UC Berkeley Summer Undergraduate Research Program 2015
July 7 Lecture

Today we will see how to use representation theory to produce embeddings of the flag variety (and partial flag varieties) inside some projective space $\mathbb{P}^M$.

Fix $G = \mathrm{GL}_n$ and let $T \subset B \subset G$ be the diagonal matrices sitting inside the upper triangular matrices.

Recall that the irreducible (finite dimensional) representations of G are indexed by partitions $\lambda : \lambda_1 \geq \ldots \geq \lambda_n \geq 0$. In general, it's not so easy to determine V_λ! However, we have seen that the representations $\bigwedge^k \mathbb{C}^n$ are irreducible with highest weight

$$\omega_i \overset{def}{=} \chi_1 + \ldots + \chi_k :$$

we will now write $V_{\omega_k} = \bigwedge^k \mathbb{C}^n$. These weights are called the **fundamental weights**. Observe that if we define the 1-parameter subgroup

$$\alpha_i^\vee : t \mapsto \mathrm{diag}(1, \ldots, t, t^{-1}, 1, \ldots, 1), \quad i = 1, \ldots, n-1,$$

then $\omega_k \circ \alpha_i^\vee(t) = \delta_{ik}$.

Fix $1 \leq k \leq n$. We know that the line $\mathbb{C}e_1 \wedge \cdots \wedge e_k \subset V_{\omega_k}$ is B-invariant: this is the same as saying that the point $[e_1 \wedge \cdots \wedge e_k] \in \mathbb{P}(V_{\omega_k}) \cong \mathbb{P}^{\binom{n}{k}-1}$ is fixed by the action of B on $\mathbb{P}(V_{\omega_k})$ (why is this well-defined action?). However, via the induced action of G on $\mathbb{P}(V_{\omega_k})$, it can be seen that

$$\mathrm{Stab}_G([e_1 \wedge \cdots \wedge e_k]) = \left\{ \begin{bmatrix} *_k & * \\ 0 & *_{n-k} \end{bmatrix} \right\},$$

a **parabolic subgroup of** G. Denote this parabolic subgroup P_k. It is a fact that P_k **is a maximal parabolic subgroup**.

Hence, we get an identification

$$G/P_k \cong G \cdot [e_1 \wedge \cdots \wedge e_k] \subset \mathbb{P}(V_{\omega_k}).$$

Recall the weight basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n\}$ of V_{ω_k}. For a subset $J \subset \{1, \ldots, n\}$ of cardinality k, denote the element of the dual basis $x_J \in V_{\omega_k}^*$. Hence, we can consider

$$\mathbb{P}(V_{\omega_k}) \longleftrightarrow \mathbb{P}^{\binom{n}{k}-1} \; ; \; [v] \longleftrightarrow [x_J(v)].$$

For example, when $n = 4$ we have $V_{\omega_2} = \bigwedge^2 \mathbb{C}^4$ and we get an isomorphism

$$\mathbb{P}(V_{\omega_2}) \longleftrightarrow \mathbb{P}^5 \; ; \; [v] \longleftrightarrow [x_{12}(v) : x_{13}(v) : x_{14}(v) : x_{23}(v) : x_{24}(v) : x_{34}(v)]$$

In particular, we identify

$$[e_1 \wedge e_2] \leftrightarrow [1:0:0:0:0:0],$$

$$[e_1 \wedge e_3] \leftrightarrow [0:1:0:0:0:0] \quad etc.$$

We have already seen how to identify G/P_k with $\mathrm{Gr}(k, n)$ (although in a slightly different way): we can think of a k-dimensional subspace $U \subset \mathbb{C}^n$ as the **column-span** of the first k columns of an invertible matrix $A = [a_1 \cdots a_n]$, so $U = \mathrm{col}(a_1, \ldots, a_k)$.

Given a matrix A we will write $\mathrm{col}_k(A)$ for the column span of the first k columns.

If we define an action of G on $\mathrm{Gr}(k,n)$ by

$$g \cdot U \overset{def}{=} \mathrm{col}_k(gA), \quad \text{where } U = \mathrm{col}_k(A),$$

the stabiliser of $U = \mathrm{span}(e_1, \dots, e_k)$ is P_k. **Note that this action is well-defined**. Hence, as the action of G is transitive we obtain $\mathrm{Gr}(k,n) \equiv G/P_k$. We can realise this equivalence of G-sets through the map

$$G \to \mathrm{Gr}(k,n)\ ;\ g \mapsto g \cdot \mathrm{span}(e_1, \dots, e_k).$$

Namely, we can factor

$$\begin{array}{ccc} G & \to & \mathrm{Gr}(k,n) \\ \downarrow & \nearrow & \\ G/P_k & & \end{array}$$

The inverse of the dashed arrow is the map

$$\mathrm{Gr}(k,n) \to G/P_k\ ;\ U = \mathrm{col}_k(A) \mapsto AP_k,$$

This is well-defined.

