CHOICELESS CARDINALS AND THE CONTINUUM PROBLEM

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Abstract. Under large cardinal hypotheses beyond the Kunen inconsistency — hypotheses so strong as to contradict the Axiom of Choice — we solve several variants of the generalized continuum problem and identify structural features of the levels V_α of the cumulative hierarchy of sets that are eventually periodic, alternating according to the parity of the ordinal α. For example, if there is an elementary embedding from the universe of sets to itself, then for sufficiently large ordinals α, the supremum of the lengths of all wellfounded relations on V_α is a strong limit cardinal if and only if α is odd.

1. Introduction

Cantor's continuum problem poses a fundamental set-theoretic question: how many real numbers exist? Equivalently, what is the cardinality of the set of all subsets of the natural numbers? The generalized continuum problem extends this inquiry to arbitrary infinite sets, asking how the cardinality of a set X compares to that of its powerset $P(X)$. A special case, which will be the main focus of this paper, is the question of the cardinality of the infinite levels of the cumulative hierarchy, the sequence of sets defined by iterating the powerset operation along the class of ordinal numbers, starting with $V_0 = \emptyset$, and letting $V_{\alpha+1} = P(V_\alpha)$, and $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$ when λ is a limit ordinal.

The generalized continuum hypothesis suggests a possible answer to the continuum problem: the cardinality of the powerset of X is as small as possible; it is the least cardinal greater than $|X|$. Equivalently, for all ordinals α, the cardinality of the α-th infinite level of the cumulative hierarchy (namely, $V_{\omega+\alpha}$) is precisely the α-th infinite cardinal number $\aleph_\alpha$.

These questions cannot be resolved within any accepted mathematical framework, such as the ZFC axioms. In other words, the (generalized) continuum hypothesis can be neither proved nor refuted. This limitation highlights a compelling issue: our foundational intuitions about sets fail to clarify the relationships among the key concepts of set theory: arbitrary sets, power sets, and cardinality. This unresolved issue motivates a search for new axioms that might provide answers to these questions or at least suggest a path forward.

The program for new axioms has succeeded in identifying a hierarchy of set theoretic hypotheses that far surpass the strength of the traditional axioms and settle many instances of the generalized continuum problem (though not the continuum problem itself). These *large cardinal hypotheses* postulate the existence of very large sets (or equivalently, very large cardinal numbers), so large that their existence is unprovable from the ZFC axioms, or indeed from any weaker large cardinal hypothesis.

A theorem of Solovay establishes that assuming the existence of a strongly compact cardinal κ, a relatively strong large cardinal hypothesis, the generalized continuum hypothesis holds for V_λ whenever $\lambda > \kappa$ is the limit of fewer than $|V_\lambda|$ smaller ordinals. Large cardinal hypotheses as currently conceived yield very little information about the sizes of the other levels of the cumulative hierarchy.

The large cardinal hypotheses we will be concerned with here go far beyond the current conception, hypotheses so strong as to contradict the Axiom of Choice. The earliest of these hypotheses was introduced by Reinhardt in the late 1960s, who postulated the existence of a nontrivial elementary embedding from the universe of sets to itself. Not long after, Kunen refuted Reinhardt's hypothesis using the Axiom of Choice. His paper raises the question of whether the principle can be refuted without appeal to the Axiom of Choice, a question which has gained prominence in recent years, with most evidence pointing towards the consistency of Reinhardt's hypothesis in the absence of the Axiom of Choice, raising the possibility that the large cardinal hierarchy extends past Kunen's bound, beyond Reinhardt's hypothesis, with no end in sight.

This paper studies the ramifications of Reinhardt's hypothesis for the generalized continuum problem. We do not use the Axiom of Choice, of course, which complicates matters considerably. First of all, in the choiceless context, the generalized continuum problem can be made precise in several inequivalent ways. Here, we will focus on the following problem. For each ordinal α, let θ_α denote the supremum of all ordinals η such that there is a surjective function from some element of V_α to η. So $\theta_\omega = \omega$ and $\theta_{\omega+1} = \omega_1$. Assuming the Axiom of Choice, for all ordinals α,

$$\theta_\alpha = \begin{cases} |V_{\alpha-1}|^+ & \text{if } \alpha \text{ is a successor ordinal} \\ |V_\alpha| & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

The generalized continuum hypothesis is therefore equivalent, in the context of ZFC, to the statement that $\theta_{\omega+\alpha} = \aleph_\alpha$ for all ordinals α.

In the context of Reinhardt's hypothesis, we will place nontrivial restrictions on the behavior of the θ_α function for all sufficiently large ordinals α, yielding an analysis of the type of level of the cumulative hierarchy that prior to this work had been impermeable by any large cardinal hypothesis.

A second complication introduced by dropping the Axiom of Choice is that most of the theory of large cardinals — indeed, most of the theory of uncountable sets — relies heavily on this axiom. It would be one thing if one could retain some weaker choice principle, and perhaps this is possible, but it is unclear at this point what a reasonable candidate is. We are therefore left to wander the desert that is bare ZF set theory, a theory known mainly for all that it *cannot* prove.

What saves us is that we can prove certain consequences of the Axiom of Choice from the very large cardinal hypotheses that refute it. This idea traces back to Woodin [1], who used supercompact cardinals for a similar purpose. Woodin's ideas were developed by Cutolo [2], who used them to prove some structural consequences of hypotheses much stronger than Reinhardt's, and later Usuba [3, 4], who introduced the notion of a Lowenheim-Skolem cardinal, which is essentially a weak choice principle abstracted from Woodin's applications of supercompact cardinals, and Schlutzenberg [5], who analyzed large cardinal hypotheses beyond choice in the context of Lowenheim-Skolem cardinals. The author [6] was the first to bring these ideas to bear on Reinhardt's hypothesis itself, assuming neither additional large cardinal hypotheses as in [7, 2] nor weak choice principles as in [3, 4, 5], and this paper relies extensively on the techniques developed there.

We finally come to the third and, in our view, the most significant drawback of choiceless large cardinal hypotheses. The Axiom of Choice is widely accepted and used throughout mathematics; the same cannot be said even for the weakest large cardinal hypotheses. Expert consensus aside, it is undeniable that the Axiom of Choice has a far greater intuitive appear than the large cardinal hypothesis proposed by Reinhardt. Moreover, if one grants that the "true universe of sets" satisfies the Axiom of Choice, then what basis could one possibly have for believing in the consistency of Reinhardt's hypothesis? Could Kunen's inconsistency result be a precursor to a deeper result refuting Reinhardt's hypothesis without the Axiom of Choice?

We will only briefly respond to these philosophical questions. First, there are reasons to study the consequences of choiceless large cardinal hypotheses even if one accepts the Axiom of Choice. Large cardinal hypotheses remain our best tools for gauging the consistency of theories beyond ZFC, and at this point there are no natural hypotheses consistent with choice that match the choiceless large cardinal hypotheses in strength. This is a situation that demands investigation, for anyone interested in the consistency hierarchy, regardless of one's perspective on the nature of the universe of sets.

Second, given the results of Schlutzenberg [8], it is conceivable that choiceless large cardinal hypotheses may hold in inner models of certain strong large cardinal hypotheses consistent with the Axiom of Choice. In fact, this is what Schlutzenberg proved for the weakest choiceless large cardinal hypothesis; namely, the existence of an elementary embedding from $V_{\lambda+2}$ to itself. If this is the case, then the situation is quite similar to determinacy axioms, where a choiceless hypothesis is important because it axiomatizes a restricted subclass of the universe.

Finally, while it is entirely reasonable to question the consistency of choiceless large cardinal hypotheses, the only way to settle the question is to achieve a deeper understanding of them. The recent research into these hypotheses seems to suggest that their theory is rich and coherent, perhaps a sign that they are consistent after all. While there are other avenues for investigating the consistency of choiceless large cardinals (in one direction, for example, the attempt to prove Woodin's HOD conjecture and in the other, the attempt to extend Schlutzenberg's consistency results to stronger hypotheses), the most direct approach is to develop the theory of these hypotheses. This paper is a contribution to that project.

1.1. **The main theorems.**

The *Lindenbaum number* of a set X, denoted by $\aleph^*(X)$, is the strict supremum of the lengths of all wellfounded relations on X. Equivalently,

$$\aleph^*(X) = \sup\{\eta + 1 : \eta \in \text{Ord and } \eta \leq^* X\}$$

where $A \leq^* B$ abbreviates the statement that there is a surjective partial function from B to A. (Of course if A is nonempty and $A \leq^* B$, then there will be a surjective total function from B to A; as it happens, though, the natural surjection is often partial.)

This notion is dual to the *Hartogs number*, the ordinal

$$\aleph(X) = \sup\{\eta + 1 : \eta \in \text{Ord and } \eta \leq X\}$$

where $A \leq B$ abbreviates the statement that there is an injective function from A to B. Clearly $\aleph(X) \leq \aleph^*(X)$, and equality holds assuming the Axiom of Choice, in which case $\aleph(X) = \aleph^*(X) = |X|^+$ for all sets X. In the choiceless context, Lindenbaum and Hartogs numbers both provide a rough ordinal measure of the cardinality of a set, but the two values may be quite different.

For each ordinal α, we define the *α-th Lindenbaum number*:

$$\theta_\alpha = \sup\{\aleph^*(x) : x \in V_\alpha\}$$

If α is a successor ordinal, then $\theta_\alpha = \aleph^*(V_{\alpha-1})$ is the supremum of the lengths of all wellfounded relations on $V_{\alpha-1}$, and if γ is a limit ordinal, then $\theta_\gamma = \sup_{\xi<\gamma} \theta_\xi$. For example, $\theta_\omega = \omega$ and $\theta_{\omega+1} = \omega_1$.

The Lindenbaum numbers provide a rough measure of the size of the levels of the cumulative hierarchy. Assuming the Axiom of Choice,

$$\theta_{\omega+\alpha} = \sup_{\xi<\alpha} \beth_\xi^+$$

and so the generalized continuum hypothesis is equivalent to the assertion that $\theta_{\omega+\alpha} = \aleph_\alpha$ for all ordinals α. (It is unclear whether this equality implies the generalized continuum hypothesis in the context of ZF alone.)

In this paper, however, we will avoid the Axiom of Choice in order to study large cardinal hypotheses so strong as to contradict it. Our main results show that these choiceless large cardinal hypotheses imply a dramatic failure of the generalized continuum hypothesis at alternating levels of the cumulative hierarchy. An ordinal κ is a *strong limit cardinal* if for all ordinals $\eta < \kappa$, there is no surjection from $P(\eta)$ to κ; in other words, $\aleph^*(P(\eta)) \leq \kappa$.

Theorem. *Assume there is an elementary embedding from the universe of sets to itself. Then for all sufficiently large limit ordinals γ and all natural numbers n, the following hold:*

(1) $\theta_{\gamma+2n}$ is a strong limit cardinal.
(2) $\theta_{\gamma+2n+1}$ is not a strong limit cardinal: in fact, $\theta_{\gamma+2n+1} \leq^* P(\theta_{\gamma+2n})$.

(1) is established by Theorem 8.1 and (2) by Theorem 8.3. An outline of the proof of (1) appears in Section 1.2.

An ordinal ϵ is *even* if $\epsilon = \gamma + 2n$ for some limit ordinal γ and natural number n; all other ordinals are *odd*. The theorem above highlights a key difference between the structure of the even and odd levels of the cumulative hierarchy under choiceless large cardinal hypotheses. It also indicates an analogy with the Axiom of Determinacy (AD), which has similar ramifications for much smaller Lindenbaum numbers: under AD, the cardinal $\theta_{\omega+2}$, which is usually denoted by Θ, is a strong limit cardinal whereas $\theta_{\omega+3} = \Theta^+$. The former is a theorem of Moschovakis [9], while the latter is Proposition 2.2, a simple observation of the author's (and maybe others').

The first instances of the periodicity phenomena around choiceless cardinals were discovered by Schlutzenberg and the author independently [10]: if α is an ordinal and $j : V_\alpha \to V_\alpha$ is an elementary embedding, then j is definable over V_α from parameters if and only if α is odd. The results of this paper go beyond those of Goldberg–Schlutzenberg by identifying periodic properties of the cumulative hierarchy that make no reference to metamathematical notions such as elementary embeddings and definability.

The calculation of the Lindenbaum numbers requires the development of some machinery for choiceless cardinals (continuing the work of [6]), which yields some other theorems. For example, the proof passes through the following result on the relationship between Hartogs numbers and Lindenbaum numbers:

Theorem 5.1. *If ϵ is even and there is an elementary embedding from $V_{\epsilon+3}$ to itself, then $\aleph(V_{\epsilon+2}) = \theta_{\epsilon+2}$.*

In other words, there is no $\theta_{\epsilon+2}$-sequence of distinct subsets of $V_{\epsilon+1}$. Note that under the Axiom of Choice, this equality is false since $\theta_{\epsilon+2} = |V_{\epsilon+1}|^+ \leq |V_{\epsilon+2}|$, whereas $\aleph(V_{\epsilon+2}) = |V_{\epsilon+2}|^+$. The question of whether $\aleph(V_\alpha) = \theta_\alpha$ for odd ordinals α is open; in general, $\theta_\alpha \leq \aleph(V_\alpha)$, but the reverse inequality is far from clear.

Theorem 7.1. *If there is an elementary embedding from the universe of sets to itself, then for some cardinal κ, for all even $\epsilon \geq \kappa$, if $\eta < \theta_{\epsilon+2}$, then the set of κ-complete ultrafilters on η has cardinality less than $\theta_{\epsilon+2}$.*

These theorems are analogous to consequences of the determinacy of real games ($\text{AD}_\mathbb{R}$), which implies that $\aleph(V_{\omega+2}) = \Theta$ and that for all $\eta < \Theta$, the set of ultrafilters on η has cardinality less than Θ. The latter is a theorem of Kechris [11], while the former is part of the folklore, perhaps first observed by Solovay. Both these theorems require the hypothesis $\text{AD}_\mathbb{R}$ since their conclusions are false in $L(\mathbb{R})$. Similarly, the conclusions of Theorem 5.1 and Theorem 7.1 are false in $L(V_{\epsilon+1})$, or even in Schlutzenberg's model $L(V_{\epsilon+1}, j)$, and so they cannot be proved assuming just the existence of an elementary embedding from $V_{\epsilon+2}$ to $V_{\epsilon+2}$.

1.2. **Outline of the paper.** In this section, we outline the proof of the theorem that all sufficiently large even Lindenbaum numbers are strong limit cardinals, and then we outline the structure of the paper. (The proof that odd Lindenbaum numbers are not strong limit cardinals involves similar ideas, so we do not provide an outline here.)

The key theorems we need to establish to analyze the even Lindenbaum numbers are two choiceless results concerning the structure of rank-to-rank embeddings (Lemma 3.3 and Theorem 3.4) and a bound on the number of ultrafilters on an ordinal (Corollary 7.13).

Our results on rank-to-rank embeddings derive from the notes [12]. They concern the following question: if $j : V_\alpha \to V_\alpha$ is an elementary embedding and $\beta < \alpha$ is an ordinal, how hard is it to define $j[\beta]$ over V_α? Lemma 3.3, which is very easy to prove, shows that if $X \leq^* Y$ are sets in V_α, then $j[X]$ is definable over V_α from $j[Y]$ and parameters in $\text{ran}(j)$. Theorem 3.4 shows that in certain important cases, this is optimal. Fix an even ordinal $\epsilon > \text{crit}(j)$ such that $\theta_{\epsilon+2} \leq \alpha$ and $j(\epsilon) = \epsilon$. On the one hand, Lemma 3.3 implies that for any $\beta < \theta_{\epsilon+2}$, $j[\beta]$ is definable in V_α from $j[V_\epsilon]$ and parameters in $\text{ran}(j)$. (We use the main theorem of [10] here (see Theorem 3.1 below) to define $j[V_{\epsilon+1}]$ from $j[V_\epsilon]$.) On the other hand, Theorem 3.4 shows that $j[\theta_\epsilon]$ is not definable over V_α from parameters in $\text{ran}(j) \cup V_{\epsilon+2}$.

Corollary 7.13, our bound on the number of ultrafilters on an ordinal, asserts that for sufficiently large even ordinals ϵ, if $\eta < \theta_{\epsilon+2}$, then there is a surjection from $V_{\epsilon+1}$ onto the set of wellfounded ultrafilters on η. The hypothesis here is the existence of a nontrivial elementary embedding from the universe of sets to itself. (We will actually use the slightly weaker hypothesis that there is a rank Berkeley cardinal, Section 6.) Here, an ultrafilter U on a set X is *wellfounded* if whenever

$(A, \prec)$ is a wellfounded structure, $(A, \prec)^X/U$ is again wellfounded. In the context of the Axiom of Choice (or just the Axiom of Dependent Choice), an ultrafilter is wellfounded if and only if it is countably complete. The bulk of this paper is spent establishing this theorem.

These results can be combined to prove that if there is an elementary embedding from the universe of sets to itself, then all sufficiently large even Lindenbaum numbers are strong limit cardinals. The case of limit Lindenbaum numbers is handled easily in Section 2, so let us focus on proving that $\theta_{\epsilon+2}$ is a strong limit cardinal. Our large cardinal hypothesis provides us with an ordinal $\alpha > \epsilon$ and an elementary embedding $j : V_\alpha \to V_\alpha$ with $\text{crit}(j) < \epsilon$ and $j(\epsilon) = \epsilon$. (See the discussion at the beginning of Section 6 for details.) We can choose α so that in addition V_α is a Σ_2-elementary substructure of the universe of sets.

By Theorem 3.4, $j[\theta_{\epsilon+2}]$ is not definable over V_α from parameters in $\text{ran}(j) \cup V_{\epsilon+2}$. Combining this with Lemma 3.3, if X is a set such that $j[X]$ is definable over V_α from parameters in $\text{ran}(j) \cup V_{\epsilon+2}$, then we can conclude that X does not surject onto $\theta_{\epsilon+2}$. Therefore to prove that $\theta_{\epsilon+2}$ is a strong limit cardinal, it suffices to show that for arbitrarily large ordinals $\eta < \theta_{\epsilon+2}$, $j[P(\eta)]$ is definable over V_α from parameters in $\text{ran}(j) \cup V_{\epsilon+2}$.

