The $n!$ and $(n+1)^{n-1}$ conjectures

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Abstract

We provide an introduction to the statements of the $n!$ and $(n+1)^{n-1}$ conjectures of Garsia, Haiman. We introduce several bigraded S_n modules (indexed by partitions) and provide evidence to support the fact that their Frobenius series is given by the transformed Macdonald polynomials, thereby showing that the Kostka-Macdonald polynomials count the bigraded multiplicities of isotypic components. This exposition was prepared for a talk given in the *Macdonald Polynomials Seminar*, Spring 2013, UC Berkeley, run by Maria Monks and Steven Sam.

1 Introduction

Fix $n \geq 1$ and let S_n be the symmetric group on n letters. For a partition μ of n, recall the (transformed) Macdonald polynomials $\tilde{H}_\mu(Z; q, t)$: these are the (two parameter) symmetric functions uniquely characterised by the following properties

i) $\tilde{H}_\mu[(1-q)Z; q, t] \in \mathbb{Q}(q, t)\{S_\lambda \mid \lambda \geq \mu\}$,

ii) $\tilde{H}_\mu[(1-t)Z; q, t] \in \mathbb{Q}(q, t)\{S_\lambda \mid \lambda \geq \mu'\}$,

iii) $\tilde{H}_\mu[1; q, t] = 1$.

Moreover, we have the fundamental identity

$$\tilde{H}_\mu(Z; q, t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q, t) S_\lambda(Z),$$

where S_λ are the Schur functions. The rational functions $\tilde{K}_{\lambda\mu}(q, t)$ are the *(transformed) Kostka-Macdonald 'polynomials'*. Of course, we don't know that they are polynomials yet, but it turns out that this is the case.

After computing some small examples of $\tilde{K}_{\lambda\mu}(q, t)$, Macdonald was led to conjecture

$$\boxed{\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t] \quad (\textbf{Macdonald positivity, integrality conjecture})}$$

In this talk we will outline a proof of this conjecture via the $n!$ conjecture of Garsia, Haiman.

An outline of the talk is as follows:

1) *An example*: we give an example that highlights some of the salient points we will discuss.

2) *The $n!$ conjecture and why it seems reasonable*: we will describe the $n!$ conjecture and indicate how Garsia, Haiman were led to formulate the $n!$ conjecture and why it seems reasonable to do so.

3) *The $(n+1)^{n-1}$ conjecture*: we will give a (very!) brief outline of the $(n+1)^{n-1}$ conjecture. This is a statement about the dimension of the diagonal coinvariant ring.

Note: we will adopt the French-style for partitions and tableau, so that rows are decreasing from bottow to top. We define the **biexponents** of a partition μ to be the cells (i, j) appearing in the diagram of μ, where there bottom left cell is $(0, 0)$. We fix an ordering on the biexponents

$$(p_1, q_1) > \ldots > (p_n, q_n),$$

arising from reading the tableau from left to right and from bottom to top. For example, if $\mu = \ydiagram{1,2}$ then we have the ordering on biexponents

$$(0, 0) > (1, 0) > (0, 1).$$

2 An example

In this section we fix $n = 3$. Consider the partition

$$\mu = \ydiagram{1,2}.$$

Denote the biexponents of μ by $(p_1, q_1) > (p_2, q_2) > (p_3, q_3)$. Define the bigraded polynomial $\Delta_\mu \in \mathbb{C}[X_3, Y_3]$ to be

$$\Delta_\mu(x, y) = \det \left[x_j^{p_i} y_j^{q_i} \right] = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1.$$

Denote the $\mathbb{C}$-span of all partial derivatives of Δ_μ by A_μ. Thus,

$$A_\mu = \mathbb{C}\{1, x_1 - x_2, x_2 - x_3, y_1 - y_2, y_2 - y_3, \Delta_\mu\}.$$

