mathcal{D}}$. Then,

$$\bar\nu=\sup i[\kappa_0^{+H_0}] < i(\kappa_0^{+H_0}).$$

Note that $\mathcal{D}$ is ordinal definable in the structure $\langle V[G],V,G,\in\rangle$, and hence $H_1\subseteq N$, by the same argument as for H_0. Since the ultrapower embedding $i: H_0\to H_1$ is definable over $\langle V[G],V,G,\in\rangle$ from ordinal parameters,

$$i\restriction\kappa_0^{+H_0}\in N.$$

The next claim yields the desired contradiction with Claim 3.4.2:

Claim 3.4.5. $N\models|(\kappa_0^+)^{H_0}|\leq\kappa$.

Proof of claim. Since $i[\kappa_0^{+H_0}]\subseteq\bar\nu$ it follows that $|\kappa_0^{+H_0}|^N\leq|\bar\nu|^N$. Also, $\bar\nu<i(\kappa_0^{+H_0})=i(\kappa_0)^{+H_1}$. Since $H_1\subseteq N$ we have the following inequalities:

$$|\kappa_0^{+H_0}|^N\leq|\bar\nu|^N\leq|i(\kappa_0)|^N=\kappa.$$

The latter equality being true in that $i(\kappa_0)$ is κ-like, as it embeds into κ_*. □

Since we get a contradiction our initial assumption that "$(\kappa^+)^{\mathrm{HOD}}<\kappa^+$" was false, and this proves the theorem. □

Let us now prove our second compactness theorem. This uses a slightly different technique (due to Casey–Goldberg) to prove the theorem for an arbitrary singular cardinal of uncountable cofinality; a direct adaptation of the argument from Theorem 3.4 would only prove the result for strong limit cardinals.

Theorem 3.5. *If κ is a singular cardinal of uncountable cofinality and $\{\delta<\kappa\mid\mathrm{cf}^{\mathrm{HOD}}(\delta)<\delta\}$ is stationary in κ, then $\mathrm{cf}^{\mathrm{HOD}}(\kappa)<\kappa$.*

Proof. Assume towards a contradiction that κ is a regular cardinal in HOD. For the rest of the proof $\mathcal{F}$ denotes the closed unbounded filter on κ.

We claim that in HOD, the filter $\bar{\mathcal{F}}=\mathcal{F}\cap\mathrm{HOD}$ is weakly normal in the sense that every regressive function $f: A\to\kappa$ in HOD defined on a set $A \in \bar{\mathcal{F}}$ admits some $\gamma < \kappa$ such that $\{\alpha \in A \mid f(\alpha) < \gamma\} \in \bar{\mathcal{F}}$.[7] Fix $f \in \mathrm{HOD}$, and assume towards a contradiction that no such γ exists.

By Fodor's Lemma, it is not hard to see that any regressive function defined on a stationary subset of κ is bounded on a stationary set. (This is the argument used in Claim 3.1.1.) Therefore let γ_0 be least such that the function $f : A \to \kappa$ is bounded by γ_0 on a stationary subset of A. By our assumption, the set A_1 of ordinals $\alpha \in A$ such that $f(\alpha) \geq \gamma_0$ is stationary as well. Let γ_1 be least such that $f \restriction A_1$ is bounded below γ_1 on a stationary set. Continuing this way, we can produce a continuous sequence $\langle \gamma_i \mid i < \kappa \rangle$ such that for all $i < \delta$,

$$\{\alpha \in A \mid f(\alpha) \in [\gamma_i, \gamma_{i+1})\}$$

is stationary. We use our assumption that κ is regular in HOD to ensure that the process can be continued at limit ordinals $i < \kappa$. (Note that the entire construction is internal to HOD.) But since $\mathrm{cf}(\kappa) < \kappa$, there cannot be κ-many disjoint stationary subsets of κ.

A similar argument shows that if $\gamma < \kappa$ is regular in HOD, greater than $\mathrm{cf}(\kappa)$, and of a different V-cofinality from κ, then $\bar{\mathcal{F}}$ is γ-indecomposable in HOD in the following sense: Working in HOD, any function $f \colon B \to \gamma$ with $B \in \bar{\mathcal{F}}$ is bounded below γ on a set in $\bar{\mathcal{F}}$.

Until further notice let us work in HOD and denote

$$S := \{\delta < \kappa \mid \mathrm{cf}(\delta)^{\mathrm{HOD}} < \delta\}.$$

Since $S \in \bar{\mathcal{F}}^+$, there is an ultrafilter U extending

$$\bar{\mathcal{F}} \cup \{S\}.$$

Since $\bar{\mathcal{F}}$ is weakly normal, U is weakly normal, and since $S \in U$, U concentrates on singular cardinals. Therefore by [Ket72, Theorem 1.3], U is (ν, κ)-regular for some $\nu < \kappa$.[8]

Claim 3.5.1. *U is γ-decomposable for every regular cardinal in (ν, κ).*

Proof of claim. Let $\langle A_\alpha \rangle_{\alpha < \kappa}$ be a witness for "U is (ν, κ)-regular". Namely, this is a collection of U-measure one sets such that $\bigcap_{\alpha \in I} A_\alpha = \emptyset$ for all $I \subseteq \kappa$ with $|I| = \nu$. Let $\gamma \in (\nu, \kappa)$ be regular, and define a function $f \colon \kappa \to \gamma$ as $f(\alpha) := \sup\{\beta < \kappa \mid \alpha \in A_\beta\}$. The fact that $\langle A_\alpha \rangle_{\alpha < \kappa}$ witnesses (κ, ν)-regularity ensures that f is well-defined. Note that f cannot be bounded below γ on a set in U: Otherwise, $A := \{\alpha < \kappa \mid f(\alpha) < \beta\} \in U$, for some $\beta < \gamma$, but by definition $A \cap A_\beta = \emptyset$. Therefore, U is γ-decomposable. $\square$

[7] Note that this is a stronger form of normality than the one proved in Claim 3.1.1.