Fix an ordinal $\eta < \theta_{\epsilon+2}$ such that $\eta \in \text{ran}(j)$. Since $\text{ran}(j)$ is an elementary substructure of V_α containing η and $j(\eta)$, $\text{ran}(j)$ contains a surjection

$$f : V_{\epsilon+1} \to j(\eta) \times \mathscr{B}$$

where $\mathscr{B}$ is the set of wellfounded ultrafilters on η. Consider the sequence of ultrafilters $\vec{U} = \langle U_\xi \rangle_{\xi < j(\eta)}$ where $U_\xi = \{S \subseteq \eta : \xi \in j(S)\}$ is the ultrafilter derived from j using ξ. For each $\xi < j(\eta)$, U_ξ is wellfounded in V_α since the ultrapower of α by U_ξ can be embedded back into V_α. Since $V_\alpha \preceq_{\Sigma_2} V$, it follows that U_ξ is wellfounded. Therefore the sequence $\vec{U}$, which we identify with its graph, is a subset of $j(\eta) \times \mathscr{B}$. Let $A \subseteq V_{\epsilon+1}$ be the preimage of $\vec{U}$ under f. Then $\vec{U}$ is definable over V_α from parameters in $\text{ran}(j) \cup V_{\epsilon+2}$; namely, from $f \in \text{ran}(j)$ and $A \in V_{\epsilon+2}$. But $j \restriction P(\eta)$ is interdefinable with $\vec{U}$: for $S \subseteq \eta$,

$$j(S) = \{\xi < j(\eta) : S \in U_\xi\}$$

Hence $j[P(\eta)]$ is definable from parameters in $\text{ran}(j) \cup V_{\epsilon+2}$, as desired.

We now outline the contents of the paper. Section 2 proves the limit case of the main theorem and proves a special case of the main theorem from the Axiom of Determinacy. Section 3 reviews the periodicity phenomena for rank-to-rank embeddings from [10] and proves the definability results Lemma 3.3 and Theorem 3.4 mentioned above. The latter result has a more interesting proof that involves ordinal definability and forcing.

Section 4 turns to certain analogs of the Moschovakis coding lemma from the theory of determinacy that can be proved from choiceless large cardinal axioms. The Moschovakis coding lemma is the key to the proof that $\Theta = \theta_{\omega+2}$ is a strong limit cardinal assuming the Axiom of Determinacy, and Woodin proved a coarse generalization of this lemma in order to establish that under I_0, in $L(V_{\lambda+1})$, $\theta_{\lambda+2}$ is a strong limit cardinal. We generalize Woodin's result to certain more complicated inner models.

Section 5 establishes our bound on the Hartogs number of $V_{\epsilon+2}$ under the assumption of an elementary embedding from $V_{\epsilon+3}$ to $V_{\epsilon+3}$. In Section 6, we review some of the large cardinal machinery from [6], especially concerning Schlutzenberg's notion of a rank Berkeley cardinal and the pseudo-supercompactness properties that follow from it. Then, in Section 7, the proof of Section 5 is combined with the coding lemmas from Section 4 and the results of Section 6 to establish our bound on the number of wellfounded ultrafilters on an ordinal (Corollary 7.13). This requires the development of some ultrafilter theory for choiceless cardinals, which is why Section 7 is the longest section of this paper.

Finally Section 8 establishes the main theorems, and Section 9 lists some related open questions.

2. Limit Lindenbaum numbers and AD

The following lemma shows that for limit levels of the cumulative hierarchy, our main theorem is a simple consequence of ZF.

Lemma 2.1. *Suppose γ is a limit ordinal. Then θ_γ is a strong limit cardinal and $\theta_{\gamma+1} = \theta_\gamma^+$.*

Proof. To see that θ_γ is a strong limit cardinal, fix $\eta < \theta_\gamma$. For some $\alpha < \gamma$, $\eta \leq^* V_\alpha$, and so $P(\eta) \leq V_{\alpha+1}$, Therefore if $\nu \leq^* P(\eta)$, then $\nu \leq^* V_{\alpha+1}$, and hence $\nu < \theta_\gamma$.

To see that $\theta_{\gamma+1} = \theta_\gamma^+$, suppose $f : V_\gamma \to \nu$ is a surjection, and we will show that $|\nu| \leq \theta_\gamma$. Note that

$$\nu = \bigcup_{\alpha<\gamma} f[V_\alpha]$$

As a consequence, we have

$$|\nu| \leq \left|\gamma \times \sup_{\alpha<\gamma} \text{ot}(f[V_\alpha])\right|$$

Indeed, letting g_α denote the unique increasing bijection from $\text{ot}(f[V_\alpha])$ to $f[V_\alpha]$, one can define a partial surjection $g : \gamma \times \sup_{\alpha<\gamma} \text{ot}(f[V_\alpha])$ by setting $g(\alpha, \xi) = g_\alpha(\xi)$ whenever $\xi < \text{ot}(f[V_\alpha])$. For each $\alpha < \gamma$, $f[V_\alpha]$ is a wellorderable set that is the surjective image of V_α, and so $\text{ot}(f[V_\alpha]) < \theta_\gamma$. Hence $\sup_{\alpha<\gamma} \text{ot}(f[V_\alpha]) \leq \theta_\gamma$, which implies that $|\nu| \leq |\gamma \times \theta_\gamma| = \theta_\gamma$. □

If ϵ is an infinite even successor ordinal, we do not have nearly enough slack to generalize the proof above that θ_γ is a strong limit cardinal to θ_ϵ. Moreover, to prove $\theta_{\epsilon+1} = \theta_\epsilon^+$ would seem to require a hierarchy on V_ϵ similar to the rank hierarchy on V_γ.

The Axiom of Determinacy allows us to generalize Lemma 2.1 in the case that $\epsilon = \omega+2$, providing the necessary slack in the strong limit result via the Moschovakis coding lemma and the hierarchy for $V_{\omega+2}$ via Wadge's semi-linear ordering theorem.

Proposition 2.2 (AD)**.** *Let $\Theta = \theta_{\omega+2}$. Then Θ is a strong limit cardinal and $\theta_{\omega+3} = \Theta^+$.*

Proof. That Θ is a strong limit cardinal is a consequence of the Moschovakis Coding Lemma [9], which implies that for all $\eta < \Theta$, there is a surjection from $V_{\omega+1}$ onto $P(\eta)$, and hence $\aleph^*(P(\eta)) \leq \aleph^*(V_{\omega+1}) = \Theta$.

We now turn to the proof that $\theta_{\omega+3} = \Theta^+$. For simplicity, we first prove the result assuming the Axiom of Dependent Choice. Then by the Martin-Monk theorem [13], the subsets of $V_{\omega+1}$ are arranged in a wellfounded hierarchy according to their position in the Wadge order of continuous reducibility. (The Martin-Monk theorem requires the Axiom of Dependent Choice for reals; it is open whether the Axiom of Determinacy suffices to prove the Axiom of Dependent Choice for reals or whether it implies the Martin-Monk theorem.) For $\alpha < \Theta$, let Γ_α be the set of subsets of $V_{\omega+1}$ of rank α in the Wadge order. By Wadge's lemma, $\Gamma_\alpha \leq^* V_{\omega+1}$: indeed, for any $A \subseteq V_{\omega+1}$ outside Γ_α, Γ_α is contained in the set of continuous preimages of A.

Suppose $\nu \leq^* V_{\omega+2}$ is an ordinal, and we will show $\nu < \Theta^+$. Fixing a surjection $f : V_{\omega+2} \to \nu$, $\nu = \bigcup_{\alpha<\Theta} f[\Gamma_\alpha]$. As in the proof of Lemma 2.1, this implies that

$$|\nu| \leq \left|\Theta \times \sup_{\alpha<\Theta} \operatorname{ot}(f[\Gamma_\alpha])\right|$$

Again, for all $\alpha < \Theta$, since $\Gamma_\alpha \leq^* V_{\omega+1}$, we have $f[\Gamma_\alpha] \leq^* V_{\omega+1}$ and hence $\operatorname{ot}(f[\Gamma_\alpha]) < \Theta$. It follows that $|\nu| \leq |\Theta \times \Theta| = \Theta$, which implies that $\nu < \Theta^+$.

We now prove the proposition without appealing to the Axiom of Dependent Choice using a trick from [14]. For each $A \subseteq V_{\omega+1}$, let $\delta(A)$ be the supremum of the lengths all wellfounded binary relations on $V_{\omega+1}$ that are continuously reducible to A (when viewed as subsets of $V_{\omega+1}$). For each ordinal $\alpha < \Theta$, let Λ_α be the set of all $A \subseteq V_{\omega+1}$ with $\delta(A) \leq \alpha$. As above, Wadge's lemma implies that for $\alpha < \Theta$, $\Lambda_\alpha \leq^* V_{\omega+1}$. The argument of the previous paragraph can now be pushed through by replacing Γ_α with Λ_α. □

Numerology suggests that this pattern should extend to $\theta_{\omega+4}$ and $\theta_{\omega+5}$, but nothing of the sort can be proved from AD alone, which seems to have little to say about the structure of $V_{\omega+3}$. Still, it is tempting to fantasize about some extension of AD that answers such questions in the expected way.

3. Periodic properties of rank-to-rank embeddings

Recall the main theorem of [10], already mentioned in the introduction:

Theorem 3.1 (Goldberg, Schlutzenberg)**.** *Suppose α is an ordinal. If $j : V_\alpha \to V_\alpha$ is an elementary embedding, j is definable over V_α from parameters if and only if α is odd.* □

The analysis of the Lindenbaum numbers requires not only this theorem but also the details of its proof. The proof of Theorem 3.1 rests on the natural attempt to define, for every ordinal α, the extension "by continuity" of a Σ_1-elementary embedding $j : V_\alpha \to V_\alpha$ to $V_{\alpha+1}$, which we now describe. (The extension operation will be defined without regard to the parity of α, but the operation will only successfully extend j to an elementary embedding of $V_{\alpha+1}$ in the case that α is even. Even in this case, not every elementary embedding of V_α will extend to an elementary embedding of $V_{\alpha+1}$; the point is just that *if* j extends, then its unique extension is its canonical extension.)

First, let $\mathcal{H}_\alpha$ be the union of all transitive sets M such that $M \leq^* V_\gamma$ for some $\gamma < \alpha$. A Σ_1-elementary embedding $j : V_\alpha \to V_\alpha$ can be extended to act on $\mathcal{H}_\alpha$ as follows. Fix $x \in \mathcal{H}_\alpha$, and let $R \in V_\alpha$ code a wellfounded relation whose transitive collapse is the transitive closure of $\{x\}$. Then define $j(x)$ to be the unique set y such that the transitive collapse of $j(R)$ is the transitive closure of $\{y\}$. This extends j to a well-defined Σ_0-elementary embedding $j : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$. If j is Σ_{n+1}-elementary on V_α, then its extension to $\mathcal{H}_\alpha$ is Σ_n-elementary. Throughout this paper, when faced with an elementary embedding of V_α, we will often make use of its extension to $\mathcal{H}_\alpha$ without comment.

Second, we attempt to extend j to $V_{\alpha+1}$. If $j : V_\alpha \to V_\alpha$ is a Σ_1-elementary embedding, the *canonical extension* of j is the map $j^+ : V_{\alpha+1} \to V_{\alpha+1}$ defined by

$$j^+(A) = \bigcup_{N \in \mathcal{H}_\alpha} j(A \cap N)$$

The point here is that if $N \in \mathcal{H}_\alpha$, then $A \cap N$ is also in $\mathcal{H}_\alpha$. (For the purposes of extending j to $V_{\alpha+1}$, we could restrict our attention to $N \in \mathcal{H}_\alpha \cap V_{\alpha+1}$, or equivalently to $N \in V_{\alpha+1}$ such that $N \leq^* V_\alpha$; this is how the canonical extension is defined in [10].)

It is not hard to show that the graph of j^+ is definable over $V_{\alpha+1}$ from (a code for) j. Theorem 3.1 is proved by an induction that establishes:

Theorem 3.2. *Suppose ϵ is an even ordinal.*

(1) If $i : V_\epsilon \to V_\epsilon$ is elementary, then its extension to $\mathcal{H}_\epsilon$ is cofinal: $\mathcal{H}_\epsilon = \bigcup_{N \in \mathcal{H}_\epsilon} i(N)$.
(2) If $j : V_{\epsilon+1} \to V_{\epsilon+1}$ is elementary, then $j = (j \restriction V_\epsilon)^+$. □

Next, we establish a strong form of undefinability for elementary embeddings of even levels of the cumulative hierarchy (Theorem 3.4), which will be used extensively in our analysis of Lindenbaum numbers. For context, we first state a trivial definability theorem to which the aforementioned undefinability theorem can be viewed as a converse:

Lemma 3.3. *Suppose $j : M \to N$ is an elementary embedding between two transitive structures. Suppose $X, Y \in M$ and M satisfies $X \leq^* Y$. If $j[Y] \in N$, then $j[X]$ is definable over N from $j[Y]$ and parameters in $j[M]$.*

Proof. Let $f : Y \to X$ be a surjection in M. Then $j[X] = \{j(f)(a) : a \in j[Y]\}$, and so $j[X]$ is definable over N from $j[Y]$ and the parameter $j(f)$, which belongs to $j[M]$. □

In particular, for all ordinals $\eta < \aleph^*(Y)$, $j[\eta]$ is definable over N from $j[Y]$ and parameters in $j[M]$. Our strong undefinability theorem reads as follows:

Theorem 3.4. *If $\epsilon \leq \alpha$ are ordinals, ϵ is even, and $j : V_\alpha \to V_\alpha$ is elementary, then for all $\gamma < \epsilon$, $j[\theta_\epsilon]$ is not definable over $\mathcal{H}_\alpha$ from $j[V_\gamma]$ and parameters in $j[\mathcal{H}_\alpha]$.*

Theorem 3.4 will follow from a general ultrafilter-theoretic fact:

Theorem 3.5. *Suppose U is an ultrafilter on a set X such that $X \times X \leq^* X$ and $\kappa = \text{crit}(j_U)$. Then for any ordinal η, if $j_U[\eta] \in M_U$, then $\eta \leq^* X$.*

We now show how to deduce Theorem 3.4 from Theorem 3.5, which we will establish shortly.

Proof of Theorem 3.4. Assume towards a contradiction that for some $\gamma < \epsilon$ and $p \in \mathcal{H}_\alpha$, $j[\theta_\epsilon]$ is definable over $\mathcal{H}_\alpha$ from $j[V_\gamma]$ and $j(p)$. If $\epsilon = \gamma + 1$, then γ is odd and so by Theorem 3.1, $j[V_\gamma]$ is definable over $\mathcal{H}_\alpha$ from $j[V_{\gamma-1}]$. By replacing γ by $\gamma - 1$ if necessary, we can assume without loss of generality that $\gamma + 1 < \epsilon$.

Let U be the ultrafilter on $V_{\gamma+1}$ derived from j using $j[V_\gamma]$, and let $k : \text{Ult}(\mathcal{H}_\alpha, U) \to \mathcal{H}_\alpha$ be the usual factor embedding defined by

$$k([f]_U) = j(f)(j[V_\gamma])$$

Fix a formula φ such that

$$j[\theta_\epsilon] = \{\xi \in \mathcal{H}_\alpha : \mathcal{H}_\alpha \vDash \varphi(\xi, j[V_\gamma], j(p))\}$$

For each $x \in V_{\gamma+1}$, let $f(x) = \{\xi \in \mathcal{H}_\alpha : \mathcal{H}_\alpha \vDash \varphi(\xi, x, p)\}$. Then

$$j(f)(j[V_\gamma]) = \{\xi \in \mathcal{H}_\alpha : \mathcal{H}_\alpha \vDash \varphi(\xi, j[V_\gamma], j(p)\}$$

and therefore

$$[f]_U = k^{-1}[j[\theta_\epsilon]] = j_U[\theta_\epsilon]$$

But then by Theorem 3.5, $\theta_\epsilon \leq^* V_{\gamma+1}$, contrary to the fact that $\gamma + 1 < \epsilon$. □

Though Theorem 3.5 itself is purely combinatorial, its proof uses the techniques of ordinal definability and forcing. Suppose Y is a set and X is ordinal definable from Y. Let $\aleph^*_Y(X)$ denote the least ordinal δ such that there is no surjection from X to δ that is definable from Y and ordinal parameters.

We use the following variant of Vopenka's theorem [15] that every set of ordinals belongs to a set forcing extension of HOD. To avoid ambiguity, let us first establish some notation. For any S, we denote by OD_S the class of sets that are ordinal definable using S as a parameter. We denote by HOD_S the union of all transitive subsets of OD_S. Then HOD_S is an inner model of ZFC.

Lemma 3.6 (Vopenka)**.** *Suppose Y is a set and X is ordinal definable from Y. Then for any $x \in X$, $\text{HOD}_{Y,x}$ is a $\aleph^*_Y(X)$-cc generic extension of HOD_Y.* □

We will only use Lemma 3.6 to conclude that $\aleph^*_Y(X)$ is regular in $\text{HOD}_{Y,x}$ for all $x \in X$ and every subset of $\aleph^*_Y(X)$ in HOD_Y that is stationary in HOD_Y remains stationary in $\text{HOD}_{Y,x}$. We verify these properties combinatorially assuming $\aleph^*_Y(X)$ is regular in HOD_Y.

