Worksheet 12/2. Math 110, Fall 2015. Solution

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Finding Jordan form/basis

Throughout this worksheet V will always be a finite dimensional vector space over $F = \mathbb{C}$.

1. Let $T \in L(V)$, where $\dim V = 5$. Suppose that T admits exactly two distinct eigenvalues λ, μ. What are the allowed Jordan forms:

a) if $\dim(T - \lambda I) = 1, \dim(T - \mu I) = 1$?

b) if $\dim(T - \lambda I) = 2, \dim(T - \mu I) = 2$?

c) if $\dim(T - \lambda I) = 2, \dim(T - \mu I) = 1$?

Solution:

a) There is one λ-Jordan block A of size $k \times k$ and one μ-Jordan block of size $j \times j$. We need $k + j = 5$ and $1 \leq k, j \leq 4$. Thus, the allowed pairs are

$$(k, j) \in \{(1, 4), (2, 3), (3, 2), (4, 1)\}.$$

b) There are two λ-Jordan blocks of size $1 \leq k_1 \leq k_2$ and two μ-Jordan blocks of size $1 \leq j_1 \leq j_2$. Also, we must have $k_1 + k_2 + j_1 + j_2 = 5$, so that $k_1 + k_2 \leq 3$, and similarly $j_1 + j_2 \leq 3$. Hence, the allowed block sizes are

$$(k_1, k_2, j_1, j_2) \in \{(1, 1, 1, 2), (1, 2, 1, 1)\}.$$

c) There are two λ-Jordan blocks of size $1 \leq k_1 \leq k_2$ and one μ-Jordan block of size $1 \leq j_1$. We need $k_1 + k_2 + j_1 = 5$ so that $k_1 + k_2 \leq 4$. Hence, the allowed block sizes are

$$(k_1, k_2, j_1) \in \{(1, 3, 1), (2, 2, 1), (1, 2, 2), (1, 1, 3)\}.$$

2.

a) Give an example of an operator $T \in L(\mathbb{C}^6)$ with precisely three 2-Jordan blocks, one (-1)-Jordan block, and such that $\text{null}(T - 2)^2 = 4$.

b) Can there exist an operator $T \in L(\mathbb{C}^6)$ with precisely three 2-Jordan blocks, one (-1)-Jordan block, and such that $\text{null}(T - 2)^2 = 5$?

Solution:

a) As there are three 2-Jordan blocks this implies $\dim \text{null}(T - 2I) = 3$. We are also told that $\dim \text{null}(T - 2I)^2 = 4$ so that there must exist a 2-Jordan block of size $k \times k$, where

$k \geq 2$. So it suffices to give a 6×6 matrix J in Jordan form, containing two 1×1 2-Jordan blocks, one 2×2 2-Jordan block, and one 2×2 (-1)-Jordan block eg

$$J = \begin{bmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 2 & 1 & & \\ & & & 2 & & \\ & & & & -1 & 1 \\ & & & & & -1 \end{bmatrix}$$

Then, define $T : \mathbb{C}^6 \to \mathbb{C}^6$; $v \mapsto Jv$.

b) Yes. A similar analysis as the previous problem leads to

$$J = \begin{bmatrix} 2 & & & & & \\ & 2 & 1 & & & \\ & & 2 & & & \\ & & & 2 & 1 & \\ & & & & 2 & \\ & & & & & -1 \end{bmatrix}.$$

3. Determine the Jordan form of the following operators $T \in L(V)$.

a) $T : \mathbb{C}^3 \to \mathbb{C}^3$; $x \mapsto \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x$

b) $T : \mathbb{C}^4 \to \mathbb{C}^4$; $x \mapsto \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} x$

(*You may use without proof that* $\lambda = 1, -1$ *are the only eigenvalues of* T.)

Solution:

a) We have eigenvalues $\lambda = 1, 2$. The characteristic polynomial of T is $(x-1)(x-2)^2$ and the minimal polynomial is $(x-1)^a(x-2)^b$. The minimal polynomials divides the characteristic polynomial so that we must have $a = 1$. Hence, the size of the largest 1-Jordan block is 1×1. We have yet to determine b.

Compute $\dim \text{null}(T - I) = 1$ and $\dim \text{null}(T - 2) = 1$, by row-reducing $A - I$ and $A - 2I$, where A is the matrix defining T. Hence, there is one 2×2 2-Jordan block, one 1×1 1-Jordan block. The Jordan form is

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) We have eigenvalues $\lambda = -1, 1$. Compute $\dim \text{null}(T - I) = 1$ and $\dim \text{null}(T + 1) = 1$, by row-reducing $A - I$ and $A + I$, where A is the matrix defining T. Hence, there is one 1-Jordan block, one (-1)-Jordan block. Compute $\dim \text{null}(T + 1)^2 = 1 = \dim \text{null}(T + I)$, so that the largest (-1) Jordan block must be 1×1. The Jordan form is thus

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Find Jordan bases for the operators in Problem 3.

Solution:

a) We need a basis (v_1, v_2, v_3) such that

$$T(v_3) = v_3 \implies v_3 \in \text{null}(T - I),$$

$$T(v_2) = 2v_2 + v_1 \implies (T - 2I)(v_2) = v_1.$$

Hence, once we have determined v_2 we know v_1. Also, $(T - 2I)(v_1) = 0$. So,

$$(T - 2I)^2(v_2) = (T - 2I)(v_1) = 0 \implies v_2 \in \text{null}(T - 2I)^2.$$

As $v_2 \notin \text{null}(T - 2I)$ (because $(T - 2I)(v_2) = v_1 \neq 0$), we can choose any v_2 satisfying this property. Now,

$$(A - 2I)^2 = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $\text{null}(T - 2I)^2 = \text{span}(e_1, e_3)$, and $\text{null}(T - 2I) = \text{span}(e_1)$. Thus, take $v_2 = e_3$, so that $(T - 2I)(v_2) = -e_1 = v_1$. Also, $\text{null}(T - I) = \text{span}(-3e_1 + e_2)$, so that $v_3 = -3e_1 + e_2$ will work.

b) We need a basis (v_1, v_2, v_3, v_4) such that $v_1 \in \text{null}(T + I)$, and v_4 determines v_2, v_3: we have

$$(T - I)(v_4) = v_3, \quad (T - I)^2(v_4) = (T - I)(v_3) = v_2.$$

Further, we need $v_4 \in \text{null}(T - I)^3$ but $v_4 \notin \text{null}(T - I)^2$. We have

$$(A - I)^2 = \begin{bmatrix} 0 & 2 & -1 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad (A - I)^3 = \begin{bmatrix} 0 & -4 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, $\text{null}(T - I)^3 = \text{span}(e_1, e_3, e_4)$ and $\text{null}(T - I)^2 = \text{span}(e_1, e_4)$. So, we take $v_4 = e_3$. Then,

$$v_3 = (T - I)(v_4) = -e_4, \quad v_2 = (T - I)(v_3) = -e_1.$$

Also, we can take $v_1 = e_1 + 2e_2$.