Math 110, Fall 2015.

Homework 3, due Sept 16.

Prob 1. Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : \int_{-2}^{2} p(x)\,dx = 0$.

(a) Find a basis for U.

(b) Extend your basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

Prob 2. Suppose $v_1, \ldots, v_m$ are linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 - w, v_2 - w, \ldots, v_m - w) \geq m - 1.$$

Prob 3. Does the 'inclusion-exclusion formula' hold for three subspaces, i.e., is it always true that

$$\begin{aligned}\dim(U_1 + U_2 + U_3) \quad = \quad & \dim(U_1) + \dim(U_2) + \dim(U_3) \\ & - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ & + \dim(U_1 \cap U_2 \cap U_3)?\end{aligned}$$

Prove this formula or provide a counterexample.

Prob 4. Let $a, b \in \mathbb{R}$. Define $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^2$ by

$$Tp := (2p(4) + 5p'(2) + ap(1)p(3), \int_{-1}^{2} x^3 p(x)\,dx + b \cos p(0)).$$

Show that T is linear if and only if $a = b = 0$.

Prob 5. Suppose $T \in \mathcal{L}(V, W)$, $v_1, \ldots, v_m \in T$ and the list/collection $Tv_1, Tv_2, \ldots Tv_m$ is linearly independent (in W). Prove that $v_1, \ldots, v_m$ must be linearly independent in V. What is the contrapositive of this statement?

Prob 6. Suppose V is a nonzero finite-dimensional vector space and W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.