UC Berkeley Summer Undergraduate Research Program 2015
July 2 Lecture

We introduced some of the basic structure theory for the general linear group $G = \mathrm{GL}(V)$, where V is a finite dimensional $\mathbb{C}$-vector space. We introduced several classes of subgroups of G:

$$T, U \subset B \subset P \subset G$$

,

and introduced some representations for G, the exterior powers $\bigwedge^k V$. Note that these representations have the property that there exists a simultaneous eigenbasis for T; in fact, this is always the case. We will discuss the internal structure of a representation of G today by investigating what the representation looks like as a representation of T, U and B (and B_- (the opposite standard Borel)).

Highest weight theory and roots: Fix a maximal torus T in GL_n - we may as well assume that this is the standard torus, since all maximal tori are conjugate. The **group of characters of** T (also called the **weight lattice**) is the set of group homomorphisms

$$X^*(T) \stackrel{def}{=} \mathrm{Hom}_{\textbf{alg. gp}}(T, \mathbb{C}^\times),$$

that are also morphisms of algebraic varieties. We have already seen that $X^*(T) \cong \mathbb{Z}^n$: in fact, we observe that the projections $x_{jj} : T \to \mathbb{C}^*$ provide a basis for the free abelian group $X^*(T)$. This is the same as saying that $A_T \cong \mathbb{C}[x_{11}^{\pm}, \dots, x_{nn}^{\pm}]$. We will denote these (special) characters $\chi_j \stackrel{def}{=} x_{jj}$.

Fact: for any representation (W, ρ)**, every element** $\rho(t) \in \mathrm{GL}(W)$**,** $t \in T$**, is diagonalisable.**

Hence, since $\rho(T) \subset \mathrm{GL}(W)$ is a commutative subgroup consisting of diagonalisable elements, we can find a simultaneous eigenbasis $\{w_1, \dots, w_m\}$ of W. This means that

$$t \cdot w_i = \alpha_i(t) w_i,$$

where $\alpha_i : T \to \mathbb{C}^\times$ takes $t \in T$ to the eigenvalue $\alpha_i(t)$ of the linear operator $\rho(t)$ associated with the eigenvector w_i. Furthermore, since

$$t \cdot (t' \cdot w_i) = (tt') \cdot w_i,$$

we have $\alpha_i \in X^*(T)$. The set $\Lambda(W) \stackrel{def}{=} \{\alpha_i\} \subset X^*(T)$ is called **the set of weights of** W; elements of the eigenbasis are called **weight vectors**.

1. Let $\mathbb{C}^n$ be the defining representation of GL_n, $(e_1, \dots, e_n)$ the standard basis. Then, in any of the wedge products $W_{(k)} = \bigwedge^k \mathbb{C}^n$, the basis of 'standard' wedges $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq n\}$ is a simultaneous eigenbasis; the set of weights is

$$\Lambda(W_{(k)} = \left\{ \sum_{j=1}^{k} \chi_{i_j} \mid \{i_1, \dots, i_k\} \subset [n] \right\}$$

   since

$$t \cdot e_{i_1} \wedge \cdots \wedge e_{i_k} = t_{i_1} \cdots t_{i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$$
$$= \chi_{i_1}(t) \cdots \chi_{i_k}(t) e_{i_1} \wedge \cdots \wedge e_{i_k} = (\chi_{i_1} + \dots + \chi_{i_k})(t) e_{i_1} \wedge \cdots \wedge e_{i_k}$$

   for any $t = \mathrm{diag}(t_1, \dots, t_n)$. Note that we are using additive notation for the (abelian) group of characters.

2. (**Important Example**) Consider the **adjoint representation**

$$\text{Ad} : G \to \text{GL}(M_n) \; ; \; g \mapsto \left[X \mapsto gXg^{-1}\right]$$

You can check that this is a well-defined representation: it's linear, and a morphism of algebraic varieties.

The matrices E_{ij}, with 0s everywhere except for a 1 in the ij-entry, are a simultaneous eigenbasis for T. For example, we have

$$t \cdot E_{ij} = tE_{ij}t^{-1} = t_i t_j^{-1} E_{ij} = (\chi_i - \chi_j)(t)E_{ij}, \quad t = \text{diag}(t_1, \dots, t_n) \in T$$

This representation is <u>fundamental</u>: it can be shown that we can identify M_n with the **Lie algebra $\mathfrak{gl}_n$ of** G, the tangent space of G at the identity. This representation can be thought of as the derivative of the action of G on itself by conjugation. It's nice that it turns out be conjugation again!

Due to their central role in the representation theory of G, we elements in $\Lambda(M_n)$ the **roots of** G, and denote $R = \Lambda(M_n)$.

The weights $\alpha_i \overset{def}{=} \chi_i - \chi_{i+1} \in X^*(T)$, $i = 1, \dots, n-1$, which we'll call the **simple roots of** G **(relative to** B**)**, play an important role: observe that

$$\chi_i - \chi_j = \alpha_i + \dots + \alpha_{j-1}, \quad \text{whenever } i < j,$$

and

$$\chi_i - \chi_j = -(\alpha_i + \dots + \alpha_{j-1}), \quad \text{whenever } j < i.$$

In particular, **every root of G can be written as an integral sum $\sum_{\alpha \in R} n_\alpha \alpha$, with every n_α either nonnegative, or nonpositive**.

The lattice generated by R inside $X^*(T)$, denoted Σ, is called the **root lattice of** G: observe that the root lattice is not equal to the character lattice. In fact, Σ sits inside the lattice $\{\sum_i a_i \chi_i \mid \sum_i a_i = 0\}$.

Fact: consider the $\mathbb{R}$-span $\Sigma_{\mathbb{R}}$ of Σ, sitting inside the $\mathbb{R}$-span of X^*. The roots $R \subset \Sigma_{\mathbb{R}}$ are a (reduced) root system:

(a) $\text{span}_{\mathbb{R}}(R) = \Sigma_{\mathbb{R}}$,

(b) if $\alpha \in R$, and $c\alpha \in \mathbb{R}$, then $c = \pm 1$,

(c) R is preserved under the linear operators

$$s_\alpha : v \mapsto v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha, \quad \alpha \in R$$

(d) the real numbers $2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ are integers.

For $G = \text{GL}_n$, the numbers $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \{0, \pm 1\}$. Moreover, s_α is the reflection through the hyperplane orthogonal to α in $\Sigma_{\mathbb{R}}$.

Given $\alpha = \chi_i - \chi_j \in R$, $i \neq j$, definte the **root subgroup** $U_\alpha \subset G$ to be the unipotent group

$$U_\alpha = I_n + \mathbb{C}E_{ij}.$$

The root subgroups are subgroups isomorphic to $\mathbb{C}$, normalised by T so that

$$t(I_n + cE_{ij})t^{-1} = I_n + \alpha(t)cE_{ij}.$$

The representation theory of G can be described as follows:

1. every irreducible representation V of G admits a unique line $L = \text{span}(v) \subset V$, with v a weight vector of weight λ, with $\lambda = \sum_i \lambda_i \chi_i$, and where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ is a nondecreasing sequence of integers; we call λ the highest weight of V. Moreover, to any such sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ there is an irreducible representation with highest weight λ. We denote this irreducible representation by V_λ

2. the weights appearing in an irreducible representation V_λ are of the form

$$\mu = \lambda - \sum_i n_i \alpha_i, \quad n_i \in \mathbb{Z}_{\geq 0}.$$