The $\mathbb{C}$-vector space A_μ admits an S_3 action by the **diagonal** permutation of variables; moreover, A_μ is *bigraded*: we have

$x \setminus y$	0	1
0	$\mathbb{C}\{1\}$	$\mathbb{C}\{y_1 - y_2, y_2 - y_3\}$
1	$\mathbb{C}\{x_1 - x_2, x_2 - x_3\}$	$\mathbb{C}\{\Delta_\mu\}$

so that, for example, $\mathcal{H}_{(1,0)}(A_\mu) = \mathbb{C}\{x_1 - x_2, x_2 - x_3\}$. Furthermore, each bigraded component is S_3-invariant and we have a decomposition into S_3-modules

$x \setminus y$	0	1
0	$V_{\ydiagram{3}}$	$V_{\ydiagram{1,2}}$
1	$V_{\ydiagram{1,2}}$	$V_{\ydiagram{1,1,1}}$

so that is apparent that $A_\mu \simeq \mathbb{C}S_3$, the left regular representation.

Recall that for a bigraded S_n-module $A = \bigoplus_{r,s} A_{r,s}$, the *Frobenius series of A* is the symmetric function

$$F(A) = \sum_{r,s,\lambda} q^r t^s m^\lambda_{r,s} S_\lambda(Z_n) \in \Lambda_{\mathbb{C}(q,t)}.$$

where $m^\lambda_{r,s} = [A_{r,s} : V^\lambda]$. Hence, we have

$$F(A_\mu) = S_{\ydiagram{3}} + (q+t)S_{\ydiagram{2,1}} + qtS_{\ydiagram{1,1,1}},$$

and this is the transformed Macdonald polynomial $\tilde{H}_\mu(Z\ q,t)$.

Hence, we can obtain the Kostka-Macdonald polynomials $\tilde{K}_{\lambda\mu}(q,t)$, for $\mu = \ydiagram{2,1}$.

By defining

$$\Delta_{\ydiagram{3}} = \Delta(x_1,x_2,x_3), \quad \Delta_{\ydiagram{1,1,1}} = \Delta(y_1,y_2,y_3),$$

the Vandermonde determinants in the alphabets X_3 and Y_3 respectively, and proceeding as above, you can determine the Kostka-Macdonald polynomials $\tilde{K}_{\lambda\mu}(q,t)$ to be

$\mu \setminus \lambda$	$\ydiagram{3}$	$\ydiagram{2,1}$	$\ydiagram{1,1,1}$
$\ydiagram{3}$	1	$q+q^2$	q^3
$\ydiagram{2,1}$	1	$q+t$	qt
$\ydiagram{1,1,1}$	1	$t+t^2$	t^3

Of course, that the Frobenius series of the bigraded S_3-module A_μ of partial derivatives of Δ_μ is the transformed Macdonald polynomial $\tilde{H}_\mu(Z;q,t)$ is a Theorem; this is the content of the $n!$ **conjecture**.

3 The $n!$ conjecture

The statement

For arbitrary n, let μ be a partition of n. Recall the ordering of the biexponents of μ

$$(p_1,q_1) > \ldots > (p_n,q_n).$$

Define the bihomogeneous polynomial $\Delta_\mu(X_n,Y_n) \in \mathbb{C}[X_n,Y_n]$ as

$$\Delta_\mu = \det\left[x_i^{p_j} y_i^{q_j}\right],$$

and consider the $\mathbb{C}$-span of all partial derivatives of Δ_μ

$$A_\mu = \mathbb{C}\{\partial^r_x \partial^s_y \Delta_\mu\}.$$

Here, $\partial^r_x = \frac{\partial^{r_1}}{\partial x_1^{r_1}} \cdots \frac{\partial^{r_n}}{\partial x_n^{r_n}}$ and similarly for ∂^s_y.

