Math 110, Fall 2015.

Homework 11, due Nov 13.

Prob 1. Let T be a self-adjoint operator on a finite-dimensional inner product space (real or complex) such that $\lambda_1, \lambda_2 \in \mathbb{R}$ are the only eigenvalues of T. Prove that $p(T) = 0$ where $p(\lambda) := (\lambda - \lambda_1)(\lambda - \lambda_2)$. Give a counterexample to this statement for an operator which is not self-adjoint.

Prob 2. Let T be a normal operator on a complex finite-dimensional inner product space V whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$. For any $v \in V$ such that $\|v\| = 1$, show that

$$\langle Tv, v \rangle = \sum_{j=1}^{k} a_j \lambda_j$$

for some nonnegative numbers a_j, $j = 1, \ldots, k$, that sum up to 1.

Prob 3. Let $T \in \mathcal{L}(V)$. Show that

$$\langle v, u \rangle_T := \langle Tv, u \rangle$$

is an inner product on V if and only if T is positive (per our definition of positivity).

Prob 4. We already know (how?) that the operator $T = -D^2$ is nonnegative on the space $V := \operatorname{span}(1, \cos x, \sin x)$ over $\mathbb{R}$, with the inner product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x) g(x) dx.$$

Find

(a) its square root operator $\sqrt{T}$;

(b) an example of a self-adjoint operator $R \neq \sqrt{T}$ such that $R^2 = T$;

(c) an example of a non-self-adjoint operator S such that $S^* S = T$.

Prob 5. Let T_1 and T_2 be normal operators on an n-dimensional inner product space V. Suppose both have n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that there is an isometry $S \in \mathcal{L}(V)$ such that $T_1 = S^* T_2 S$.

Prob 6. Find the singular values of the operator $T \in \mathcal{P}_2(\mathbb{C}) : p(x) \mapsto x p'(x) + 2x^2 p''(x)$ if the inner product on $\mathcal{P}_2(\mathbb{C})$ is defined as

$$\langle p, q \rangle := \int_{-1}^{1} p(x) \, \overline{q(x)} \, dx.$$