Then, the adjoint of $f$ is the morphism
\[ f^+: \mathbb{Q}^3 \to \mathbb{Q}^3 : x \mapsto \begin{bmatrix} 1 & -3 & -1 \\ 1 & 5 & 0 \\ 0 & 2 & 3 \end{bmatrix} x. \]

As a verification, you can check that
\[ B \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -3 & -1 \\ 1 & 5 & 0 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = B \begin{pmatrix} 1 & 0 & 1 \\ -1 & 3 & 0 \\ -3 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \]

### 3.2 Real and complex symmetric bilinear forms

Throughout the remainder of these notes we will assume that $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Throughout this section we will assume that all bilinear forms are **symmetric**.

When we consider symmetric bilinear forms on real or complex vector spaces we obtain some particularly nice results. For a $\mathbb{C}$-vector space $V$ and symmetric bilinear form $B \in \text{Bil}_{\mathbb{C}}(V)$ we will see that there is a basis $B \subset V$ such that
\[ [B]_B = l_{\dim V}. \]

First we introduce the important **polarisation identity**.

**Lemma 3.2.1** (Polarisation identity). Let $B \in \text{Bil}_{\mathbb{K}}(V)$ be a symmetric bilinear form. Then, for any $u, v \in V$, we have
\[ B(u, v) = \frac{1}{2} \left( B(u + v, u + v) - B(u, u) - B(v, v) \right). \]

**Proof:** Left as an exercise for the reader. \qed

**Corollary 3.2.2.** Let $B \in \text{Bil}_{\mathbb{K}}(V)$ be symmetric and nonzero. Then, there exists some nonzero $v \in V$ such that $B(v, v) \neq 0$.

**Proof:** Suppose that the result does not hold: that is, for every $v \in V$ we have $B(v, v) = 0$. Then, using the polarisation identity (Lemma 3.2.1) we have, for every $u, v \in V$,
\[ B(u, v) = \frac{1}{2} \left( B(u + v, u + v) - B(u, u) - B(v, v) \right) = \frac{1}{4} (0 - 0 - 0) = 0. \]

Hence, we must have that $B = 0$ is the zero bilinear form, which contradicts our assumption on $B$. Hence, ther must exist some $v \in V$ such that $B(v, v) \neq 0$. \qed

This seemingly simple result has some profound consequences for nondegenerate complex symmetric bilinear forms.

**Theorem 3.2.3** (Classification of nondegenerate symmetric bilinear forms over $\mathbb{C}$). Let $B \in \text{Bil}_{\mathbb{C}}(V)$ be symmetric and nondegenerate. Then, there exists an ordered basis $B \subset V$ such that
\[ [B]_B = l_{\dim V}. \]

**Proof:** By Corollary 3.2.2 we know that there exists some nonzero $v_1 \in V$ such that $B(v_1, v_1) \neq 0$ (we know that $B$ is nonzero since it is nondegenerate). Let $E_1 = \text{span}_{\mathbb{C}} \{v_1\}$ and consider $E_1^\perp \subset V$.

We have $E_1 \cap E_1^\perp = \{0\}$: indeed, let $x \in E_1 \cap E_1^\perp$. Then, $x = cv_1$, for some $c \in \mathbb{C}$. As $x \in E_1^\perp$ we must have
\[ 0 = B(x, v_1) = B(cv_1, v_1) = cB(v_1, v_1), \]
so that $c = 0$ (as $B(v_1, v_1) \neq 0$). Thus, by Proposition 3.1.17 we must have
\[ V = E_1 \oplus E_1^\perp. \]

\footnote{Actually, all results that hold for $\mathbb{C}$-vector space also hold for $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is an algebraically closed field. To say that $\mathbb{K}$ is algebraically closed means that the Fundamental Theorem of Algebra holds for $\mathbb{K}[t]$: equivalently, every polynomial $f \in \mathbb{K}[t]$ can be written as a product of linear factors.}
Moreover, $B$ restricts to a nondegenerate symmetric bilinear form on $E_1^\perp$: indeed, the restriction is

$$B|_{E_1^\perp} : E_1^\perp \times E_1^\perp \to \mathbb{C}; (u, u') \mapsto B(u, u'),$$

