is a block diagonal matrix, with $A_i \in \text{Mat}_{\dim U_i}(\mathbb{C})$. In fact, we can assume that $B = B_1 \cup \ldots \cup B_k$, with $B_i$ an ordered basis of $U_i$, and that

$$A_i = [f|_{U_i}]_{B_i},$$

where $f|_{U_i} : U_i \to U_i$ is the restriction of $f$ to $U_i$.40

2.3 Nilpotent endomorphisms

(II, p.133-136)

In this section we will consider those linear endomorphisms $f \in \text{End}_C(V)$ whose only eigenvalue is 0. This necessarily implies that

$$\chi_f(\lambda) = \lambda^n.$$

We will see that for such endomorphisms there is a (ordered) basis $B$ of $V$ such that $[f]_B$ is ‘nearly diagonal’.41

**Definition 2.3.1.** An endomorphism $f \in \text{End}_C(V)$ is called *nilpotent* if there exists $r \in \mathbb{N}$ such that $f^r = 0_{\text{End}_C(V)}$, so that $f^r(v) = 0_V$, for every $v \in V$.

A matrix $A \in \text{Mat}_n(\mathbb{C})$ is called *nilpotent* if the endomorphism $T_A \in \text{End}_C(\mathbb{C}^n)$ is nilpotent.

**Lemma 2.3.2.** Let $f \in \text{End}_C(V)$ be a nilpotent endomorphism. Then, the only eigenvalue of $f$ is $\lambda = 0$, so that $\chi_f(\lambda) = \lambda^\dim V$.

*Proof:* Suppose that $v \in V$ is an eigenvector of $f$ with associated eigenvalue $\lambda$. Therefore, we have $v \neq 0$ and $f(v) = \lambda v$. Suppose that $f^r = 0$. Then,

$$0 = f^r(v) = f \circ \cdots \circ f(v) = f \circ \cdots \circ f(\lambda v) = \lambda^r v.$$

Thus, as $v \neq 0$ we must have $\lambda^r = 0$ (Proposition 1.2.5) implying that $\lambda = 0$. \hfill \Box

For a nilpotent endomorphism $f$ (resp. matrix $A \in \text{Mat}_n(\mathbb{C})$) we define the exponent of $f$ (resp. of $A$), denoted $\eta(f)$ (resp. $\eta(A)$), to be the smallest $r \in \mathbb{N}$ such that $f^r = 0$ (resp. $A^r = 0$). Therefore, if $\eta(f) = r$ then there exists $v \in V$ such that $f^{r-1}(v) \neq 0_V$.

For $v \in V$ we define the height of $v$ (with respect to $f$), denoted $\text{ht}(v)$, to be the smallest integer $m$ such that $f^m(v) = 0_V$, while $f^{m-1}(v) \neq 0_V$. Hence, for every $v \in V$ we have $\text{ht}(v) \leq \eta(f)$.

Define $H_k = \{v \in V \mid \text{ht}(v) \leq k\}$, the set of vectors that have height no greater than $k$; this is a subspace of $V$.41

Let $f \in \text{End}_C(V)$ be a nilpotent endomorphism. Then, we obviously have $H_{\eta(f)} = V$, $H_0 = \{0_V\}$ and a sequence of subspaces

$$\{0_V\} = H_0 \subset H_1 \subset \cdots \subset H_{\eta(f)-1} \subset H_{\eta(f)} = V.$$

Let us denote

$$\dim H_i = m_i,$$

so that we have

$$0 = m_0 \leq m_1 \leq \ldots \leq m_{\eta(f)-1} \leq m_{\eta(f)} = \dim V.$$

**We are going to construct a basis of** $V$: for ease of notation we let $\eta(f) = k$. Assume that $k \neq 1$, so that $f$ is not the zero endomorphism of $V$.

1. Let $G_k$ be a complementary subspace of $H_{k-1}$ so that

$$H_k = H_{k-1} \oplus G_k,$$

and let $(z_1, \ldots, z_{p_k})$ be an ordered basis of $G_k$. Then, since $z_j \in H_k \setminus H_{k-1}$ we have that $f^{k-1}(z_j) \neq 0_V$, for each $j$.40

