The \( n! \) and \( (n + 1)^{n-1} \) conjectures

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Abstract

We provide an introduction to the statements of the \( n! \) and \( (n + 1)^{n-1} \) conjectures of Garsia, Haiman. We introduce several bigraded \( S_n \) modules (indexed by partitions) and provide evidence to support the fact that their Frobenius series is given by the transformed Macdonald polynomials, thereby showing that the Kostka-Macdonald polynomials count the bigraded multiplicities of isotypic components. This exposition was prepared for a talk given in the Macdonald Polynomials Seminar, Spring 2013, UC Berkeley, run by Maria Monks and Steven Sam.

1 Introduction

Fix \( n \geq 1 \) and let \( S_n \) be the symmetric group on \( n \) letters. For a partition \( \mu \) of \( n \), recall the (transformed) Macdonald polynomials \( \tilde{H}_\mu(Z; q, t) \): these are the (two parameter) symmetric functions uniquely characterised by the following properties

i) \( \tilde{H}_\mu[(1 - q)Z; q, t] \in \mathbb{Q}(q, t)\{S_\lambda \mid \lambda \geq \mu\} \),

ii) \( \tilde{H}_\mu[(1 - t)Z; q, t] \in \mathbb{Q}(q, t)\{S_\lambda \mid \lambda \geq \mu'\} \),

iii) \( \tilde{H}_\mu[1; q, t] = 1 \).

Moreover, we have the fundamental identity

\[
\tilde{H}_\mu(Z; q, t) = \sum_{\lambda} \tilde{K}_{\lambda \mu}(q, t) S_\lambda(Z),
\]

where \( S_\lambda \) are the Schur functions. The rational functions \( \tilde{K}_{\lambda \mu}(q, t) \) are the (transformed) Kostka-Macdonald ‘polynomials’. Of course, we don’t know that they are polynomials yet, but it turns out that this is the case.

After computing some small examples of \( \tilde{K}_{\lambda \mu}(q, t) \), Macdonald was led to conjecture

\[
\tilde{K}_{\lambda \mu}(q, t) \in \mathbb{N}[q, t] \quad \text{(Macdonald positivity, integrality conjecture)}
\]

In this talk we will outline a proof of this conjecture via the \( n! \) conjecture of Garsia, Haiman. An outline of the talk is as follows:

1) An example: we give an example that highlights some of the salient points we will discuss.
2) The $n!$ conjecture and why it seems reasonable: we will describe the $n!$ conjecture and indicate how Garsia, Haiman were led to formulate the $n!$ conjecture and why it seems reasonable to do so.

3) The $(n+1)^{n-1}$ conjecture: we will give a (very!) brief outline of the $(n+1)^{n-1}$ conjecture. This is a statement about the dimension of the diagonal coinvariant ring.

Note: we will adopt the French-style for partitions and tableau, so that rows are decreasing from bottom to top. We define the biexponents of a partition $\mu$ to be the cells $(i, j)$ appearing in the diagram of $\mu$, where the bottom left cell is $(0, 0)$. We fix an ordering on the biexponents $(p_1, q_1) > \cdots > (p_n, q_n)$, arising from reading the tableau from left to right and from bottom to top. For example, if $\mu = \begin{array}{c}
\end{array}$ then we have the ordering on biexponents $(0, 0) > (1, 0) > (0, 1)$.

2 An example

In this section we fix $n = 3$. Consider the partition

$$\mu = \begin{array}{c}
\end{array}.$$  
Denote the biexponents of $\mu$ by $(p_1, q_1) > (p_2, q_2) > (p_3, q_3)$. Define the bigraded polynomial $\Delta_{\mu} \in \mathbb{C}[X_3, Y_3]$ to be

$$\Delta_{\mu}(x, y) = \det \begin{bmatrix}
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3
\end{bmatrix} = x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1.$$ 

Denote the $\mathbb{C}$-span of all partial derivatives of $\Delta_{\mu}$ by $A_{\mu}$. Thus,

$$A_{\mu} = \mathbb{C}\{1, x_1 - x_2, x_2 - x_3, y_1 - y_2, y_2 - y_3, \Delta_{\mu}\}.$$ 