and this is a symmetric bilinear form. We need to check that it is nondegenerate. Suppose that $w \in E_1^\perp$ is such that, for every $z \in E_1^\perp$ we have

$$B(z, w) = 0.$$

Then, for any $v \in V$, we have $v = cv_1 + z, z \in E_1^\perp, c \in \mathbb{C}$, so that

$$B(v, w) = B(cv_1 + z, w) = cB(v_1, w) + B(z, w) = 0 + 0 = 0,$$

where we have used the assumption on $w$ and that $w \in E_1^\perp$. Hence, using nongeneracy of $B$ on $V$ we see that $w = 0_V$. Hence, we have that $B$ is also nondegenerate on $E_1^\perp$.

As above, we can now find $v_2 \in E_1^\perp$ such that $B(v_2, v_2) \neq 0$ and, if we denote $E_2 = \text{span}_\mathbb{C}\{v_2\}$, then

$$E_1^\perp = E_2 \oplus E_2^\perp,$$

where $E_2^\perp$ is the $B$-complement of $E_2$ in $E_1^\perp$. Hence, we have

$$V = E_1 \oplus E_2 \oplus E_2^\perp.$$

Proceeding in the manner we obtain

$$V = E_1 \oplus \cdots \oplus E_n,$$

where $n = \dim V$, and where $E_i = \text{span}_\mathbb{C}\{v_i\}$. Moreover, by construction we have that

$$B(v_i, v_j) = 0, \text{ for } i \neq j.$$

Define

$$b_i = \frac{1}{\sqrt{B(v_i, v_i)}} v_i,$$

we know that the square root $\sqrt{B(v_i, v_i)}$ exists (and is nonzero) since we are considering $\mathbb{C}$-scalars. Then, it is easy to see that

$$B(b_i, b_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Finally, since

$$V = \text{span}_\mathbb{C}\{b_1\} \oplus \cdots \oplus \text{span}_\mathbb{C}\{b_n\},$$

we have that $B = (b_1, \ldots, b_n)$ is an ordered basis such that

$$[B]_B = I_n.$$

**Corollary 3.2.4.** Let $A \in \text{GL}_n(\mathbb{C})$ be a symmetric matrix (so that $A = A^t$). Then, there exists $P \in \text{GL}_n(\mathbb{C})$ such that

$$P^t A P = I_n.$$

*Proof:* This is just Theorem 3.2.3 and Proposition 3.1.8 applied to the bilinear form $B_A \in \text{Bil}_\mathbb{C}(\mathbb{C}^n)$. The assumptions on $A$ ensure that $B_A$ is symmetric and nondegenerate. \qed

**Corollary 3.2.5.** Suppose that $X, Y \in \text{GL}_n(\mathbb{C})$ are both symmetric. Then, there is a nondegenerate bilinear form $B \in \text{Bil}_\mathbb{C}(\mathbb{C}^n)$ and bases $B, C \subset \mathbb{C}^n$ such that

$$X = [B]_B, \ Y = [B]_C.$$

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61 This is a consequence of the Fundamental Theorem of Algebra: for any $c \in \mathbb{C}$ we have that $t^2 - c = 0$, has a solution.
Proof: By the previous Corollary we can find \( P, Q \in \text{GL}_n(\mathbb{C}) \) such that
\[
P^tXP = I_n = Q^tYQ \implies (Q^{-1})^tP^tXPQ^{-1} = Y \implies (PQ)^{-1})^tXPQ^{-1} = Y.
\]
Now, let \( B = B_X \in \text{Bil}_C(\mathbb{C}^n), B = S(n) \) and \( C = (c_1, \ldots, c_n) \), where \( c_i \) is the \( i \)th column of \( PQ^{-1} \). Then, the above identity states that
\[
[B]_C = P_{B \rightarrow C}^t[B]_B P_{B \rightarrow C} = Y.
\]
The result follows. \( \square \)

The situation is not as simple for an \( \mathbb{R} \)-vector space \( V \) and nondegenerate symmetric bilinear form \( B \in \text{Bil}_\mathbb{R}(V) \), however we can still obtain a nice classification result.