40 This is a well-defined function since $U_i$ is $f$-invariant.

41 Exercise: show this.
2. Consider the vectors \( f(z_1), f(z_2), \ldots, f(z_{p_1}) \). We have, for each \( j \),
\[
f^{k-1}(f(z_j)) = f^k(z_j) = 0_V,
\]
since \( z_j \in H_k \),
so that \( f(z_j) \in H_{k-1} \) for each \( j \). In addition, we can’t have \( f(z_j) \in H_{k-2} \), else
\[
0_V = f^{k-2}(f(z_j)) = f^{k-1}(z_j),
\]
implying that \( z_j \in H_{k-2} \).
Moreover, the set \( S_1 = \{ f(z_1), f(z_2), \ldots, f(z_{p_1}) \} \subset H_{k-1} \setminus H_{k-2} \) is linearly independent: indeed, suppose that there is a linear relation
\[
c_1 f(z_1) + \cdots + c_{p_1} f(z_{p_1}) = 0_V,
\]
with \( c_1, \ldots, c_{p_1} \in \mathbb{C} \). Then, since \( f \) is a linear morphism we obtain
\[
f(c_1 z_1 + \cdots + c_{p_1} z_{p_1}) = 0_V,
\]
so that \( c_1 z_1 + \cdots + c_{p_1} z_{p_1} \in H_1 \subset H_{k-1} \).
Hence, we have \( c_1 z_1 + \cdots + c_{p_1} z_{p_1} \in H_{k-1} \cap G_k = \{ 0_V \} \), so that \( c_1 z_1 + \cdots + c_{p_1} z_{p_1} = 0_V \). Hence, because \( \{ z_1, \ldots, z_{p_1} \} \) is linearly independent we must have \( c_1 = \cdots = c_{p_1} = 0 \in \mathbb{C} \). Thus, \( S_1 \) is linearly independent.

3. \( \text{span}_\mathbb{C} S_1 \cap H_{k-2} = \{ 0_V \} \): otherwise, we could find a linear combination
\[
c_1 f(z_1) + \cdots + c_{p_1} f(z_{p_1}) \in H_{k-2},
\]
with some \( c_j \neq 0 \). Then, we would have
\[
0_V = f^{k-2}(c_1 f(z_1) + \cdots + c_{p_1} f(z_{p_1})) = f^{k-1}(c_1 z_1 + \cdots + c_{p_1} z_{p_1}),
\]
so that \( c_1 z_1 + \cdots + c_{p_1} z_{p_1} \in H_{k-1} \cap G_k = \{ 0_V \} \) which gives all \( c_j = 0 \), by linear independence of the \( z_j \)’s. But this contradicts that some \( c_j \) is nonzero so that our initial assumption that \( \text{span}_\mathbb{C} S_1 \cap H_{k-2} \neq \{ 0_V \} \) is false.

Hence, we have
\[
\text{span}_\mathbb{C} S_1 + H_{k-2} = \text{span}_\mathbb{C} S_1 \oplus H_{k-2} \subset H_{k-1}.
\]
In particular, we see that \( m_k - m_{k-1} \leq m_{k-1} - m_{k-2} \).

4. Let \( G_{k-1} \) be a complementary subspace of \( H_{k-2} \oplus \text{span}_\mathbb{C} S_1 \) in \( H_{k-1} \), so that
\[
H_{k-1} = H_{k-2} \oplus \text{span}_\mathbb{C} S_1 \oplus G_{k-1},
\]
and let \( (z_{p_1+1}, \ldots, z_{p_2}) \) be an ordered basis of \( G_{k-1} \).

5. Consider the subset \( S_2 = \{ f^2(z_1), \ldots, f^2(z_{p_1}), f(z_{p_1+1}), \ldots, f(z_{p_2}) \} \). Then, as in 2, 3, 4 above we have that
\[
S_2 \subset H_{k-2} \setminus H_{k-3},
\]
\( S_2 \) is linearly independent and \( \text{span}_\mathbb{C} S_2 \cap H_{k-3} = \{ 0_V \} \). Therefore, we have
\[
\text{span}_\mathbb{C} S_2 + H_{k-3} = \text{span}_\mathbb{C} S_2 \oplus H_{k-3} \subset H_{k-2},
\]
so that \( m_{k-1} - m_{k-2} \leq m_{k-2} - m_{k-3} \).
6. Let \( G_{k-2} \) be a complementary subspace of \( \text{span}_C \{ S_2 \oplus H_{k-3} \} \) in \( H_{k-2} \), so that
\[
H_{k-2} = H_{k-3} \oplus \text{span}_C \{ S_2 \oplus G_{k-2} \},
\]
and \( (z_{p_0+1}, \ldots, z_{p_1}) \) be an ordered basis of \( G_{k-2} \).