Let $\theta = \aleph^*_Y(X)$. The key point is that Bukovsky's *uniform θ-covering property* holds between HOD_Y and $\text{HOD}_{Y,x}$ [16]: if $I \in \text{HOD}_Y$ and $f : I \to \text{HOD}_Y$ is in $\text{HOD}_{Y,x}$, then there is some $F : I \to \text{HOD}_Y$ in HOD_Y such that for all $i \in I$, $f(i) \in F(i)$ and $|F(i)|^{\text{HOD}_Y} < \theta$. To see this, let φ be a formula in the language of set theory such that for some $\nu \in \text{Ord}$, for all $i \in I$,

$$f(i) = z \iff V \vDash \varphi(i, z, Y, x, \nu)$$

For each $i \in I$, let $F(i)$ be the set of $z \in \text{HOD}_Y$ such that for some $x \in X$, z is the unique set satisfying $\varphi(i, z, Y, x, \nu)$; $f(i) \in F(i)$, and there is an OD_Y surjection from X to $F(i)$, so $|F(i)|^{\text{HOD}_Y} < \theta$. Moreover, F is definable from Y and ν, so $F \in \text{HOD}_Y$. This establishes the uniform θ-covering property between HOD_Y and $\text{HOD}_{Y,x}$.

Bukovsky's theorem [17, 18] allows us to conclude from the uniform covering property that $\text{HOD}_{Y,x}$ is a θ-cc generic extension of HOD_Y, which yields the version of Vopenka's theorem stated in Lemma 3.6. But it is actually easy to deduce the combinatorial consequences of this that we will use directly from the uniform covering property. Suppose $M \subseteq N$ has the θ-uniform covering property. If $\gamma \geq \theta$ is regular in M, it remains regular in N for the following reason. Assume towards a contradiction that γ is singular in N. Then for some $\delta < \gamma$, there is a cofinal function $f : \delta \to \theta$ that belongs to N. Then there is a function $F : \delta \to P_\theta(\theta)$ in M such that $f[\delta] \subseteq \bigcup_{\alpha<\delta} F(\alpha)$. Thus $\bigcup_{\alpha<\delta} F(\alpha)$ is cofinal in θ. But in M, $|\bigcup_{\alpha<\delta} F(\alpha)| \leq \delta \cdot \sup_{\alpha<\delta} |F(\alpha)| < \gamma$, contradicting that γ is regular in M.

Similarly, one can prove that if $\gamma \geq \theta$ is regular, then any M-stationary set $S \subseteq \gamma$ remains stationary in N. Assume not, towards a contradiction, and let $f : \gamma \to \gamma$

be a function in N such that S is disjoint from the set of closure points of f. Let $F : \gamma \to P_\theta(\gamma)$ be a function in M witnessing the θ-cover property with respect to f. Then define a function $g : \gamma \to \gamma$ in M by $g(\alpha) = \sup F(\alpha)$. Then the set of closure points of g are a subset of the set of closure points of f, and so S is disjoint from the closure points of g, contrary to our assumption that S is stationary in M.

The HOD_Y-regularity of $\theta = \aleph^*_Y(X)$ seems to require some assumption on Y; we will assume that there is an OD_Y surjection from X onto $X \times X$. Then if $\nu < \theta$ and $f : \nu \to \theta$, we can prove that f is bounded as follows. First, for each $\alpha < \nu$, let $g_\alpha : X \to f(\alpha)$ be the OD_Y-least surjection. Stringing these together yields an OD_Y-surjection $g : \nu \times X \to \sup_{\alpha<\nu} f(\alpha)$; namely, $g(\alpha, x) = g_\alpha(x)$. Since $\nu \times X$ is an OD_Y-surjective image of $X \times X$ which is an OD_Y-surjective image of X, we obtain that $\sup_{\alpha<\nu} f(\alpha) < \theta$.

We now return to our ultrafilter undefinability result.

Proof of Theorem 3.5. Fix a function f on X such that $[f]_U = j_U[\eta]$ and a surjection $p : X \to X \times X$, and let Y be a set from which f and p are ordinal definable. Let $\delta = \aleph^*_Y(X)$. Then δ is regular in $M = \mathrm{HOD}_Y$. Assume towards a contradiction that $\eta \geq \delta$.

For each $x \in X$, let $M_x = \mathrm{HOD}_{Y,x}$, so that M_x is a δ-cc generic extension of M by Lemma 3.6. Let $N = \prod_{x\in X} M_x/U$ be the ultraproduct formed using only those functions on X that are ordinal definable from Y. For each such function g, let $[g]$ denote the element of N it represents. Then this ultraproduct satisfies Łoś's theorem in the sense that $N \vDash \varphi([g])$ if and only if $M_x \vDash \varphi(g(x))$ for U-almost all $x \in X$. This is because for any OD_Y function $F \in \prod_{x\in X} M_x$ such that $F(x) \neq \emptyset$ for U-almost all $x \in X$, we can obtain an OD_Y function $f \in \prod_{x\in X} M_x$ that uniformizes F by letting $f(x)$ be the $\mathrm{OD}_{Y,x}$-least element of $F(x)$ for each $x \in X$ such that $F(x) \neq \emptyset$.

Since $M \subseteq M_x$ for all $x \in X$, we can define a function i on M by $i(B) = [c_B]$ where $c_B : X \to \{B\}$ is the constant function. Then i is an elementary embedding from M to the substructure H of N given by

$$H = \{a \in N : \exists B \in M\,(N \vDash a \in i(B))\}$$

Essentially, we are defining H to be the union $\bigcup_{B\in M} i(B)$, but since we do not assume that N is wellfounded (or equivalently transitive), making this precise is a little bit cumbersome.

Identifying the wellfounded part of N with its transitive collapse, $\delta + 1 \subseteq N$ since $i[\delta] = [f] \cap i(\delta)$ belongs to N. By our remarks above (or Lemma 3.6), δ is regular in M_x for all $x \in X$, and so $i(\delta)$ is regular in N. Moreover $i(\delta) = \sup i[\delta]$ since there is no OD_Y function from X to an unbounded subset of δ. (The range of such a function would be an OD_Y cofinal subset of δ of ordertype less than δ, which contradicts that δ is regular in HOD_Y.) Since $i[\delta]$ belongs to N and is an unbounded subset of the N-regular cardinal $i(\delta)$, N contains an isomorphism between $i(\delta)$ and $i[\delta]$. Therefore $i(\delta) = \delta$.

Working in M, fix a partition $\mathcal{S} = \langle S_\alpha : \alpha < \delta\rangle$ into stationary sets of the set S of ordinals less than δ that have countable cofinality in M. Let $\mathcal{T} = i(\mathcal{S})$, so $\mathcal{T} = \langle T_\alpha : \alpha < \delta\rangle$ is a partition of $i(S)$ into H-stationary sets. For each $x \in X$, any set in $P(\delta) \cap M$ that is stationary in M remains stationary in M_x by Lemma 3.6, and so by Łoś's theorem, any set in $P(\delta) \cap H$ that is stationary in H remains stationary in N. Since i is continuous at ordinals of cofinality ω, the set $i[\delta]$ is an

ω-closed unbounded subset of δ in V and in N. It follows that $T_\kappa \cap i[\delta] \neq \emptyset$ where $\kappa = \text{crit}(j_U) = \text{crit}(i)$. But if $i(\xi) \in T_\kappa$, then $\xi \in S_\alpha$ for some $\alpha < \delta$, and hence $\xi \in T_{i(\alpha)}$. Since $\mathcal{T}$ is a partition, the fact that $T_\kappa \cap T_{i(\alpha)} \neq \emptyset$ implies that $\kappa = i(\alpha)$, which is impossible since the critical point of an elementary embedding is never in that embedding's range. □

4. The coding lemma

In this section, we establish an analog of the Moschovakis coding lemma [9] under choiceless large cardinal hypotheses. The proof is a slight twist on an argument of Woodin [19] establishing a similar property of $L(V_{\lambda+1})$ under I_0 in the context of ZFC.

Recall that the Moschovakis coding lemma states that if η is an ordinal and $\varphi : V_{\omega+1} \to \eta$ is a surjective function, then for any binary relation R on $V_{\omega+1}$ with $\varphi[\text{dom}(R)] = \eta$, there is some $S \subseteq R$ with $\varphi[\text{dom}(S)] = \eta$ such that S is Σ_1-definable from parameters over the structure $(V_{\omega+1}, \in, \prec)$, where $\prec$ is the prewellordering of $V_{\omega+1}$ induced by φ.

The coarse version of the *coding lemma* in which we will be interested states that if η is an ordinal and $\varphi : V_{\epsilon+1} \to \eta$ is a surjective function, then there is a set $\Gamma \subseteq V_{\epsilon+2}$ such that $\Gamma \leq^* V_{\epsilon+1}$ and every binary relation R on $V_{\epsilon+1}$ with $\varphi[\text{dom}(R)] = \eta$ has a subrelation $S \in \Gamma$ such that $\varphi[\text{dom}(S)] = \eta$. Thus unlike Moschovakis's theorem, there is no sharp bound on the definability of the subrelation S in terms of φ; there seems to be little one can say about the structure of Γ except that $\Gamma \leq^* V_{\epsilon+1}$, which is already substantive. This is the variant of the coding lemma that Woodin established in $L(V_{\epsilon+1})$ assuming I_0. If this could be proved in V under choiceless large cardinal hypotheses, it would easily imply the result that $\theta_{\epsilon+2}$ is a strong limit cardinal:

Proposition 4.1. *Assuming the coding lemma, $P(\eta) \leq^* V_{\epsilon+1}$ for every $\eta < \theta_{\epsilon+2}$.*

Proof. Let $\varphi : V_{\epsilon+1} \to \eta$ be a surjective function and let $\Gamma \subseteq V_{\epsilon+2}$ be a set such that $\Gamma \leq^* V_{\epsilon+1}$ and every binary relation R on $V_{\epsilon+1}$ with $\varphi[\text{dom}(R)] = \eta$ has a subrelation $S \in \Gamma$ such that $\varphi[\text{dom}(S)] = \eta$. For each $S \in \Gamma$ such that $S \subseteq V_{\epsilon+1} \times \{0, 1\}$, let $A_S \subseteq \eta$ be defined by

$$A_S = \{\varphi(x) : S(x, 1)\}$$

For all $A \subseteq \eta$, we will show there is some $S \in \Gamma$ such that $A = A_S$, which will imply that $P(\eta) \leq^* \Gamma \leq^* V_{\epsilon+1}$, completing the proof.

Fix $A \subseteq \eta$. Let $\chi_A : \eta \to \{0, 1\}$ denote the characteristic function of A (so $\chi_A(\alpha) = 1$ iff $\alpha \in A$), and let $R : V_{\epsilon+1} \to \{0, 1\}$ be the composition $\chi_A \circ \varphi$. We can view R as a binary relation on $V_{\epsilon+1}$. If $S \in \Gamma$ is a subrelation of R such that $\varphi[\text{dom}(S)] = \eta$, then $A_S = \{\varphi(x) : S(x, 1)\} = A$, as desired. □

We do not know how to prove a coding lemma in anything like this level of generality. Instead we will prove coding lemmas for certain inner models of V. Still this involves improving on Woodin's theorem in a nontrivial way.

For ordinals $\lambda \leq \epsilon$, let $I(\lambda, \epsilon)$ abbreviate the statement that for all $\alpha < \lambda$ and all $A \subseteq V_{\epsilon+1}$, there is some $A' \subseteq V_{\epsilon+1}$ admitting elementary embeddings $j_0, j_1 : (V_{\epsilon+1}, A') \to (V_{\epsilon+1}, A)$ whose distinct critical points lie between α and λ. Admittedly, this is a somewhat contrived large cardinal hypothesis, but it is useful because if ϵ is even, $I(\lambda, \epsilon)$ is downwards absolute to inner models containing $V_{\epsilon+1}$:

Lemma 4.2. *Suppose $\lambda \leq \epsilon$ are ordinals, ϵ is even, and M is an inner model containing $V_{\epsilon+1}$. If $I(\lambda,\epsilon)$ holds, then $I(\lambda,\epsilon)$ holds in M.*

Proof. Suppose $\alpha < \lambda$ and $A \subseteq V_{\epsilon+1}$ belongs to M. We must find a set $A' \in M$ and elementary embeddings $j_0, j_1 : (V_{\epsilon+1}, A') \to (V_{\epsilon+1}, A)$ in M whose distinct critical points lie between α and λ. Applying $I(\lambda,\epsilon)$, there are elementary embeddings $j_0, j_1 : (V_{\epsilon+1}, A') \to (V_{\epsilon+1}, A)$ whose distinct critical points lie between α and λ. Note that $j_0 \in M$ since $j_0 \restriction V_\epsilon$ is essentially an element of $V_{\epsilon+1}$ and Theorem 3.1 implies that j_0 is definable over $V_{\epsilon+1}$ from $j_0 \restriction V_\epsilon$. Similarly, $j_1 \in M$. Because of this, $A' \in M$: note that $A' = j_0^{-1}[A]$, which belongs to M since A and j_0 belong to M. Thus we have found set $A' \in M$ and elementary embeddings $j_0, j_1 : (V_{\epsilon+1}, A') \to (V_{\epsilon+1}, A)$ in M whose distinct critical points lie between α and λ, as desired. □

A much more natural large cardinal hypothesis with the same absoluteness property as $I(\lambda,\epsilon)$ simply states that for all $\alpha < \lambda$ and all $A \subseteq V_{\epsilon+1}$, there is some $A' \subseteq V_{\epsilon+1}$ admitting an elementary embedding $j : (V_{\epsilon+1}, A') \to (V_{\epsilon+1}, A)$ such that $\alpha < \text{crit}(j) < \lambda$. It is unclear whether $I(\lambda,\epsilon)$ follows from this weaker principle or whether this principle implies the coding lemma.

On the other hand, $I(\lambda,\epsilon)$ does follow from the existence of an elementary embedding $j : V_{\epsilon+2} \to V_{\epsilon+2}$ with critical point κ such that $\lambda = \sup\{\kappa, j(\kappa), j(j(\kappa)), \dots\}$:

Lemma 4.3. *If $\lambda = \sup\{\kappa, j(\kappa), j(j(\kappa)), \dots\}$ for some elementary $j : V_{\epsilon+2} \to V_{\epsilon+2}$ with critical point κ, then $I(\lambda,\epsilon)$ holds.*

Proof. Let $\alpha < \lambda$ be the least ordinal such that $I(\lambda,\epsilon)$ fails for α in the sense that there is some set $A \subseteq V_{\epsilon+1}$ such that there is no $A' \subseteq V_{\epsilon+1}$ admitting elementary embeddings $j_0, j_1 : (V_{\epsilon+1}, A') \to (V_{\epsilon+1}, A)$ whose distinct critical points lie between α and λ. Note that α is definable over $V_{\epsilon+2}$, and so $j(\alpha) = \alpha$. Since λ is the least fixed point of j above κ, $\kappa > \alpha$.

Fix a set $A \subseteq V_{\epsilon+1}$ such that (α, A) witnesses the failure of $I(\lambda,\epsilon)$. Let $A_1 = j(A)$ and $A_2 = j(j(A))$. By the elementarity of $j \circ j$, (α, A_2) also witnesses the failure of $I(\lambda,\epsilon)$. Since $j_0 = j \restriction V_{\epsilon+1}$ is an elementary embedding from $(V_{\epsilon+1}, A)$ to $(V_{\epsilon+1}, A_1)$, the elementarity of j implies that $j_1 = j(j_0)$ is an elementary embedding from $(V_{\epsilon+1}, A_1)$ to $(V_{\epsilon+1}, A_2)$. But note that j_0 is also an elementary embedding from $(V_{\epsilon+1}, A_1)$ to $(V_{\epsilon+1}, A_2)$ since $j(A_1) = A_2$. We have $\alpha < \text{crit}(j_0) = \kappa < j(\kappa) = \text{crit}(j_1) < \lambda$. Therefore j_0 and j_1 witness $I(\lambda,\epsilon)$ for (α, A_2), which contradicts that (α, A_2) witnesses the failure of $I(\lambda,\epsilon)$. □

We prove one last lemma about the principle $I(\lambda,\epsilon)$.

Lemma 4.4. *If ϵ is an even ordinal, then the following are equivalent:*

(1) $I(\lambda,\epsilon)$ holds.
(2) For any transitive set $M \in \mathcal{H}_{\epsilon+2}$ with $V_{\epsilon+1} \subseteq M$, there is a transitive set $M' \in \mathcal{H}_{\epsilon+2}$ with $V_{\epsilon+1} \subseteq M'$ admitting elementary embeddings $j_0, j_1 : M' \to M$ such that $\text{crit}(j_0) < \text{crit}(j_1) < \lambda$.
(3) For any transitive set $M \in \mathcal{H}_{\epsilon+2}$ with $V_{\epsilon+1} \subseteq M$ and any $a \in M$, there is a transitive set $M' \in \mathcal{H}_{\epsilon+2}$ with $V_{\epsilon+1} \subseteq M'$ admitting elementary embeddings $j_0, j_1 : M' \to M$ such that $\text{crit}(j_0) < \text{crit}(j_1) < \lambda$ and for some $a' \in M'$, $j_0(a') = j_1(a') = a$.

(4) *For any $\eta < \theta_{\epsilon+2}$ and any transitive set $M \in \mathcal{H}_{\epsilon+2}$ with $V_{\epsilon+1} \cup \{\eta\} \subseteq M$ and any $a \in M$, there is a transitive set $M' \in \mathcal{H}_{\epsilon+2}$ with $V_{\epsilon+1} \cup \{\eta\} \subseteq M'$ admitting elementary embeddings $j_0, j_1 : M' \to M$ such that $\text{crit}(j_0) < \text{crit}(j_1) < \lambda$, $j_0(\eta) = j_1(\eta) = \eta$, and for some $a' \in M'$, $j_0(a') = j_1(a') = a$.*

Proof. The implication (1) implies (2) is proved by applying $I(\lambda, \epsilon)$ to a wellfounded relation A on $V_{\epsilon+1}$ whose transitive collapse is M.

The implication (2) implies (3) is proved by applying (2) to the transitive closure of $\{M, a\}$.

