UC Berkeley Summer Undergraduate Research Program 2015
July 8 Lecture

This lecture is intended to tie up some (potential) loose ends that we have encountered on the road during the past couple of weeks. We will focus on root subgroups and some connections with Lie algebras.

Root Subgroups Fix $G = \mathrm{GL}_n$ and let $T \subset B \subset G$ be the diagonal matrices sitting inside the upper triangular matrices.

Let $R = \{\chi_i - \chi_j \mid i \neq j\}$ be the set of roots, $R_+ = \{\chi_i - \chi_j \mid i < j\}$ the **positive roots (relative to B)**. For $\alpha \in R$, we define the unipotent subgroup

$$U_\alpha = \{I_n + aE_{ij} \mid a \in \mathbb{C}\}.$$

This is a subgroup of G isomorphic to $\mathbb{C}$ - there exists a group isomorphism $\mathbb{C} \to U_\alpha$ that is also a morphism of algebraic varieties. Let's write

$$u_\alpha(a) = I_n + aE_{ij} \in U_\alpha.$$

Exercise: $u_\alpha(a) = \exp(aE_{ij})$**.**

T normalises U_α and

$$t(I_n + aE_{ij})t^{-1} = I_n + \alpha(t)aE_{ij} \implies tu(a)t^{-1} = u(\alpha(t)a).$$

Hence, identifying U_α with $\mathbb{C}$, we find that conjugation by T induces an action of T on $\mathbb{C}$ via the root α: $t \cdot a = \alpha(t)a$.

Now, suppose that W is a representation of G and $w \in W(\mu)$ is a (nonzero) weight vector of weight μ: this means that $tw = \mu(t)w$, for $t \in T$. Fix $\alpha \in R$, some root of G.

Important Proposition: $u_\alpha(a)w = \sum_i a^i w_i$, where $w_0 = w$, $w_i \in W(\mu + i\alpha)$, and the sum is finite.

Proof: Restrict the representation of G on W to $U_\alpha \cong \mathbb{C}$: this amounts to giving a morphism of varieties that is also a group homomorphism

$$F : \mathbb{C} \cong U_\alpha \to G \to \mathrm{GL}(W).$$

Such a morphism of affine varieties is determined by the induced morphism on coordinate rings

$$F^* : A_{\mathrm{GL}(W)} \to A_\mathbb{C} = \mathbb{C}[t].$$

Fix a weight basis $\mathcal{B} = (w, u_2, \dots, u_m)$ of W (relative to T), such that w is a basis vector. Then, we get $\mathrm{GL}(W) \cong \mathrm{GL}_m$, and $A_{\mathrm{GL}(W)} \cong \mathbb{C}[x_{ij}, \det^{-1}]$, so that any $\mathbb{C}$-algebra morphism is determined by $F^*(x_{ij}) = f_{ij}(t) \in \mathbb{C}[t]$: we have $[u_\alpha(a)]_\mathcal{B} = [f_{ij}(a)]$ with respect to this basis. Hence, we have

$$u_\alpha(a)w = \sum_i f_{i1}(a)u_i = \sum_k a^k v_k, \quad v_k \in W.$$

Note that the v_k's above are not weight vector a priori. We will show that, in fact, they are! Let $t \in T$. Then,

$$t(u_\alpha(a)w) = \sum_k a^k tv_k.$$

Also, we have

$$\begin{aligned} t(u_\alpha(a)w) &= \left(tu_\alpha(a)t^{-1}\right)(tw), \\ &= u_\alpha(\alpha(t)a)\mu(t)w, \\ &= \mu(t)\sum_i f_{i1}(\alpha(t)a)u_i, \\ &= \mu(t)\sum_k \alpha(t)^k a^k v_k. \end{aligned}$$

Hence, for every $a \in \mathbb{C}$, $t \in T$,

$$\mu(t)\sum_k \alpha(t)^k a^k v_k = \sum_k a^k t v_k \implies \sum_k a^k(\mu(t)\alpha(t)^k v_k - tv_k) = 0 \in W$$

Remark: a nicer proof of the following discussion was provided by Mark during lecture; I'll keep what I had for prosperity... Suppose that $\sum_k a^k w_k$ is a polynomial with coefficients in W (ie $w_k \in W$). Fix any basis of W, $(x_1, \dots, x_m)$. Then, we have, for every $a \in \mathbb{C}$,

$$0 = \sum_k a^k w_k = \sum_{j=1}^m f_j(a)u_j \implies f_j(a) = 0 \implies f_j = 0 \in \mathbb{C}[t]$$

Note that the coefficients of $f_j = \sum_{i=0}^{n_j} c_{ij}T^i$ are such that $w_k = \sum_{j=1}^m c_{kj}u_j$, so that we must have $w_k = 0 \in W$, for all k.
In fact, we have then that

$$\mu(t)\alpha(t)^k v_k = tv_k, \quad \text{for all } t \in T, \text{ and each } k \text{ such that } v_k \neq 0.$$