7. Consider the subset \( S_3 = \{ f^3(z_1), \ldots, f^3(z_{p_0}), f^2(z_{p_0+1}), \ldots, f^2(z_{p_1}), f(z_{p_1+1}), \ldots, f(z_{p_1}) \} \). Again, it can be shown that
\[
S_3 \subset H_{k-3} \setminus H_{k-4}.
\]
\( S_3 \) is linearly independent and \( \text{span}_C S_3 \cap H_{k-4} = \{ 0_V \} \). We obtain \( m_{k-2} - m_{k-3} \leq m_{k-3} - m_{k-4} \).

8. Proceed in this fashion to obtain a basis of \( V \). We denote the vectors we have obtained in a table (2.3.1)
\[
\begin{array}{cccc}
    z_1, & \ldots & z_{p_1}, \\
    f(z_1), & \ldots & f(z_{p_1}), & z_{p_0+1}, & \ldots & z_{p_1}, \\
    \vdots & & \vdots & \vdots \\
    f^{k-1}(z_1), & \ldots & f^{k-1}(z_{p_1}), & f^{k-2}(z_{p_0+1}), & \ldots & f^{k-2}(z_{p_1}), & \ldots & z_{p_1+1}, & \ldots & z_{p_1},
\end{array}
\]
where the vectors in the \( i \)-th row have height \( k - i + 1 \), so that vectors in the last row have height 1.

Also, note that each column determines an \( f \)-invariant subspace of \( V \), namely the span of the vectors in the column.

**Lemma 2.3.3.** Let \( W_i \) denote the span of the \( i \)-th column of vectors in the table above. Set \( p_0 = 1 \). Then,
\[
\dim W_i = k - j, \quad \text{if } p_j + 1 \leq i \leq p_{j+1}.
\]

**Proof:** Suppose that \( p_j + 1 \leq i \leq p_{j+1} \). Then, we have
\[
W_i = \text{span}_C \{ z_i, f(z_i), \ldots, f^{k-1-j}(z_i) \}.
\]

Suppose that there exists a linear relation
\[
c_0 z_i + c_1 f(z_i) + \ldots + c_{k-j-1} f^{k-1-j}(z_i) = 0_V.
\]
Then, applying \( f^{k-1-j} \) to both sides of this equation gives
\[
c_0 f^{k-1-j}(z_i) + c_1 f^{k-j}(z_i) + \ldots + c_{k-j-1} f^{2k-2j-2}(z_i) = 0_V.
\]
Now, as \( z_i \) has height \( k - j \) (this follows because the vector at the top of the \( i \)-th column is in the \( (k-j) \)-th row, therefore as height \( (k-j) \)) the previous equation gives
\[
c_0 f^{k-1-j}(z_i) + 0_V + \ldots + 0_V = 0_V,
\]
so that \( c_0 = 0 \), since \( f^{k-1-j}(z_i) \neq 0_V \). Thus, we are left with a linear relation
\[
c_1 f(z_i) + \ldots + c_{k-j-1} f^{k-1-j}(z_i) = 0_V,
\]
and applying \( f^{j-k-2} \) to this equation will give \( c_1 = 0 \), since \( f(z_i) \) has height \( k - j - 1 \). Proceeding in this manner we find that \( c_0 = c_1 = \ldots = c_{k-j-1} = 0 \) and the result follows. \( \square \)

Thus, the information recorded in (2.3.1) and Lemma 2.3.3 proves the following

**Theorem 2.3.4.** Let \( f \in \text{End}_C(V) \) be a nilpotent endomorphism with exponent \( \eta(f) = k \). Then, there exists integers \( d_1, \ldots, d_k \in \mathbb{Z}_{\geq 0} \) so that
\[
kd_1 + (k - 1)d_2 + \ldots + 2d_{k-1} + d_k = \dim V.
\]
and \(f\)-invariant subspaces

\[
W_1^{(k)} , \cdots , W_d^{(k)} , W_1^{(k-1)} , \cdots , W_d^{(k-1)} , \cdots , W_1^{(1)} , \cdots , W_d^{(1)} \subset V ,
\]

with \(\dim \mathbb{C} W_j^{(i)} = j\), such that

\[
V = W_1^{(k)} \oplus \cdots \oplus W_d^{(k)} \oplus W_1^{(k-1)} \oplus \cdots \oplus W_d^{(k-1)} \oplus \cdots \oplus W_1^{(1)} \oplus \cdots \oplus W_d^{(1)} .
\]

Moreover, there is an ordered basis \(B_j^{(i)}\) of \(W_j^{(i)}\) such that

\[
[f|_{W_j^{(i)}}]_{B_j^{(i)}} = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix}.
\]

We call such matrices \(0\)-Jordan blocks. Hence, we can write the matrix of \(f\) relative to \(B = \bigcup_{i,j} B_j^{(i)}\) as a block diagonal matrix for which all of the blocks are \(0\)-Jordan blocks and are of nonincreasing size as we move from left to right.