The $\mathbb{C}$-vector space $A_{\mu}$ admits an $S_3$ action by the diagonal permutation of variables; moreover, $A_{\mu}$ is bigraded: we have

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<td>$\mathbb{C}{y_1 - y_2, y_2 - y_3}$</td>
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<tr>
<td>1</td>
<td>$\mathbb{C}{x_1 - x_2, x_2 - x_3}$</td>
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so that, for example, $\mathcal{H}_{(1,0)}(A_{\mu}) = \mathbb{C}\{x_1 - x_2, x_2 - x_3\}$. Furthermore, each bigraded component is $S_3$-invariant and we have a decomposition into $S_3$-modules

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so that is apparent that $A_\mu \simeq \mathbb{C}S_3$, the left regular representation.

Recall that for a bigraded $S_n$-module $A = \bigoplus_{r,s} A_{r,s}$, the Frobenius series of $A$ is the symmetric function

$$F(A) = \sum_{r,s,\lambda} q^r t^s m_{r,s}^\lambda S_\lambda(Z_n) \in \Lambda_\mathbb{C}(q,t),$$

where $m_{r,s}^\lambda = [A_{r,s} : V^\lambda]$. Hence, we have

$$F(A_\mu) = S_{\bullet\bullet\bullet} + (q + t)S_{\bullet\bullet\bullet} + qtS_{\bullet\bullet\bullet},$$

and this is the transformed Macdonald polynomial $\tilde{H}_\mu(Z; q, t)$.

Hence, we can obtain the Kostka-Macdonald polynomials $\tilde{K}_{\lambda\mu}(q, t)$, for $\mu = \square$.

By defining

$$\Delta_{\square\square\square} = \Delta(x_1, x_2, x_3), \quad \Delta_{\square\square\square} = \Delta(y_1, y_2, y_3),$$

the Vandermonde determinants in the alphabets $X_3$ and $Y_3$ respectively, and proceeding as above, you can determine the Kostka-Macdonald polynomials $\tilde{K}_{\lambda\mu}(q, t)$ to be

$$\begin{array}{c|ccc}
\mu \setminus \lambda & \square\square\square & \square\square\square & \square\square\square \\
\square\square\square & 1 & q + q^2 & q^3 \\
\square\square\square & 1 & q + t & qt \\
\square\square\square & 1 & t + t^2 & t^3 \\
\end{array}$$

Of course, that the Frobenius series of the bigraded $S_3$-module $A_\mu$ of partial derivatives of $\Delta_\mu$ is the transformed Macdonald polynomial $\tilde{H}_\mu(Z; q, t)$ is a Theorem; this is the content of the $n!$ conjecture.

3 The $n!$ conjecture

The statement

For arbitrary $n$, let $\mu$ be a partition of $n$. Recall the ordering of the biexponents of $\mu$

$$(p_1, q_1) > ... > (p_n, q_n).$$

Define the bihomogeneous polynomial $\Delta_\mu(X_n, Y_n) \in \mathbb{C}[X_n, Y_n]$ as

$$\Delta_\mu = \det \begin{bmatrix} x_i^{p_j} y_i^{q_j} \end{bmatrix},$$

and consider the $\mathbb{C}$-span of all partial derivatives of $\Delta_\mu$

$$A_\mu = \mathbb{C}\{\partial_{x_1}^{p_1} \partial_{x_2}^{p_2} \cdots \partial_{x_n}^{p_n} \Delta_\mu\}.$$ 

Here, $\partial_{x_1}^p = \frac{\partial^p}{\partial x_1^p}$ and similarly for $\partial_{y_j}^q$. 

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Example 3.1. In particular, we always have
\[ \Delta \left( \binom{n}{r} \right) = \Delta \left( \binom{n}{r} \right), \quad \Delta \left( \binom{1}{n} \right) = \Delta \left( \binom{1}{n} \right), \]
the Vandermonde polynomials in the alphabets \(X_n\) and \(Y_n\), respectively. In this case, \(A_\mu\) are the Garsia-Procesi modules \(R_\mu\) and \(R_\mu\) respectively (defined in my last talk); that is, they are the cohomology rings of the full flag variety in type A equipped with their (Borel-Springer) \(S_n\)-action.