Otherwise, let

$$J = \{k \mid \mu(t)\alpha(t)^k v_k \neq tv_k, \text{ for some } t \in T\} \neq \varnothing.$$

Then, for $k \in J$ we observe that

$$v_k = \sum_j c_j u_j,$$

where c_j is the coefficient of t^k in $f_{j1}(t)$, and

$$tv_k = \sum_j c_j t u_j = \sum_j c_j \mu_j(t) u_j \quad (u_j \in W(\mu_j))$$

Moreover,

$$\mu(t)\alpha(t)^k v_k = \sum_j c_j \mu(t)\alpha(t)^k u_j,$$

so that we must have $\mu(t_0)\alpha(t_0)^k \neq \mu_{j_0}(t_0)$, for some $t_0 \in T$, some j_0 such that $c_{j_0} \neq 0$, and any $k \in J$.

Hence, for every $a \in \mathbb{C}$,

$$W \ni 0 = \sum_{k\in J} a^k \sum_j c_j(\mu(t_0)\alpha(t_0)^k - \mu_j(t_0))u_j$$

$$\implies 0 = \sum_{k\in J} a^k c_j(\mu(t_0)\alpha(t_0)^k - \mu_j(t_0)), \quad \text{for every } j,\ a \in \mathbb{C}.$$

In particular, for every $a \in \mathbb{C}$,

$$0 = c_{j_0} \sum_{k \in J} a^k (\mu(t_0)\alpha(t_0)^k - \mu_{j_0}(t_0)) \implies c_{j_0} = 0, \quad \text{which is absurd.}$$

The result follows! (*Can you think of a 'nicer' proof? I feel like that there is a more elegant way to prove this, without the juggling of quantifiers.*)

Why do we care? We have just shown that the root subgroups **move weight vectors in the direction of** α. Namely, if $w \in W(\mu)$ then

$$u_\alpha(a)w \in \sum_{i \geq 0} W(\mu + i\alpha).$$

Recall that we have defined a partial order on weights: for $\lambda, \mu \in X^*(T)$ we define

$$\lambda \geq \mu \iff \lambda - \mu = \sum_i n_i \alpha_i, \quad n_i \in \mathbb{Z}_{\geq 0}.$$

In particular, if $\alpha \in R_+$ so that $\alpha = \sum_j m_j \alpha_j$, with $m_j \in \mathbb{Z}_{\geq 0}$, then for $0 \neq w \in W(\mu)$,

$$U_\alpha w \in w + \sum_{\lambda > \mu} W(\lambda).$$

Similarly, if $\alpha \in -R^+$, and $0 \neq w \in W(\mu)$, then

$$U_\alpha w \in w + \sum_{\lambda < \mu} W(\lambda).$$

Draw some A_2 examples: fundamental represenations, Ad, decompose some symmetric products

The above results tell us more. If we choose a weight basis $\mathcal{B} = (w_1, \ldots, w_m)$, $w_i \in W(\mu_i)$, for a representation W and order the basis elements so that they (linearly) refine the above partial ordering - this means $\mu_i \geq \mu_j \implies i < j$ - **then the matrix of an element** $u \in U_\alpha$**, for any** $\alpha \in \mathbb{R}^+$**, is upper triangular, unipotent**: we must always have

$$[u_\alpha(a)]_\mathcal{B} = \begin{bmatrix} 1 & a^* \\ 0 & 1 \end{bmatrix},$$

with powers of a appearing above the diagonal. Similarly, **the matrix of an element** $u \in U_\alpha$**, for any** $\alpha \in -R^+$**, is lower triangular, unipotent**.

For example, fix the basis $(e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34})$ of $\bigwedge^2 \mathbb{C}^4$. We have the set of weights of this representation

$$\Lambda(\bigwedge^2 \mathbb{C}^4) = \{\chi_i + \chi_j \mid i < j\}.$$

Note that

$$\mu_{12} = \chi_1 + \chi_2 = \omega_2,$$

$$\mu_{13} = \chi_1 + \chi_3 = \omega_2 - \alpha_2,$$

$$\mu_{14} = \chi_1 + \chi_4 = \omega_2 - \alpha_2 - \alpha_3,$$

$$\mu_{23} = \chi_2 + \chi_3 = \omega_2 - \alpha_1 - \alpha_2,$$

$$\mu_{24} = \chi_2 + \chi_4 = \omega_2 - \alpha_1 - \alpha_2 - \alpha_3,$$
$$\mu_{34} = \chi_3 + \chi_4 = \omega_2 - \alpha_1 - 2\alpha_2 - \alpha_3.$$

