Worksheet 11/18. Math 110, Fall 2015.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Nilpotent Operators; Jordan Form

Throughout this worksheet V will always be a finite dimensional vector space over $F = \mathbb{C}$.

1.

a) Give two distinct examples of a operators $T_1, T_2 \in L(\mathbb{C}^3)$ such that $\dim G(2, T_1) = 2 = \dim G(2, T_2)$, and $\dim G(-1, T_1) = 1 = \dim G(-1, T_2)$.

b) Give an example of a non-diagonalisable operator $T \in L(\mathbb{C}^4)$ with distinct eigenvalues $\lambda = 1, -1$ such that $(e_1, e_1 + e_2)$ is a basis of $G(1, T)$ and $(e_3 - e_4, e_3 + e_4)$ is a basis of $G(-1, T)$.

c) Give an example of a nonzero nilpotent operator $T : \mathbb{C}^4 \to \mathbb{C}^4 \, ; \, v \mapsto Av$, where A is <u>not</u> an upper-triangular matrix and $T^2 = 0$.

2. Let $\dim V = 5$ and $T \in L(V)$. Suppose that T has three distinct eigenvalues λ, μ, ν, and that T is <u>not</u> diagonalisable. Prove that T admits a generalised eigenvector that is not an eigenvector.

3. Let $\dim V = 4$ and $T \in L(V)$ be such that $(T - 2)^2(T - 1) = 0 \in L(V)$. Assume further that $\dim G(2, T) = 2$. Prove that T admits exactly two distinct eigenvalues.

4. Let $T \in L(V)$, $\dim V = 8$, be such that $\dim \operatorname{null} T^4 = 8$.

a) Suppose that $\dim \operatorname{null} T^3 = 6$. Prove that there exists linearly independent vectors $(v, u) \subset V$ such that $B = (v, Tv, T^2v, T^3v, u, Tu, T^2u, T^3u)$ is linearly independent (and hence a basis!). Write down that matrix of T with respect to B.

b) Is it possible that $\dim \operatorname{null} T^3 = 4$? Explain your answer.

5. Prove or give a counterexample:

a) Let $\dim V = 6$, $T \in L(V)$. If $\dim G(0, T) = \dim G(1, T) = \dim G(2, T) = 2$ then T is not diagonalisable.

b) Let $\dim V = 8$, $T \in L(V)$. If $\dim \operatorname{null} T^7 = 7$ then T admits a nonzero eigenvalue.

c) Let $\dim V = 8$, $T \in L(V)$. If $\dim \operatorname{null} T^7 = 8$, $\dim \operatorname{null} T^6 = 7$ then there exists three linearly independent eigenvectors for T.

6*. (*Harder*) Consider the operator

$$T : \mathbb{C}^4 \to \mathbb{C}^4 \, ; \, v \mapsto \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} v$$

Find a basis (v_1, v_2, v_3, v_4) consisting of generalised eigenvectors of T, and such that

$$T(v_1) = v_1, \;\; T(v_2) = v_2 + v_1, \;\; T(v_3) = v_3, \;\; T(v_4) = 0.$$