UC Berkeley Summer Undergraduate Research Program 2015
June 30 Lecture

We introduced the notion of a toric variety: an irreducible variety X is a **toric variety** if

- there is a dense open $T \subset X$ such that T is isomorphic to a complex torus $(\mathbb{C}^\times)^n$,
- the action of T on itself via multiplication $T \times T \to T$ extends to an action $T \times X \to X$.

For example, the variety $V = V(y^4 - x^2zw) \subset \mathbb{C}^4$ contains a torus

$$T = \{(s, t, u, t^4s^{-2}u^{-1}) \mid (s, t, u) \in (\mathbb{C}^\times)^3\} \subset V,$$

and this extends to an action

$$(s, t, u) \cdot (x, y, z, w) \overset{def}{=} (sx, ty, uz, t^4s^{-2}u^{-1}w).$$

We observe that $T = V - V(xyzw)$, or that $T = V \cap U(y)$, where $U(y) = \mathbb{C}^4 - V(y)$; hence, T is open in V.

We have the following orbits:

- the dense $3 - d$ orbit T,
- the 2-d orbits $T \cdot (0, 0, z, w)$, with $z \neq 0, w \neq 0$, $T \cdot (x, 0, 0, w)$, with $x \neq 0, w \neq 0$, $T \cdot (x, 0, z, 0)$, with $x \neq 0, z \neq 0$,
- the 1-d orbits $T \cdot (x, 0, 0, 0)$, $T \cdot (0, 0, z, 0)$, $T \cdot (0, 0, 0, w)$,
- the (unique, closed) 0-d orbit $T \cdot (0, 0, 0, 0)$.

We will see that this orbit structure is encoded in a **(strongly) convex polyhedral cone**, from which we can reconstruct V.

Recall that a convex polyhedral cone was a subset $\sigma \subset \mathbb{R}^n$ of the form

$$\sigma = \text{cone}(S) = \sum \mathbb{R}_{\geq 0} v_i, \quad S = \{v_1, \dots, v_r\}.$$

We say that σ is **strongly convex** if σ contains no positive dimensional linear subspace. This is equivalent to $\{0\}$ being a face of σ, or that $\dim \sigma^\vee = n$.

For strongly convex polyhedral cones there is a particularly nice way to obtain generators of σ: recall that a face of σ is a subset $\tau = H_u \cap \sigma$, where $H_u = \ker u$, $u \in (\mathbb{R}^n)^*$, and that faces are themselves convex polyhedral cones (in fact, they are strongly convex). We define an **edge (or ray) of** σ to be a 1-dimensional face of σ. It is not too hard to see that a generating set of σ can be chosen from (nonzero) elements in edges, and that such a generating set is minimal.

We say that a cone is **rational** if $\sigma = \text{cone}(S)$, for $S \subset \mathbb{Z}^n \subset \mathbb{R}^n$. Hence, we see that in a strongly convex rational polyhedral cone we can find a **unique minimal generating set**.

We now come to an important construction: given a (strongly) convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, we define

$$S_\sigma = \sigma^\vee \cap M.$$

For example, if $\sigma = \mathbb{R}_{\geq 0} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then

$$S_\sigma = \mathbb{Z}_{\geq 0}[2\ 1] + \mathbb{Z}_{\geq 0}[1\ 1] + \mathbb{Z}_{\geq 0}[0\ 1]$$

In particular, we see that S_σ is a finitely generated semigroup: in fact, this will always be the case.

Gordan's Lemma: if $\sigma \subset N_{\mathbb{R}}$ is a rational polyhedral cone then $S_\sigma = \sigma^\vee \cap M$ is finitely generated semigroup (monoid).

Hence, the semigroup algebra $\mathbb{C}[S_\sigma]$ (= subalgebra of the group algebra of M) is finitely generated and we can consider the corresponding affine variety $\mathrm{Spec}(\mathbb{C}[S_\sigma])$.

In the above example we get

$$\mathbb{C}[x^\pm, y^\pm] \supset \mathbb{C}[S_\sigma] = \mathbb{C}[x^2y, xy, y] \cong \mathbb{C}[u, v, w]/(v^2 - uw)$$

and we see that $\mathrm{Spec}(\mathbb{C}[S_\sigma]) = V(v^2 - uw) \subset \mathbb{C}^3$, with the action of the torus $T = (\mathbb{C}^\times)^2$,

$$(x, y) \cdot (u, v, w) \stackrel{def}{=} (x^2yu, xyv, yw).$$

We see T sitting inside V via the isomorphism

$$(x, y) \mapsto (x^2y, xy, y), \quad (u, v, w) \mapsto (vw^{-1}, w).$$

Given a face $\tau \subset \sigma$ we get an orbit of T on X as follows: if $\tau = \mathbb{R}_{\geq 0} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = H_{[2\ 1]} \cap \sigma$, then consider the projection

$$S_\sigma \to S_\sigma(\tau) \stackrel{def}{=} \sigma^\vee \cap \tau^\perp \cap M = \mathbb{Z}_{\geq 0}[2\ 1]\ ;\ a[2\ 1] + b[1\ 1] + c[0\ 1] \mapsto a[2\ 1]$$

which gives rise to the surjective morphism of semigroups algebras

$$\mathbb{C}[S_\sigma] \to \mathbb{C}[S_\sigma(\tau)],$$

which in turn corresponds to the morphism of varieties

$$V(v, w) \cap \mathrm{Spec}(\mathbb{C}[S_\sigma]) \subset \mathrm{Spec}(\mathbb{C}[S_\sigma]).$$

The subvariety on the left hand side above is an **orbit closure**: it is the union of the orbit $T \cdot (u, 0, 0) \subset \mathrm{Spec}(\mathbb{C}[S_\sigma])$ and $T \cdot (0, 0, 0) \subset \mathrm{Spec}(\mathbb{C}[S_\sigma])$.

Observe that $S_\sigma(\sigma) = \sigma^\vee \cap \sigma^\perp \cap M = \{0\} \subset M$, because $\dim \sigma = 2$, so that we have an embedding of varieties

$$V(u, v, w) \cap \mathrm{Spec}(\mathbb{C}[S_\sigma]) = \{(0, 0, 0)\} \to \mathrm{Spec}(\mathbb{C}[S_\sigma]).$$

Hence, to the face $\tau \subset \sigma$ we get an orbit

$$\mathcal{O}_\tau = \mathrm{Spec}(\mathbb{C}[S_\sigma(\tau)]) - \mathrm{Spec}(\mathbb{C}[S_\sigma(\sigma)]).$$

In general, we have

$$\mathrm{Spec}(\mathbb{C}[S_\sigma(\tau)]) = \bigsqcup_{\tau \prec \gamma} \mathcal{O}_\gamma$$

where we have partial ordering on faces

$$\tau \prec \gamma \iff \tau \subset \gamma.$$

There is an order-reversing correspondence between faces of σ and orbits in $\mathrm{Spec}(\mathbb{C}[S_\sigma])$**; denote that orbit corresponding to the face τ by $\mathcal{O}_\tau$. Moreover, $\mathcal{O}_\tau \subset \overline{\mathcal{O}_{\tau'}}$ if and only if $\tau' \prec \tau$.**