# Math 110, Fall 2015. Funny Fields

A **field** is a set $F$, together with two operations $+$, $\cdot$, called **addition** and **multiplication**, such that a bunch of axioms hold. In particular, there must exist two **distinct** elements $0, 1 \in F$ such that

$$\text{for every } x \in F, \quad 0 + x = x + 0 = x, \text{ and, } x \cdot 1 = 1 \cdot x = x.$$

Note that the elements $0, 1 \in F$ need not be the same 'numbers' you are used to thinking about when seeing these symbols.

Fields should be thought of as collections of objects, together with well-defined notions of addition and multiplication, such that all of the 'usual' laws of arithmetic hold (eg. for any nonzero $a \in F$, there is another element $b \in F$ such that $a \cdot b = b \cdot a = 1 \in F$).

Examples of fields are

$$\mathbb{Q} = \{\text{rational numbers}\}, \quad \mathbb{R} = \{\text{real numbers}\}, \quad \mathbb{C} = \{\text{complex numbers}\}.$$

Now, let's talk about some 'funny fields': consider the set of numbers appearing on a standard clockface, let's call this set

$$C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

If we consider 'clock addition' - for example, $7 + 10$ should be interpreted as 'add 10 hours to 7 o'clock', with answer $7 + 10 = 5$ - then you can see that, for any $x \in C$, we have

$$x + 12 = x.$$

Hence, the element 12 satisfies the conditions required of an element 0 in a 'field'. Thus, we can relabel $12 \leftrightarrow 0$, so we now have

$$C = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}.$$

We have simply relabelled the object 12 as the object 0 - we are not saying (for now) that there is any relation between the (genuine, real) numbers 0 and 12.

You can check (directly!) that 'clock addition' is commutative (ie, it doesn't matter if I add 10 hours to 5 o'clock, or 5 hours to 10 o'clock), associative (ie, it doesn't matter if I add 10 hours to 5 o'clock (to get 3 o'clock) and then add 4 hours to $(5 + 10)$ (this is $(5 + 10) + 4$), or if I add 4 hours to 10 o'clock (to get 2 o'clock) and then add 5 hours to $(10 + 4)$ (this is $5 + (10 + 4)$)), and that there exists 'clock additive' inverses (for any $x \in C$, if we let $y = 12 - x$ then $x + y = 0 \in C$).

In particular, we see that $11 + 1 = 0 \in C$, so that $11 \in C$ possesses the properties that we expect of '$-1$'. Thus, we could relabel $11 \leftrightarrow -1$, and so on... Of course, have only discussed 'clock addition', and would hope for things to work out well for 'clock multiplication'...

If we define 'clock multiplication' as follows:

$$\text{for any } a, b \in C, \quad a \cdot b \stackrel{def}{=} \overbrace{b + \ldots + b}^{a \text{ times}}$$

For example, $3 \cdot 7 = 7 + 7 + 7 = 2 + 7 = 9 \in C$ (recalling what 'clock addition' means).Then, $1 \in C$ is a multiplicative identity: for any $a \in C$ we have $1 \cdot a = a$ (by definition), and $a \cdot 1 = 1 + \ldots + 1 = a$.

However, $C$ is not a field: let's show that $2 \in C$ does not possess a multiplicative inverse. Suppose, in order to obtain a contradiction, $C$ is a field so that 2 does have an inverse, let's call it $x \in C$. Thus, we must have $2 \cdot x = 1 \in C$. Notice that $6 \cdot 2 = 2 + ... + 2 = 12 = 0 \in C$. Thus, we would have

$$6 = 6 \cdot 1 = 6 \cdot (2 \cdot x) \overset{(1)}{=} (6 \cdot 2) \cdot x = 0 \cdot x \overset{(2)}{=} 0,$$

which is absurd. Here we have used (1) associativity of multiplication, (2) the fact that $0 \cdot x = 0 \in C$, for any $x \in C$.

Now, there is nothing special about having a clock with 12 numbers on it... We could define an $n$-clock to be a clock with $n$ hours appearing, and define 'clock addition/multiplication' as above.

**Fact:** An $n$-clock (with clock addition/multiplication defined similarly to above) is a field if and only if $n$ is a prime number (ie $n = 2, 3, 5, 7, 11, ...$).

Consider the 5-clock $C_5 = \{0, 1, 2, 3, 4\}$. In order to understand the '5-clock addition/multiplication' we encode it in tables: the row $i$, column $j$ entry is the result of adding/mutiplying $i$ with $j$

| $+$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 0 |
| 2 | 2 | 3 | 4 | 0 | 1 |
| 3 | 3 | 4 | 0 | 1 | 2 |
| 4 | 4 | 0 | 1 | 2 | 3 |

| $\cdot$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 2 | 4 | 1 | 3 |
| 3 | 0 | 3 | 1 | 4 | 2 |
| 4 | 0 | 4 | 3 | 2 | 1 |

Note that $2 \cdot 3 = 1 \in C_5$, and $4 \cdot 4 = 1$, so that we could consider 2 as the multiplicative inverse of 3, and 4 is its own multiplicative inverse...!

**Remark:** Note that the entire discussion above is based on the (given) notions of clock addition/multiplication; this arithmetic is called **modular arithmetic**. It may be possible to define new (weirder) notions of addition/multiplication on the set of numbers appearing on an $n$-clock face, such that the resulting algebraic object **is/is not** a field!

Problem 1 on HW1 is asking you to prove that there is **precisely one notion of addition/multiplication** on $\{0, 1, x\}$ that gives rise to a field structure on $\{0, 1, x\}$.