The poset $(\Lambda(\bigwedge^2 \mathbb{C}^4), \leq)$ is

$$\begin{array}{ccccc} & & \mu_{12} & & \\ & & \uparrow & & \\ & & \mu_{13} & & \\ & \nearrow & & \nwarrow & \\ \mu_{14} & & & & \mu_{23} \\ & \nwarrow & & \nearrow & \\ & & \mu_{24} & & \\ & & \uparrow & & \\ & & \mu_{34} & & \end{array}$$

We see that, for example,

$$[u_{\alpha_3}(a)] = \begin{bmatrix} 1 & & & & & \\ & 1 & a & & & \\ & & 1 & & & \\ & & & 1 & a & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

$$[u_{\alpha_1+\alpha_2}(a)] = \begin{bmatrix} 1 & & & -a & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

The Weyl Group Suppose that $w \in W(\mu)$ is a weight vector of weight μ (with respect to T). Recall that the Weyl group of G (relative to T) is defined to be $W = N_G(T)/T \cong S_n$. For $\sigma \in W$, let $x_\sigma \in N_G(T)$ be a representative. Then,

1. W acts on $X^*(T)$ as follows: for $\sigma \in W$, $\alpha \in X^*(T)$,

$$\sigma \cdot \alpha \ : \ t \mapsto \alpha(x_\sigma^{-1} t x_\sigma).$$

2. if $\alpha \in R$ and $s_\alpha \in W$ is the reflection in the hyperplane orthogonal to α, then

$$t(x_{s_\alpha} w) = x_{s_\alpha}(x_{s_\alpha}^{-1} t x_{s_\alpha}) w = x_{s_\alpha} \mu(x_{s_\alpha}^{-1} t x_{s_\alpha}) w = (\sigma \cdot \mu)(t) x_{s_\alpha} w.$$
$$\implies x_{s_\alpha} w \in W(\sigma \cdot \mu).$$

Hence, W **permutes the weight spaces**; in particular, for any $\sigma \in W$, $\dim W(\mu) = \dim W(\sigma \cdot \mu)$.

The Lie algebra The **Lie algebra** of G is defined as $\mathfrak{g} \overset{def}{=} T_e G$, the tangent space at the identity. For us, $\mathfrak{g} = M_n$. We can think of tangent vectors as **derivatives of curves passing through** $e \in G$ **at** $t = 0$. For example,

$$\gamma_\alpha : [0,1] \to U_\alpha \ ; \ t \mapsto u_\alpha(t),$$

and we have, assuming $\alpha = \chi_i - \chi_j$, $\gamma'(0) = E_{ij}$.

Given a rational representation $\rho : G \to \mathrm{GL}(W)$, we get an induced map of tangent spaces:

$$d\rho : T_eG \to T_{\mathrm{id}}\mathrm{GL}(W) \; ; \; \gamma'(0) \mapsto (\rho \circ \gamma)'(0)$$

For example, if we choose a weight basis $\mathcal{B}$ for $W = \bigwedge^2 \mathbb{C}^4$ refining $\leq$, as above, then we can identify $\mathrm{GL}(W)$ with GL_6, and for any $u \in U_\alpha$ we have seen that the i^{th} column of $[u_\alpha(a)]_{\mathcal{B}}$ is of the form

$$\begin{bmatrix} c_1 a^{i-1} \\ c_2 a^{i-2} \\ \vdots \\ c_{i-1} a \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad c_i \in \{0, 1\}$$

Hence, if we restrict ρ to U_α then we find that $(\rho \circ \gamma_\alpha)'(0)$ is a matrix with 0s everywhere away from the main superdiagonal, where there may appear 1s.

For example, in the above examples, we see that

$$d\rho(\gamma'_{\alpha_3}(0)) = \begin{bmatrix} 0 & & & & & \\ & 0 & 1 & & & \\ & & 0 & & & \\ & & & 0 & 1 & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}$$

$$d\rho(\gamma'_{\alpha_1+\alpha_2}(0)) = \begin{bmatrix} 0 & & & -1 & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}.$$

Observing that

$$\gamma'_{\alpha_3}(0) = E_{34} \in M_4, \quad \gamma'_{\alpha_1+\alpha_2}(0) = E_{13} \in M_4,$$

we can think of the above matrices as showing us how the **root operators** E_{34}, E_{13} act on the **induced Lie algebra representation** (whatever this means). For example, E_{34} does nothing to any weight space $W(\mu_{ij})$, except for $W(\mu_{14})$ and $W(\mu_{24})$, and

$$E_{34} \cdot W(\mu_{14}) \subset W(\mu_{13}), \quad E_{34} \cdot W(\mu_{24}) \subset W(\mu_{23}).$$

Observe that we have $\mu_{13} = \mu_{14} + \alpha_3, \mu_{23} = \mu_{24} + \alpha_3$; this is no accident.

Fact: If $\alpha \in R$, then $E_\alpha \cdot W(\mu) \subset W(\mu + \alpha)$.