## Worksheet 11/18. Math 110, Fall 2015. Solution

*These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!*

**Nilpotent Operators; Jordan Form**

*Throughout this worksheet $V$ will always be a finite dimensional vector space over $F = \mathbb{C}$.*

1.

a) Give two distinct examples of a operators $T_1, T_2 \in L(\mathbb{C}^3)$ such that $\dim G(2, T_1) = 2 = \dim G(2, T_2)$, and $\dim G(-1, T_1) = 1 = \dim G(-1, T_2)$.

b) Give an example of a non-diagonalisable operator $T \in L(\mathbb{C}^4)$ with distinct eigenvalues $\lambda = 1, -1$ such that $(e_1, e_1 + e_2)$ is a basis of $G(1, T)$ and $(e_3 - e_4, e_3 + e_4)$ is a basis of $G(-1, T)$.

c) Give an example of a nonzero nilpotent operator $T : \mathbb{C}^4 \to \mathbb{C}^4$ ; $v \mapsto Av$, where $A$ is <u>not</u> an upper-triangular matrix and $T^2 = 0$.

*Solution:*

a) To define the operators $T_i$ we need only give a $3 \times 3$ matrix $A_i$: then we set $T_i(x) = A_i x$. We let

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

b) Define $T \in L(\mathbb{C}^4)$ as follows:

$$T(e_1) = 2e_1 + e_2, \; T(e_1 + e_2) = e_1 + e_2, \; T(e_3 - e_4) = -e_3 + e_4, \; T(e_3 + e_4) = -e_3 - e_4.$$

$T$ is not diagonalisable since $(T - I)(e_1) \neq 0$, but $(T - I)^2(e_1) = 0$. So, $T$ admits generalised eigenvectors that are not eigenvectors.

c) Consider

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

2. Let $\dim V = 5$ and $T \in L(V)$. Suppose that $T$ has three distinct eigenvalues $\lambda, \mu, \nu$, and that $T$ is <u>not</u> diagonalisable. Prove that $T$ admits a generalised eigenvector that is not an eigenvector.

*Solution:* Since $T$ is not diagonalisable there is an eigenvalue, $\lambda$ say, such that $\text{null}(T - \lambda) \neq \text{null}(T - \lambda)^2$. Hence, $T$ admits a generalised eigenvector that is not an eigenvector.

3. Let $\dim V = 4$ and $T \in L(V)$ be such that $(T - 2)^2(T - 1) = 0 \in L(V)$. Assume further that $\dim G(2, T) = 2$. Prove that $T$ admits exactly two distinct eigenvalues.

*Solution:* The polynomial relation on $T$ gives that the allowed eigenvalues of $T$ are $\lambda = 1, 2$. Since we know that $G(2, T) \neq 0$, this gives that 2 is an eigenvalue of $T$. However, $V$ is a direct sum of its generalised eigenspaces, so that 1 must also be an eigenvalue.

4. Let $T \in L(V)$, $\dim V = 8$, be such that $\dim \text{null}\, T^4 = 8$.

a) Suppose that $\dim \operatorname{null} T^3 = 6$. Prove that there exists linearly independent vectors $(v, u) \subset V$ such that $B = (v, Tv, T^2v, T^3v, u, Tu, T^2u, T^3u)$ is linearly independent (and hence a basis!). Write down that matrix of $T$ with respect to $B$.

b) Is it possible that $\dim \operatorname{null} T^3 = 4$? Explain your answer.

*Solution:*

a) Let $W$ be 2-dimensional subspace such that $W \oplus \operatorname{null} T^3 = \operatorname{null} T^4$. Choose a basis $(u, v) \subset W$. Then, $T^4(u) = 0$, $T^4(v) = 0$. We claim that the desired list is linearly independent: suppose that there is a linear relation

$$a_0 u + a_1 T(u) + a_2 T^2(u) + a_3 T^3(u) + b_0 v + b_1 T(v) + b_2 T^2(v) + b_3 T^3(v) = 0.$$

Applying $T^3$ to this relation gives

$$a_0 T^3(u) + 0 + b_0 T^3(v) = 0 \implies a_0 u + b_0 v \in \operatorname{null} T^3 \cap W = \{0\},$$

so that $a_0 u + b_0 v = 0 \implies a_0 = b_0 = 0$. In a similar way, on can show that $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$.

The matrix has two $4 \times 4$ 0-Jordan blocks.

b) No: since $\operatorname{null} T^3 \neq \operatorname{null} T^2 \neq \operatorname{null} T$, the given dimension restriction on $\dim \operatorname{null} T^3$ would imply that $\dim \operatorname{null} T$ can be at most 2. Let $W \subset V$ be such that $W \oplus \operatorname{null} T^3 = \operatorname{null} T^4 = V$ (so that $\dim W = 4$). Take three linearly indpendent vectors $u, v, w \in W$. Then, $(T^3(u), T^3(v), T^3(w))$ is linearly independent: if we have a linear relation

$$0 = aT^3(u) + bT^3(v) + cT^3(w) \implies 0 = T^3(au + bv + cw)$$

and $au + bv + cw \in W \cap \operatorname{null} T^3 = \{0\}$, so that $a = b = c = 0$. Moreover, $T(T^3(u)) = T^4(u) = 0$ so that $T^3(u) \in \operatorname{null} T$. Similarly, $T^3(v), T^3(w) \in \operatorname{null} T$. This contradicts that $\dim \operatorname{null} T \leq 2$.

5. Prove or give a counterexample:

a) Let $\dim V = 6$, $T \in L(V)$. If $\dim G(0, T) = \dim G(1, T) = \dim G(2, T) = 2$ then $T$ is not diagonalisable.

b) Let $\dim V = 8$, $T \in L(V)$. If $\dim \operatorname{null} T^7 = 7$ then $T$ admits a nonzero eigenvalue.

c) Let $\dim V = 8$, $T \in L(V)$. If $\dim \operatorname{null} T^7 = 8$, $\dim \operatorname{null} T^6 = 7$ then there exists three linearly independent eigenvectors for $T$.

*Solution:*

a) False: let $T \in L(\mathbb{C}^6)$ be given by a $6 \times 6$ matrix with $0, 1, 2$ appearing on the diagonal exactly twice, and 0s everywhere else.

b) False: Take $T$ defined by an $8 \times 8$ matrix that is a single 0-Jordan block.

c) False: the information implies that $\operatorname{null} T^{i-1} \neq \operatorname{null} T^i$, for each $i \leq 7$. Hence, $\dim \operatorname{null} T$ can be at most 2.

6*. (*Harder*) Consider the operator

$$T : \mathbb{C}^4 \to \mathbb{C}^4 \; ; \; v \mapsto \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} v$$

Find a basis $(v_1, v_2, v_3, v_4)$ consisting of generalised eigenvectors of $T$, and such that

$$T(v_1) = v_1, \;\; T(v_2) = v_2 + v_1, \;\; T(v_3) = v_3, \;\; T(v_4) = 0.$$