Math 110, Fall 2015.

Homework 10, due Nov 4.

Prob 1. Let $T \in \mathcal{L}(V, W)$. Prove

(a) T is injective if and only if T^* is surjective;

(b) T^* is injective if and only if T is surjective.

Prob 2. Suppose S, $T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

Prob 3. Let $P \in \mathcal{L}(V)$ be such that $P^2 = P$. Prove that there is a subspace U of V such that $P_U = P$ if and only if P is self-adjoint.

Prob 4. Let $n \in \mathbb{N}$ be fixed. Consider the real space $V := = \operatorname{span}(1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx)$ with inner product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Show that the differentiation operator $D \in \mathcal{L}(V)$ is *anti-Hermitian*, i.e., satisfies $D^* = -D$.

Prob 5. Let T be a normal operator on V. Evaluate $\|T(v - w)\|$ given that

$$Tv = 2v, \qquad Tw = 3w, \qquad \|v\| = \|w\| = 1.$$

Prob 6. Suppose T is normal. Prove that, for any $\lambda \in \mathbb{F}$,

$$\operatorname{Null}(T - \lambda I)^k = \operatorname{Null}(T - \lambda I).$$