[8] Here one cannot directly use Ketonen's result since U is not countably complete. Ketonen's result refers to the "first function" of an ultrafilter, which in the case of a weakly normal ultrafilter is the identity. As Ketonen remarks after the proof of Theorem 1.3, the result only requires the existence of a first function, not the countable completeness of U. Since our ultrafilter does have a first function, namely the identity, the required result is true.

Now we return to V. Since $\bar{\mathcal{F}}$ is γ-indecomposable in HOD for all ordinals γ that are regular in HOD, greater than $\mathrm{cf}(\kappa)$, and of different V-cofinality from κ, U is γ-indecomposable for such ordinals.

It follows that for all ordinals $\gamma \in (\max\{\nu, \mathrm{cf}(\kappa)\}, \kappa)$, if γ is regular in HOD, then $\mathrm{cf}(\gamma) = \mathrm{cf}(\kappa)$; otherwise the previous paragraph implies U is γ-indecomposable while the paragraph preceding it implies U is γ-decomposable. But κ is a limit of V-regular cardinals, and these are certainly regular in HOD and do not have the same cofinality as κ. This is a contradiction. □

3.1. **Optimality.** In this section we discuss the optimality of Theorems 3.4 and 3.5. Some of our arguments require rather technical Prikry-type forcings. Instead of elaborating on their precise definitions (which are fairly long) we give appropriate references. Let us begin with Theorem 3.4. The next shows that the cofinality assumption is necessary in Theorem 3.4:

Proposition 3.6. *Assume that κ is a κ^+-supercompact cardinal. Then, there is a generic extension where*

(1) κ is a strong limit cardinal with $\mathrm{cf}(\kappa) = \omega$,
(2) $\delta^{+\,\mathrm{HOD}} = \delta^+$ for all $\delta < \kappa$,
(3) and $(\kappa^{+\mathrm{HOD}}) < \kappa^+$.

Proof. By forcing with McAloon iteration we may assume that "$V = \mathrm{HOD}$" holds (see p.6). Let $\mathcal{U}$ be a κ-complete, normal and fine ultrafilter over $\mathcal{P}_\kappa(\kappa^+)$. Let us force with the *Supercompact Prikry forcing* with respect to $\mathcal{U}$ ([Git10, §1]). This forcing is easily shown to be cone-homogeneous so that $\mathrm{HOD}^{V[G]} = V$ holds for all V-generic $G \subseteq \mathbb{P}$. This forcing does not introduce bounded subsets of κ so, in $V[G]$, $V[G]_\kappa = V_\kappa$. Also, κ becomes a strong limit cardinal with $\mathrm{cf}(\kappa) = \omega$. This gives (1) and (2) above. Finally, in $V[G]$, $(\kappa^+)^V$ is collapsed to κ, hence $(\kappa^+)^{\mathrm{HOD}^{V[G]}} = (\kappa^+)^V < \kappa^+$. □

The hypothesis "$\{\delta < \kappa \mid \delta^{+\,\mathrm{HOD}} = \delta^+\}$ is stationary" is also necessary:

Theorem 3.7. *Suppose that κ is a κ^{+2}-supercompact cardinal such that $2^{\kappa^{+n}} = \kappa^{n+1}$ for $n < \omega$. Then, for each regular uncountable cardinal $\mu < \kappa$ there is a generic extension where:*

(1) κ is a strong limit cardinal with cofinality μ.
(2) $\kappa^{+\,\mathrm{HOD}} < \kappa^+$.
(3) There is a club $C \subseteq \kappa$ with $\mathrm{otp}(C) = \mu$ such that

$$\delta^{+\,\mathrm{HOD}} = \delta^+ \textit{ for all cardinals } \delta < \kappa \textit{ not in } \mathrm{acc}(C).$$

Proof. By preliminarily forcing with McAloon's iteration we may assume that $V \subseteq \mathrm{HOD}^{V^{\mathbb{Q}}}$ for any set-sized forcing $\mathbb{Q}$. If this iteration is started at a sufficiently large regular cardinal our hypothesis on κ are maintained.

Suppose that $j\colon V \to M$ is a κ^{+2}-supercompact embedding. Let $\mu < \kappa$ be a regular cardinal and let u be the (κ, κ^+)-measure sequence of length μ derived from j. Namely, $u = \langle u_\alpha \mid \alpha < \mu\rangle$ where $u_0 := j``\kappa^+$ and

$u_\alpha := \{X \subseteq V_\kappa \mid u \restriction \alpha \in j(X)\}$ for $\alpha > 0$. Notice that M contains every (κ, κ^+)-measure sequence of length less than μ so u above indeed exists. In addition, by the argument in [CFG15, Lemma 3.2], u belongs to $\mathcal{U}^{\sup}_\infty$.[9]

Let $\mathbb{R}^{\sup}_u$ be the supercompact Radin forcing defined from u [CFG15]. Let $G \subseteq \mathbb{R}^{\sup}_u$ a V-generic filter. Combining [CFG15, Corollary 4.2] with our forcing preparation

$$V \subseteq \mathrm{HOD}^{V[G]} \subseteq V[G^\phi],$$

where G^ϕ is a V-generic for a plain Radin forcing $\mathbb{R}_u$ – hence, for a cardinal-preserving poset. In particular, the following inequalities hold:

$$(\kappa^+)^V \leq (\kappa^+)^{\mathrm{HOD}^{V[G]}} \leq (\kappa^+)^{V[G^\phi]} = (\kappa^+)^V < (\kappa^+)^{V[G]}.$$

The above yields item (2) of the theorem.

