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Today we will begin talking about algebraic geometry, with the ultimate goal of playing with toric varieties and concrete examples.

There are many, many references for algebraic geometry and we will mostly use the Introduction to Cox's 'Toric Varieties' notes. However, more complete introductions (at the undergraduate level) are

1. M. Reid *Undergraduate Algebraic Geometry* (available online at Reid's website)
2. Shafarevich *Basic Algebraic Geometry*

To be able to study algebraic geometry effectively you need to have taken/concurrently take a strong commutative algebra class. My thoughts are the following:

commutative algebra is to algebraic geometry as calculus is to differential geometry

An **affine algebraic variety** is a subset $V = V(A) \subset \mathbb{C}^n$ defined by the vanishing of some collection of polynomials $A \subset \mathbb{C}[X] \stackrel{def}{=} \mathbb{C}[x_1, \dots, x_n]$. There are some basic observations:

1. $V = \{x \in \mathbb{C}^n \mid f(x) = 0, \text{ for every } f \in (A)\}$, where (A) is the ideal generated by A
2. Hilbert's Basis Theorem implies that we can always describe V as the zero locus of finitely many polynomials (simply take a finite generating set for (A)). As such, we may also sometimes write $V = V(f_1, \dots, f_r)$, where f_i generate (A).

Given a subset $X \subset \mathbb{C}^n$, we define

$$I_X \stackrel{def}{=} \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f(x) = 0, \text{ for every } x \in X\}.$$

So, I_X consists of all polynomial functions on $\mathbb{C}^n$ that vanish on X; it's an ideal in $\mathbb{C}[X]$.

We can think of I and V as defining functions

$$V : \{\text{ideals in } \mathbb{C}[X]\} \to \{\text{subsets of } \mathbb{C}^n\}, \quad I : \{\text{subsets of } \mathbb{C}^n\} \to \{\text{ideals in } \mathbb{C}[X]\}.$$

For example, we have

1. $V(x^2) = \{(0, y) \mid y \in \mathbb{C}\} \subset \mathbb{C}^2$,
2. $V(f) = \{x \in \mathbb{C}^n \mid f(x) = 0\} \subset \mathbb{C}^n$,
3. $V(f, g) = \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ and } g(x) = 0\} = V(f) \cap V(g)$,
4. $V(fg) = \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ or } g(x) = 0\} = V(f) \cup V(g)$,
5. $I_{\mathbb{Z}} = \mathbb{C}[x]$,
6. $I_{(0,0)} = (x, y)$,
7. $I_{\mathbb{Z}^2} = \mathbb{C}[x, y]$,
8. $V(\mathbb{C}[x, y]) = \varnothing$,
9. $V(0) = \mathbb{C}^n$.

Check:

- $V(I_X) = X$, for any affine variety $X \subset \mathbb{C}^n$;
- $I_X \subset I_{X'} \implies X \supset X'$;
- $X \subset X' \implies I_X \supset I_{X'}$

Note that if $X = V(x^2) \subset \mathbb{C}$, then $I_X = (x) \supsetneq (x^2)$.

The fundamental relationship between V and I is given by **Hilbert's Nullstellensatz**: let $I \subset \mathbb{C}[x_1, \dots, x_n]$, $X = V(I)$. Then, $I_X = \sqrt{I} = \{f \in \mathbb{C}[X] \mid f^m \in I\}$, that radical of I. In particular, points $(a_1, \dots, a_n) \in \mathbb{C}^n$ correspond to maximal ideals $(x_1 - a_1, \dots, x_n - a_n) \subset \mathbb{C}[X]$; this says, moreover, that every maximal ideal is of this form.

Given an affine variety $X \subset \mathbb{C}^n$, its **coordinate ring** is $A_X \stackrel{def}{=} \mathbb{C}[X]/I_X$. Elements of A_X should be thought of as polynomial functions on $\mathbb{C}^n$ that restrict to give equal functions on X. There is a one-to-one correspondence between points in X and maximal ideals of A_X.

A **subvariety of the affine variety** X is a subset $V \subset X$, where $V = V(I)$ for some $I \supset I_X$.

A **morphism of affine varieties** $F : \mathbb{C}^n \supset X \to Y \subset \mathbb{C}^m$ is a map

$$F = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)).$$

Any morphism $F : X \to Y$ of affine varieties gives rise to a (well-defined!) map

$$F^* : A_Y \to A_X \; ; \; f \mapsto F^*(f) = f \circ F.$$

We say that X and Y are isomorphic if and only if there exists morphisms $F : X \to Y$ and $G : Y \to X$ that are inverse to each other.

Fact: X is isomorphic to Y (both affine varieties) if and only if $A_X \cong A_Y$ (as $\mathbb{C}$-algebras).

1. We have $X = V(y - x^2)$ is isomorphic to $\mathbb{C}$: indeed we have $A_X = \mathbb{C}[x, y]/(y - x^2)$, $A_{\mathbb{C}} = \mathbb{C}[u]$ and we have an isomorphism
$$\alpha : A_Y \to A_X \; ; \; f(u) \mapsto f(x), \quad \beta : A_X \to A_Y \; ; \; f(x, y) \mapsto f(u, u^2).$$
2. The map $F : V(y^2 - x^3) \to \mathbb{C} \; ; \; (x, y) \mapsto x$ is a bijective morphism, but is not an isomorphism: the inverse function $(u \mapsto (u, u^{3/2}))$ is not a morphism.