The only slightly nontrivial implication is that (3) implies (4). Assume towards a contradiction that (4) fails, and let η be the least ordinal such that for some transitive set $M \in \mathcal{H}_\epsilon + 2$ containing $V_{\epsilon+1} \cup \{\eta\}$, the instance of (4) for M and η is false. (We will ignore the parameter a, which should be easy enough for the reader to handle.) Let $N \in \mathcal{H}_{\epsilon+2}$ be a transitive model of Kripke-Platek set theory such that $M \in N$. Let $j_0, j_1 : N' \to N$ witness (3) for N with $a = (M, \eta)$. Fix M' and η' in N' such that $j_0(M') = j_1(M') = M$ and $j_0(\eta') = j_1(\eta') = \eta$. We must have $\eta' \leq \eta$, and therefore $\eta' < \eta$ by our assumption. By the minimality of η, the instance of (4) for M' and η' is true, and by the proof of Lemma 4.2, it follows that this instance holds in N'. But then by the elementarity of j_0, the instance of (4) for M and η holds in N, which easily implies that this instance holds in V, contradicting our assumption.

Finally, for the implication (4) implies (1), one proves the instance of $I(\lambda, \epsilon)$ for a set $A \subseteq V_{\epsilon+1}$ by applying (4) (or even just (2)) to the transitive closure of $\{V_{\epsilon+1}, A\}$. □

For the rest of this section, we fix an even ordinal ϵ. We will establish a (slightly technical) general result with the following consequence:

Theorem 4.5. *Fix a class A and let M be the minimal inner model of* ZF *containing $V_{\epsilon+1}$ such that $x \cap A \in M$ for all $x \in M$. If M satisfies $I(\lambda, \epsilon)$ for some $\lambda \leq \epsilon$, then for all $\eta < \theta^M_{\epsilon+2}$, M satisfies $P(\eta) \cap M \leq^* V_{\epsilon+1}$.*

If A is the Σ_2-satisfaction predicate of $(V, \in, B)$, the model M of Theorem 4.5 is equal to the class of sets hereditarily ordinal definable from B and parameters in $V_{\epsilon+1}$. This is the model to which we will typically apply Theorem 4.5.

Definition 4.6.

- A set $\Gamma \subseteq V_{\epsilon+2}$ is an *elementary pointclass* if $\Gamma \leq^* V_{\epsilon+1}$ and for all Σ_1-elementary embeddings $j : V_\epsilon \to V_\epsilon$ and all $A \in \Gamma$, the image and preimage of A under the canonical extension of j to $V_{\epsilon+1}$ belong to Γ.
- Let EP denote the set of all elementary pointclasses.
- If λ is an ordinal, then λ-*pointclass choice* holds if for any any total relation R on $\text{EP} \times V_{\epsilon+2}$, there is a sequence $\langle \Gamma_\alpha \rangle_{\alpha<\lambda} \in \text{EP}^\lambda$ such that $\bigcup_{\alpha<\lambda} \Gamma_\alpha \leq^* V_{\epsilon+1}$ and for all $\beta < \lambda$, there is some $A \in \Gamma_\beta$ such that $R(\bigcup_{\alpha<\beta} \Gamma_\alpha, A)$.
- The *weak coding lemma* states that for any surjective $\varphi : V_{\epsilon+1} \to \eta$, there is a set $\Gamma \subseteq V_{\epsilon+2}$ such that $\Gamma \leq^* V_{\epsilon+1}$ and every binary relation R on $V_{\epsilon+1}$ with $\sup \varphi[\text{dom}(R)] = \eta$ has a subrelation $S \in \Gamma$ such that $\sup \varphi[\text{dom}(S)] = \eta$.

Note that we can assume in the conclusion of the (weak) coding lemma that Γ is an elementary pointclass.

Any model M as in Theorem 4.5 satisfies λ-pointclass choice since M contains a wellordered set of elementary pointclasses whose union is $V_{\lambda+2}$; see the proof of Theorem 4.5 below for details. Also note that λ-pointclass choice follows from the principle of λ-Dependent Choice (DC_λ), which states that any $<\lambda$-closed tree with no maximal nodes has a branch of length λ.

Lemma 4.7. *Assume $I(\lambda,\epsilon)$ and λ-pointclass choice. Then the weak coding lemma holds.*

Proof. Suppose the weak coding lemma fails. Fix a surjection $\varphi : V_{\epsilon+1} \to \eta$ witnessing this.

By λ-pointclass choice, there is a sequence $\langle \Gamma_\alpha \rangle_{\alpha<\lambda} \in \mathcal{H}_{\epsilon+2}$ such that for each $\beta < \lambda$, there is a binary relation R on $V_{\epsilon+1}$ in Γ_β with $\sup \varphi[\mathrm{dom}(R)] = \eta$ that has no subrelation $S \in \bigcup_{\alpha<\beta} \Gamma_\alpha$ such that $\sup \varphi[\mathrm{dom}(S)] = \eta$.

Let $M \in \mathcal{H}_{\epsilon+2}$ be a transitive set such that $V_{\epsilon+1}, \eta \in M$ and $\langle \Gamma_\alpha \rangle_{\alpha<\lambda}$ and φ belong to M. By Lemma 4.4, $I(\lambda,\epsilon)$ implies the existence of a transitive set M' with $V_{\epsilon+1}, \eta \in M'$ admitting elementary embeddings $j_0, j_1 : M' \to M$ such that $\mathrm{crit}(j_0) < \mathrm{crit}(j_1) < \lambda$, $j_0(\eta) = j_1(\eta) = \eta$, and for some $\langle \Gamma'_\alpha \rangle_{\alpha<\lambda}$ and φ' in M', $j_i(\langle \Gamma'_\alpha \rangle_{\alpha<\lambda}) = \langle \Gamma_\alpha \rangle_{\alpha<\lambda}$ and $j_i(\varphi') = \varphi$ for $i = 0, 1$.

Let $\kappa = \mathrm{crit}(j_0)$. By elementarity, there is a relation R in Γ'_κ with $\sup \varphi'[\mathrm{dom}(R)] = \eta$ that has no subrelation $S \in \bigcup_{\alpha<\kappa} \Gamma'_\alpha$ such that $\sup \varphi'[\mathrm{dom}(S)] = \eta$. Let $R_0 = j_0(R)$, and note that $R_0 \in \Gamma_{j_0(\kappa)}$ has no subrelation in Γ_κ such that $\sup \varphi[\mathrm{dom}(S)] = \eta$.

Since $j_1(\kappa) = \kappa$ and $R \in \Gamma'_\kappa$, $j_1(R) \in \Gamma_\kappa$. Since Γ_κ is an elementary pointclass, $R = j_1^{-1}[j_1(R)] \in \Gamma_\kappa$ and hence $j_0[R] \in \Gamma_\kappa$. But $S = j_0[R]$ is a subrelation of $j_0(R)$ such that

$$\sup \varphi[\mathrm{dom}(S)] = \sup j_0 \circ \varphi'[\mathrm{dom}(R)] = \eta$$

and this is a contradiction. □

The *local collection principle* states that every total relation on $V_{\epsilon+1} \times V_{\epsilon+2}$ has a total subrelation S such that $\mathrm{ran}(S) \leq^* V_{\epsilon+1}$. Any model M as in Theorem 4.5 satisfies the local collection principle; see the proof of Theorem 4.5 below for details.

Theorem 4.8. *Assume the local collection principle and the weak coding lemma. Then the coding lemma holds.*

Proof. The proof is by induction on η. Let $\varphi : V_{\epsilon+1} \to \eta$ be a surjection. By the local collection principle and our induction hypothesis, there is an elementary pointclass $\Gamma = \{A_e\}_{e\in V_{\epsilon+1}}$ that witnesses the coding lemma for all $\gamma < \eta$.

Let Λ be an elementary pointclass containing

$$U = \{(e, y) \in V_{\epsilon+1} \times V_{\epsilon+1} : y \in A_e\}$$

that witnesses the weak coding lemma for η and is closed under compositions of binary relations. We will show that Λ witnesses the coding lemma for η.

Let R be a binary relation on $V_{\epsilon+1}$ such that $\varphi[\mathrm{dom}(R)] = \eta$. Let $\tilde{R}(x, e)$ hold if A_e is a subrelation of R such that $\varphi[\mathrm{dom}(A_e)] = \varphi(x)$. By our induction hypothesis, $\varphi[\mathrm{dom}(R)] = \eta$, and so by the weak coding lemma, $\tilde{R}$ has a subrelation $\tilde{S} \in \Lambda$ such that $\varphi[\mathrm{dom}(\tilde{S})] = \eta$. Now $S = U \circ \tilde{R}$ is a subrelation of R in Λ such that $\varphi[\mathrm{dom}(S)] = \eta$. □

We finally establish Theorem 4.5.

Proof of Theorem 4.5. Work in M. We first establish λ-pointclass choice. Note that there is an ordinal η and a sequence $\langle \Lambda_\alpha \rangle_{\alpha<\eta}$ of elementary pointclasses such that $\bigcup_{\alpha<\eta} \Lambda_\alpha = V_{\epsilon+2}$. This is because every set is ordinal definable from $\bar{A} = A \cap M$

and parameters in $V_{\epsilon+1}$, and so we can let Λ_α be the set of all subsets of $V_{\epsilon+1}$ definable in $(V_\alpha, \bar{A} \cap V_\alpha)$ from parameters in $V_{\epsilon+1}$. Since there is an OD-sequence of surjections $f_\alpha : V_{\epsilon+1} \to \Lambda_\alpha$, any $<\theta_{\epsilon+2}$-sized union of Λ_αs is again an elementary pointclass. Therefore the instance of λ-pointclass choice for a relation R can be established by letting $\Gamma_\alpha = \Lambda_\beta$ where β is least such that for some $A \in \Lambda_\beta$, $R(\bigcup_{\xi<\alpha} \Gamma_\xi, A)$.

Next we establish the local collection principle. Fix a relation R on $V_{\epsilon+1} \times V_{\epsilon+2}$. For each $x \in V_{\epsilon+1}$, let α_x be least such that for some $A \in \Lambda_{\alpha_x}$, $R(x, A)$. Let $\Gamma = \bigcup_{x \in V_{\epsilon+1}} \Lambda_{\alpha_x}$. Then $S = R \cap (V_{\epsilon+1} \times \Gamma)$ is a total subrelation of R and $\operatorname{ran}(S) \subseteq \Gamma \leq^* V_{\epsilon+1}$.

It follows from Theorem 4.8 that M satisfies the coding lemma, and the theorem follows by applying Proposition 4.1 inside M. □

5. The Hartogs number of $V_{\epsilon+2}$

The *Hartogs number* of a set X, denoted by $\aleph(X)$, is the least ordinal η such that there is no η-sequence of distinct elements of X. The main theorem of this section computes the Hartogs number of the even levels of the cumulative hierarchy:

Theorem 5.1. *Suppose ϵ is an even ordinal and there is an elementary embedding from $V_{\epsilon+3}$ to itself. Then $\aleph(V_{\epsilon+2}) = \theta_{\epsilon+2}$.*

It is easily provable in ZF that $\aleph(V_{\epsilon+2}) \geq \aleph^*(V_{\epsilon+1}) = \theta_{\epsilon+2}$, so the main content of Theorem 5.1 is that there is no $\theta_{\epsilon+2}$-sequence of distinct subsets of $V_{\epsilon+1}$. This does not follow from the existence of an elementary embedding from $V_{\epsilon+2}$ to itself.

We begin by proving the following weak version of Theorem 5.1.

Proposition 5.2. *Suppose ϵ is an even ordinal and there is an elementary $j : V_{\epsilon+3} \to V_{\epsilon+3}$. Then there is no sequence $\vec{\varphi} = \langle \varphi_\eta : \eta < \theta_{\epsilon+2} \rangle$ such that for all $\eta < \theta_{\epsilon+2}$, φ_η is a surjection from $V_{\epsilon+1}$ onto η.*

Proof. Assume towards a contradiction that there is such a sequence. This implies $\theta_{\epsilon+2}$ is regular by the standard ZFC argument that $\aleph^*(X)$ is regular for any set X.

Let $\vec{\psi} = j(\vec{\varphi})$. For any $\eta \in j[\theta_{\epsilon+2}]$,

$$j[\theta_{\epsilon+2}] \cap \eta = \psi_\eta \circ j[V_{\epsilon+1}]$$

It follows that $j[\theta_{\epsilon+2}]$ is the unique ω-closed unbounded subset $C \subseteq \theta_{\epsilon+2}$ such that for all $\eta \in C$, $C \cap \eta = \psi_\eta \circ j[V_{\epsilon+1}]$; here we use that any other such ω-closed unbounded set has unbounded intersection with $j[\theta_{\epsilon+2}]$, which is a consequence of the regularity of $\theta_{\epsilon+2}$.

It follows that $j[\theta_{\epsilon+2}]$ is definable in $\mathcal{H}_{\epsilon+3}$ from $j[V_{\epsilon+1}]$ and $\vec{\psi} \in \operatorname{ran}(j)$, which contradicts Theorem 3.4. □

While we motivated this section with Theorem 5.1, our ulterior motive is to establish the following bound on definable Lindenbaum numbers, which will be important in the proof of the bound on the number of ultrafilters given in Theorem 7.1.

Theorem 5.3. *Suppose ϵ is an even ordinal and there is an elementary $j : V_{\epsilon+3} \to V_{\epsilon+3}$. Then for any class A, if M is the minimal inner model containing $V_{\epsilon+1}$ such that $A \cap x \in M$ for all $x \in M$, then $\theta^M_{\epsilon+3} < \theta_{\epsilon+2}$.*

For the proof, we need the concept of extenders and their associated ultrapower embeddings, a subject that requires some extra care if one does not assume the Axiom of Choice. We will not develop this theory in much generality, but rather work only in the setting that will be required here, the setting of *derived extenders*. Suppose $j : M \to N$ is an elementary embedding between transitive sets and $\eta \leq \sup j[\text{Ord} \cap M]$. For each $a \in [\eta]^{<\omega}$, let δ_a denote the least ordinal δ such that $a \subseteq j(\delta)$. The *M-extender of length η derived from j* is the sequence

$$E_{j,\eta} = \langle U_a : a \in [\eta]^{<\omega} \rangle$$

where U_a is the M-ultrafilter on $[\delta]^{|a|}$ derived from j using a.

For $a \in [\eta]^{<\omega}$, let $M_a = \text{Ult}(M, U_a)$ and let $k_{a,N} : M_a \to N$ denote the canonical factor embedding defined by

$$k_{a,N}([f]_{U_a}) = j(f)(a)$$

This map is a well-defined, injective homomorphism, but need not be elementary. If $a \subseteq b \in [\eta]^{<\omega}$, we can define $k_{a,b} : M_a \to M_b$ as the composition

$$k_{a,b} = k_{b,N}^{-1} \circ k_{a,N}$$

noting that $\text{ran}(k_{a,N}) \subseteq \text{ran}(k_{b,N})$. The *ultrapower of M by the derived extender $E = E_{j,\eta}$* is the direct limit M_E of the system of ultrapowers

$$\langle M_a, k_{a,b} : a \subseteq b \in [\eta]^{<\omega} \rangle$$

For $a \in [\eta]^{<\omega}$, let $k_{a,E} : M_a \to M_E$ denote the direct limit map. The *ultrapower embedding associated to E* is the map

$$j_E : M \to M_E$$

defined by $j_E = k_{\emptyset,E}$. The canonical factor maps $k_{a,N} : M_a \to N$ along with the universal property of the direct limit induce an embedding $k_{E,N} : M_E \to N$.

Every element of the structure M_E is of the form $k_{a,E}([f]_{U_a})$ for some $f \in M$ and $a \in [\eta]^{<\omega}$, and so for ease of notation we denote

$$[f, a]_E = k_{a,E}([f]_{U_a})$$

With this notation, the factor map from M_E to N may be expressed as

$$k_{E,N}([f, a]_E) = j(f)(a)$$

Taking f equal to the identity, we see that $[\eta]^{<\omega} \subseteq \text{ran}(k_{E,N})$, so in particular $\text{crit}(k_{E,N}) \geq \eta$.

Note that the extender E is essentially the same object as the function $i : P([\eta]^{<\omega}) \to N$ defined by $i(A) = j(A) \cap [\eta]^{<\omega}$. If $\eta = \omega^\eta$, in terms of ordinal arithmetic, then each element $a \in [\eta]^{<\omega}$ can be coded by a single ordinal, namely $\omega^{\alpha_{n-1}} + \cdots + \omega^{\alpha_0}$ where $\langle \alpha_m \rangle_{m<n}$ is the increasing enumeration of a. Therefore in this case, the extender E is essentially the same as the sequence of ultrafilters $\langle U_{\{\alpha\}} \rangle_{\alpha<\eta}$, and essentially the same as the function $i : P(\eta) \to N$ given by $i(A) = j(A) \cap \eta$.

Proof of Theorem 5.3. By enlarging A, we may assume without loss of generality that $j \restriction (V_{\epsilon+3} \cap M) \in M$. We then have $j(V_{\epsilon+2} \cap M) = V_{\epsilon+2} \cap M$: on the one hand, $j(V_{\epsilon+2} \cap M) \subseteq V_{\epsilon+2} \cap M$ since we have ensured $j \restriction (V_{\epsilon+3} \cap M) \in M$. For the reverse inclusion, we use that $V_{\epsilon+1} \subseteq j(V_{\epsilon+2} \cap M)$, and so $j \restriction V_\epsilon \in j(V_{\epsilon+2} \cap M)$, which implies that $j \restriction V_{\epsilon+1} \in j(V_{\epsilon+2} \cap M)$ by Theorem 3.1. It follows that for all $A \in V_{\epsilon+2} \cap M$, $A \in j(V_{\epsilon+2} \cap M)$, since

$$A = (j \restriction V_{\epsilon+1})^{-1}(j(A))$$

and $j(A) \in j(V_{\epsilon+2} \cap M)$.