Let $\langle w_\alpha \mid \alpha < \mu\rangle$ be an injective enumeration of $\{w \mid w$ appears in $p \in G\}$. Denote $\kappa_{w_\alpha} := \min(\mathrm{Ord} \setminus w_\alpha(0))$ and $\lambda_{w_\alpha} := \mathrm{otp}(w_\alpha(0))$. The increasing enumeration of $\{\kappa_{w_\alpha} \mid \alpha < \mu\}$ yields a club $C \subseteq \kappa$ of order-type μ and by forcing below an appropriate condition μ remains regular in $V[G]$. In addition, standard arguments show that κ remains strong limit in $V[G]$ (see [CFG15, Lemma 3.10(6)]). These two observations combined yield item (1) of the theorem. Finally, [CFG15, Lemma 3.10(8)] shows that the only V-cardinals $\leq\kappa$ that survive after passing to $V[G]$ are those outside

$$\bigcup_{\alpha \in \mathrm{Lim} \cap \mu} (\kappa_{w_\alpha}, \lambda_{w_\alpha}].$$

Thus, for every $V[G]$-cardinal $\delta \notin \mathrm{acc}(C)$ we have that $(\delta^+)^V$ does not belong to the above union and thus $(\delta^+)^V = (\delta^+)^{V[G]}$. By our previous observations this yields $(\delta^+)^{\mathrm{HOD}^{V[G]}} = (\delta^+)^V = (\delta^+)^{V[G]}$, as claimed. □

Remark 3.8. The exact consistency strength of the configuration described above is unclear to us. Since the configuration violates the weak covering theorem for K [JS13, MSS97], one obtains the lower bound of a Woodin cardinal, but presumably one can obtain a stronger lower bound.

Theorem 3.9. *Suppose that κ is Mahlo with $\{\delta < \kappa \mid o(\delta) = \tau\}$ stationary for all $0 < \tau < \kappa$. Then, there is a cardinal-preserving generic extension $V[G * C]$ where the following properties hold:*

1. *(1) κ remains inaccessible, hence $V[G * C]_\kappa \models \mathrm{ZFC}$;*
2. *(2) For each regular $\omega_1 \leq \lambda < \kappa$ there is a stationary set $S_\lambda \subseteq E^\kappa_\lambda$ consisting of strong limit cardinals θ such that:*
   - *(a) θ is inaccessible in HOD;*
   - *(b) $\{\delta < \theta \mid \mathrm{cf}(\delta)^{\mathrm{HOD}} < \delta\}$ is unbounded in θ.*

[9] Our assumption that "$2^{\kappa^{+n}} = \kappa^{+n+1}$ holds for all $n < \omega$" is used precisely at this stage. A close inspection of the proof of [CFG15, Lemma 3.2] indicates that less instances of the GCH suffice to run the argument. However, we opted for this slightly stronger assumption for the sake of a more neat presentation.

Proof. Let $\mathcal{U} = \langle \mathcal{U}(\alpha,\tau) \mid \alpha < \kappa, \tau < o^{\mathcal{U}}(\alpha)\rangle$ be a coherent sequence of normal measures. Using $\mathcal{U}$ one inductively defines an iteration of Prikry-type forcings with non-stationary support $\mathbb{P} = \langle \mathbb{P}_\alpha; \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa\rangle$ such that:

(1) $\mathbb{Q}_\alpha$ is non-trivial if and only if $o(\alpha) > 0$;
(2) For non-trivial stages $\alpha < \kappa$, $\mathbb{Q}_\alpha$ has the following properties:
  (a) $\langle \mathbb{Q}_\alpha, \leq_\alpha, \leq^*_\alpha\rangle$ is an α^+-cc Prikry-type forcing and α-closed with respect to its pure-extension order $\leq^*_\alpha$.
  (b) $\mathbb{Q}_\alpha$ adds a club $c_\alpha \subseteq \alpha$ of order-type $\omega^{o(\alpha)}$ which is a Prikry/Magidor sequence for the sequence $\langle \mathcal{U}(\alpha,\tau) \mid \tau < o^{\mathcal{U}}(\alpha)\rangle$.

The forcing (in the above form) is due to Ben-Neria and Unger [BNU17], which in turn is inspired by a classical construction due to Gitik [Git86].

Let G a $\mathbb{P}$-generic filter. Following [BNU17] we conveniently denote:

$$\Delta_\tau := \{\delta < \kappa \mid \delta \in \mathrm{Inac} \vee o(\delta) = \tau\} \text{ and } \Delta := \textstyle\bigcup_{\tau<\kappa} \Delta_\tau.^{10}$$