More generally, we say that a subset $V \subset \mathbb{C}^n$ is an affine variety if it is isomorphic to some affine variety.

In fact, **any finitely generated $\mathbb{C}$-algebra A without nilpotent elements (ie $f^n = 0 \implies f = 0$) is of the form $A = A_X$ for some affine variety X.** Given such a $\mathbb{C}$-algebra A we will write (by abuse of notation/meaning) $X = \mathrm{Spec}(A)$, for the variety A for which A is its coordinate ring.

Let X be an affine variety. The **Zariski topology on X** is the topology with **closed sets** precisely the subsets of the form $V(I)$ for some ideal $I \subset A_X$. The complements of subsets of the form $V(I)$ are called **open**. The Zariski topology is weird and does not capture any of the usual topological information: for example, it is almost never Hausdorff.

Given a subset $S \subset X$ we then the **Zariski closure of** S is the smallest subvariety of X that contains S; this is the closure $\overline{S}$ with respect to the Zariski topology. In fact, we have $\overline{S} = V(I_S)$.

Essentially, the algebraic geometry of affine varieties is completely captured by coordinate rings; asking a question about an affine variety is equivalent to asking a question about its coordinate ring.

Projective Geometry: define n-**dimensional projective space** to be

$$\mathbb{P}^n \stackrel{def}{=} \left(\mathbb{C}^{n+1} - \{0\}\right) / \sim,$$

where $u \sim v$ if and only if there exists $\lambda \neq 0$ such that $u = \lambda v$. Hence,

$$\mathbb{P}^n = \{\text{lines in } \mathbb{C}^{n+1}\} = \mathrm{Gr}(1, n+1).$$

We will denote the equivalence class of $v = (v_0, \ldots, v_n)$ by $[v]$ or $[v_0 : \ldots : v_n]$. We call the v_i **homogeneous coordinates** - they are not well defined functions on $\mathbb{P}^n$; hence, neither are polynomials!

However, if $f \in \mathbb{C}[V] \stackrel{def}{=} \mathbb{C}[v_0, \ldots, v_n]$ is homogeneous of degree d, and we define $f([v]) = 0$, then asking whether $f([v]) = 0$ is well-defined: if $[u] = [v]$ then $u = \lambda v$, for some nonzero λ, and

$$f([u]) = f(u) = f(\lambda v) = \lambda^d f(v) = 0, \quad \text{assuming } f(v) = 0.$$

Hence, we can talk about the zero locus of a collection of **homogeneous polynomials** $f_1, \ldots, f_r \in \mathbb{C}[V]$. Define

$$V(f_1, \ldots, f_r) = \{[v] \mid f_i([v]) = 0, \text{ for each } i\} \subset \mathbb{P}^n.$$

Such a subset V as above is called a **projective variety**. A **subvariety of** V is a subset $W \subset V$ defined by the vanishing of (the restriction to V of) some homogeneous polynomials: we declare the subvarieties of V to be **closed** to obtain the **Zariski topology on** V.

If $I \subset \mathbb{C}[V]$ is a homogeneous ideal (ie generated by homogeneous polynomials) then we can define

$$V(I) \stackrel{def}{=} \{[v] \mid f([v]) = 0, \text{ for every } f \in I\}.$$

Similarly, given a projective variety $V \subset \mathbb{P}^n$ we get a homogeneous (check!) ideal

$$I_X \stackrel{def}{=} \{f \in \mathbb{C}[V] \mid f([v]) = 0, \text{ for every } [v] \in X\}.$$

The **projective nullstellensatz** now states: let $I \subset \mathbb{C}[V]$ be a homogeneous ideal. Then,

$$V(I) = \varnothing \leftrightarrow I \supset (v_0, \ldots, v_n)^m, \text{ for some } m \geq 0,$$

and

$$V(I) \neq \varnothing \implies I_{V(I)} = \sqrt{I}.$$

Given a projective variety $V \subset \mathbb{P}^n$ we define its **homogeneous coordinate ring (with respect to the embedding** $V \subset P^n$**)** to be $B_V = \mathbb{C}[V]/I_V$. Note that we have emphasized the dependence on realising V as a subvariety of $\mathbb{P}^n$.

A **morphism of projective varieties** $F : \mathbb{P}^n \supset U \to V \subset \mathbb{P}^m, [u] \mapsto (f_0([u]), \ldots, f_m([u]))$, with each of the f's being homogeneous of the same degree, can only be defined everywhere on U if $U \cap V(f_1, \ldots, f_m) = \varnothing$.

Important Fact: if $F : U \to V$ is a morphism of projective varieties then $F(U)$ is a projective subvariety of V.

There is a major difference between affine and projective algebraic geometry: a projective variety is not uniquely determined by its coordinate ring.