UC Berkeley Summer Undergraduate Research Program 2015
July 1 Lecture

We are going to introduce some of the basic structure of the **general linear group** $\mathrm{GL}(V)$, where V is a finite dimensional $\mathbb{C}$-vector space. (*And properties of the symplectic group* $\mathrm{Sp}(V)$*, if there's time.*)

References for this material can be found:

1. Goodman-Wallach 'Symmetries, Representations, Invariants' ('freely' available online)

2. Any book on linear algebraic groups eg. Humphreys 'Linear Algebraic Groups; Springer 'Linear Algebraic Groups'; Borel 'Linear Algebraic Groups' (2nd Edition) (*these are in increasing order of difficulty, in my opinion.*)

3. Online notes by: Dave Anderson 'Linear Algebraic Groups: a crash course'; Kleschev 'Lectures on Algebraic Groups'; Szamuely 'Lectures on Algebraic Groups'... lots of references.

Be warned: you can spend a whole semester/year studying the structure of algebraic groups. We will see the main ideas illuminated for $\mathrm{GL}(V)$ (and $\mathrm{Sp}(V)$).

Basic structure results: Fix a basis of V; this gives $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$. Inside $\mathrm{GL}_n(\mathbb{C})$ there are some special (and well-known) subgroups:

$$T = \{\text{diagonal}\} \subset B = \{\text{upper-triangular}\} \subset \mathrm{GL}_n.$$

We will call T the **standard maximal torus** and B the **standard (upper) Borel**.

We can characterise these subgroups in an intrinsic manner:

- first, observe that all of these subgroups are closed subgroups (in the Zariski topology); hence, they are subvarieties of the (affine) algebraic variety GL_n.

- T is commutative and consists of diagonalisable elements; it is maximal with respect to this property. Indeed, if $S \supset T$ and S is commutative and contains diagonalisable elements then we can find a simultaneous eigenbasis for $\mathbb{C}^n$, containing $(e_1, \dots, e_n)$. Hence, this eigenbasis must be equal to the standard basis and $S = T$.

- B admits a simultaneous eigenvector, namely e_1; or, we can say that B fixes a point (=line) in $\mathbb{P}(V)$. B also has the property that it is **solvable**: the descending chain terminates in $\{e\}$

  $$B = B_0 \supset B_1 \stackrel{def}{=} (B_0, B_0) \supset B_2 \stackrel{def}{=} (B_1, B_1) \supset \cdots \supset B_i \stackrel{def}{=} (B_{i-1}, B_{i-1}) \supset \cdots$$

  where $(H, H) = \{ghg^{-1}h^{-1} \mid g, h \in H\}$ is the commutator of H. This is an interesting (but tedious) exercise.

We can now define certain types of subgroups in GL_n (generalising the above fixed subgroups):

- a **maximal torus** $S \subset \mathrm{GL}_n$ is a commutative, connected, closed subgroup containing diagonalisable elements, that is contained in no other such group in GL_n,

- a **Borel subgroup** $B' \subset \mathrm{GL}_n$ is a maximal solvable, connected, closed subgroup,

- a **parabolic subgroup** $P \subset \mathrm{GL}_n$ is a connected, closed subgroup containing some Borel subgroup B'; a parabolic containing the standard Borel is called a **standard parabolic**.
- a **unipotent subgroup** $U \subset \mathrm{GL}_n$ is a closed subgroup containing **unipotent elements**: these are elements $u \in \mathrm{GL}_n$ that have characteristic polynomial $\chi(t) = \pm(t-1)^n$,
- the **Weyl group of** GLn is defined to be $W \stackrel{def}{=} N_G(T)/T$: you can check that this is isomorphic to S_n.

Here are some basic (but nontrivial!) facts:

1. All Borel subgroups are conjugate.
2. $N_G(B) = B$, for any Borel B; $N_G(P) = P$, for any parabolic P.
3. The coset space G/H can be given the structure of a projective variety (so that the natural quotient map $G \to G/H$ is a morphism of algebraic varieties) if and only if H is parabolic.
4. (**Lie-Kolchin**) Any Borel subgroup fixes a unique flag $V_\bullet$.
5. (**Bruhat decomposition**) $\mathrm{GL}_n = \bigsqcup_{w \in S_n} U_- wB$,
6. All maximal tori are conjugate (so they have the same dimension).
7. The union of all maximal tori is dense in GL_n; ie, the set of al diagonalisable elements in GL_n is dense.
8. Let B be a Borel. Then, B admits a unique maximal normal unipotent subgroup (called the **unipotent radical of** B; the quotient B/U is isomorphic to some maximal torus in B. Hence, we have $B \cong T \times U$ as a variety (but a semidirect product of groups).
9. Any standard parabolic subgroup $P \subset \mathrm{GL}_n$ is of the form

$$P = \left\{ \begin{bmatrix} *_{d_1} & * & \cdots & * \\ 0 & *_{d_2} & \cdots & * \\ 0 & 0 & \ddots & \vdots \\ & & & *_{d_r} \end{bmatrix} \right\}$$

   where $d_1 + d_2 + ... + d_r = n$ and $*_{d_j}$ represents an invertible $d_j \times d_j$ block.