Claim 3.9.1. *Δ is a fat stationary subset of κ in $V[G]$.*

Proof of claim. By [BNU17, Corollary 3.7], Δ is stationary in $V[G]$. Let $\eta < \kappa$ and $E \subseteq \kappa$ be a club in $V[G]$. We have to show that $E \cap \Delta$ contains a closed set of order-type η. Working in V, let $\tau \in (0,\kappa)$ big enough so that $\eta < \omega^\tau$. By [BNU17, Corollary 3.7] there is a V-club $\bar{E} \subseteq E$. Let $\delta \in \mathrm{acc}(\bar{E}) \cap \Delta_\tau$. (This exists because Δ_τ is V-stationary.) The iteration $\mathbb{P}$ introduces at stage δ a club $c_\delta \subseteq \delta$ with $\mathrm{otp}(c_\delta) = \omega^\tau$. Moreover, c_δ is a Prikry/Magidor club so it is eventually contained in every V-club on δ (e.g., in $\bar{E} \cap \delta$) and it eventually consists of V-inaccessible cardinals. Therefore,

$$c_\delta \setminus \chi \subseteq \bar{E} \cap \Delta_0 \subseteq E \cap \Delta$$

for some $\chi < \delta$. It suffices to take an initial segment of $c_\delta \setminus \chi$ with order-type η to conclude that $E \cap \Delta$ contains a closed set with that order-type. $\square$

Let us force over $V[G]$ with $\mathbb{C}(\Delta)$ the poset adding a (generic) club subset $C \subseteq \Delta$ (see [AS83]). Since Δ is fat, results of Abraham and Shelah [AS83] guarantee that $\mathbb{C}(\Delta)$ is κ-distributive and thus κ remains inaccessible in $V[G * C]$. Thanks to the analysis made by Ben-Neria and Unger (specially, [BNU17, Theorem 4.6]) we know that $\mathbb{P} * \dot{\mathbb{C}}(\Delta)$ is weakly homogeneous. So, a fortiori, let us asssume that "$V \subseteq \mathrm{HOD}^{V^{\mathbb{Q}}}$" holds for all set-sized poset $\mathbb{Q} \in V$ (by preliminarily forcing with McAloon's iteration). In this case we have $\mathrm{HOD}^{V[G*C]} = V = \mathrm{HOD}^V$ holds in $V[G * C]$.

Claim 3.9.2. *Item (1) holds; namely, κ remains inaccessible in $V[G * C]$.*

Proof of claim. Since $\mathbb{C}(\Delta)$ is κ-distributive in $V[G]$ it suffices to argue that the above holds in $V[G]$: κ remains strong limit in $V[G]$ by a standard factoring argument; κ remains regular by virtue of [BNU17, Corollary 3.7], which says that every $V[G]$-club on κ contains a V-club. $\square$

[10] For $\tau > 0$ the requirement $o(\delta) = \tau$ subsumes '$\delta \in \mathrm{Inac}$".

Let $\omega_1 \leq \lambda < \kappa$ be a regular cardinal and set

$$S_\lambda := \mathrm{acc}(C) \cap (E^\kappa_\lambda)^{V[G]}.$$

Fix $\theta \in S_\lambda$. Since $\theta \in S_\lambda \subseteq \Delta$ this is V-inaccessible, hence $\mathrm{HOD}^{V[G*C]}$-inaccessible, this disposes with (2)(a) above. Standard factoring arguments with $\mathbb{P}$ also show that θ remains strong limit in $V[G]$ and thus in $V[G*C]$ as well. Finally, $\{\delta < \theta \mid \mathrm{cf}(\delta)^{\mathrm{HOD}^{V[G*C]}} < \delta\} = \{\delta < \theta \mid \mathrm{cf}(\delta)^V < \delta\}$ is unbounded in θ because this latter is a V-inaccessible. $\square$

3.2. **On ω-club amenability.** The first author showed that many of the known results on HOD, for example the HOD dichotomy theorem, can actually proved for an arbitrary inner model that is ω-club amenable [Gol23].

A set $C \subseteq \delta$ is an *ω-club in δ* (for $\mathrm{cf}(\delta) \geq \omega_1$) if it is unbounded (in δ) and whenever S is a countable subset of C, $\sup(S) \in C$. The ω-club filter on δ, denoted by $\mathcal{C}_\delta$, is the collection of all subsets of δ that contain an ω-club.

Definition 3.10. An inner model M is *ω-club amenable* if $\mathcal{C}_\delta \cap M \in M$ for all ordinals δ with uncountable cofinality.

The truth is that very little is known about HOD that does not already hold of any ω-club amenable model, so it is natural to seek properties that are more specific to HOD. In this section, we show that Theorem 3.4 does not generalize to an arbitrary ω-club amenable model.

Let $\vec{\mathcal{C}}$ denote the proper class $\{(\delta, S) \mid \mathrm{cf}(\delta) \geq \omega_1, S \in \mathcal{C}_\delta\}$. If one builds the constructible universe relative to the sequence $\vec{\mathcal{C}}$, then one obtains an ω-club amenable model. More generally:

Lemma 3.11. *For any class A, $M = L[A, \vec{\mathcal{C}}]$ is ω-club amenable.* $\square$

To construct ω-club amenable models that do not satisfy the conclusion of Theorem 3.4 we need a fairly mild large cardinal hypothesis. In terms of consistency strength, the hypothesis is a bit weaker than the assumption of a measurable cardinal of Mitchell order 2, or equivalently the existence of a normal ultrafilter that concentrates on measurable cardinals. The specific hypothesis we use is the notion of the *sword* of a set, which is a generalization of the sharp of a set. If X is a set, then $X^\ddagger$ will denote X-sword, which is roughly the minimal model of set theory M such that $X \cap M \in M$ and there is some $\kappa > \mathrm{rank}(X)$ that carries a normal ultrafilter of Mitchell order 1. The precise definition of $X^\ddagger$ involves a bit of inner model theory, all of which can be found in Zeman's textbook [Zem11] except that we must relativize the theory developed there to an arbitrary predicate.

