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This is an exposition of the paper: Kogan-Miller 'Toric Degenerations of Schubert varieties and Gelfand-Tsetlin polytopes', Adv. Math 193 (2005) p1-17.

By considering the Plucker algebra $\mathcal{P}$ and exhibiting a SAGBI basis for $\mathcal{P}$ (consisting of Plucker coordinates), we obtained a toric degeneration of the Flag variety (algebraically, at least).

In Kogan-Miller they construct the same toric degeneration (ie, the special fibre is the toric variety coming from the Gelfand-Tsetlin polytope). Moreover, their construction allows to see what happens to certain subvarieties of the flag variety under this degeneration - these are the Schubert varieties (=closures of Schubert cells), and are important in understanding the topology of flag varieties.

One of the projects for this summer is to try and perform a similar construction as in Kogan-Miller for a toric degeneration of a Grassmannian.

Again, we will focus on an example in order to highlight the main points.

Let $M = M_3(\mathbb{C})$, the set of 3×3 matrices, $G = \mathrm{GL}_3$, $B \subset G$ the set of lower triangular matrices in G.

There is an action of $G^3 = G \times G \times G$ on M by column-wise multiplication - this means that the usual action of G on M is obtained via restriction to *the diagonal subgroup of* G^3. Namely, if $g = (g_1, g_2, g_3) \in G^3$ and $m = [v_1\ v_2\ v_3] \in M$ then we define

$$g \cdot m \stackrel{def}{=} [g_1 v_1\ g_2 v_2\ g_3 v_3].$$

Denote by $\underline{B} \subset G^3$ the image of B under the diagonal embedding of G in G^3 - so elements of $\underline{B}$ are of the form (b, b, b), for $b \in B$.

Fix the following (generically arbitrary) 3×3 matrix

$$W = [w_{ij}] = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There is a homomorphism from $\mathbb{C}^* = \{z \in \mathbb{C} \mid z \neq 0\}$, the multiplicative group of nonzero complex numbers, into G^3

$$T : \mathbb{C}^* \to G^3\ ;\ z \mapsto (Z_1, Z_2, Z_3),$$

where $Z_j = \mathrm{diag}(z^{w_{1j}}, z^{w_{2j}}, z^{w_{3j}})$. Denote $T_W \stackrel{def}{=} \mathrm{im}\, T \subset G^3$, and if $z \in \mathbb{C}^*$ we will write $\tilde{z} \stackrel{def}{=} T(z) \in T_W$

Exercise: is T_W a subgroup of $\underline{B}$? Write down some elements in T_W.

Define a **family of subgroups** $B^* \subset B^3 \times \mathbb{C}^*$ as follows:

$$B^* \stackrel{def}{=} \{(\tilde{z}^{-1} b \tilde{z}, z) \mid b \in \underline{B}, z \in \mathbb{C}^*\}.$$

The fibre over $z \in \mathbb{C}^*$ is denoted $B(z) \cong \underline{B} \cong B$. (**Show this last group isomorphism.**)

Exercise: check that if $b = (b_1, b_2, b_3) \in B^3$ then $\tilde{z}^{-1} b \tilde{z}$ is the element of B^3 obtained from multiplying entries in b by the sequence

$$\begin{bmatrix} 1 & & \\ z^2 & 1 & \\ z^3 & z & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ z & 1 & \\ z & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix}. \qquad (*)$$

For example, if

$$b = \begin{bmatrix} 1 & & \\ -2 & 2 & \\ 1 & 1 & 1 \end{bmatrix},$$

then

$$\tilde{z}^{-1}(b, b, b)\tilde{z} = \left(\begin{bmatrix} 1 & & \\ -2z^2 & 2 & \\ z^3 & z & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ -2z & 2 & \\ z & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ -2 & 2 & \\ 1 & 1 & 1 \end{bmatrix} \right) \in B(z)$$

Note that each power of z appearing in the sequence $(*)$ is positive; hence, we can also extend our family 'over 0', in the sense that we can set

$$B(0) = \{((b_1, b_2, b_3), 0)\},$$

where b_j obtained from b_3 by setting to 0 all entries in columns $1, \dots, 3-j$ strictly below main diagonal. That is, we are considering $\lim_{z \to 0} B(z) = B(0)$.

For example, we have

$$\left(\begin{bmatrix} 1 & & \\ & 2 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 2 & \\ & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ -2 & 2 & \\ 1 & 1 & 1 \end{bmatrix} \right) \in B(0)$$

Denote $p : B^* \to \mathbb{C}$ the resulting projection.

