Math 110, Fall 2015.

Homework 6, due Oct 7.

Prob 1. Let $V = \mathcal{P}_2(\mathbb{R})$ and suppose $\varphi_j(p) = p(j)$, $j = 0, 1, 2$. Prove that $(\varphi_0, \varphi_1, \varphi_2)$ is a basis for $\mathcal{P}_2(\mathbb{R})'$ and find a basis (p_0, p_1, p_2) of $\mathcal{P}_2(\mathbb{R})$ whose dual is $(\varphi_0, \varphi_1, \varphi_2)$.

Prob 2. Let V be a finite-dimensional vector space and let U be its proper subspace (i.e., $U \neq V$). Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for all $u \in U$ but $\varphi \neq 0$.

Prob 3. Let $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}) : p(x) \mapsto (x-1)^3 p(x) + p''(x)$.

(a) Let $\varphi \in \mathcal{P}(\mathbb{R})' : \varphi(p) = p'(1)$. Give a formula for $T'(\varphi)$.

(b) Let $\varphi \in \mathcal{P}(\mathbb{R})' : \varphi(p) = \int_0^1 p(x)\,dx$. Evaluate $T'(\varphi)(x^2)$.

Prob 4. Suppose V is finite-dimensional and U, W are its subpaces. Prove that

$$(U \cap W)^0 = U^0 + W^0.$$

Prob 5. Suppose V and W are finite-dimensional, $T \in \mathcal{L}(V, W)$, and $\operatorname{null} T' = \operatorname{span}(\varphi)$ for some $\varphi \in W'$. Prove that $\operatorname{range} T = \operatorname{null} \varphi$.