If X is a family of subsets of an ordinal λ, an *X-oracle mouse* is a structure $(M, \mathbb{E})$ satisfying the usual mousehood conditions relative to a predicate for X. Thus $\mathbb{E}$ is a fine sequence of partial extenders with critical point above λ and $M = J_\alpha[X, \mathbb{E}]$ is iterable.[11] This notion is slightly different from the

[11] Note that the definition of an X-oracle mouse technically depends on the ordinal λ.

usual concept of an X-mouse because we do not demand that X belongs to M, only that $X \cap M \in M$. The analogy is: X-oracle mice are to X-mice as $L[X]$ is to $L(X)$.

We will only discuss mice at the level of a measurable cardinal of Mitchell order 2, so all the extenders on $\mathbb{E}$ are normal measures (of Mitchell order at most 1) and iterability simply asserts that any iterated ultrapower formed using these measures is well-founded. In fact, the only X-oracle mice we consider are *s-mice* in the sense of Zeman [Zem11, Page 199], which means that they are active and their last extender is the unique measure of Mitchell order 1. We will call such a structure an *X-oracle s-mouse.*

If X belongs to an inner model containing a measurable cardinal of Mitchell order 2 above the rank of X, then an X-oracle s-mouse exists, and a Skolem hull argument yields one that is in addition sound:

Lemma 3.12. *Suppose X is a family of subsets of the ordinal λ and there is an X-oracle s-mouse. Then there is an X-oracle s-mouse M such that $M = \mathrm{Hull}^M_{\Sigma_1}(\lambda)$.* □

In fact, by a comparison argument, one can show that there is a unique such X-mouse.

Definition 3.13. If X is a family of subsets of the ordinal λ, the X-oracle mouse *X-sword*, denoted by $X^\ddagger$, is the unique X-oracle s-mouse M such that $M = \mathrm{Hull}^M_{\Sigma_1}(\lambda)$.

Although X-oracle mice seem like a natural generalization of relativized constructibility, the detailed fine structure of these objects does not seem to have been considered in the literature. We are confident that, for example, the comparison lemma can be generalized to this context, but we have not checked this. Moreover, thanks to a suggestion of one of the referees, the proofs below do not require any deep fine-structural analysis: we do not really make use of the comparison theorem for X-oracle mice, but rather we will just define one particular iteration of one particular mouse.

The only place the comparison lemma appears to show up is in Definition 3.13 above: the typical proof of the uniqueness of $X^\ddagger$ requires a comparison argument. However, we do not really need this uniqueness: instead of using $X^\ddagger$, we could run the proofs of Lemmas 3.14 and 3.16 with any X-oracle s-mouse M such that $M = \mathrm{Hull}^M_{\Sigma_1}(\alpha)$, and we would still obtain the desired conclusions. In spite of this, we will use the notation $X^\ddagger$ for convenience.

The following lemmas, which are due to one of the anonymous referees, vastly simplify the proof of Lemma 3.16 below.

Lemma 3.14. *Suppose X is a family of subsets of an ordinal λ. Then $\mathrm{cf}(o(X^\ddagger)) = \omega$.*

Proof. Let $\theta > \lambda$ be a regular cardinal, and let N be a countable transitive set admitting an elementary embedding $\pi : N \to H(\theta)$ with $(X, \lambda) \in \mathrm{ran}(\pi)$. Let $\bar{X} = \pi^{-1}(X)$, and note that $\bar{X}^\ddagger$ is a countable model and π restricts to an elementary embedding $i : \bar{X}^\ddagger \to X^\ddagger$. Let $\bar{U} = F^{\bar{X}^\ddagger}$ be the top measure of $\bar{X}^\ddagger$, let $\nu = \sup i[o(\bar{X}^\ddagger)]$, and let $M = (X^\ddagger|\nu, U)$ where $U = \bigcup_{A\in \bar{X}^\ddagger} i(\bar{U}\cap A)$. Then $i : \bar{X}^\ddagger \to (X^\ddagger|\nu, U)$ is Σ_1-elementary and $(X^\ddagger|\nu, U) \preceq_{\Sigma_1} X^\ddagger$, so since $X^\ddagger = \mathrm{Hull}_{\Sigma_1}^{X^\ddagger}(\lambda)$ where $\lambda < \nu$, we have $M = X^\ddagger$. It follows that $\mathrm{cf}(o(X^\ddagger)) = \mathrm{cf}(o(M)) = \mathrm{cf}(o(\bar{X}^\ddagger)) = \omega$. $\square$

Lemma 3.15. *Suppose X is a family of subsets of an ordinal λ and M is an X-oracle s-mouse. If $M = \mathrm{Hull}_{\Sigma_1}^M(\alpha)$ where $\alpha \geq \lambda$, and $\delta > \alpha$ is regular in M, then $\mathrm{cf}(\delta) = \mathrm{cf}(o(M))$.*

Proof. Let U be the top measure of M, and for $\xi < o(M)$, let $M_\xi = (M|\xi, U\cap M|\xi)$. Then for any Σ_1-formula $\varphi(x)$ and any $a \in M$, $M \vDash \varphi(a)$ if and only if $M_\xi \vDash \varphi(a)$ for all sufficiently large ξ. It follows that $M = \mathrm{Hull}_{\Sigma_1}^M(\alpha) = \bigcup_{\xi<o(M)} \mathrm{Hull}_{\Sigma_1}^{M_\xi}(\alpha)$. Note that for all $\xi < o(M)$, $\mathrm{Hull}_{\Sigma_1}^{M_\xi}(\alpha) \in M$ and $M \vDash |\mathrm{Hull}_{\Sigma_1}^{M_\xi}(\alpha)| \leq \alpha$.