**Theorem 3.2.6 (Sylvester’s law of inertia).** Let \( V \) be an \( \mathbb{R} \)-vector space, \( B \in \text{Bil}_\mathbb{R}(V) \) a nondegenerate symmetric bilinear form. Then, there is an ordered basis \( B \subset V \) such that \([B]_B \) is a diagonal matrix
\[
[B]_B = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix},
\]
where \( d_i \in \{1, -1\} \).

Moreover, if \( p \) = the number of 1s appearing on the diagonal and \( q \) = the number of \(-1\)s appearing on the diagonal, then \( p \) and \( q \) are invariants of \( B \): this means that if \( C \subset V \) is any other basis of \( V \) such that
\[
[B]_C = \begin{bmatrix} e_1 & & & \\ & e_2 & & \\ & & \ddots & \\ & & & e_n \end{bmatrix},
\]
where \( e_j \in \{1, -1\} \), and \( p' \) (resp. \( q' \)) denotes the number of 1s (resp. \(-1\)s) on the diagonal. Then,
\[
p = p', \ q = q'.
\]

**Proof:** The proof is similar to the proof of Theorem 3.2.3: we determine \( v_1, \ldots, v_n \in V \) such that
\[
V = \text{span}_\mathbb{R}\{v_1\} \oplus \cdots \oplus \text{span}_\mathbb{R}\{v_n\},
\]
and with \( B(v_i, v_j) = 0 \), whenever \( i \neq j \). However, we now run into a problem: what if \( B(v_i, v_i) < 0 \)? We can’t find a real square root of a negative number so we can’t proceed as in the complex case. However, if we define
\[
\delta_i = \sqrt{|B(v_i, v_i)|}, \ \text{for every} \ i,
\]
then we can obtain a basis \( B = (b_1, \ldots, b_n) \), where we define
\[
b_i = \frac{1}{\delta_i} v_i.
\]

Then, we see that
\[
B(b_i, b_j) = \begin{cases} 
0, & i \neq j, \\
\pm 1, & i = j,
\end{cases}
\]
and \([B]_B \) is of the required form.

Let us reorder \( B \) so that, for \( i = 1, \ldots, p \), we have \( B(b_i, b_i) > 0 \). Then, if we denote
\[
P = \text{span}_\mathbb{R}\{b_1, \ldots, b_p\}, \ \text{and} \ Q = \text{span}_\mathbb{R}\{b_{p+1}, \ldots, b_n\},
\]
we have
\[
\dim P = p, \ \dim Q = q = (n - p).
\]
We see that the restriction of $B$ to $P$ satisfies

$$B(u, u) > 0, \text{ for every } u \in P,$$

and that if $P \subset P', P \neq P'$, with $P' \subset V$ a subspace, then there is some $v \in P'$ such that $B(v, v) \leq 0$: indeed, as $v \notin P$ then we have

$$v = \lambda_1 b_1 + \ldots + \lambda_p b_p + \mu_1 b_{p+1} + \ldots + \mu_q b_n,$$

and some $\mu_j \neq 0$. Then, since $P \subset P'$ we must have $b_{p+j} \in P'$ and

$$B(b_{p+j}, b_{p+j}) < 0.$$

Hence, we can see that $p$ is the dimension of the largest subspace $U$ of $V$ for which the restriction of $B$ to $U$ satisfies $B(u, u) > 0$, for every $u \in U$.

Similarly, we can define $q$ to be the dimension of the largest subspace $U' \subset V$ for which the restriction of $B$ to $U'$ satisfies $B(u', u') < 0$, for every $u' \in U'$.

Therefore, we have defined $p$ and $q$ only in terms of $B$ so that they are invariants of $B$. \qed

**Corollary 3.2.7.** For every symmetric $A \in \text{GL}_n(\mathbb{R})$, there exists $X \in \text{GL}_n(\mathbb{R})$ such that

$$X^t AX = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \end{bmatrix},$$

with $d_i \in \{1, -1\}$.

**Definition 3.2.8.** Suppose that $B \in \text{Bil}_R(V)$ is nondegenerate and symmetric and that $p, q$ are as in Theorem 3.2.6. Then, we define the **signature of $B$**, denoted $\text{sig}(B)$, to be the number

$$\text{sig}(B) = p - q.$$

It is an invariant of $B$: for any basis $B \subset V$ such that

$$[B]_B = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \end{bmatrix},$$

with $d_i \in \{1, -1\}$, the quantity $p - q$ is the same.