Then, \(A_\mu\) is a finite dimensional bigraded \(S_n\)-module with the \(S_n\)-action arising from the diagonal permutation of variables \((x_1, \ldots, x_n; y_1, \ldots, y_n)\).

**Theorem 3.2** (\(n!\) conjecture; Haiman (2001)). The Frobenius series of \(A_\mu\) is the transformed Macdonald polynomial \(\widetilde{H}_\mu(Z; q, t)\). In particular, letting \(q = t = 1\), we find that \(A_\mu\) affords the left regular representation of \(S_n\) and \(\dim A_\mu = n!\).

**Corollary 3.3** (Macdonald positivity). The (transformed) Macdonald-Kostka polynomials \(\tilde{K}_{\lambda\mu}(q, t)\) are polynomials and have integral, positive coefficients.

**Proof:** Indeed, by the Theorem, and the identity
\[ \tilde{H}_\mu(Z; q, t) = \sum_{\lambda} \tilde{K}_{\lambda\mu}(q, t) S_\lambda, \]
the Macdonald-Kostka polynomials give the bigraded count of the multiplicities of \(V^\lambda\) as a submodule of \(A_\mu\).

**What's to come:** we will see over the next couple of weeks how Haiman approached a proof of the \(n!\) conjecture. The argument comes down to finding some (connected) geometric object \(X\) and some fibre bundle \(P\) over \(X\) such that \(P\) admits a fibre \(P_x\) (with \(S_n\)-action) isomorphic to \(A_\mu\). Then, by showing that \(P\) is a vector bundle, so that all fibres have the same dimension, and showing that one fibre has dimension \(n!\), we obtain the dimension statement. The identification of the Frobenius series requires a bit more explanation.

**Why \(n!\)?**

Recall the perfect pairing
\[ \langle \cdot, \cdot \rangle : \mathbb{C}[X_n, Y_n] \times \mathbb{C}[X_n, Y_n] \to \mathbb{C} ; (f, g) \mapsto ev_0(f(\partial)(g)) \]
Note that this restricts to a perfect pairing on each bihomogeneous piece of \(\mathbb{C}[X_n, Y_n]\).

Fix \(\mu\), a partition of \(n\) and define the ideal
\[ I_\mu = \{ f \in \mathbb{C}[X_n, Y_n] \mid f(\partial_x, \partial_y)(\Delta_\mu) = 0 \} \subset \mathbb{C}[X_n, Y_n]. \]
This is a (bi)homogeneous ideal in \(\mathbb{C}[X_n, Y_n]\), so that we have
\[ \mathbb{C}[X_n, Y_n] = I_\mu \oplus I_\mu^\perp, \quad \text{where} \quad I_\mu^\perp = \{ g \in \mathbb{C}[X_n, Y_n] \mid f(\partial_x, \partial_y)(g) = 0, \forall f \in I_\mu \}. \]
In particular, we have \(A_\mu \subset I_\mu^\perp\). The \(n!\) conjecture is stating that we have equality \(A_\mu = I_\mu^\perp\) (although this is not so apparent!) However, this still doesn’t give a reasonable explanation for why \(A_\mu\) should have dimension \(n!\) as both of these modules seem quite difficult to get to grips with.

In fact, the ideal \(I_\mu\) admits the following characterisation
Lemma 3.4. \( f \in I_{\mu} \) if and only if the principal ideal \((f)\) generated by its image in \( R_{\mu}(X_n) \otimes_{\mathbb{C}} \) \( R_{\mu'}(Y_n) \) does not intersect the (unique) copy of the sign representation.