Now suppose $\delta > \alpha$ is regular in M. For $\xi < o(M)$, let

$$\beta_\xi = \sup(\delta \cap \mathrm{Hull}_{\Sigma_1}^{M_\xi}(\alpha))$$

Then $\beta_\xi < \delta$ since δ is regular in M and $M \vDash |\mathrm{Hull}_{\Sigma_1}^{M_\xi}(\alpha)| \leq \alpha < \delta$. Since $\langle\beta_\xi\rangle_{\xi<o(M)}$ is a weakly increasing cofinal sequence in δ, it follows that $\mathrm{cf}(\delta) = \mathrm{cf}(o(M))$. $\square$

Lemma 3.16. *Assume that for all X, $X^\ddagger$ exists. Then for any cardinal λ and any set $A \subseteq \lambda$, $L[A, \vec{\mathcal{C}}]$ does not correctly compute λ^+.*

Proof. Fix a set $A \subseteq \lambda$. Using a pairing function on λ, it is not hard to construct a family X of subsets of λ such that for any class E, $L[X, E] = L[A, \mathcal{C} \restriction (\lambda + 1), E]$. In particular,

$$L[X, \vec{\mathcal{C}} \restriction (\lambda, \infty)] = L[A, \vec{\mathcal{C}}]$$

where $(\lambda, \infty) = \{\xi \in \mathrm{Ord} : \xi > \lambda\}$.

Let $M = X^\ddagger$. We claim that $L[A, \vec{\mathcal{C}}]$ is contained in a proper initial segment P of an iterate N of M. (In fact, we will have $L[A, \vec{\mathcal{C}}] = P$ and $P = N\restriction\mathrm{Ord}$ where N is a non-dropping iterate of M that is not set-like.) By Lemmas 3.15 and 3.14, $\lambda^{+P} = \lambda^{+M}$ has countable cofinality, and it follows that $\lambda^{+L[A,\vec{\mathcal{C}}]} \leq \lambda^{+P} < \lambda^+$.

The idea is to iterate M to a model N with the following property. For each ordinal $\delta > \lambda$, there is a total measure on δ on the sequence of N if and only if δ has uncountable cofinality in V; moreover, in this case the least total measure on δ on the sequence of N is equal to $\mathcal{C}_\delta \cap N$.

The iteration is defined by selecting at each stage the first total measure on the sequence of the current iterate that lies on an ordinal of countable cofinality. More formally, we define an iterated ultrapower

$$\langle(M_\alpha, U_\alpha) \mid \alpha \in \mathrm{Ord}\rangle$$

of M by setting U_α equal to the first total measure W on the M_α-sequence that lies on an ordinal κ_α of countable cofinality in V; if there is no such measure W, set U_α equal to the top measure of M_α.

Let N be the direct limit of the linearly directed system $\langle (M_\alpha, U_\alpha) \mid \alpha \in \mathrm{Ord}\rangle$, and let $P = N|\mathrm{Ord}$ be the set-like part of N. It is clear that if for some ordinal δ, there is a total measure W on δ on the sequence of P, then δ has uncountable cofinality.

We claim that if $\delta > \lambda$ is a regular cardinal of P that has uncountable cofinality in V, then there is a total measure on δ on the sequence of P and the least such measure is equal to $\mathcal{C}_\delta \cap P$. We prove the claim by showing by induction on $\alpha \in \mathrm{Ord}$ that if $\delta > \lambda$ is a regular cardinal of M_α that has uncountable cofinality in V, then there is a total measure on δ on the sequence of M_α and the least such measure equal to $\mathcal{C}_\delta \cap M_\alpha$.

For the case that $\alpha = 0$, note that by the definition of $M = X^\ddagger$, we have $M = \mathrm{Hull}^M_{\Sigma_1}(\lambda)$, and so by the referee's Lemmas 3.14 and 3.15, if $\delta > \lambda$ is a regular cardinal of M, then δ has countable cofinality in V. Thus the base case holds vacuously.

Now assume the induction hypothesis holds for M_α, and we claim it is true for $M_{\alpha+1}$. Suppose therefore that $\delta > \lambda$ is a regular cardinal of $M_{\alpha+1}$. Note that $M_\alpha|\kappa_\alpha = M_{\alpha+1}|\kappa_\alpha$, so for ordinals δ in the open interval (λ, κ_α), then the induction hypothesis for M_α easily implies the induction hypothesis for $M_{\alpha+1}$. A slightly more complicated variation of this argument establishes the induction hypothesis for $M_{\alpha+1}$ in the case that $\delta = \kappa_\alpha$: by the definition of the iteration, either κ_α has countable cofinality, in which case we have nothing to show, or else U_α is the top measure of M_α, in which case the fact that M_α and $M_{\alpha+1}$ have the same least measure on κ_α is enough to verify the induction hypothesis for $\delta = \kappa_\alpha$ in $M_{\alpha+1}$.

To finish the successor case, we show that if $\delta > \kappa_\alpha$ is regular in $M_{\alpha+1}$, then $\mathrm{cf}(\delta) = \omega$, so the induction hypothesis holds vacuously in the interval $(\kappa_\alpha, o(M_{\alpha+1}))$. Since $M_{\alpha+1}$ is generated by the critical points of the iteration along with λ, $M_{\alpha+1} = \mathrm{Hull}^{M_{\alpha+1}}_{\Sigma_1}(\kappa_\alpha + 1)$. Since there is a cofinal embedding from M to $M_{\alpha+1}$, Lemma 3.14 implies that $\mathrm{cf}(o(M_{\alpha+1})) = \omega$. Therefore we can apply Lemma 3.15 to obtain that $\mathrm{cf}(\delta) = \omega$, as desired.