### 3.2.1 Computing the canonical form of a real nondegenerate symmetric bilinear form

([1], p.185-191)

Suppose that $B \in \text{Bil}_R(V)$ is symmetric and nondegenerate, with $V$ a finite dimensional $\mathbb{R}$-vector space. Suppose that $B \subset V$ is an ordered basis such that

$$[B]_B = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \end{bmatrix},$$

where $d_i \in \{1, -1\}$. Such a basis exists by Theorem 3.2.6. How do we determine $B$?

Suppose that $C \subset V$ is any ordered basis. Then, we know that

$$P^t_C \cdot [B]_C \cdot P^f_C = [B]_B,$$
by Proposition 3.1.8. Hence, the problem of determining $B$ is equivalent to the problem of determining $P_{C \leftarrow B}$ (since we already know $C$ and we can use $P_{C \leftarrow B}$ to determine $B$).

Therefore, suppose that $A = [a_{ij}] \in \text{GL}_n(\mathbb{R})$ is symmetric. We want to determine $P \in \text{GL}_n(\mathbb{R})$ such that

$$P^tAP = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \\ & & & d_n \end{bmatrix},$$

where $d_i \in \{1, -1\}$.

Consider the column vector of variables

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Then, we have

$$\mathbf{x}^tA\mathbf{x} = a_{11}x_1^2 + \ldots + a_{nn}x_n^2 + 2\sum_{i<j} a_{ij}x_ix_j.$$\(^{63}\)

By performing the ‘completing the square’ process for each variable $x_i$, we will find variables

$$y_1 = q_{11}x_1 + q_{12}x_2 + \ldots + q_{1n}x_n,$$

$$y_2 = q_{21}x_1 + q_{22}x_2 + \ldots + q_{2n}x_n,$$

$$\vdots$$

$$y_n = q_{n1}x_1 + q_{n2}x_2 + \ldots + q_{nn}x_n$$

such that

$$\mathbf{y}^tA\mathbf{y} = y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_n^2.$$

Then, $P = [q_{ij}]^{-1}$ is the matrix we are looking for.

Why? The above system of equations corresponds to the matrix equation

$$\mathbf{y} = Q\mathbf{x}.$$\(^{63}\) $Q = [q_{ij}] \in \text{GL}_n(\mathbb{R})$, which we can consider as a change of coordinate transformation $P_{B \leftarrow S^{(n)}}$ from the standard basis $S^{(n)} \subset \mathbb{R}^n$ to a basis $B$ (we consider $\mathbf{x}$ to be the $S^{(n)}$-coordinate vector of the corresponding element of $\mathbb{R}^n$).

Then, we see that

$$(P\mathbf{y})^tA(P\mathbf{y}) = \mathbf{y}^tA\mathbf{y} = y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_n^2,$$

where $P = Q^{-1}$. As

$$\mathbf{y}^tP^tAP\mathbf{y} = (P\mathbf{y})^tA(P\mathbf{y}) = y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_n^2 = \mathbf{y}^t \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix} \mathbf{y},$$

we see that $P^tAP$ is of the desired form. Moreover, $B$ is the required basis.

It is better to indicate this method through an example.

\(^{62}\)Why? \(^{63}\)The assignment $\mathbf{x} \mapsto \mathbf{x}^tA\mathbf{x}$ is called a quadratic form. The study of quadratic forms and their properties is primarily determined by the symmetric bilinear forms defined by $A$. 

97
Example 3.2.9. 1. Let

\[ A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & 1 & -2 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & -1 \end{bmatrix}, \]

so that \( A \) is symmetric and invertible. Consider the column vector of variable \( x \) as above. Then, we have

\[ x^t A x = x_1^2 + 2x_2^2 - x_3^2 - 2x_1x_3 + 4x_1x_4 + 2x_2x_3 - 4x_2x_4. \]

Let’s complete the square with respect to \( x_1 \): we have

\[ x_1^2 - 2x_1x_3 + (x_3 - 2x_4)^2 = (x_1 - (x_3 - 2x_4))^2 + 2x_2^2 - x_3^2 - 2x_1x_3 - 4x_2x_4 + 4x_3x_4 \]