Remark 3.5. Here, \( R_{\mu}(X_n) \) (resp. \( R_{\mu'}(Y_n) \)) are the (singly) graded Garsia-Procési \( S_n \)-modules in the alphabets \( X_n \), (resp. \( Y_n \)), defined in my last talk. Since these modules only contain those submodules of the form \( V^{\lambda} \), for \( \lambda \geq \mu \) (resp. \( \lambda \geq \mu' \)), and the sign representation \( V^{(1^n)} \) can only appear (once) in the product \( V^{\lambda} \otimes V^{\lambda'} \), we must have that the sign representation appears (uniquely) in the top degree component of \( R_{\mu} \otimes R_{\mu'} \) (recalling that \( V^\mu \) (resp. \( V^{\mu'} \)) appears once in \( R_{\mu} \) (resp. \( V^{\mu'} \)) in the top degree.)

Now, consider the (diagonal) \( S_n \)-orbit \( O = S_n \cdot b \) of the point
\[
b = (1, 2, \ldots, n; -1, -2, \ldots, -n) \in \mathbb{C}^{2n}.
\]
If we let \( J_b \subset \mathbb{C}[X_n, Y_n] \) be the ideal of leading forms of polynomials vanishing on \( O \) (with respect to some ordering) then \( H_b = \mathbb{C}[X_n, Y_n]/J_b \) is an \( S_n \)-module affording the left regular representation; in particular, \( \dim H_b = n! \). Then, we have \( J_b \subset I_{\mu} \) (this is a consequence of the Lemma; for details, see [3]). Hence, we obtain
\[
\dim A_{\mu} \leq \dim I_{\mu}^- = \dim \mathbb{C}[X_n, Y_n]/I_{\mu} \leq \dim H_b = n!.
\]

Remark 3.6. The representation \( S_n \)-module \( H_b \) is (a prior) only singly graded. It is a consequence of the \( n! \) conjecture that we must have \( J_b = I_{\mu} \), so that \( H_b \) is bigraded (as \( I_{\mu} \) is) and independent of our choice of \( b \). Moreover, this implies that \( A_{\mu} = I_{\mu}^- \) affords the left regular representation.

4 The \((n + 1)^{n-1}\) conjecture

We see that the bigraded \( S_n \)-modules defined above (which are all isomorphic as bigraded \( S_n \)-modules) can be viewed as quotients of \( \mathbb{C}[X_n, Y_n] \) that afford the left regular representation; in particular, they contain a single copy of the trivial representation \( \mathbb{C}\{1\} \).

The diagonal coinvariant ring is the ring
\[
R_n = \mathbb{C}[X_n, Y_n]/\mathbb{C}[X_n, Y_n]^{S_n},
\]
where \( \mathbb{C}[X_n, Y_n]^{S_n} \) is the ideal of \( S_n \)-invariant polynomials without constant term. \( R_n \) has the property that any \( S_n \)-quotient of \( \mathbb{C}[X_n, Y_n] \) that contains the trivial representation exactly once, appearing as \( \mathbb{C}\{1\} \), is a quotient of \( R_n \). Hence, each of the rings \( A_{\mu} \cong \mathbb{C}[X_n, Y_n]/I_{\mu} \) may be considered as a quotient of \( R_n \).

The space of diagonal harmonics for \( S_n \) is the set
\[
DH_n = \{ g \in \mathbb{C}[X_n, Y_n] \mid f(\partial_x, \partial_y)(g) = 0, \forall f \in \mathbb{C}[X_n, Y_n]^{S_n}\};
\]
since \( \mathbb{C}[X_n, Y_n]^{S_n} \subset I_{\mu} \), for each \( \mu \), we see that \( \Delta_\mu \subset DH_n \), for each \( \mu \). Hence, \( A_{\mu} \subset DH_n \) for each \( \mu \). In some sense, the space of diagonal harmonics controls all of the bigraded modules introduced:

Proposition 4.1. The canonical projection \( DH_n \rightarrow R_n \) is an isomorphism and induces the isomorphism \( A_{\mu} \cong \mathbb{C}[X_n, Y_n]/I_{\mu} \).
Now, for the punchline:

**Theorem 4.2** (Master conjecture). The Frobenius series of the diagonal coinvariant ring is

\[ F(R_n)(Z; q, t) = \nabla e_n(Z). \]

**Corollary 4.3** \((n + 1)^{n-1} \) conjecture).

\[ \dim R_n = (n + 1)^{n-1}. \]

For details, see [3], sections 3.5, 4.2.

**References**