Moreover, the geometric multiplicity of \(0\) as an eigenvalue of \(f\) is equal to the number of blocks of the matrix \([f]_B\) and this number equals the sum

\[d_1 + d_2 + \ldots + d_k = \dim E_0.\]

\textbf{Proof:} Everything except for the final statement follows from the construction of the basis \(B\) made prior to the Theorem.

The last statement is shown as follows: we have that \(E_0 = H_1\), so that the \(0\)-eigenspace of \(f\) consists of the set of all height 1 vectors in \(V\). Moreover, the construction of the basis \(B\) shows that a basis of \(H_1\) is given by the bottom row of the table (2.3.1) and that this basis has the size specified.

\textbf{Corollary 2.3.5.} Let \(A \in \text{Mat}_{n}(\mathbb{C})\) be a nilpotent matrix. Then, \(A\) is similar to a block diagonal matrix for which all of the blocks are \(0\)-Jordan blocks.

\textbf{Proof:} Consider the endomorphism \(T_A \in \text{End}_{\mathbb{C}}(\mathbb{C}^n)\) and apply Theorem 2.3.4. Then, we have a basis \(B\) such that \([T_A]_B\) takes the desired form. Now, use Corollary 1.7.7 and \([T_A]_{S(n)} = A\) to deduce the result.

\textbf{Definition 2.3.6.} Let \(n \in \mathbb{N}\). A \textit{partition} of \(n\) is a decomposition of \(n\) into a sum of positive integers.

If we have a partition of \(n\)

\[n = n_1 + \ldots + n_l,\] with \(n_1, \ldots, n_l \in \mathbb{N}, \ n_1 \leq n_2 \leq \ldots \leq n_l,\]

then we denote this partition

\[1^{n_1}2^{n_2} \cdots \ell^{n_l},\]

where we are assuming that \(1\) appears \(r_1\) times in the partition of \(n\), \(2\) appears \(r_2\) times etc.

For example, consider the partition of 13

\[13 = 1 + 1 + 1 + 2 + 4 + 4,\]

then we denote this partition

\[1^32^14^2.\]

\[\text{Check this.}\]
For a nilpotent endomorphism \( f \in \text{End}_\mathbb{C}(V) \) we define its \textit{nilpotent class} to be the set of all nilpotent endomorphisms \( g \) of \( V \) for which there is some ordered basis \( C \subset V \) with

\[
[f]_B = [g]_C,
\]

where \( B \) is the basis described in Theorem 2.3.4.

We define the \textit{partition associated to the nilpotent class of} \( f \), denoted \( \pi(A) \), to be the partition \( 2^{d_1}2^{d_2} \ldots 2^{d_l} \) obtained in Theorem 2.3.4. We will also call this partition the \textit{partition associated to} \( f \).

For a matrix \( A \in \text{Mat}_n(\mathbb{C}) \) we define its nilpotent class (or \textit{similarity class}) to be the nilpotent class of the endomorphism \( T_A \). We define the \textit{partition associated to} \( A \) to be the partition associated to \( T_A \).

\textbf{Theorem 2.3.7} (Classification of nilpotent endomorphisms). Let \( f, g \in \text{End}_\mathbb{C}(V) \) be nilpotent endomorphisms of \( V \). Then, \( f \) and \( g \) lie in the same nilpotent class if and only if the partitions associated to \( f \) and \( g \) coincide.

\textbf{Corollary 2.3.8}. Let \( A, B \in \text{Mat}_n(\mathbb{C}) \) be nilpotent matrices. Then, \( f \) and \( g \) are similar if and only if the partitions associated to \( A \) and \( B \) coincide.