Using Theorem 4.5, fix a sequence $\langle \varphi_\eta : \eta < \theta^M_{\epsilon+2}\rangle \in M$ such that $\varphi_\eta : V_{\epsilon+1} \to P(\eta) \cap M$ is a surjection. Note that $j \restriction P(\eta) \cap M$ is uniformly definable from $j(\varphi_\eta)$ and $j[V_\epsilon]$ in $\mathcal{H}_{\epsilon+2}$ for $\eta < \theta^M_{\epsilon+2}$, and as a consequence $j \restriction P_{\text{bd}}(\theta^M_{\epsilon+2}) \cap M$ is definable from $j(\langle \varphi_\eta : \eta < \theta^M_{\epsilon+2}\rangle)$ and $j \restriction \theta^M_{\epsilon+2}$ in $\mathcal{H}_{\epsilon+2}$. Since $\theta^M_{\epsilon+2} < \theta_{\epsilon+2}$, it follows that $j \restriction P_{\text{bd}}(\theta^M_{\epsilon+2}) \cap M$ is definable in $\mathcal{H}_{\epsilon+2}$ from $j[V_\epsilon]$ and parameters in the range of j.

Let E be the M-extender of length $\theta^M_{\epsilon+2}$ derived from j, so E and $j \restriction P_{\text{bd}}(\theta^M_{\epsilon+2}) \cap M$ are essentially the same object. Let $i : \theta^M_{\epsilon+3} \to \theta^M_{\epsilon+3}$ be the ultrapower associated to E (using only functions in M). We claim $i = j \restriction \theta^M_{\epsilon+3}$. Let $k : \text{Ult}(\theta^M_{\epsilon+3}, E) \to \text{Ord}$ be the factor embedding defined by $k([f, a]_E) = j(f)(a)$ for $a \in [\theta^M_{\epsilon+2}]^{<\omega}$ and $f \in M$ a function from an ordinal less than $\theta^M_{\epsilon+2}$ into $\theta^M_{\epsilon+3}$. We will show that k is the identity, or equivalently that k is surjective.

Fix $\xi < \theta^M_{\epsilon+3}$, and let us show $\xi \in \text{ran}(k)$. Let $R \in M$ be a prewellorder of $V_{\epsilon+2} \cap M$ of length greater than ξ. Then $j(R) \in M$ is a prewellorder of $V_{\epsilon+2} \cap M$ of length greater than ξ. Fix $A \in V_{\epsilon+2} \cap M$ such that $\text{rank}_{j(R)}(A) = \xi$. For each extensional $\sigma \subseteq V_\epsilon$, let $j_\sigma : V_\epsilon \to V_\epsilon$ denote the inverse of the transitive collapse of σ, and assuming j_σ is Σ_1-elementary, let $A_\sigma \subseteq V_{\epsilon+1}$ denote the preimage of A under the canonical extension $j^+_\sigma : V_{\epsilon+1} \to V_{\epsilon+1}$. Define a partial function $g : V_{\epsilon+1} \to \theta^M_{\epsilon+3}$ by $g(\sigma) = \text{rank}_R(A_\sigma)$. Then $g \in M$ and $j(g)(j[V_\epsilon]) = \xi$. The ordertype ν of $\text{ran}(g)$ is less than $\theta^M_{\epsilon+2}$. Let $h : \nu \to \text{ran}(g)$ be the increasing enumeration. Then $h \in M$ is a function from an ordinal less than $\theta^M_{\epsilon+2}$ into $\theta^M_{\epsilon+3}$, and since $\xi \in \text{ran}(j(g)) = \text{ran}(j(h))$, there is some $\alpha < j(\nu)$ such that $j(h)(\alpha) = \xi$. This shows $\xi \in \text{ran}(k)$, since $\xi = k([h, \alpha]_E)$.

It follows that $i = j \restriction \theta^M_{\epsilon+3}$. We claim that $j[\theta^M_{\epsilon+3}]$ is definable over $\mathcal{H}_{\epsilon+3}$ from $j[V_\epsilon]$ and parameters in the range of j. If we could show this, then we could conclude the theorem by appealing to Theorem 3.4 to deduce that $\theta^M_{\epsilon+3} < \theta_{\epsilon+2}$. In fact, it at first seems straightforward to show that $j[\theta^M_{\epsilon+3}]$ is so definable because it is just the range of i, and i is the ultrapower embedding associated to the extender E, which is itself definable in $\mathcal{H}_{\epsilon+2}$ from $j[V_\epsilon]$ and parameters in the range of j since $j \restriction P_{\text{bd}}(\theta^M_{\epsilon+2}) \cap M$ is. The issue, however, is that defining i from E requires a parameter for the set $\mathcal{F}$ defined by

$$\mathcal{F} = \{f \in M : \exists \eta < \theta^M_{\epsilon+2}\, f : \eta \to \theta^M_{\epsilon+3}\}$$

and we must show that this parameter is in the range of j.

We will show that $j(\mathcal{F}) = \mathcal{F}$. A slight hitch here is that it is not clear at first that $\mathcal{F} \in \mathcal{H}_{\epsilon+3}$, and therefore it is not obvious that it makes sense to apply j to $\mathcal{F}$ at all. Note, however, that for all $\xi < \theta^M_{\epsilon+3}$,

$$\mathcal{F}_\xi = \{f \in \mathcal{F} : \text{ran}(f) \subseteq \xi\}$$

is in $\mathcal{H}_{\epsilon+3}$, since fixing a surjection $\varphi : V_{\epsilon+2} \cap M \to \xi$ in M,

$$\mathcal{F}_\xi = \{\varphi \circ g : \exists \eta < \theta^M_{\epsilon+2}\, g \in V^\eta_{\epsilon+2} \cap M\}$$

yielding a surjective function from $V^\eta_{\epsilon+2} \cap M$ onto $\mathcal{F}_\xi$. We will first show that $j(\mathcal{F}_\xi) = \mathcal{F}_{j(\xi)}$.

This is because

$$j(\mathcal{F}_\xi) = \{j(\varphi) \circ g : \exists \eta < j(\theta^M_{\epsilon+2})\, g \in V^\eta_{\epsilon+2} \cap j(M)\}$$

and since $j(V_{\epsilon+2} \cap M) = V_{\epsilon+2} \cap M$, we have $j(\theta^M_{\epsilon+2}) = \theta^M_{\epsilon+2}$ and $V^\eta_{\epsilon+2} \cap j(M) = V^\eta_{\epsilon+2} \cap M$, and since moreover $j(\varphi) \in M$, we can conclude that $j(\mathcal{F}_\xi) = \mathcal{F}_{j(\xi)}$ as desired. Now $i \restriction \xi$ is definable via the ultrapower construction from E and $\mathcal{F}_\xi$, and so $i \restriction \xi$ is definable in $\mathcal{H}_{\epsilon+3}$ using $j[V_\epsilon]$ and parameters in the range of j, and as a consequence $\xi < \theta_{\epsilon+2}$.

Having shown that an arbitrary $\xi < \theta^M_{\epsilon+3}$ must be less than $\theta_{\epsilon+2}$, we can conclude that $\theta^M_{\epsilon+3} \leq \theta_{\epsilon+2}$. It follows that $\mathcal{F} \in \mathcal{H}_{\epsilon+3}$ after all, and now we can apply the argument showing $j(\mathcal{F}_\xi) = \mathcal{F}_{j(\xi)}$ to $\mathcal{F}$. This yields that $j(\mathcal{F}) = \mathcal{F}$, and so we can finally conclude that $j[\theta^M_{\epsilon+3}]$ is definable over $\mathcal{H}_{\epsilon+3}$ from $j[V_\epsilon]$ and parameters in the range of j as indicated above. By Theorem 3.4, this implies that $\theta^M_{\epsilon+3} < \theta_{\epsilon+2}$, which proves the theorem. □

Proof of Theorem 5.1. Assume towards a contradiction that $\langle A_\alpha : \alpha < \theta_{\epsilon+2}\rangle$ is a sequence of distinct subsets of $V_{\epsilon+1}$. Let

$$A = \{(\alpha, x) : \alpha < \theta_{\epsilon+2} \text{ and } x \in A_\alpha\}$$

and let M be the minimal inner model containing $V_{\epsilon+1}$ such that $A \cap x \in M$ for all $x \in M$. Then $A \in M$ since $A \subseteq \theta_{\epsilon+2} \times V_{\epsilon+1} \in M$. It follows that $\langle A_\alpha : \alpha < \theta_{\epsilon+2}\rangle \in M$, and so $\theta^M_{\epsilon+3} \geq (\aleph(V_{\epsilon+2}))^M \geq \theta_{\epsilon+2}$, contradicting Theorem 5.3. □

The proof of Theorem 5.3 raises a question: if ϵ is even and there is an elementary embedding $i : V_{\epsilon+2} \to V_{\epsilon+2}$, what is the value of $\theta_{\epsilon+3}$ in $\text{HOD}_{V_{\epsilon+1}}$? Let $\delta = (\theta_{\epsilon+2})^{\text{HOD}_{V_{\epsilon+1}}}$, so δ is the least positive ordinal that is not the surjective image of an ordinal definable function on $V_{\epsilon+1}$. We will show that

$$(\theta_{\epsilon+3})^{\text{HOD}_{V_{\epsilon+1}}} = \delta^{+\text{HOD}}$$

(To dispel a bit of mystery here, we point out that $\delta^{+\text{HOD}} = \delta^{+\text{HOD}_{V_{\epsilon+1}}}$ since for any function $f : \delta \to \text{Ord}$ in $\text{HOD}_{V_{\epsilon+1}}$, there is a function $F : \delta \to [\text{Ord}]^{<\delta}$ in HOD such that $f(\alpha) \in F(\alpha)$ for all $\alpha < \delta$.) We will also show that there is an ordinal definable surjection from $\delta \times V_{\epsilon+1}$ onto $V_{\epsilon+2} \cap \text{HOD}_{V_{\epsilon+1}}$. (This turns out to be stronger than the result that $(\theta_{\epsilon+3})^{\text{HOD}_{V_{\epsilon+1}}} = \delta^{+\text{HOD}}$.) The main interest here is less in the results that are proved than in their proofs, which constitute the only successful application to date of ideas from Wadge theory (especially Wadge's semi-linear ordering principle) outside the domain of descriptive set theory.

For each ordinal ξ, let T_ξ be the satisfaction predicate of V_ξ with parameters from $V_{\epsilon+1}$; we can view T_ξ as a subset of $V_{\epsilon+1}$. Since each set in $V_{\epsilon+2} \cap \text{HOD}_{V_{\epsilon+1}}$ is definable from parameters in the structure $(V_{\epsilon+1}, T_\xi)$ for some ξ, there is an ordinal definable surjection from $\{T_\xi : \xi \in \text{Ord}\} \times V_{\epsilon+1}$ onto $V_{\epsilon+2} \cap \text{HOD}_{V_{\epsilon+1}}$. Therefore to show that there is an ordinal definable surjection from $\delta \times V_{\epsilon+1}$ onto $V_{\epsilon+2} \cap \text{HOD}_{V_{\epsilon+1}}$, it suffices to show that $|\{T_\xi : \xi \in \text{Ord}\}|^{\text{HOD}_{V_{\epsilon+1}}} \leq \delta$.

Define a binary relation $\preceq$ on the ordinals by setting $\xi_0 \preceq \xi_1$ if there is an elementary embedding $j : V_{\epsilon+1} \to V_{\epsilon+1}$ such that $T_{\xi_0} = j^{-1}[T_{\xi_1}]$. Say an ordinal ξ is *original* if for all $\nu < \xi$, $\xi \not\preceq \nu$. Then the restriction of $\preceq$ to the original ordinals is wellfounded, and each original ordinal has fewer than δ original predecessors: there is an ordinal definable partial surjection from $V_{\epsilon+1}$ to the set of predecessors of ξ, given by sending an elementary embedding $j : V_\epsilon \to V_\epsilon$ to the unique original ordinal $\nu \prec \xi$ such that $T_\nu = (j^+)^{-1}[T_\xi]$, if there is one. Therefore the $\preceq$ relation has rank at most δ.

Next we claim that every level of the $\preceq$ order has cardinality less than $\kappa_\omega(i)$. In other words, for any $\alpha < \theta$, the set S of original ordinals ξ of rank α in the $\preceq$ order has cardinality less than $\kappa_\omega(i)$. By iterating i, we may assume that $i(\alpha) = \alpha$. Then $i(S) = S$. If $|S| \geq \text{crit}(i)$, then there is some $\xi \in S$ such that $i(\xi) \neq \xi$. But then $T_\xi = i^{-1}[T_{i(\xi)}]$, and hence $\xi \prec i(\xi)$, contradicting that ξ and $i(\xi)$ have the same rank in the $\preceq$ order.

Since the set of original ordinals carries a wellfounded relation of rank at most δ each of whose levels has cardinality less than $\kappa_\omega(i)$, there are at most δ original ordinals. For any ordinal ξ, if ν is the least ordinal such that there is an elementary embedding $j : V_{\epsilon+1} \to V_{\epsilon+1}$ with $T_\xi = j^{-1}[T_\nu]$, then ν is original. Let A_ν be the set of all T_ξ which are equal to $j^{-1}[T_\nu]$ for some elementary embedding $j : V_{\epsilon+1} \to V_{\epsilon+1}$. Then $|A_\nu|^{\text{HOD}_{V_{\epsilon+1}}} < \delta$ and $\{T_\xi : \xi \in \text{Ord}\} = \bigcup\{A_\nu : \nu \text{ is original}\}$. Thus $|\{T_\xi : \xi \in \text{Ord}\}|^{\text{HOD}_{V_{\epsilon+1}}} \leq \delta$, as desired.

Finally, work in $\text{HOD}_{V_{\epsilon+1}}$, and suppose $\beta < \theta_{\epsilon+3}$. Then there is a surjection from $\delta \times V_{\epsilon+1}$ onto β. It follows that β is the union of δ-many sets, each of cardinality less than δ, and hence $\beta < \delta^+$.

6. Rank Berkeley and rank reflecting cardinals

A cardinal λ is *rank Berkeley* if for all ordinals $\alpha < \lambda \leq \beta$, there is an elementary embedding from V_β to itself with critical point between α and λ. The term is due to Schlutzenberg who noticed that if there is an elementary embedding from the universe of sets to itself, then there is a rank Berkeley cardinal. (This was realized independently and earlier by Woodin.)

Indeed, if $j : V \to V$ is an elementary embedding with critical point κ, then we claim $\lambda = \sup\{\kappa, j(\kappa), j(j(\kappa)), \dots\}$ is rank Berkeley. Assume not, towards a contradiction, and note that j fixes the lexicographically least pair (α, β) of ordinals $\alpha < \lambda \leq \beta$ such that V_β admits no elementary embedding into itself with critical point between α and λ. Since λ is the least fixed point of j above its critical point κ, we have $\alpha < \kappa$. Therefore $j \restriction V_\beta$ witnesses that there is an elementary embedding from V_β to itself with critical point between α and λ, contrary to our choice of (α, β).

We prefer to work with rank Berkeley cardinals rather than elementary embeddings from the universe of sets to itself, partly because the former notion is first-order and seems to capture all of the set-theoretic content of the latter. Another reason for our preference is that the least rank Berkeley cardinal is an important threshold in the choiceless theory of large cardinals, as we now explain.

A cardinal κ is *supercompact* if for all ordinals $\beta \geq \kappa$, for some ordinal $\bar{\beta} < \kappa$, there is an elementary embedding $\pi : V_{\bar{\beta}} \to V_\beta$ such that $\pi(\text{crit}(\pi)) = \kappa$. A cardinal κ is *almost supercompact* if for all ordinals $\xi < \kappa \leq \beta$, for some ordinal $\bar{\beta}$ between ξ and κ, there is an elementary embedding from $V_{\bar{\beta}}$ into V_β that fixes ξ. In the context of ZFC, every almost supercompact cardinal is either a supercompact cardinal or a limit of supercompact cardinals.

A much weaker notion is that of a *rank reflecting cardinal*, a cardinal κ such that for all ordinals $\xi < \kappa$ and all formulas $\varphi(x)$ in the language of set theory, if there is an ordinal $\beta > \xi$ such that $V_\beta \vDash \varphi(\xi)$, then there is an ordinal β between ξ and κ such that $V_\beta \vDash \varphi(\xi)$. Every almost supercompact cardinal is rank reflecting, and every supercompact cardinal is a limit of rank reflecting cardinals. The existence of a proper class of rank reflecting cardinals is provable in ZF as an easy consequence of the Lévy-Montague reflection theorem. (See the discussion below Proposition 6.2.)

The following property of reflecting ordinals should be a standard exercise in set theory, but the author does not know a reference:

Proposition 6.1. *If α is an ordinal and κ is the least rank reflecting cardinal above α, then* $\text{cf}(\kappa) = \omega$.

Proof. For every ordinal ξ and every formula $\varphi(x)$ in the language of set theory, let $\beta_{\varphi,\xi}$ denote the least ordinal β such that $V_\beta \vDash \varphi(\xi)$. For each ordinal γ, let $\beta_\gamma = \sup_{\xi<\gamma,\varphi} \beta_{\varphi,\xi}$ and let $\nu_\gamma = \text{ot}(\{\beta_{\varphi,\xi} : \xi < \gamma,\, \varphi\})$. If $\delta_0 \leq \delta_1$, then $(\nu_\gamma)^{V_{\delta_0}} \leq (\nu_\gamma)^{V_{\delta_1}}$ with strict inequality if and only if there is some $\xi < \gamma$ such that $\delta_0 \leq \beta_{\varphi,\xi} < \delta_1$. Moreover $\nu_\gamma < \gamma^+ < \kappa$, and so by rank reflection, there is an ordinal $\delta < \gamma$ such that $(\nu_\gamma)^{V_\delta} = \nu_\gamma$. It follows that there is no $\delta' > \delta$ such that $(\nu_\gamma)^{V_{\delta'}} > (\nu_\gamma)^{V_\delta}$, and so there is no $\xi < \gamma$ such that $\beta_{\varphi,\xi} \geq \delta$. In other words, $\beta_\gamma \leq \delta$.