By (1), (2), (3) we see that GL_n/B is a projective variety, called **the flag variety of** GL_n, and

$$\mathrm{GL}_n/B = \{\text{Borel subgroups of } \mathrm{GL}_n\} \leftrightarrow \mathrm{Fl}_n = \{\text{flags in } \mathbb{C}^n\}.$$

Moreover, (5) tells us that there are finitely U_- orbits on the flag variety, indexed by elements of the symmetric group. This is precisely the cell decomposition we saw last week.

Some representation theory: We are now going to introduce some of the representation theory of GL_n; a good reference is Goodman-Wallach.

A **(rational) representation of** GL_n is a morphism of algebraic varieties $\rho : \mathrm{GL}_n \to \mathrm{GL}(W)$, with W a finite dimensional $\mathbb{C}$-vector space, that is also a group homomorphism. If we fix a basis of W then this becomes a map

$$\rho : \mathrm{GL}_n \to \mathrm{GL}_m \; ; \; g \mapsto \rho(g) = [\rho_{ij}(g)],$$

where the **matrix coefficients** $\rho_{ij}(g)$ are elements in the coordinate ring of GL_n; hence, they are polynomials in x_{ij}, $1 \leq i, j \leq n$, and $\det^{-1}$. We will also sometimes call W a representation of GL_n when the map ρ is understood; and simply write $g \cdot v$, by abuse of notation.

1. The **defining representation** is the map $\rho = \mathrm{id} : \mathrm{GL}_n \to \mathrm{GL}_n$.
2. The **determinant representation** is the representation $\det : \mathrm{GL}_n \to \mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times$.
3. If $(V, \rho_1), (W, \rho_2)$ are representations then $(V \oplus W, \rho)$ is a representation, where $\rho(g)(v, w) = (\rho_1(g)v, \rho_2(g)w)$.
4. If W is a representation then so is $\bigwedge^k W$, where $g \cdot (w_1 \wedge \cdots \wedge w_k) = (g \cdot w_1) \wedge \cdots \wedge (g \cdot w_k)$, and we extend linearly.
5. If W is a representation then so is W^*: for $\alpha \in W^*$ we define $g \cdot \alpha$ to be the linear function
$$g \cdot \alpha : w \mapsto \alpha(g^{-1} \cdot w).$$

Let W be a representation of GL_n.

- a subspace $U \subset W$ that is GL_n-invariant is called a **subrepresentation**.
- W is **irreducible** if the only subrepresentations of W are $\{0\}$ and W.
- a **morphism of representations** W, W' is a linear map $T : W \to W'$ such that $T(gw) = gT(w)$, for every $g \in \mathrm{GL}_n$, $w \in W'$. A morphism is an **isomorphism** if T is an isomorphism.

Complete irreducibility: let W be a finite dimensional representation of GL_n. Then, there exist irreducible subrepresentations $W_1, \ldots, W_r$ (not necessarily distinct, nor unique) such that $W = W_1 \oplus \cdots \oplus W_r$.

Complete reducibilty means that in order to understand the (finite dimensional) representations of GL_n we need only determine all of the irreducible representations.

It can be checked that the representations $\bigwedge^k \mathbb{C}^n$ are irreducible representations, for $1 \leq k \leq n$. (Do an example) Note that a basis for $\bigwedge^k \mathbb{C}^n$ is given by

$$\{e_J \mid J \subset \{1, \ldots, n\},\ |J| = k\}.$$

Observe that $\bigwedge^n \mathbb{C}^n$ is just the determinant representation. In fact, the determinant representation is what constitutes (essentially) the only difference between the representation theory of $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$.

Highest weight theory and roots: Fix a maximal torus T in GL_n - we may as well assume that this is the standard torus, since all maximal tori are conjugate. The **group of characters of** T (sometimes called the **weight lattice**) is the set of group homomorphisms

$$X^*(T) \stackrel{def}{=} \mathrm{Hom}_{\textbf{alg. gp}}(T, \mathbb{C}^\times),$$

that are also morphisms of algebraic varieties. We have already seen that $X^*(T) \cong \mathbb{Z}^n$: in fact, we observe that the projections $x_{jj} : T \to \mathbb{C}^*$ provide a basis for the free abelian group $X^*(T)$. We will denote these characters χ_j.

Fact: for any representation (W, ρ), every element $\rho(t) \in \mathrm{GL}(W)$, $t \in T$, is diagonalisable. Hence, since $\rho(T) \subset \mathrm{GL}(W)$ is a commutative subgroup consisting of diagonalisable elements, we can find a simultaneous eigenbasis $\{w_1, \dots, w_m\}$ of W. This means that

$$t \cdot w_i = \alpha_i(t) w_i,$$

where $\alpha_i : T \to \mathbb{C}^\times$ takes $t \in T$ to the eigenvalue $\alpha_i(t)$ of the linear operator $\rho(t)$ associated with the eigenvector w_i. Furthermore, since

$$t \cdot (t' \cdot w_i) = (tt') \cdot w_i,$$

we have $\alpha_i \in X^*(T)$. The set $\{\alpha_i\} \subset X^*(T)$ is called **the set of weights of** W.

Examples.