Finally, we consider the limit case. Suppose α is an infinite limit ordinal and $\delta > \lambda$ is a regular cardinal of M_α. Assume the induction hypothesis is true for all $\beta < \alpha$. For each $\beta < \alpha$, let $\delta_\beta = j^{-1}_{\beta,\alpha}[\delta]$ where $j_{\beta,\alpha} : M_\beta \to M_\alpha$ is the iteration map. Then for some $\beta_0 < \alpha$, for all $\beta \in (\beta_0, \alpha)$, $j_{\beta,\alpha}(\delta_\beta) = \delta$.

First assume that for some $\beta_1 > \beta_0$, for all $\beta > \beta_1$, $\kappa_\beta \neq \delta_\beta$. Then $j_{\beta,\alpha}$ is continuous at δ_β for all $\beta \in (\beta_1, \alpha)$. Using this, it is easy to propagate the induction hypothesis for M_β with respect to δ_β, where $\beta \in (\beta_1, \alpha)$, to M_α, with respect to δ.

Now assume instead that $\kappa_\beta = \delta_\beta$ for cofinally many $\beta < \alpha$. Then Since the critical points in the iteration are increasing, it easily follows that $C_0 = \{\beta < \alpha : \kappa_\beta = \delta_\beta\}$ is a closed unbounded subset of α, and hence

$C = \{\kappa_\beta : \beta \in C_0\}$ is a closed unbounded subset of δ. If $\mathrm{cf}(\delta) = \omega$, there is nothing to prove, so assume $\mathrm{cf}(\delta)$ is uncountable. For $A \in P(\delta) \cap M_\alpha$, note that A belongs to the least measure W on δ if and only if $C \setminus \eta \subseteq A$ for some $\eta < \delta$. It now follows that $W = \mathcal{C}_\delta \cap M_\alpha$, as desired.

This finishes the induction, proving that for all $\alpha \in \mathrm{Ord}$, if $\delta > \lambda$ is a regular cardinal of M_α that has uncountable cofinality in V, then there is a total measure on δ on the sequence of M_α, and the least such measure is equal to $\mathcal{C}_\delta \cap M_\alpha$. But for all $\gamma \in \mathrm{Ord}$, for all sufficiently large ordinals α, $P|\gamma = M_\alpha|\gamma$. It follows that if $\delta > \lambda$ is a regular cardinal of P that has uncountable cofinality in V, then there is a total measure on δ on the sequence of P, and the least such measure is equal to $\mathcal{C}_\delta \cap P$. (In fact, this measure is unique.)

From this, it follows that $L[A, \vec{\mathcal{C}}]]$ is an inner model of P, since in fact $L[A, \vec{\mathcal{C}}] = L[X, \vec{\mathcal{C}} \restriction (\lambda, \infty)]$ is definable over P using the predicates $X \cap P$ and the extender sequence of P. $\square$

Putting everything together we arrive at the following corollary:

Corollary 3.17. *Suppose that for every set X, $X^\ddagger$ exists. Then for every cardinal λ, there is an ω-club inner model M that is correct about cardinals and cofinalities below λ while $(\lambda^+)^M < \lambda^+$.*

Proof. Fix a sequence $\langle a_\alpha \rangle_{\alpha<\lambda}$ such that for every limit ordinal $\alpha < \lambda$, a_α is a cofinal subset of α ordertype $\mathrm{cf}(\alpha)$. Let $A \subseteq \lambda \times \lambda$ be the set $\{(\alpha, \beta) \mid \beta \in a_\alpha\}$. Using the Gödel pairing function, A can be viewed as a subset of λ, and by Lemma 3.16, the inner model $M = L[A, \vec{\mathcal{C}}]$ is an ω-club amenable model such that $\lambda^{+M} < \lambda^+$. $\square$

4. Open questions and remarks

The following is a configuration not handled by our arguments:

Question 1. *Suppose that κ is a strong limit singular cardinal of uncountable cofinality and that $\{\delta < \kappa \mid (\delta^{++})^{\mathrm{HOD}} \geq \delta^+\}$ is stationary. Is it true that $(\kappa^{++})^{\mathrm{HOD}} \geq \kappa^+$?*

We do not know either if other HOD-related properties behave in a compact-like way. For instance, the following is open.

Question 2. *Suppose that κ is a singular strong limit cardinal with uncountable cofinality and that $\{\delta < \kappa \mid \delta^+$ is not ω-strongly measurable in* HOD$\}$ *is stationary. Is it true that κ^+ is not ω-strongly measurable in* HOD*?*

There is another intriguing question connecting Woodin's HOD Conjecture with Theorem 3.4. Assuming the existence of strong enough large cardinals, in [Pov23] it was proved that a cardinal κ can consistently be $<\lambda$-extendible yet $(\lambda^+)^{\mathrm{HOD}_x} < \lambda^+$ for a strong limit cardinal λ with $\mathrm{cf}(\lambda) = \omega$ and $x \subseteq \lambda$. In simple terms, the HOD Conjecture can *fail locally.*[12]

A natural speculation is whether this failure can take place at a strong limit singular of cofinality $\geq\omega_1$. Namely,