Let $\kappa_0 = \beta_{\alpha+1}$, and for $n < \omega$, let $\kappa_{n+1} = \beta_{\kappa_n}$. Then $\kappa_\omega = \sup_{n<\omega} \kappa_n$ is a rank reflecting cardinal and $\kappa_\omega \leq \kappa$, so by minimality, $\kappa = \kappa_\omega$. Since $\kappa_n < \kappa$ for all $n < \omega$ by the previous paragraph, $\text{cf}(\kappa) = \omega$. □

The following ZF fact shows that the restriction to ordinal parameters ξ in the definition of a rank reflecting cardinal can be removed:

Proposition 6.2. *If κ is rank reflecting, then for all $a \in V_\kappa$ and all formulas $\varphi(x)$ in the language of set theory, if there is an ordinal $\beta >$ rank(a) such that $V_\beta \vDash \varphi(a)$, then there is an ordinal β between* rank(a) *and κ such that $V_\beta \vDash \varphi(a)$.*

Proof. Fix a set $a \in V_\kappa$ and a formula $\varphi(x)$ in the language of set theory such that for some $\beta > \text{rank}(a)$, $V_\beta \vDash \varphi(a)$. Let $\rho = \text{rank}(a)$, and for each $a' \in V_{\rho+1}$ such that $V_\beta \vDash \varphi(a')$ for some $\beta > \rho$, let $\beta_{a'} = \min\{\beta : V_\beta \vDash \varphi(a')\}$. We will show that $\beta_{a'} < \kappa$ for all $a' \in V_{\rho+1}$, and for the particular case $a = a'$ this will establish the conclusion of the proposition.

Let $B = \{\beta_{a'} : a' \in V_{\rho+1}\}$ and let $\nu = \text{ot}(B)$. Then $\nu < \theta_{\rho+2} < \kappa$; here we use that rank reflecting cardinals are θ-fixed points in the sense that $\kappa = \theta_\kappa$. Since κ is rank reflecting, there is an ordinal β between $\max\{\nu, \rho\}$ and κ such that $\text{ot}(B \cap \beta) = \nu$. Note that for any $a' \in V_{\rho+1}$, $\beta_{a'} < \beta$ since otherwise $\text{ot}(B \cap (\beta_{a'} + 1)) > \nu$ contradicting the fact that $\text{ot}(B) = \nu$. This proves the proposition. □

Note that if κ is an ordinal such that $V_\kappa \preceq_{\Sigma_2} V$, then κ is rank reflecting. In ZFC, the converse holds since every Σ_2-formula $\varphi(x)$ is equivalent to one of the form $\exists \beta\, V_\beta \vDash \psi(x)$ for some formula $\psi(x)$ in the language of set theory. This normal form for Σ_2-formulas is apparently open under ZF, and so the relationship between these two forms of reflection is unclear.

We will take advantage of the following classification of almost supercompact cardinals, which shows that rank reflection need not be so much weaker than almost supercompactness after all:

Theorem 6.3. *Let λ denote the least rank Berkeley cardinal.*

(1) A cardinal $\kappa \leq \lambda$ is almost supercompact if and only if it is either supercompact or a limit of supercompact cardinals.

(2) A cardinal $\kappa \geq \lambda$ is almost supercompact if and only if it is rank reflecting.

Justified by Theorem 6.3, the results of the following sections are stated in terms of rank reflecting cardinals even though the proofs will employ theorems from [6] concerning almost supercompact cardinals.

Proof of Theorem 6.3. Any cardinal that is either supercompact or a limit of supercompact cardinals is almost supercompact, so we focus on the converse. The idea of the proof is as follows. For any ordinal ξ, let η_ξ denote the least ordinal η greater than ξ such that for all $\beta \geq \eta$, there is some $\bar{\beta}$ between ξ and η admitting an elementary embedding $\pi : V_{\bar{\beta}} \to V_\beta$ such that $\pi(\xi) = \xi$. Assuming there is no rank Berkeley cardinal below η_ξ, we will show that η_ξ is supercompact. Since any almost supercompact cardinal κ is the supremum of the set $\{\eta_\xi : \xi < \kappa\}$, it will follow that any almost supercompact cardinal less than or equal to the least rank Berkeley cardinal is either a supercompact cardinal or a limit of supercompact cardinals.

Let $\eta = \eta_\xi$. We first claim that for all $\delta < \eta$, for all sufficiently large ordinals α, there is no $\bar{\alpha}$ between ξ and δ admitting an elementary embedding $\pi : V_{\bar{\alpha}} \to V_\alpha$ such that $\pi(\xi) = \xi$ and $\delta \in \text{ran}(\pi)$. To see this, let β be the least ordinal such that for all $\bar{\beta}$ between ξ and δ, there is no elementary embedding $\pi' : V_{\bar{\beta}} \to V_\beta$ such that $\pi(\xi) = \xi$. This ordinal β exists by the minimality of $\eta = \eta_\xi$, since we are assuming that $\delta < \eta$. Suppose $\alpha > \beta$. We will show that there is no $\bar{\alpha}$ between ξ and δ admitting an elementary embedding $\pi : V_{\bar{\alpha}} \to V_\alpha$ such that $\pi(\xi) = \xi$ and $\delta \in \text{ran}(\pi)$. Assume towards a contradiction that one can find such an ordinal $\bar{\alpha}$ and such an elementary embedding π. Since V_α satisfies that β is the least ordinal such that for all for all $\bar{\beta}$ between ξ and δ, there is no elementary embedding $\pi' : V_{\bar{\beta}} \to V_\beta$ such that $\pi(\xi) = \xi$, β is definable in V_α from ξ and δ. Since ξ and δ belong to the range of π, β belongs to the range of π. Let $\bar{\beta} = \pi^{-1}(\beta)$. Then $\pi' = \pi \restriction V_{\bar{\beta}}$ contradicts the definition of β.

Fix an ordinal α large enough that no cardinal less than η is rank Berkeley in V_α and for all $\delta < \eta$, there is no $\bar{\alpha}$ between ξ and δ admitting an elementary embedding $\pi : V_{\bar{\alpha}} \to V_\alpha$ fixing ξ with $\delta \in \text{ran}(\pi)$. Suppose $\bar{\eta} < \bar{\alpha} < \eta$ and $\pi : V_{\bar{\alpha}+1} \to V_{\alpha+1}$ is an elementary embedding with $\pi(\bar{\eta}) = \eta$. We will show that $\text{crit}(\pi) = \bar{\eta}$.

We first claim that $\pi[\bar{\eta}] \subseteq \bar{\eta}$. Otherwise, let $\bar{\delta} < \bar{\eta}$ be such that $\pi(\bar{\delta}) > \bar{\eta}$. Let $\delta = \pi(\bar{\delta})$. Since $\delta < \eta$, our choice of α implies that $\bar{\alpha} > \delta$. Since $\pi \restriction V_{\bar{\alpha}} \in V_{\alpha+1}$ is an elementary embedding from $V_{\bar{\alpha}}$ to V_α with $\delta = \pi(\bar{\delta})$ in its range, the elementarity of $\pi : V_{\bar{\alpha}+1} \to V_{\alpha+1}$ implies that there is some $\bar{\bar{\alpha}} < \bar{\eta}$ admitting an elementary $\bar{\pi} : V_{\bar{\bar{\alpha}}} \to V_{\bar{\alpha}}$ with $\bar{\delta} \in \text{ran}(\pi)$. Now $\bar{\bar{\alpha}} < \bar{\eta} < \delta$ and $\pi \circ \bar{\pi} : V_{\bar{\bar{\alpha}}} \to V_\alpha$ is an elementary embedding with $\delta \in \text{ran}(\pi \circ \bar{\pi})$, contrary to our choice of α.

Assume towards a contradiction that $\text{crit}(\pi) < \bar{\eta}$. We will show that in V_α, there is a rank Berkeley cardinal less than $\bar{\eta}$, contrary to our choice of α. Let $\nu = \text{crit}(\pi)$ and $\gamma = \sup\{\nu, \pi(\nu), \pi(\pi(\nu)), \dots\}$. Since $\pi[\bar{\eta}] \subseteq \bar{\eta}$, $\gamma \leq \bar{\eta}$. Also since π is discontinuous at $\bar{\eta}$ (as $\pi[\bar{\eta}] \subseteq \bar{\eta}$ while $\pi(\bar{\eta}) = \eta$), $\bar{\eta}$ must have uncountable cofinality; since γ has countable cofinality, $\gamma < \bar{\eta}$. The proof that γ is rank Berkeley in $V_{\bar{\eta}}$ is just like the proof that if $j : V \to V$ is elementary, then the supremum of its critical sequence is rank Berkeley: consider the lexicographically least pair (α, β) with $\alpha < \gamma \leq \beta < \bar{\eta}$ such that there is no elementary embedding from V_β to itself with critical point between α and γ, and note that $\pi(\alpha) = \alpha$ and $\pi(\beta) = \beta$, and so $\pi \restriction V_\beta$ contradicts the definition of α and β. By elementarity (since $\pi(\gamma) = \gamma$), γ is rank Berkeley in V_η, and so since $\bar{\alpha} \in (\gamma, \eta)$, γ is rank Berkeley in $V_{\bar{\alpha}}$. By elementarity, γ is rank Berkeley in V_α, and this is a contradiction.

Turning to (2), the fact that almost supercompact cardinals are rank reflecting is immediate from the definition. Towards the converse, let us make some definitions and observations.

Fix an ordinal ξ. We define an increasing continuous sequence of ordinals $(\nu_i)_{i<\gamma_\xi}$ by letting ν_0 be ξ and ν_{i+1} the least ordinal ν greater than ν_i such that for all $\bar{\nu}$ strictly between ξ and ν_i, there is no elementary embedding from $V_{\bar{\nu}}$ into V_ν that fixes ξ. The process terminates at stage i if such an ordinal ν does not exist, in which case $\gamma_\xi = i+1$.

We claim that γ_ξ is defined, and in fact γ_ξ is less than the least rank Berkeley cardinal λ. To see this, assume towards a contradiction that the process does not terminate at any $i < \lambda$, so that ν_λ is defined. Let $j : V_{\nu_\lambda} \to V_{\nu_\lambda}$ be an elementary embedding fixing ξ whose critical point η is less than λ. Then $j \restriction V_{\nu_{\eta+1}}$ witnesses that there is an elementary embedding from $V_{\nu_{\eta+1}}$ to $V_{\nu_{j(\eta)+1}}$ that fixes ξ. This contradicts the definition of $\nu_{j(\eta)+1}$ as an ordinal such that for all $\bar{\nu}$ between ξ and $\nu_{j(\eta)+1}$, there is no elementary embedding $V_{\bar{\nu}}$ into $V_{\nu_{j(\eta)+1}}$ that fixes ξ.

Now suppose $\kappa \geq \lambda$ is a rank reflecting cardinal above ξ. Since κ is rank reflecting, there is some ordinal $\beta < \kappa$ such that V_β correctly computes γ_ξ. It follows that V_β correctly computes $(\nu_i)_{i<\gamma_\xi}$, and hence $\delta = \nu_{\gamma_\xi - 1}$ is less than κ. But δ has the property that for all $\alpha \geq \delta$, there is some $\bar{\alpha} < \delta$ and an elementary embedding $\pi : V_{\bar{\alpha}} \to V_\alpha$ that fixes ξ. Since $\xi < \kappa$ was arbitrary, κ is almost supercompact. □

The proof of Theorem 6.3 identifies what might be the correct analog of a supercompact cardinal above a rank Berkeley cardinal (as opposed to almost supercompact cardinals): a cardinal δ such that for all $\beta \geq \delta$, there exist $\bar{\delta} \leq \bar{\beta} < \delta$ and an elementary embedding $\pi : V_{\bar{\beta}} \to V_\beta$ such that $\pi[\bar{\delta}] \subseteq \bar{\delta}$ and $\pi(\bar{\delta}) = \delta$. The proof above shows that, for example, the least almost supercompact above the first rank Berkeley cardinal is a limit of such cardinals.

In order to state our theorems (particularly Theorem 7.2) in terms of rank reflection, we need to improve some of the large cardinal results from [6]. These involve analyzing the closure or completeness of embeddings associated with an almost supercompact cardinal. If X is a set, a cardinal λ is *X-closed rank Berkeley* if for all ordinals $\alpha < \lambda \leq \beta$, there is an elementary embedding $j : V_\beta \to V_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$ and $j(X) = j[X]$. A cardinal κ is *X-closed almost supercompact* if for all ordinals $\xi < \kappa \leq \beta$, for some ordinal $\bar{\beta}$ between ξ and κ and some set $\bar{X} \in V_{\bar{\beta}}$, there is an elementary embedding $\pi : V_{\bar{\beta}} \to V_\beta$ such that $\pi(\xi) = \xi$ and $\pi[\bar{X}] = \pi(\bar{X}) = X$.

The *Scott ordinal* of a set X, denoted by $\mathrm{scott}(X)$, is the least rank of a set Y that is in bijection with X. The proof of the following proposition is exactly like that of Theorem 6.3, using Proposition 6.2 to handle the parameter X.

Proposition 6.4. *If λ is X-closed rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and $\mathrm{scott}(X) < \kappa$, then κ is X-closed almost supercompact.* □

The next lemma is an improvement of [6, Lemma 2.11].

Lemma 6.5. *If λ is X-closed rank Berkeley and $\kappa \geq \lambda$ is rank reflecting, then $\mathrm{scott}(X) < \kappa$.*

Proof. We may assume that κ is the least rank reflecting cardinal greater than or equal to λ. Let S be the set of Scott ordinals of subsets of X. Then there is an elementary embedding $j : V_\alpha \to V_\alpha$ such that $j(S) = j[S]$, and hence $|S| < \lambda$.

Let $\beta = \text{rank}(X)$. By the proof of Theorem 6.3, one can find an ordinal $\bar{\beta}$ between λ and κ, a set $\bar{X} \in V_{\bar{\beta}+1}$, and an elementary embedding $\pi : V_{\bar{\beta}+1} \to V_{\beta+1}$ with $\text{crit}(\pi) > |S|$ and $\pi(\bar{X}) = X$.

Let $\bar{S}$ denote the set of Scott ordinals of subsets of $\bar{X}$. Note that $\bar{S} \subseteq S$ since every set $A \subseteq \bar{X}$ is in bijection with $\pi[A]$, which is a subset of X. But $\pi[\bar{S}] = S$ since $\pi(\bar{S}) = S$ and $\text{crit}(\pi) > |S|$. It follows that $\pi(\nu) = \nu$ for all $\nu \in \bar{S}$. In particular,

$$\text{scott}(X) = \pi(\text{scott}(\bar{X})) = \text{scott}(\bar{X}) < \kappa \qquad \square$$

As an immediate corollary of Proposition 6.4 and Lemma 6.5, we obtain:

Corollary 6.6. *If λ is X-closed rank Berkeley and $\kappa \geq \lambda$ is rank reflecting, then κ is X-closed almost supercompact.* □

7. The number of ultrafilters on an ordinal

If X is a set and κ is a cardinal, then $\beta_\kappa(X)$ denotes the set of κ-complete ultrafilters on X. The main theorem of this section bounds the size of this set when X is an ordinal:

Theorem 7.1. *Suppose λ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and $\epsilon \geq \kappa$ is an even ordinal. Then for all $\eta < \theta_\epsilon$, $|\beta_\kappa(\eta)| < \theta_\epsilon$.*

This should be contrasted with the following fact [6, Theorem 4.14], which combined with Theorem 7.11 implies that $\beta_\kappa(\eta)$ is quite large:

Theorem 7.2. *Suppose λ is rank Berkeley and $\kappa \geq \lambda$ is rank reflecting. Then every κ-complete filter on an ordinal extends to a κ-complete ultrafilter.*

In the context of ZFC, if every κ-complete filter on an ordinal extends to a κ-complete ultrafilter, then κ is strongly compact, and in particular for all cardinals η of cofinality at least κ, $|\beta_\kappa(\eta)| = 2^{2^\eta}$. Theorem 7.1 tells a very different story.

[6, Theorem 4.14] is actually proved assuming a stronger large cardinal hypothesis on κ than rank reflection (or equivalently almost supercompactness, see Theorem 6.3), though it is shown in [6, Theorem 2.9] this stronger hypothesis also holds at a closed unbounded class of cardinals κ. The assumption is that κ is X-closed almost extendible for all sets X such that λ is X-closed rank Berkeley. To prove Theorem 7.2 as it is stated here, one observes that the proof can be carried out assuming only that κ is X-closed almost supercompact for all sets X such that λ is X-closed rank Berkeley, and to obtain this, one appeals to Corollary 6.6.

The tendency of sufficiently complete ultrafilters on ordinals to behave like ordinals themselves is the key to the results of this section. The following theorem [6, Lemma 3.6] is one example of this behavior:

Theorem 7.3. *Suppose λ is rank Berkeley and $\kappa \geq \lambda$ is rank reflecting. Then for all $\eta \geq \kappa$, $\beta_\kappa(\eta)$ is wellorderable.* □

Lemma 7.4. *Suppose λ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and $\epsilon \geq \kappa$ is an even ordinal. Then for all $\eta < \theta_{\epsilon+2}$ and all surjections $\varphi : V_{\epsilon+1} \to \eta$, there is a wellordered sequence $\mathscr{F} \in \text{HOD}_{V_{\epsilon+1},\varphi}$ of subsets of $P(\eta)$ with the following properties:*

1. *Every $U \in \beta_\kappa(\eta)$ extends some $B \in \mathscr{F}$.*
2. *For each $B \in \mathscr{F}$, $|\{U \in \beta_\kappa(\eta) : B \subseteq U\}| < \lambda$.*

The proof involves an ordering of ultrafilters called the *Ketonen order*, whose behavior under choiceless large cardinal hypotheses partly explains the ordinal-like behavior of complete ultrafilters in this context. Because this order is really the key to Lemma 7.4, we give the definition of the Ketonen order and state some of its basic properties here.

An ultrafilter U is *wellfounded* if the ultrapower of any wellfounded structure by U is wellfounded; this is equivalent to the statement that the ultrapower of any wellorder by U is again a wellorder. For each ordinal η, let $\mathscr{B}(\eta)$ denote the set of wellfounded ultrafilters on η. Of course, assuming DC, an ultrafilter is wellfounded if and only if it is countably complete. In the context of choiceless cardinals, we must make do with the following analogous result [6, Lemma 3.6]:

Theorem 7.5. *If κ is almost supercompact and $U \in \beta_\kappa(\eta)$, then U is wellfounded.* □

Of course, much would be simplified by assuming DC, which feels harmless in this context.