Remark: we have had to be careful here so that we get an identification of the **left cosets** gP_k **with k-dimensional subspaces of $\mathbb{C}^n$**.

Given a subset $J = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ of cardinality k, and an $n \times n$ matrix A, we can form the $k \times k$ minor $\Delta_J(A)$ with columns $1, \dots, k$ and rows $i_1, \dots, i_k$. As we saw in the first week for $n = 3, k = 2$, we can **coordinatise** $\mathrm{Gr}(k,n)$ using these **Plucker coordinates**

$$\mathrm{Gr}(k,n) \to \mathbb{P}^{\binom{n}{k}-1}\ ;\ U = \mathrm{col}_k(A) \mapsto [\Delta_J(A)].$$

In particular, the **Plucker embedding** above is injective. (*We can even give a description of the image of this map as a projective variety*)

Let's see how these coordinates arrive in the representation-theoretic framework: denote the natural pairing between a vector space W and its dual W^* by

$$\langle , \rangle : W^* \times W \to \mathbb{C}\ ;\ (\alpha, v) \mapsto \langle \alpha, v \rangle.$$

Consider the following functions on G:

$$\xi_J : g \mapsto \langle x_J, g(e_1 \wedge \cdots \wedge e_k) \rangle.$$

If we consider the represenation of G defined by V_{ω_k}, then $\xi_J(g)$ is the $(J,1)$ entry of the matrix of the action of g on V_{ω_k}, with respect to the weight basis.

Since V_{ω_k} is a rational representation, the $\xi_J \in A_G$ are regular functions on G, and are P_k-invariant in the sense that $\xi_J(gp) = \xi_J(g)$, for any $p \in P_k$. Thus, they descend to the quotient G/P_k, so that we obtain functions $\xi_J : G/P_k \to \mathbb{C}$.

If the columns of g are $g_1, \dots, g_n$ then we have

$$\xi_J(g) = \langle x_J, g_1 \wedge \cdots \wedge g_k \rangle = \Delta_J(g).$$

One way to check this without performing a calculation is to use the following characterisation of the determinant: there exists a unique function $f : \mathbb{C}^k \times \cdots \mathbb{C}^k \to \mathbb{C}$ (k copies of $\mathbb{C}^k$, though of as row vectors) such that

1. f is alternating, $f(u_1, \dots, u_i, \dots, u_j, \dots, u_k) = -f(u_1, \dots, u_j, \dots, u_i, \dots, u_k)$,
2. f is multilinear, ie, it is linear in each of its arguments,
3. $f(e_1, \dots, e_k) = 1$.

In summary: **the Plucker embedding of the Grassmannian has an intrinsic characterisation (without reference to minors). The Grassmannian is the image in $\mathbb{P}(V_{\omega_k})$ of the morphism**

$$G/P_k \to \mathbb{P}(V_{\omega_k})\ ;\ gP_k \mapsto [gu],$$

where $u \in V_{\omega_k}$ is any highest weight vector (so that $tu = \omega_k(t)u$).

We observe that the homogenous coordinate ring of G/P_k under this embedding is isomorphic to the **Plucker algebra** $\mathbb{C}[\Delta_J]$.

Now, let $\lambda = \sum_i \lambda_i \chi_i$ be a weight such that $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ and $u \in V_\lambda$ a highest weight vector (so that $tu = \lambda(t)u$, for $t \in T$). If we set $a_i = \lambda_i - \lambda_{i+1} \geq 0$, $\mathrm{a}_n = \lambda_n$, then we can write

$$\lambda = a_1\omega_1 + a_2\omega_2 + \dots + a_n\omega_n.$$

The action of G on V_λ is linear so that it descends to an action on $\mathbb{P}(V_\lambda)$

$$g \cdot [v] \overset{def}{=} [gv].$$

Recall that $[v]$ is B-invariant, so that $B \subset \mathrm{Stab}_G([v]]$ and $P_\lambda \overset{def}{=} \mathrm{Stab}_G([v])$ is a parabolic subgroup of G. Hence, G/P_λ is a projective variety and we have determined an embedding

$$G/P_\lambda \to \mathbb{P}(V_\lambda)\ ;\ gP_\lambda \mapsto [gv].$$

Fact: if $\lambda = a_1\omega_1 + \dots + a_n\omega_n$ and $I = \{i \mid a_i \neq 0\} = \{d_1 < \dots < d_k\}$ then $\mathrm{Stab}_G([v])$ is a parabolic subgroup consisting of block upper triangular matrices of size $d_1, d_2 - d_1, \dots, d_k - d_{k-1}$. Moreover, G/P_λ is isomorphic to the multistep Grassmannian $\mathrm{Gr}(d_1, d_2, \dots, d_k)$. In particular, whenever $I = \{1, \dots, n\}$ we obtain an embedding of the full flag variety G/B in some projective space.