Question 3. *Consider the following configuration:*

(1) κ *is* $<\lambda$*-extendible.*
(2) λ *strong limit with* $\mathrm{cf}(\lambda) \geq \omega_1$.
(3) $(\lambda^+)^{\mathrm{HOD}} < \lambda^+$

Is it consistent with ZFC*?*

Granting the HOD Conjecture, Theorem 3.4 suggests that the answer to Question 3 is negative. For suppose Clause (3) above holds. Then, by Theorem 3.4, the set $\{\delta < \lambda \mid (\delta^+)^{\mathrm{HOD}} < \delta^+\}$ contains a club. In particular, the degree of extendibility of κ overlaps a singular cardinal $\delta < \lambda$ witnessing $(\delta^+)^{\mathrm{HOD}} < \delta^+$. This is on the verge of refuting the HOD Conjecture. Note, however, that it does not outright preclude it. The reason being that V_λ may no satisfy ZF. A positive answer would point out yet another difference between singular cardinals of countable and uncountable cofinality.

Acknowledgments. Poveda was supported by the Department of Mathematics and the Center of Mathematical Sciences and Applications at Harvard University. Goldberg was supported in part by NSF Grant No. 2401789.

References

[AS83] Uri Abraham and Saharon Shelah. Forcing closed unbounded sets. *The Journal of Symbolic Logic*, 48(3):643–657, 1983.

[BNU17] Omer Ben-Neria and Spencer Unger. Homogeneous changes in cofinalities with applications to HOD. *Journal of Mathematical Logic*, 17(02):1750007, 2017.

[BP23] Joan Bagaria and Alejandro Poveda. More on the preservation of large cardinals under class forcing. *The Journal of Symbolic Logic*, 88(1):290–323, 2023.

[Buk73] Lev Bukovskỳ. Characterization of generic extensions of models of set theory. *Fundamenta Mathematicae*, 83(1):35–46, 1973.

[CFG15] James Cummings, Sy David Friedman, and Mohammad Golshani. Collapsing the cardinals of hod. *Journal of Mathematical Logic*, 15(02):1550007, 2015.

[Coh63] Paul J. Cohen. The independence of the continuum hypothesis. *Proc. Nat. Acad. Sci. U.S.A.*, 50:1143–1148, 1963.

[Eas70] William B. Easton. Powers of regular cardinals. *Ann. Math. Logic*, 1:139–178, 1970.

[FHR15] Gunter Fuchs, Joel David Hamkins, and Jonas Reitz. Set-theoretic geology. *Annals of Pure and Applied Logic*, 166(4):464–501, 2015.

[For09] Matthew Foreman. Ideals and generic elementary embeddings. In *Handbook of set theory*, pages 885–1147. Springer, 2009.

[Git86] Moti Gitik. Changing cofinalities and the nonstationary ideal. *Israel Journal of Mathematics*, 56:280–314, 1986.

[12] A related result, yet this time requiring the failure of AC, has been proved in [Sch22, Theorem 3.7].

[Git10] Moti Gitik. Prikry-type forcings. In *Handbook of set theory. Vols. 1, 2, 3*, pages 1351–1447. Springer, Dordrecht, 2010.

[Göd39] Kurt Gödel. Consistency-proof for the generalized continuum-hypothesis. *Proceedings of the National Academy of Sciences*, 25(4):220–224, 1939.

[Göd46] Kurt Gödel. Remarks before the princeton bicentennial conference on problems in mathematics. 1946.

[Gol23] Gabriel Goldberg. A note on Woodin's HOD dichotomy. *To apper in Journal of Mathematical Logic*, 2023.

[Jec03] Thomas Jech. *Set theory: The third millennium edition, revised and expanded.* Springer, 2003.

[JS13] Ronald Jensen and John Steel. K without the measurable. *The Journal of Symbolic Logic*, 78(3):708–734, 2013.

[Kan76] Akihiro Kanamori. Weakly normal filters and irregular ultrafilters. *Transactions of the American Mathematical Society*, 220:393–399, 1976.

[Ket72] Jussi Ketonen. Strong compactness and other cardinal sins. *Annals of Mathematical Logic*, 5(1):47–76, 1972.

[Kön05] Julius König. Zum Kontinuum-Problem. *Mathematische Annalen*, 60(2):177–180, 1905.

[MSS97] W.J. Mitchell, E. Schimmerling, and J.R. Steel. The covering lemma up to a Woodin cardinal. *Annals of Pure and Applied Logic*, 84(2):219–255, 1997.

[Pov23] A. Poveda. Two results on extendible cardinals. *Accepted in Proc. Amer. Math. Soc.*, 2023.

[Sch20] Ralf Schindler. From set theoretic to inner model theoretic geology. *Preprint*, 2020.

[Sch22] Farmer Schlutzenberg. Reinhardt cardinals and iterates of V. *Annals of Pure and Applied Logic*, 173(2):103056, 2022.

[She74] Saharon Shelah. Infinite abelian groups, Whitehead problem and some constructions. *Israel Journal of Mathematics*, 18:243–256, 1974.

[Sil75] Jack Silver. *On the Singular Cardinals Problem*, pages 435–438. 1975.

[Zem11] Martin Zeman. *Inner models and large cardinals*, volume 5. Walter de Gruyter, 2011.

(Goldberg) Department of Mathematics, UC Berkeley, CA 94720, USA
Email address: ggoldberg@berkeley.edu

(Poveda) Harvard University, Department of Mathematics and Center of Mathematical Sciences and Applications, Cambridge, MA 02138, USA
Email address: alejandro@cmsa.fas.harvard.edu
URL: www.alejandropovedaruzafa.com