The Ketonen order is defined on $\mathscr{B}(\eta)$ by setting $U <_{\Bbbk} W$ if there is a set $I \in W$ and a sequence $\langle U_\alpha \rangle_{\alpha\in I} \in \prod_{\alpha\in I} \mathscr{B}(\alpha)$ such that for all $A \in U$, $\{\alpha \in I : A \cap \alpha \in U_\alpha\} \in W$. The Ketonen order is wellfounded, and this is a theorem of ZF [6, Theorem 4.2].

If j is an elementary embedding, let $\text{Fix}(j)$ denote the set of ordinals fixed by j.

Lemma 7.6. *Suppose $j : V_{\epsilon+3} \to V_{\epsilon+3}$ is an elementary embedding, $\eta < \theta_{\epsilon+2}$, and $U \in \mathscr{B}(\eta)$ has Ketonen rank ξ. If $j(\xi) = \xi$ and $j(\eta) = \eta$, then $j(U) = U$ and $\text{Fix}(j) \cap \eta \in U$.*

Proof. Let $I = \eta \setminus \text{Fix}(j)$ be the set of ordinals $\alpha < \eta$ such that $j(\alpha) > \alpha$, and for $\alpha \in I$, let $D_\alpha = \{A \subseteq \alpha : \alpha \in j(A)\}$. We will show that $I \notin j(U)$. Assume towards a contradiction that $I \in j(U)$. Then $U <_{\Bbbk} j(U)$, since for all $A \in U$, $\{\alpha \in I : A \in D_\alpha\} = j(A) \cap I \in j(U)$. But this contradicts that $j(U)$ has rank ξ in the Ketonen order, so it cannot have a predecessor that also has rank ξ.

Therefore $I \notin U$, and hence its complement $\text{Fix}(j) \cap \eta \in j(U)$. From this it follows that $U = j(U)$: for each $A \in U$, $j(A) \cap \text{Fix}(j) \in j(U)$, but $j(A) \cap \text{Fix}(j) = A \cap \text{Fix}(j) \subseteq A$, so $A \in j(U)$. This shows that $U \subseteq j(U)$, but since U and $j(U)$ are proper filters on η, it follows that $U = j(U)$. □

Despite its simplicity, this result has nontrivial consequences.

Proposition 7.7. *Suppose ϵ is an even ordinal and there is an elementary embedding $j : V_{\lambda+3} \to V_{\lambda+3}$. If U is a wellfounded ultrafilter on an ordinal $\eta < \theta_{\epsilon+2}$, then $\text{Ult}(\eta, U) < \theta_{\epsilon+2}$.*

Proof. By iterating j, we may assume that j fixes both η and the Ketonen rank of U, and hence $j(U) = U$ and $\text{Fix}(j) \in U$. Then for any $f : \eta \to \eta$,

$$j([f]_U) = [j(f)]_U = [j \circ f]_U$$

with the final equality holding because $j(f) \restriction \text{Fix}(j) = (j \circ f) \restriction \text{Fix}(j)$ and $\text{Fix}(j) \in U$. It follows that $j \restriction \text{Ult}(\eta, U)$ is definable over $\mathcal{H}_{\epsilon+3}$ from $j \restriction \eta$ and parameters in the range of j; namely, $j(U) = U$. Since $\eta < \theta_{\epsilon+2}$, Theorem 3.4 and Lemma 3.3 now combine to yield $\text{Ult}(\eta, U) < \theta_{\epsilon+2}$. □

Note that Proposition 7.7 is a special case of our main result that $\theta_{\epsilon+2}$ is a strong limit cardinal since $\text{Ult}(\eta, U) \leq^* P(\eta)$.

We will also appeal to yet another result from [6]:

Theorem 7.8 ([6, Theorem 3.12])**.** *Suppose κ and η are ordinals, $\mathcal{U}$ is a nonprincipal normal fine ultrafilter on $P(\eta)$, and $F \in \mathcal{U}$ is a κ-complete filter. Suppose there is a wellfounded $\kappa_{\mathcal{U}}^+$-complete fine ultrafilter on $P(P(\eta))$ that concentrates on the set of $\sigma \in P(P(\eta))$ such that $\aleph(P(\sigma)) < \kappa$. Then η can be partitioned into fewer than $\kappa_{\mathcal{U}}$-many atoms of F.* □

The author of [6] has an unfortunate tendency to state theorems under the most general hypotheses possible, favoring combinatorial notions like fine ultrafilters over more intuitive concepts like elementary embeddings, regardless of its effect on the readability of his paper. For this reason, we are forced to state what is hopefully a more comprehensible corollary to this theorem:

Corollary 7.9. *Suppose λ is the least rank Berkeley cardinal and $\kappa \geq \lambda$ is rank reflecting. Suppose η is an ordinal and F is a κ-complete filter on η such that for some ordinal $\alpha > \eta$, for any elementary embedding $j : V_\alpha \to V_\alpha$, $j[\eta] \in j(F)$. Then η can be partitioned into fewer than λ-many atoms of F.*

Proof. We verify that the hypotheses of Theorem 7.8 hold for F.

Since λ is rank Berkeley, there is an elementary embedding $j : V_\alpha \to V_\alpha$ with critical point $\nu < \lambda$ such that $\lambda = \sup\{\nu, j(\nu), j(j(\nu)), \dots\}$. Note that F belongs to the normal fine ultrafilter $\mathcal{U}$ on $P(\eta)$ derived from j because $\mathcal{U}$ is the set of $B \subseteq P(\eta)$ such that $j[\eta] \in j(B)$, and by assumption $j[\eta] \in j(F)$.

To finish the proof, we just need to prove that there is a wellfounded ν^+-complete fine ultrafilter on $P(P(\eta))$ that concentrates on the set of $\sigma \in P(P(\eta))$ such that $\aleph(P(\sigma)) < \kappa$. This uses the fact that the embeddings witnessing that κ is almost supercompact may be taken to have critical point arbitrarily large below λ, which follows from the proof of Theorem 6.3 (or alternatively is a direct consequence of [6, Lemma 2.7]).

Now let γ be a rank reflecting ordinal above α and let $\pi : V_{\bar{\gamma}} \to V_\gamma$ be an elementary embedding with $\bar{\gamma} < \kappa$, $\text{crit}(\pi) > \nu$, and $\{\eta, \kappa\} \subseteq \text{ran}(\pi)$. Let $\bar{\eta} = \pi^{-1}(\eta)$ and $\bar{\kappa} = \pi^{-1}(\kappa)$. Let $\bar{\mathcal{W}}$ be the ultrafilter on $P(P(\bar{\eta}))$ derived from π using $\pi[P(\bar{\eta})]$. In $V_{\bar{\gamma}}$, $\bar{\mathcal{W}}$ is a wellfounded ν^+-complete fine ultrafilter on $P(P(\eta))$ that concentrates on the set of $\sigma \in P(P(\eta))$ such that $\aleph(P(\sigma)) < \bar{\kappa}$. The wellfoundedness of $\mathcal{W}$ in $V_{\bar{\gamma}}$ follows from the fact that the ultrapower of any ordinal $\beta < \bar{\gamma}$ by $\mathcal{W}$ embeds into $\pi(\beta)$ and is therefore wellfounded. That $\mathcal{W}$ concentrates on the set of $\sigma \in P(P(\eta))$ such that $\aleph(P(\sigma)) < \bar{\kappa}$ follows from the fact that, letting $\sigma = \pi[P(\bar{\eta})]$, $\aleph(P(\sigma)) = \aleph(P(P(\bar{\eta}))) < \kappa$.

Letting $\mathcal{W} = \pi(\bar{\mathcal{W}})$, we have that in V_γ, $\mathcal{W}$ is a wellfounded ν^+-complete fine ultrafilter on $P(P(\eta))$ that concentrates on the set of $\sigma \in P(P(\eta))$ such that $\aleph(P(\sigma)) < \kappa$. Since γ was chosen to be rank reflecting, it follows that $\mathcal{W}$ is truly wellfounded, which concludes the proof of the corollary. □

Proof of Lemma 7.4. For each ordinal α, let $\mathcal{E}_\alpha$ denote the set of elementary embeddings from V_α to itself. For $\xi < \theta_{\epsilon+3}$, let

$$B_\xi = \{j[\eta] : j \in \mathcal{E}_{\epsilon+3}, \varphi \in j[\mathcal{H}_{\epsilon+2}], j(\xi) = \xi, \text{ and } j(\eta) = \eta\}$$

We let $\mathscr{F} = \langle B_\xi \rangle_{\xi < \theta_{\epsilon+3}}$. Note that $\mathscr{F}$ is OD_φ. Also, each B_ξ is a subset of $\text{HOD}_{V_{\epsilon+1},\varphi}$: for $j \in \mathcal{E}_{\epsilon+3}$ with $\varphi \in j[\mathcal{H}_{\epsilon+2}]$, $j[\eta] = \text{ran}(\varphi \circ j^+)$, and j^+ is definable over $V_{\epsilon+1}$ by Theorem 3.1. It follows that $\mathscr{F} \in \text{HOD}_{V_{\epsilon+1},\varphi}$.

By Lemma 7.6, if $U \in \beta_\kappa(\eta)$ has Ketonen rank ξ, then $B_\xi \subseteq U$. This shows (1).

For any $j : V_{\epsilon+4} \to V_{\epsilon+4}$ such that $j(\xi) = \xi$ and $j(\eta) = \eta$, we have $j[\eta] \in j(B_\xi)$ since

$$j(B_\xi) = \{i[\eta] : i \in \mathcal{E}_{\epsilon+3}, j(\varphi) \in i[\mathcal{H}_{\epsilon+2}], i(\xi) = \xi, \text{ and } i(\eta) = \eta\}$$

and taking $i = j \restriction V_{\epsilon+3}$, we obtain that $j[\eta] = i[\eta] \in j(B_\xi)$ by Lemma 7.6.

Let F_ξ be the intersection of all κ-complete ultrafilters extending B_ξ. By Corollary 7.9, there is a partition $\mathcal{P}$ of η into atoms of F_ξ with $|\mathcal{P}| < \lambda$. It follows that for each $A \in \mathcal{P}$, $B_\xi \cup \{A\}$ extends uniquely to an element of $\beta_\kappa(\eta)$, namely the ultrafilter U generated by $F_\xi \cup \{A\}$. To see this, let U' be a κ-complete ultrafilter extending $B_\xi \cup \{A\}$. Since $B_\xi \subseteq U'$, $F_\xi \subseteq U'$, and hence $F_\xi \cup \{A\} \subseteq U'$, which implies that $U' = U$.

As a consequence, $|\{U \in \beta_\kappa(\eta) : B \subseteq U\}| = |\mathcal{P}| < \lambda$, establishing (2). □

Proof of Theorem 7.1. Fix a family $\mathscr{F}$ as in Lemma 7.4. In $M = \text{HOD}_{V_{\epsilon+1},\varphi}$, $\mathscr{F}$ is a wellordered family of subsets of $P(\eta)$ and $P(\eta) \leq^* V_{\epsilon+1}$ by Theorem 4.5. Therefore $|\mathscr{F}| < \theta^M_{\epsilon+3}$. By Theorem 5.3, $\theta^M_{\epsilon+3} < \theta_{\epsilon+2}$. Fix an enumeration $\langle B_\alpha \rangle_{\alpha < |\mathscr{F}|}$ of $\mathscr{F}$. For $\alpha < |\mathscr{F}|$, let $\mathscr{U}_\alpha = \{U \in \beta_\kappa(\eta) : B_\alpha \subseteq U\}$, so that $|\mathscr{U}_\alpha| < \lambda$ for all $\alpha < |\mathscr{F}|$. Then $\beta_\kappa(\eta) = \bigcup_{\alpha < |\mathscr{F}|} \mathscr{U}_\alpha$, and so, appealing to Theorem 7.3,

$$|\beta_\kappa(\eta)| \leq |\mathscr{F}| \cdot \lambda < \theta_{\epsilon+2}$$ □

We remark that the results of this section (especially Lemma 7.4) take advantage of a very simple *almost everywhere* approximation to the (local) Moschovakis coding lemma: if ϵ is an even ordinal and $\varphi : V_{\epsilon+2} \to \eta$ is a surjection, then for any normal fine Jonsson ultrafilter $\mathcal{U}$ on $V_{\epsilon+1}$, if $\mathcal{W}$ is the projection of $\mathcal{U}$ to $P(\eta)$, then $\mathcal{W}$-almost every $A \subseteq \eta$ is Σ_1-definable in the codes over $(V_{\epsilon+1}, \prec)$ where $\prec$ is the prewellorder induced by φ. Theorem 7.1 can be seen as combining this almost everywhere local coding lemma with the coarse coding lemma of Section 4 to obtain a coding lemma for ultrafilters.

For the proof of our main theorem, Theorem 8.1, we require a further bound on the cardinality of the set of ultrafilters in the context of Theorem 7.1 but without the restriction to κ-complete ultrafilters.

If U is an ultrafilter on a set X and h and g are functions on X, write $h \sqsubseteq_U g$ if there is a function e such that $h(x) = e \circ g(x)$ for U-almost all $x \in X$.

Suppose λ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and $\eta \geq \kappa$ is an ordinal. Our first lemma allows us to code any κ-wellfounded ultrafilter on η as a pair (D, F) where D is an ultrafilter on an ordinal less than κ and F is a κ-complete filter. (Recall from [6] that an ultrafilter U is γ-*wellfounded* if the ultrapower of the ordinal γ by U is wellfounded.)

Lemma 7.10. *Suppose λ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and U is a κ-wellfounded ultrafilter on an ordinal η.*

(1) There is a function g that is $\sqsubseteq_U$-maximal among all bounded functions from η to κ.

(2) Letting $D = g_*(U)$ and $k : \text{Ult}(P(\eta), D) \to \text{Ult}(P(\eta), U)$ be the factor embedding, the $\text{Ult}(P(\eta), D)$-ultrafilter $\mathcal{B}$ derived from k using $[\text{id}]_U$ generates a κ-complete filter.

This lemma is a choiceless version of a theorem of Silver [20] on the existence of finest partitions for indecomposable ultrafilters.

We will use the wellordered collection lemma [6, Corollary 2.22].

Theorem 7.11. *Suppose λ is rank Berkeley and $\kappa \geq \lambda$ is rank reflecting. Then for any family $\mathcal{F}$ of nonempty sets with $|\mathcal{F}| < \kappa$, there is a sequence $\langle a_x : x \in V_\kappa \rangle$ such that for all $A \in \mathcal{F}$, there is some $x \in V_\kappa$ with $a_x \in A$.* □

Proof of Lemma 7.10. In our applications, we will only need the special case of the lemma in which κ is the least rank reflecting cardinal greater than or equal to λ, and then $\text{cf}(\kappa) = \omega$ by Proposition 6.1. In general, we can reduce the lemma to the special case in which κ has cofinality ω. Otherwise, κ is a limit of rank reflecting cardinals of cofinality ω, and one obtains the result by applying the lemma to these smaller cardinals using a regressive function argument. Specifically, for each rank reflecting $\rho < \kappa$, let $\beta_\rho = [h]_U$ where h is a $\sqsubseteq_U$-maximal bounded function from η to ρ; note that β_ρ does not depend on the choice of h. The function $\rho \mapsto \beta_\rho$ is a monotone regressive function on the closed unbounded set of rank reflecting cardinals below κ, and therefore it is eventually constant. If $\gamma < \kappa$ is large enough that $\beta_\rho = \beta_{\rho'}$ for all rank reflecting cardinals between γ and κ, then letting g be a $\sqsubseteq_U$-maximal bounded function from η to the least rank reflecting cardinal above γ, g easily satisfies (1) and (2).

So let us establish (1) and (2) assuming $\text{cf}(\kappa) = \omega$. We begin with (1). Using the wellordered collection lemma (Theorem 7.11), fix functions $\langle f_x : x \in V_\kappa \rangle$ from η to κ such that for each $\alpha < \kappa$, there is some $x \in V_\kappa$ such that $\alpha = [f_x]_U$. Then define $g : \eta \to \kappa^{V_\kappa}$ by $g(\xi) = \langle f_x(\xi) : x \in V_\kappa \rangle$.

For any $\nu < \kappa$ and $h : \eta \to \nu$, there is some $x \in V_\kappa$ such that for U-almost all $\xi < \eta$, $h(\xi) = f_x(\xi) = \text{ev}_x \circ g(\xi)$ where $\text{ev}_x : g[\eta] \to \kappa$ is given by $\text{ev}_x(s) = s(x)$. To finish, it suffices to show that there is some $A \in U$ such that $|g[A]| < \kappa$.

We claim $\aleph(V_{\kappa+1}) = \kappa^+$. [6, Theorem 3.13] implies that κ^+ is measurable. The proof that successor cardinals cannot be measurable under the Axiom of Choice shows in the ZF context that if a set X carries a nonprincipal ultrafilter that is closed under Y-indexed intersections, then there is no injection from X to $P(Y)$. Since κ is rank reflecting, $\kappa = \theta_\kappa$, and so any κ^+-complete ultrafilter on κ^+ is closed under V_κ-indexed intersections by [6, Lemma 3.5]. It follows that there is no injection from κ^+ to $P(V_\kappa) = V_{\kappa+1}$, and hence $\aleph(V_{\kappa+1}) = \kappa^+$.

Since $\aleph(V_{\kappa+1}) = \kappa^+$, $|g[\eta]| \leq \kappa$. Since $\text{cf}(\kappa) = \omega$ and U is countably complete, $|g[A]| < \kappa$ for some $A \in U$.

We now turn to (2). Fix g as in (1). We will use the following characterization of $\mathcal{B}$: if $f : \nu \to P(\eta)$, then $[f]_D \in \mathcal{B}$ if and only if for U-almost every $\xi < \eta$, $\xi \in f \circ g(\xi)$. This is because $[f \circ g]_U = k([f]_D)$, so U-almost every ξ belongs to $f \circ g(\xi)$ if and only if $[\text{id}]_U \in k([f]_D)$; that is, $[f]_D$ is in the ultrafilter derived from k using $[\text{id}]_U$.