Lemma: there is an isomorphism of 'families of groups' $B^* \cong B \times \mathbb{C}$ in the following sense - there is a function $f : B \times \mathbb{C} \to B^*$ such that $p \circ f = \pi$, where π is the canonical projection to $\mathbb{C}$, and f restricts to group isomorphisms on fibres over $\mathbb{C}$.

Proof: Note that $B \cong \underline{B}$, sending $b \mapsto \underline{b} = (b, b, b)$. For $0 \neq z \in \mathbb{C}$ we define

$$f(b, z) = (\tilde{z}^{-1}\underline{b}\tilde{z}, z) \in B^*.$$

If $z = 0$ then we denote

$$f(b, 0) = ((b_1, b_2, b), 0) \in B(0),$$

where b_i is obtained from b by setting to 0 all entries below main diagonal in columns $1, \dots, 3-i$ (as above). (**Check this is a group homomorphism on fibres: this means that you <u>fix</u> $z \in \mathbb{C}$ and check that $f(bb', z) = f(b, z)f(b', z)$, where the group operation on the right comes from the group operation on G^3.**

In fact, we have

$$f\left(\begin{bmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, z \right) = \left(\begin{bmatrix} a_{11} & & \\ a_{21}z^2 & a_{22} & \\ a_{31}z^3 & a_{32}z & a_{33} \end{bmatrix}, \begin{bmatrix} a_{11} & & \\ a_{21}z & a_{22} & \\ a_{31}z & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right)$$

We consider B acting on $M \times \mathbb{C}$ by identifying $B \times \mathbb{C}$ with B^* through the above isomorphism and consider B^* acting 'fibrewise' on $M \times \mathbb{C}$: ie we define an action

$$b * (m, z) \stackrel{def}{=} ((\tilde{z}^{-1}\underline{b}\tilde{z}) \cdot m, z)$$

.

For example, if

$$b = \begin{bmatrix} 1 & & \\ -2 & 2 & \\ 1 & 1 & 1 \end{bmatrix},$$

and $X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ then

$$b * (X, z) = \left(\left[\begin{bmatrix} 1 & & \\ -2z^2 & 2 & \\ z^3 & z & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2z & 2 & \\ z & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 2 & \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right], z \right)$$
$$= \left(\begin{bmatrix} 1 & 1 & 0 \\ -2z^2 + 2 & -2z & 0 \\ z^3 + z & z + 1 & 1 \end{bmatrix}, z \right)$$

Remark: When $z = 0$, we have that the action of B on $M \times \{0\}$ commutes with the action of the subgroup

$$S = \left\{ \left(\begin{bmatrix} a & & \\ & b & \\ & & 1 \end{bmatrix}, \begin{bmatrix} c & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right) \in G^3 \right\} \subset G^3$$

on M. We will denote by $\overline{t} \in S$ the sequence

$$\overline{z} \stackrel{def}{=} \left(\begin{bmatrix} z^3 & & \\ & z & \\ & & 1 \end{bmatrix}, \begin{bmatrix} z & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right) \in S.$$

We are now going to consider a 'family of Plucker coordinates': for $J = \{j_1, \dots, j_k\} \subset \{1, 2, 3\}$, $k = 1, 2, 3$, we denote

$$w_J = \sum_{i=1}^{2} w_{i, 4-j_i},$$

where $W = [w_{ij}]$ is as above. For example, $w_{12} = 0 + 0$, $w_{13} = 0 + 1$, $w_{23} = 1 + 1$.

We are going to define 'Plucker coordinates on the family' $M \times \mathbb{C}$ - these will be polynomials $q_J(\underline{x}, z) \in \mathbb{C}[x_{11}, x_{12}, \dots, x_{32}, x_{33}, z]$ (think of the variable z as the degenerating parameter) such that, if we fix $z = z_0$, the polynomial $q_J(\underline{x}, z_0)$ is U-invariant by restricting the $*$-action of B to

$$U = \left\{ \begin{bmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{bmatrix} \right\} \subset B.$$

This means that $q_J(u * (m, z)) = q_J(m, z)$, for any $u \in U$.

For $J \subset \{1, 2, 3\}$, we define

$$q_J(\underline{x}, z) = z^{-w_J} \Delta_J(\overline{z} \cdot X).$$

For example,

$$q_{12}(\underline{x}, z) = z^0 \Delta_{12}(\overline{z} \cdot X) = \Delta_{12}\left(\begin{bmatrix} z^3 x_{11} & z x_{12} & x_{13} \\ z x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}\right) = z^3 x_{11} x_{22} - z^2 x_{12} x_{21}.$$

Important Theorem: The polynomials q_J generate the $\mathbb{C}[z]$-algebra of U-invariant functions inside $\mathbb{C}[\underline{x}, z]$.