DETERMINING SOLUTION SETS OF MATRIX EQUATIONS

LET $A\underline{x} = \underline{b}$ BE A MATRIX EQUATION,

WHERE A $m \times n$ MATRIX, $\underline{b} \in \mathbb{R}^m$.

m: # ROWS; n: # COLUMNS

WE WANT TO FIND <u>ALL</u> $\underline{x} \in \mathbb{R}^n$ SUCH THAT $A\underline{x} = \underline{b}$.

<u>HOW?</u>
- FORM AUGMENTED MATRIX $[A \; \underline{b}]$
- ROW-REDUCE TO <u>REDUCED-ECHELON FORM</u>, SAY $[U \; \underline{c}]$
- PIVOT IN RIGHTMOST COLUMN $\Rightarrow$ <u>NO SOLUTION</u>

- OTHERWISE; USE $[U \; \underline{c}]$ TO OBTAIN <u>ALL</u> SOLUTIONS OF $A\underline{x} = \underline{b}$

<u>RECALL:</u> $\boxed{A\underline{x} = \underline{b} \iff U\underline{x} = \underline{c}.}$

<u>eg.</u> SUPPOSE

$$[A \; \underline{b}] \sim \underbrace{\left[\begin{array}{cccccccc|c} 1 & -2 & 0 & 0 & -3 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]}_{U \qquad\qquad \underline{c}}$$

<u>THEN,</u>

$\underline{x} \in \mathbb{R}^8$ SATISFIES $U\underline{x} = \underline{c}$

$$\Rightarrow \begin{array}{rcl} x_1 - 2x_2 - 3x_5 + x_7 &=& -1 \\ x_3 + 2x_5 &=& 1 \\ x_4 + x_5 &=& 0 \\ x_6 + 2x_7 &=& 2 \\ x_8 &=& 1 \end{array}$$

SO; $\underline{x} \in \mathbb{R}^8$, $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$ SATISFIES $U\underline{x} = \underline{c}$

$$\iff \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 2x_2 + 3x_5 - x_7 - 1 \\ x_2 \\ -2x_5 + 1 \\ -x_5 + 0 \\ x_5 \\ -2x_7 + 2 \\ x_7 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

WITH $x_2, x_5, x_7 \in \mathbb{R}$ ARBITRARY (IE, FREE)

HENCE, ALL SOLUTIONS OF $A\underline{x} = \underline{b}$ ARE OF THIS FORM.