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Math54 Second Midterm Fall 2006 Instructor: D.V. Voiculescu
This is a "closed book" exam, so you may not bring in or use notes or the textbook. Calculators are not allowed.
Please write your name, SID and Discussion Section Number on everything you hand in, including this sheet of paper on which you should provide the answers to Problem IV (true/false questions)
For Problems I, II and III you must show the method and calculations to get the answers (write the solutions to these in your blue book). The Requirement is 20 points.

Problem I (2+2 pts). Compute the characteristic polynomial and the eigenvalues of the matrix: $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

Find an invertible matrix S and a diagonal matrix B so that AS = SB.

Problem II (4pts). Find the equation of the line that best fits the points (-1, 1), (0, 1), (1, -1) in the sense of least squares.

Problem III (1+1pts). a) Compute the Wronskian of the solutions $y_1 = 10$, $y_2 = x^2$ of $xy'' - y' = 0$ on $(-\infty, 0)$.
b) Solve the initial value problem $xy'' - y' = 0$, $y(-1) = y'(-1) = 1$.

Problem IV (10pts, each question 1pt). Check True or False.

10 pts

	True	False	
a) the formula $(a,b)\cdot(c,d) = ad-bc+3ac+3bd$ defines an inner product on $\mathbb{R}^2$		X	False
b) if the 3x3 matrix A is invertible and diagonalizable then A^{-1} is also diagonalizable	X		true
c) if A, B are 2x2 symmetric matrices then AB - BA is also a symmetric matrix		X	false
d) if A, B are 2x2 symmetric matrices then AB + BA is also a symmetric matrix	X		true
e) if A is a square matrix then rank A = rank(A^2)		X	false
f) two linearly independent eigenvectors of a symmetric matrix are always orthogonal		X	false
g) the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is not diagonalizable	X		true
h) the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ is not diagonalizable true	X		true
i) the matrix $\begin{pmatrix} 0.6 & 0 & -0.8 \\ 0 & 0.1 & 0 \\ 0.8 & 0 & 0.6 \end{pmatrix}$ is orthogonal		X	false
j) $\det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{pmatrix} = 123$		X	false

Problem I

Characteristic Polynomial:

$|\lambda I - A|$

$= \left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right|$

$= \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} + \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \right|$

$= \begin{vmatrix} \lambda-1 & 0 & 1 \\ 0 & \lambda+1 & 0 \\ 1 & 0 & \lambda-1 \end{vmatrix} = \cancel{0} + (\lambda+1) \begin{vmatrix} \lambda-1 & 1 \\ 1 & \lambda-1 \end{vmatrix} - 0$

$= (\lambda+1)\left((\lambda-1)^2 - 1\right)$

$= (\lambda+1)(\lambda^2 - 2\lambda)$

$= (\lambda+1)(\lambda-2)\lambda$

$\boxed{\begin{array}{l} \text{Characteristic polynomial} = (\lambda+1)(\lambda-2)\lambda \\ \text{eigenvalues: } -1, 0, 2 \end{array}}$

eigenvectors:

for $\lambda = -1$:

$\lambda I - A = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

row-reduce: $\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ -2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - 2x_3 = 0$, $x_3 = 0 \Rightarrow x_1 = 0$ $\quad \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} r$

for $\lambda = 0$

$\lambda I - A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

row-reduce: $\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_2 = 0$, $x_1 = x_3 =$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} r$

for $\lambda = 2$

$\lambda I - A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

row-reduce: $\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_2 = 0$, $x_1 = -x_3 = r$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} r$

$\therefore$ eigenvectors are $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$\boxed{S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$ ✓

to check:

$$AS = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$SB = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$ ✓

Problem II

$y = mx + b$

$1 = -m + b$

$1 = b$

$-1 = m + b$

put into matrix:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow A\vec{x} = b$$

to solve multiply by A^T

$$A^T A \vec{x} = A^T b$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1/3 \end{bmatrix}$$

$$y = -x + \frac{1}{3}$$

Problem III

a) $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 10 & x^2 \\ 0 & 2x \end{vmatrix} = 20x - 0 = \boxed{20x}$

which $\neq 0$ for all ~~x~~ x in $(-\infty, 0)$, $\therefore$ y_1 and y_2 form a fundamental set of solutions

b) total solution: $y = ay_1 + by_2 = 10a + b\,x^2$

$y(-1) = 10a + b = 1$

$y'(-1) = \cancel{10}\; 2bx\,\big]_{x=-1} = -2b = 1$

$b = -\frac{1}{2}$

$a = \frac{1-b}{10} = \frac{3/2}{10} = \frac{3}{20}$

$y = \cancel{30}\,\frac{3}{20} \cdot 10 + -\frac{1}{2}x^2$

$\frac{3}{2} - \frac{x^2}{2}$

$\boxed{y = \frac{3}{2} - \frac{x^2}{2}}$