







Then,  $\mathrm{Sp}_{2n}$  acts linearly on  $\mathfrak{sp}_{2n}$  by conjugation, and  $T$  admits a common eigenbasis: namely, they are the same basis vectors as we see for the above examples. For example, for  $\mathfrak{sp}_4$  we have eigenbasis

$$\{E_{11} - E_{44}, E_{22} - E_{33}, E_{12} - E_{34}, E_{21} - E_{43}, E_{13} + E_{24}, E_{14}, E_{23}, E_{32}, E_{41}, E_{31} + E_{42}\}$$

and the weights are

$$0, \pm\chi_1 - \chi_2, \pm\chi_1 + \chi_2, \pm 2\chi_1, \pm 2\chi_2$$

If we let  $\alpha_1 = \chi_1 - \chi_2$ ,  $\alpha_2 = 2\chi_2$ , then we can write the nonzero elements above as

$$\pm\alpha_1, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm\alpha_2;$$

these are the **roots of  $\mathfrak{sp}_4$** .

In the  $\mathfrak{sp}_6$  case we have weights

$$0, \pm\chi_i - \chi_j, \pm\chi_k + \chi_l, \quad i < j, k \leq l.$$

If we let

$$\alpha_1 = \chi_1 - \chi_2, \alpha_2 = \chi_2 - \chi_3, \alpha_3 = 2\chi_3,$$

then we can write the above nonzero weights

$$\begin{aligned} \pm\alpha_i, \pm(\alpha_1 + \alpha_2), \pm(\alpha_2 + \alpha_3), \pm(2\alpha_2 + \alpha_3) \pm(\alpha_1 + \alpha_2 + \alpha_3), \pm(\alpha_1 + 2\alpha_2 + \alpha_3), \\ \pm(\alpha_1 + 2\alpha_2 + \alpha_3), \pm(2\alpha_1 + 2\alpha_2 + \alpha_3); \end{aligned}$$

these are the **roots of  $\mathfrak{sp}_6$** .

In general, the roots of  $\mathrm{Sp}_{2n}$  (with respect to our choice of  $T$ ) are

$$R \stackrel{\text{def}}{=} \{\chi_i - \chi_j \mid i \neq j\} \cup \{\chi_i + \chi_j \mid i \leq j\} \subset X^*(T)$$

and a set of simple roots are

$$\alpha_1 = \chi_1 - \chi_2, \alpha_2 = \chi_2 - \chi_3, \dots, \alpha_n = 2\chi_n.$$

This allows us to define the **positive roots**

$$R_+ \stackrel{\text{def}}{=} \{\alpha \in \mathbb{R} \mid \alpha = \sum n_i \alpha_i, n_i \in \mathbb{Z}_{\geq 0}\},$$

and the **negative roots**  $-R_+$ .

For  $\alpha \in R$ , a **root subgroup** is a subgroup

$$U_\alpha = I_{2n} + aE_\alpha, \quad a \in \mathbb{C}$$

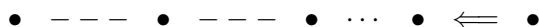
where  $E_\alpha$  is an eigenvector with  $T$ -eigenvalue  $\alpha$ .

**Proposition/Definition:** The standard Borel  $B \subset \mathrm{Sp}_{2n}$  is the subgroup generated by  $T$  and  $U_\alpha$ ,  $\alpha \in R_+$ . It is a Borel subgroup. It is the intersection  $\mathrm{Sp}_{2n} \cap B_{2n}$  of  $\mathrm{Sp}_{2n}$  with the upper triangular matrices in  $\mathrm{GL}_{2n}$ .

We define the **fundamental weights**

$$\omega_1 = \chi_1, \omega_2 = \chi_1 + \chi_2, \dots, \omega_{n-1} = \chi_1 + \dots + \chi_{n-1}, \omega_n = \chi_1 + \dots + \chi_n.$$

The corresponding **Dynkin diagram** is



This signifies that the angle between  $\alpha_i$  and  $\alpha_{i+1}$  is  $2\pi/3$ , for  $i = 1, \dots, n-1$ ; the angle between  $\alpha_{n-1}$  and  $\alpha_n$  is  $3\pi/4$ ; and all other angles between simple roots is  $\pi/2$ . All of these angles are measured in the Euclidean space  $\sum \mathbb{R}\chi_i$ , with respect to the standard inner (= dot) product.

The set of roots  $R$  is a root system: the reflections defined by each simple root  $\alpha_i$  are

$$s_i : v \mapsto v - 2 \frac{(v, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i.$$

The group generated by the  $s_i$  in  $GL(X^*(T))$  is isomorphic to the Weyl group.

**Remark:** Observe, that  $\text{rank } \mathbb{Z}R = n = \dim T$ , in contrast to the  $GL_n$  case: this is because  $Sp_{2n}$  is **(semi)simple**.

**Representation Theory** The main Theorem is

**Theorem:**

1. Let  $W$  be a finite dimensional irreducible representation of  $Sp_{2n}$ . Then, there is a unique line  $L \subset W$  that is  $B$ -invariant, and on which  $T$  acts by  $\lambda \in X^*(T)$ . Moreover, if  $\mu \in \Lambda(W)$  is a weight then  $\lambda \geq \mu$ ; we call  $\lambda$  **the highest weight in  $W$** . The weight  $\lambda = \sum_i a_i \chi_i$  satisfies  $a_1 \geq \dots \geq a_n \geq 0$ .
2. To any weight  $\lambda = \sum_i a_i \chi_i$ ,  $a_1 \geq \dots \geq a_n \geq 0$ , there is an irreducible representation of  $Sp_{2n}$  with  $\lambda$  as its highest weight.

We will focus on the example  $Sp_4$ .