Example 3.1. In particular, we always have

$$\Delta_{(n)} = \Delta(X_n), \quad \Delta_{(1^n)} = \Delta(Y_n),$$

the Vandermonde polynomials in the alphabets X_n and Y_n, respectively. In this case, A_μ are the Garsia-Procesi modules $R_{\mu'}$ and R_μ respectively (defined in my last talk); that is, they are the cohomology rings of the full flag variety in type A equipped with their (Borel-Springer) S_n-action.

Then, A_μ is a finite dimensional bigraded S_n-module with the S_n-action arising from the **diagonal** permutation of variables $(x_1, \dots, x_n; y_1, \dots, y_n)$.

Theorem 3.2 ($n!$ conjecture; Haiman (2001))**.** *The Frobenius series of A_μ is the transformed Macdonald polynomial $\tilde{H}_\mu(Z; q, t)$. In particular, letting $q = t = 1$, we find that A_μ affords the left regular representation of S_n and* $\dim A_\mu = n!$*.*

Corollary 3.3 (Macdonald positivity)**.** *The (transformed) Macdonald-Kostka polynomials $\tilde{K}_{\lambda\mu}(q, t)$ are polynomials and have integral, positive coefficients.*

Proof: Indeed, by the Theorem, and the identity

$$\tilde{H}_\mu(Z; q, t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q, t) S_\lambda,$$

the Macdonald-Kostka polynomials give the bigraded count of the multiplicities of V^λ as a submodule of A_μ. $\square$

What's to come: we will see over the next couple of weeks how Haiman approached a proof of the $n!$ conjecture. The argument comes down to finding some (connected) geometric object X and some fibre bundle P over X such that P admits a fibre P_x (with S_n-action) isomorphic to A_μ. Then, by showing that P is a vector bundle, so that all fibres have the same dimension, and showing that one fibre has dimension $n!$, we obtain the dimension statement. The identification of the Frobenius series requires a bit more explanation.

Why $n!$?

Recall the perfect pairing

$$\langle , \rangle : \mathbb{C}[X_n, Y_n] \times \mathbb{C}[X_n, Y_n] \to \mathbb{C} \; ; \; (f, g) \mapsto ev_0\left(f(\partial)(g)\right)$$

Note that this restricts to a perfect pairing on each bihomogeneous piece of $\mathbb{C}[X_n, Y_n]$.

Fix μ, a partition of n and define the ideal

$$I_\mu = \{f \in \mathbb{C}[X_n, Y_n] \mid f(\partial_x, \partial_y)(\Delta_\mu) = 0\} \subset \mathbb{C}[X_n, Y_n].$$

This is a (bi)homogeneous ideal in $\mathbb{C}[X_n, Y_n]$, so that we have

$$\mathbb{C}[X_n, Y_n] = I_\mu \oplus I_\mu^\perp, \quad \text{where } I_\mu^\perp = \{g \in \mathbb{C}[X_n, Y_n] \mid f(\partial_x, \partial_y)(g) = 0, \ \forall f \in I_\mu\}.$$

In particular, we have $A_\mu \subset I_\mu^\perp$. **The $n!$ conjecture is stating that we have equality $A_\mu = I_\mu^\perp$** (although this is not so apparent!). However, this still doesn't give a reasonable explanation for why A_μ should have dimension $n!$ as both of these modules seem quite difficult to get to grips with.

In fact, the ideal I_μ admits the following characterisation

Lemma 3.4. *$f \in I_\mu$ if and only if the principal ideal (f) generated by its image in $R_\mu(X_n) \otimes_{\mathbb{C}} R_{\mu'}(Y_n)$ does not intersect the (unique) copy of the sign representation.*

Remark 3.5. Here, $R_\mu(X_n)$ (resp. $R_{\mu'}(Y_n)$) are the (singly) graded Garsia-Procesi S_n-modules in the alphabets X_n, (resp. Y_n), defined in my last talk. Since these modules only contain those submodules of the form V^λ, for $\lambda \geq \mu$ (resp. $\lambda \geq \mu'$), and the sign representation $V^{(1^n)}$ can only appear (once) in the product $V^\lambda \otimes V^{\lambda'}$, we must have that the sign representation appears (uniquely) in the top degree component of $R_\mu \otimes R_{\mu'}$ (recalling that V^μ (resp. $V^{\mu'}$) appears once in R_μ (resp. $V^{\mu'}$) in the top degree.)