Suppose $\gamma < \kappa$ and $\langle A_\beta \rangle_{\beta < \gamma}$ belong to the filter generated by $\mathcal{B}$. We will show that there is a set $A \in \mathcal{B}$ such that $A \subseteq \bigcap_{\xi < \gamma} A_\xi$. This shows that the filter generated by $\mathcal{B}$ is κ-complete.

Applying the wellordered collection lemma (Theorem 7.11), let $\langle f_x : x \in V_\kappa \rangle$ be functions representing sets in $\mathcal{B}$ such that for all $\beta < \gamma$, there is some $x \in V_\kappa$ such that $[f_x]_D \subseteq A_\beta$. Let $h : \eta \to V_{\kappa+1}$ be the function

$$h(\xi) = \{x \in V_\kappa : \xi \in f_x \circ g(\xi)\}$$

For all $x \in V_\kappa$, the fact that $[f_x]_D \in \mathcal{B}$ implies that for U-almost all $\xi < \eta$, $x \in h(\xi)$. Since $\aleph(V_{\kappa+1}) = \kappa^+$, $|h[\eta]| \leq \kappa$, and so since g is $\sqsubseteq_U$-maximal, there is some $e : \nu \to V_{\kappa+1}$ such that $e \circ g(\xi) = h(\xi)$ for U-almost all ξ. For all $x \in V_\kappa$, since $x \in h(\xi)$ for U-almost all $\xi < \eta$, $x \in e(\alpha)$ for D-almost all $\alpha < \kappa$.

Define $f : \kappa \to P(\eta)$ by setting

$$f(\alpha) = \bigcap_{x \in e(\alpha)} f_x(\alpha)$$

and let $A = [f]_D$. For all $x \in V_\kappa$, for D-almost all $\alpha < \kappa$, $f(\alpha) \subseteq f_x(\alpha)$, and hence $A \subseteq [f_x]_D$. This means $A \subseteq \bigcap_{\beta<\gamma} A_\beta$.

Finally, we claim $A \in \mathcal{B}$. To see this, note that by the definition of h, for all $\xi < \eta$, $\xi \in \bigcap_{x\in h(\xi)} f_x \circ g(\xi)$. By our choice of e, for U-almost all $\xi < \eta$,

$$f \circ g(\xi) = \bigcap_{x \in e\circ g(\xi)} f_x \circ g(\xi) = \bigcap_{x\in h(\xi)} f_x \circ g(\xi)$$

Hence for U-almost all $\xi < \eta$, $\xi \in f \circ g(\xi)$, which means that $A = [f]_D \in \mathcal{B}$. □

Corollary 7.12. *If λ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and U is a κ-wellfounded ultrafilter on an ordinal, then U is wellfounded.*

Proof. Let $(D, \mathcal{B})$ decompose U as in Lemma 7.10. Note that $D \in V_\kappa$ and D is wellfounded in V_κ, so D is wellfounded by rank reflection. The wellfoundedness of D, along with Lemma 7.10, implies that $\mathcal{B}$ generates a κ-complete filter on an ordinal, and so applying Theorem 7.2, $\mathcal{B}$ extends to a κ-complete ultrafilter W. Note that by Theorem 7.5, W is wellfounded.

There is an order-preserving embedding

$$e : \text{Ult}(\text{Ord}, U) \to \text{Ult}(\text{Ord}, W)$$

defined by $e([f]_U) = [j_D(f)]_W$, so U is wellfounded because W is. □

Corollary 7.13. *Assume λ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and $\epsilon \geq \kappa$ is even. Then for any $\eta < \theta_{\epsilon+2}$, there is a surjection from $V_{\epsilon+1}$ onto $\mathscr{B}(\eta)$.*

Proof. We may assume κ is the least rank reflecting cardinal above λ, so that $\text{cf}(\kappa) = \omega$ by Proposition 6.1. Let $\mathscr{B}$ denote the set of wellfounded ultrafilters on ordinals less than κ, and let

$$\gamma = \sup\{\text{Ult}(\eta, D) : D \in \mathscr{B}\}$$

We claim that $\gamma < \theta_{\epsilon+2}$. Let $S = \{\text{Ult}(\eta, D) : D \in \mathscr{B}\}$, so that $\gamma = \sup(S)$. By Proposition 7.7, $S \subseteq \theta_{\epsilon+2}$, so it suffices to show that S is bounded below $\theta_{\epsilon+2}$. But S is the union of the sets $S_\alpha = \{\text{Ult}(\eta, D) : D \in \mathscr{B}(\alpha)\}$ for $\alpha < \eta$, and $|S_\alpha| < \theta_{\alpha+2} < \kappa$. Therefore $|S| \leq \kappa$. By [6, Corollary 2.17], $\text{cf}(\theta_{\epsilon+2}) \geq \kappa$, and since $\text{cf}(\kappa) = \omega$, the inequality is strict. Therefore $\gamma = \sup(S) < \theta_{\epsilon+2}$.

Consider the function $p : \mathscr{B} \times \beta_\kappa(\gamma) \to \mathscr{B}(\eta)$ defined by $p(D, W) = j_D^{-1}[W]$. We claim that p is surjective. To see this, fix $U \in \mathscr{B}(\eta)$. By Lemma 7.10, let $g : \eta \to \kappa$ be the $\sqsubseteq_U$-maximal bounded function, let $D = g_*(U)$, let $k : \text{Ult}(P(\eta), D) \to \text{Ult}(P(\eta), U)$ be the factor embedding, and let $\mathscr{B}$ be the $\text{Ult}(P(\eta), D)$-ultrafilter on $\text{Ult}(\eta, D)$ derived from k using $[\text{id}]_U$. Then by Lemma 7.10, $\mathscr{B}$ generates a κ-complete filter F on $\text{Ult}(\eta, U)$, and by Theorem 7.2, F extends to a κ-complete ultrafilter W on $\text{Ult}(\eta, U)$. Now

$$U = j_D^{-1}[\mathscr{B}] = j_D^{-1}[W] = p(D, W)$$

Since $|\beta_\kappa(\gamma)| < \theta_{\epsilon+2}$ by Theorem 7.1, $\beta_\kappa(\gamma) \leq^* V_{\epsilon+1}$. Moreover, we have $\mathscr{B} \subseteq V_\kappa \leq^* V_{\epsilon+1}$. Therefore

$$\mathscr{B}(\eta) \leq^* \mathscr{B} \times \beta_\kappa(\gamma) \leq^* V_{\epsilon+1} \times V_{\epsilon+1} \leq^* V_{\epsilon+1}$$

which proves the theorem. □

The proof above actually produces a surjection from $V_\kappa \times \rho$ onto $\mathscr{B}(\eta)$ for some ordinal $\rho < \theta_{\epsilon+2}$.

8. Periodicity in the Lindenbaum numbers

We finally come to the main theorems of this paper. We first show that sufficiently large even Lindenbaum numbers are strong limit cardinals. In fact, we will show:

Theorem 8.1. *Suppose λ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and $\epsilon \geq \kappa$ is an even ordinal. Then θ_ϵ is a strong limit cardinal.*

Proof. For simplicity, we will just prove that if $\epsilon \geq \kappa$ is even, then $\theta_{\epsilon+2}$ is a strong limit cardinal; that limit Lindenbaum numbers are strong limit cardinals is established by Lemma 2.1. Fix an ordinal $\eta < \theta_{\epsilon+2}$ and (applying Corollary 7.13) a surjection $f : V_{\epsilon+1} \to \eta \times \mathscr{B}(\eta)$. Let $j : V_{\epsilon+3} \to V_{\epsilon+3}$ be an elementary embedding such that $j(\eta) = \eta$ and $f \in j[\mathcal{H}_{\epsilon+3}]$.

We claim that $j \restriction P(\eta)$ is definable over $\mathcal{H}_{\epsilon+3}$ from $j \restriction V_\epsilon$ and parameters in the range of j. For $\alpha < \eta$, let

$$U_\alpha = \{A \subseteq \eta : \alpha \in j(A)\}$$

be the ultrafilter on η derived from j using α. It suffices to show that the sequence $E = \langle U_\alpha \rangle_{\alpha<\eta}$ (which is essentially the extender of length η derived from j) is definable over $\mathcal{H}_{\epsilon+3}$ from $j \restriction V_\epsilon$ and parameters in the range of j, since j and E are interdefinable:

$$j(A) = \{\alpha < \eta : A \in U_\alpha\}$$

Note that E, or perhaps more precisely the graph of E, is a subset of $\eta \times \mathscr{B}(\eta)$. (Here it is important that every ultrafilter of E is wellfounded, which follows from Corollary 7.12.) Let $A \subseteq V_{\epsilon+1}$ be the preimage of E under the surjection $f : V_{\epsilon+1} \to \eta \times \mathscr{B}(\eta)$. Then $A = (j \restriction V_{\epsilon+1})^{-1}[j(A)]$ is definable from $j[V_\epsilon]$ and $j(A)$ in $V_{\epsilon+3}$, using Theorem 3.1. Finally, $E = f[A]$ is definable from A and f in $\mathcal{H}_{\epsilon+3}$. Since $f \in j[\mathcal{H}_{\epsilon+3}]$ and A is definable in $\mathcal{H}_{\epsilon+3}$ from $j[V_\epsilon]$ and parameters in the range of j, it follows that E is definable in $\mathcal{H}_{\epsilon+3}$ from $j[V_\epsilon]$ and parameters in the range of j, as claimed.

It now follows from Theorem 3.4 and Lemma 3.3 that $P(\eta)$ does not surject onto $\theta_{\epsilon+2}$: if it did, then by Lemma 3.3, $j[\theta_{\epsilon+2}]$ would be definable in $\mathcal{H}_{\epsilon+3}$ from $j \restriction P(\eta)$ and parameters in the range of j, and hence by our claim, $j[\theta_{\epsilon+2}]$ would be definable in $\mathcal{H}_{\epsilon+3}$ from $j[V_\epsilon]$ and parameters in the range of j, contrary to Theorem 3.4. □

We now prove our main theorem on odd Lindenbaum numbers. The following fact is implicit in the proof of Theorem 5.3.

Lemma 8.2. *Suppose ϵ is an even ordinal and $j : V_{\epsilon+1} \to V_{\epsilon+1}$ is an elementary embedding. Let E be the extender of length θ_ϵ derived from j. Then $j_E(A) = j(A)$ for any bounded subset A of $\theta_{\epsilon+1}$.* □

Theorem 8.3. *If λ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and $\epsilon \geq \kappa$ is an even ordinal, then there is a surjection from $P(\theta_\epsilon)$ onto $\theta_{\epsilon+1}$.*

Proof. Again, for simplicity, having handled the case that ϵ is a limit ordinal in Lemma 2.1, let us just show that for all even $\epsilon \geq \kappa$, there is a surjection from $P(\theta_{\epsilon+2})$ onto $\theta_{\epsilon+3}$. Since $\beta_\kappa(<\theta_{\epsilon+2}) = \bigcup_{\eta<\theta_\epsilon} \beta_\kappa(\eta)$ is wellorderable by Theorem 7.3, and $|\beta_\kappa(\eta)| < \theta_{\epsilon+2}$ for all $\eta < \theta_{\epsilon+2}$ by Theorem 7.1, $|\beta_\kappa(<\theta_{\epsilon+2})| = \theta_{\epsilon+2}$. Following the proof of Corollary 7.13, there is a surjection from $V_{\epsilon+1}\times\theta_{\epsilon+2}$ onto $\mathscr{B} = \bigcup_{\eta<\theta_\epsilon} \mathscr{B}(\eta)$. Fix such a surjection $f : V_{\epsilon+1} \times \theta_{\epsilon+2} \to \mathscr{B}$.

Let $j : V_{\epsilon+3} \to V_{\epsilon+3}$ be an elementary embedding such that $j(\kappa) = \kappa$ and $f \in \operatorname{ran}(j)$. Let E be the extender of length $\theta_{\epsilon+2}$ derived from j. Following the proof of Theorem 8.1, one can show that E is definable in $\mathcal{H}_{\epsilon+3}$ from $j[V_\epsilon \cup \theta_{\epsilon+2}]$ and parameters in the range of j. By Lemma 8.2, it follows that $j[\theta_{\epsilon+3}]$ is definable over $\mathcal{H}_{\epsilon+3}$ from $j[V_\epsilon \cup \theta_{\epsilon+2}]$ and parameters in the range of j. By Theorem 3.4, this implies that there is a surjection from $P(V_\epsilon \cup \theta_{\epsilon+2})$ onto $\theta_{\epsilon+3}$, and hence there is a surjection $g : V_{\epsilon+1} \times P(\theta_{\epsilon+2}) \to \theta_{\epsilon+3}$.

For each $S \in P(\theta_{\epsilon+2})$, let $T_S = \{g(x, S) : x \in V_{\epsilon+1}\}$. Note that T_S is the surjective image of $V_{\epsilon+1}$, and so $\operatorname{ot}(T_S) < \theta_{\epsilon+2}$. Let $f_S : \operatorname{ot}(T_S) \to T_S$ be the increasing enumeration. Then define a partial surjection

$$f : P(\theta_{\epsilon+2}) \times \theta_{\epsilon+2} \to \theta_{\epsilon+3}$$

by setting $f(S, \xi) = f_S(\xi)$ if $\xi < \operatorname{ot}(T_S)$. We now have:

$$\theta_{\epsilon+3} \leq^* P(\theta_{\epsilon+2}) \times \theta_{\epsilon+2} \leq^* P(\theta_{\epsilon+2})$$ □

We include one last theorem on the size of the odd Lindenbaum numbers. We would like to show that these must be the successors of the even ones, but all we can currently show is that there are not too many regular cardinals in between.

Corollary 8.4. *If ϵ is an even ordinal and $j : V_{\epsilon+1} \to V_{\epsilon+1}$ is an elementary embedding with critical point κ, then the set of regular cardinals in the interval $(\theta_\epsilon, \theta_{\epsilon+1})$ has cardinality less than κ.*

Proof. The set R of regular cardinals in the interval $(\theta_\epsilon, \theta_{\epsilon+1})$ does not have cardinality exactly κ since $|R|$ is definable over $\mathcal{H}_{\epsilon+1}$ while κ, being the critical point of the elementary embedding $j : \mathcal{H}_{\epsilon+1} \to \mathcal{H}_{\epsilon+1}$, is not.

Assume towards a contradiction $|R| > \kappa$ and that δ is the κ-th regular cardinal in the interval $(\theta_\epsilon, \theta_{\epsilon+1})$. Since $j : \mathcal{H}_{\epsilon+1} \to \mathcal{H}_{\epsilon+1}$ is elementary, $j(\delta)$ is the $j(\kappa)$-th regular cardinal in the same interval.

Let E be the extender of length θ_ϵ derived from j. Then $j(\delta) = j_E(\delta)$, and j_E is continuous at δ since each measure of E lies on a cardinal smaller than δ. It follows that $j(\delta)$ has cofinality δ, which contradicts that $j(\delta)$ is a regular cardinal larger than δ. □

9. Questions

Assume there is an elementary embedding from the universe of sets to itself.

Question 9.1. For sufficiently large even ordinals ϵ, is $\theta_{\epsilon+1} = \theta_\epsilon^+$?

This is probably the most glaring question left open by our theorems here. But many other combinatorial questions remain:

Question 9.2. For sufficiently large even ordinals ϵ, if $\eta < \theta_{\epsilon+2}$, is there a surjection from $V_{\epsilon+1}$ onto $P(\eta)$? Does the coding lemma hold?

Define a sequence of ordinals $\langle \nu_\alpha : \alpha \in \text{Ord}\rangle$ by setting $\nu_{\alpha+1} = \aleph^*(P(\nu_\alpha))$ and $\nu_\gamma = \sup_{\alpha<\gamma} \nu_\alpha$ for γ a limit ordinal. The arguments of this paper show that if there is an elementary embedding from the universe of sets to itself, then ν_α is a strong limit cardinal for all sufficiently large ordinals α.

Question 9.3. Is $\theta_{\gamma+2n} = \nu_{\gamma+n}$ for all sufficiently large limit ordinals γ and all $n < \omega$?

The various formulations of the generalized continuum problem, equivalent under the Axiom of Choice, come apart when AC is dropped. Under choiceless large cardinal hypotheses, some of these variants may turn out to be just as interesting as the ones studied here, although currently there seems to be no way to approach them:

Question 9.4. Suppose ϵ is a sufficiently large even ordinal and $A \subseteq V_{\epsilon+1}$. If there is no surjection from V_ϵ onto A, must $|A| = |V_{\epsilon+1}|$?

On the one hand, it is hard to believe this could be provable from choiceless large cardinal hypotheses since it would imply that there is a bijection between $V_{\epsilon+1}$ and $P(\theta_\epsilon)$, while it seems more likely to be consistent with choiceless large cardinals that there is a set that is not the surjective image of the powerset of any ordinal. On the other hand, results of [21] show that it is consistent with choiceless large cardinals that every set is the surjective image of the powerset of an ordinal. Maybe a better question is whether the answer to Question 9.4 is *consistently* positive.

Certain arguments in this paper require simulating the Axiom of Choice using rank reflecting cardinals, and for this reason, the following question remains open:

Question 9.5. If λ is rank Berkeley — or just assuming there is an elementary embedding from $V_{\lambda+2}$ to itself — is $\theta_{\lambda+2}$ a strong limit cardinal?

It is natural to speculate that perhaps the right higher-order generalization of the Axiom of Determinacy is some kind of regularity property for subsets of $V_{\omega+2n+1}$ for $n < \omega$.

Question 9.6. Is there an extension of the Axiom of Determinacy that implies $\theta_{\omega+4}$ is a strong limit cardinal and $\theta_{\omega+5}$ is its successor? Is this theory even consistent with the Axiom of Determinacy? If $\theta_{\omega+2}$ is a strong partition cardinal, does this hold?

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