Now we set

\[ y_1 = x_1 - x_3 + 2x_4. \]

Then, complete the square with respect to the remaining \( x_2 \) terms: we have

\[ y_1^2 + 2x_2^2 - x_3^2 - 5x_4^2 + 2x_2x_3 - 4x_2x_4 + 4x_3x_4 \]

\[ = y_1^2 + 2(x_2^2 + x_2(x_3 - 2x_4) + \frac{1}{4}(x_3 - 2x_4)^2) - \frac{1}{2}(x_3 - 2x_4)^2 - x_3^2 - 2x_2x_3 - 4x_2x_4 + 4x_3x_4 \]

Now we set

\[ y_2 = \sqrt{2} (x_2 + \frac{1}{2}x_3 - x_4). \]

We obtain

\[ x_1^2 + 2x_2^2 - x_3^2 - 2x_1x_3 + 4x_1x_4 + 2x_2x_3 - 4x_2x_4 = y_1^2 + y_2^2 - \frac{3}{2}x_3^2 - 7x_4^2 - 2x_3x_4. \]

Completing the square with respect to \( x_3 \) we obtain

\[ y_1^2 + y_2^2 - \frac{3}{2}x_3^2 - 7x_4^2 - 2x_3x_4 \]

\[ = y_1^2 + y_2^2 - \frac{3}{2} (x_3^2 + \frac{14}{3}x_3x_4 + \frac{49}{9}x_4^2) + \frac{49}{6}x_4^2 \]

\[ = y_1^2 + y_2^2 - \frac{3}{2} (x_3 + \frac{7}{3}x_4)^2 + \frac{49}{6}x_4^2. \]

Then, set

\[ y_3 = \sqrt{\frac{3}{2}} (x_3 + \frac{7}{3}x_4), \]

\[ y_4 = \frac{7}{\sqrt{6}} x_4. \]

So, if we let

\[ Q = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} & -\sqrt{2} \\ 0 & 0 & \sqrt{3} & \frac{7}{\sqrt{6}} \\ 0 & 0 & 0 & \frac{7}{\sqrt{6}} \end{bmatrix}, \]

then we have

\[ y = Qx. \]
Hence, if we define $P = Q^{-1}$, then we have that

$$P^tAP = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$ 

Hence, we have that $p = 3, q = 1$ and that if $B_A \in \text{Bil}_R(\mathbb{R}^4)$ then

$$\text{sig}(B_A) = 3 - 1 = 2.$$ 

2. Consider the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

which is symmetric and invertible. Consider the column vector of variables $\mathbf{x}$ as before. Then, we have

$$\mathbf{x}^tA\mathbf{x} = -x_1^2 + 2x_2x_3.$$ 

Proceeding as before, we ‘complete the square’ with respect to $x_2$ (we don’t need to complete the square for $x_1$): we have

$$-x_1^2 + 2x_2x_3 = -x_1^2 + \frac{1}{2}(x_2 + x_3)^2 - \frac{1}{2}(x_2 - x_3)^2.$$ 

Hence, if we let

$$y_1 = x_1, \\
y_2 = \frac{1}{\sqrt{2}}(x_2 + x_3), \\
y_3 = \frac{1}{\sqrt{2}}(x_2 - x_3)$$

then we have

$$\mathbf{x}^tA\mathbf{x} = -y_1^2 + y_2^2 - y_3^2.$$ 

Furthermore, if we let

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix},$$

and defined $P = Q^{-1}$, then

$$P^tAP = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$ 

Hence, $p = 1, q = 2$ and

$$\text{sig}(B_A) = -1.$$ 

3.3 Euclidean spaces

Throughout this section $V$ will be a finite dimensional $\mathbb{R}$-vector space and $K = \mathbb{R}$.

Definition 3.3.1. Let $B \in \text{Bil}_K(V)$ be a symmetric bilinear form. We say that $B$ is an inner product on $V$ if $B$ satisfies the following property:

$$B(v, v) \geq 0, \; \text{for every} \; v \in V, \; \text{and} \; B(v, v) = 0 \iff v = 0_V.$$ 

If $B \in \text{Bil}_K(V)$ is an inner product on $V$ then we will write

$$\langle u, v \rangle \overset{\text{def}}{=} B(u, v).$$