Now, consider the (diagonal) S_n-orbit $\mathcal{O} = S_n \cdot b$ of the point

$$b = (1, 2, \dots, n; -1, -2, \dots, -n) \in \mathbb{C}^{2n}.$$

If we let $J_b \subset \mathbb{C}[X_n, Y_n]$ be the ideal of leading forms of polynomials vanishing on $\mathcal{O}$ (with respect to some ordering) then $H_b = \mathbb{C}[X_n, Y_n]/J_b$ is an S_n-module affording the left regular representation; in particular, $\dim H_b = n!$. Then, we have $J_b \subset I_\mu$ (this is a consequence of the Lemma; for details, see [3]). Hence, we obtain

$$\dim A_\mu \leq \dim I_\mu^\perp = \dim \mathbb{C}[X_n, Y_n]/I_\mu \leq \dim H_b = n!.$$

Remark 3.6. The representation S_n-module H_b is (a prior) only *singly* graded. It is a consequence of the $n!$ conjecture that we must have $J_b = I_\mu$, so that H_b is bigraded (as I_μ is) and independent of our choice of b. Moreover, this implies that $A_\mu = I_\mu^\perp$ affords the left regular representation.

4 The $(n+1)^{n-1}$ conjecture

We see that the bigraded S_n-modules defined above (which are all isomorphic as bigraded S_n-modules) can be viewed as quotients of $\mathbb{C}[X_n, Y_n]$ that afford the left regular representation; in particular, they contain a single copy of the trivial representation $\mathbb{C}\{1\}$.

The **diagonal coinvariant ring** is the ring

$$R_n = \mathbb{C}[X_n, Y_n]/\mathbb{C}[X_n, Y_n]_+^{S_n},$$

where $\mathbb{C}[X_n, Y_n]_+^{S_n}$ is the ideal of S_n-invariant polynomials without constant term. R_n has the property that any S_n-quotient of $\mathbb{C}[X_n, Y_n]$ that contains the trivial representation exactly once, appearing as $\mathbb{C}\{1\}$, is a quotient of R_n. Hence, each of the rings $A_\mu \simeq \mathbb{C}[X_n, Y_n]/I_\mu$ may be considered as a quotient of R_n.

The **space of diagonal harmonics** for S_n is the set

$$DH_n = \{g \in \mathbb{C}[X_n, Y_n] \mid f(\partial_x, \partial_y)(g) = 0, \ \forall f \in \mathbb{C}[X_n, Y_n]_+^{S_n}\};$$

since $\mathbb{C}[X_n, Y_n]_+^{S_n} \subset I_\mu$, for each μ, we see that $\Delta_\mu \in DH_n$, for each μ. Hence, $A_\mu \subset DH_n$, for each μ. In some sense, the space of diagonal harmonics controls all of the bigraded modules introduced:

Proposition 4.1. *The canonical projection $DH_n \to R_n$ is an isomorphism and induces the isomorphism $A_\mu \simeq \mathbb{C}[X_n, Y_n]/I_\mu$.*

Now, for the punchline:

Theorem 4.2 (Master conjecture)**.** *The Frobenius series of the diagonal coinvariant ring is*

$$F(R_n)(Z; q, t) = \nabla e_n(Z).$$

Corollary 4.3 ($(n+1)^{n-1}$ conjecture)**.**

$$\dim R_n = (n+1)^{n-1}.$$

For details, see [3], sections 3.5, 4.2.

References

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[3] M. Haiman; *Combinatorics, Symmetric Functions and Hilbert Schemes*, available at the Seminar webpage.