\textit{Proof:} We simply note that if \( T_A \) and \( T_B \) are in the same nilpotent class then there are bases \( B, C \subset \mathbb{C}^n \) such that

\[
[T_A]_B = [T_B]_C.
\]

Hence, if \( P_1 = P_{S(\pi) \leftrightarrow B}, P_2 = P_{S(\pi) \leftrightarrow C} \) then we must have

\[
P_1^{-1}AP_1 = P_2^{-1}BP_2,
\]

so that

\[
P_2P_1^{-1}AP_1P_2^{-1} = B.
\]

Now, since \( P_2P_1^{-1} = (P_1P_2^{-1})^{-1} \) we have that \( A \) and \( B \) are similar precisely when \( T_A \) and \( T_B \) are in the same nilpotent class. The result follows. \( \square \)

\subsection*{2.3.1 Determining partitions associated to nilpotent endomorphisms}

Given a nilpotent endomorphism \( f \in \text{End}_\mathbb{C}(V) \) (or nilpotent matrix \( A \in \text{Mat}_n(\mathbb{C}) \)) how can we determine the partition associated to \( f \) (resp. \( A \))?

Once we have chosen an ordered basis \( B \) of \( V \) we can consider the nilpotent matrix \( [f]_B \). Then, the problem of determining the partition associated to \( f \) reduces to determining the partition associated to \( [f]_B \). As such, we need only determine the partition associated to a nilpotent matrix \( A \in \text{Mat}_n(\mathbb{C}) \).

1. Determine the exponent of \( A \), \( \eta(A) \), by considering the products \( A^2, A^3 \), etc. The first \( r \) such that \( A^r = 0 \) is the exponent of \( A \).

2. We can determine the subspaces \( H_i \) since

\[
H_i = \{ x \in \mathbb{C}^n \mid \text{ht}(x) \leq i \} = \ker T_{A^i}.
\]

In particular, we have that \( \dim H_i \) is the number of non-pivot columns of \( A^i \).

3. \( d_1 = \dim H_{\eta(A)} - \dim H_{\eta(A)-1} \).

4. \( d_2 = \dim H_{\eta(A)-1} - \dim H_{\eta(A)-2} - d_1 \).

5. \( d_3 = \dim H_{\eta(A)-2} - \dim H_{\eta(A)-3} - d_2 \).

6. Thus, we can see that \( d_i = \dim H_{\eta(A)-(i-1)} - \dim H_{\eta(A)-i} - d_{i-1} \), for \( 1 \leq i \leq \eta(A) \).

Hence, the partition associated to \( A \) is

\[
\pi(A) = 1^{d_1}2^{d_2} \ldots \eta(A)^{d_l}.
\]

66
Example 2.3.9. Consider the endomorphism
\[ f : \mathbb{C}^5 \to \mathbb{C}^5 ; \]
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix}
\mapsto
\begin{bmatrix}
x_2 \\
0 \\
x_4 \\
0 \\
0 \\
\end{bmatrix}.
\]

Then, with respect to the standard basis \( S^{(5)} \) we have that
\[
A \overset{\text{def}}{=} [f]_{S^{(5)}} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

You can check that \( A^2 = 0 \) so that \( \eta(A) = 2 \). Then,
- \( d_1 = \dim H_2 - \dim H_1 = 5 - 3 = 2 \), since \( H_1 = \ker T_A \) has dimension 3 (there are 3 non-pivot columns of \( A \)).
- \( d_2 = \dim H_1 - \dim H_0 - d_1 = 3 - 0 - 2 = 1 \), since \( H_0 = \{0\} \).

Hence, the partition associated to \( A \) is
\[
\pi(A) : 12^2 \leftrightarrow 1 + 2 + 2 = 5;
\]
there are three 0-Jordan blocks - two of size 2 and one of size 1.

You can check that the following matrix \( B \) is nilpotent
\[
B =
\begin{bmatrix}
1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 \\
\end{bmatrix}
\]
and that the partition associated to \( B \) is
\[
\pi(B) : 1^3 2 \leftrightarrow 1 + 1 + 1 + 2 = 5
\]
- We have \( B^2 = 0 \) so that \( \eta(B) = 2 \).
- \( d_1 = \dim H_2 - \dim H_1 = 5 - 4 = 1 \), since \( H_1 = \ker T_B \) has dimension 4 (there are 4 non-pivot columns of \( B \)).
- \( d_2 = \dim H_1 - \dim H_0 - d_1 = 4 - 0 - 1 = 3 \), since \( H_0 = \{0\} \).

Thus, \( A \) and \( B \) are not similar, by Corollary 2.3.8. However, since the matrix
\[
C =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]
has associated partition
\[
\pi(C) : 1^3 2,
\]
then we see that \( B \) is similar to \( C \), by Corollary 2.3.8.

Moreover, there are four 0-Jordan blocks of \( B \) (and \( C \)) - one of size 2 and three of size 1.